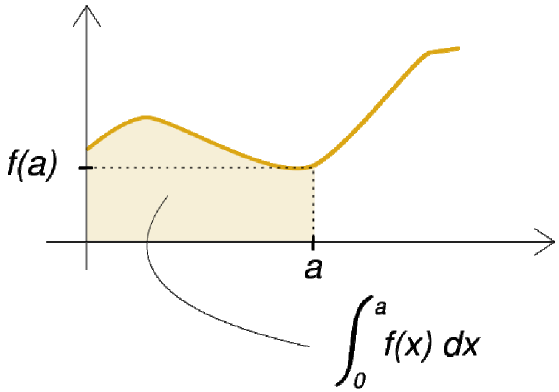
# Tutorial: Operations Research and Constraint Programming

John Hooker
Carnegie Mellon University

June 2008




# Why Integrate OR and CP?

Complementary strengths

Computational advantages

Outline of the Tutorial

# Complementary Strengths

- CP:
  - Inference methods
  - Modeling
  - Exploits local structure
- OR:
  - Relaxation methods
  - Duality theory
  - Exploits global structure

Let’s bring them together!

# Computational Advantage of Integrating CP and OR

Using CP + relaxation from MILP

| | *Problem* | *Speedup* |
|---|---|---|
| Focacci, Lodi, Milano (1999) | Lesson timetabling | 2 to 50 times faster than CP |
| Refalo (1999) | Piecewise linear costs | 2 to 200 times faster than MILP |
| Hooker & Osorio (1999) | Flow shop scheduling, etc. | 4 to 150 times faster than MILP. |
| Thorsteinsson & Ottosson (2001) | Product configuration | 30 to 40 times faster than CP, MILP |

# Computational Advantage of Integrating CP and MILP

## Using CP + relaxation from MILP

| | *Problem* | *Speedup* |
|---|---|---|
| Sellmann & Fahle (2001) | Automatic recording | 1 to 10 times faster than CP, MILP |
| Van Hoeve (2001) | Stable set problem | Better than CP in less time |
| Bollapragada, Ghattas & Hooker (2001) | Structural design (nonlinear) | Up to 600 times faster than MILP. 2 problems: <6 min vs >20 hrs for MILP |
| Beck & Refalo (2003) | Scheduling with earliness & tardiness costs | Solved 67 of 90, CP solved only 12 |

# Computational Advantage of Integrating CP and MILP

## Using CP-based Branch and Price

| | *Problem* | *Speedup* |
|---|---|---|
| Yunes, Moura & de Souza (1999) | Urban transit crew scheduling | Optimal schedule for 210 trips, vs. 120 for traditional branch and price |
| Easton, Nemhauser & Trick (2002) | Traveling tournament scheduling | First to solve 8-team instance |

# Computational Advantage of Integrating CP and MILP

## Using CP/MILP Benders methods

| | *Problem* | *Speedup* |
|---|---|---|
| Jain & Grossmann (2001) | Min-cost planning & scheduling | 20 to 1000 times faster than CP, MILP |
| Thorsteinsson (2001) | Min-cost planning & scheduling | 10 times faster than Jain & Grossmann |
| Timpe (2002) | Polypropylene batch scheduling at BASF | Solved previously insoluble problem in 10 min |

# Computational Advantage of Integrating CP and MILP

## Using CP/MILP Benders methods

| | *Problem* | *Speedup* |
|---|---|---|
| Benoist, Gaudin, Rottembourg (2002) | Call center scheduling | Solved twice as many instances as traditional Benders |
| Hooker (2004) | Min-cost, min-makespan planning & cumulative scheduling | 100-1000 times faster than CP, MILP |
| Hooker (2005) | Min tardiness planning & cumulative scheduling | 10-1000 times faster than CP, MILP |

# Outline of the Tutorial

- Why Integrate OR and CP?
- A Glimpse at CP
- Initial Example: Integrated Methods
- CP Concepts
- CP Filtering Algorithms
- Linear Relaxation and CP
- Mixed Integer/Linear Modeling
- Cutting Planes
- Lagrangean Relaxation and CP
- Dynamic Programming in CP
- CP-based Branch and Price
- CP-based Benders Decomposition

# Detailed Outline

- Why Integrate OR and CP?
  - Complementary strengths
  - Computational advantages
  - Outline of the tutorial
- A Glimpse at CP
  - Early successes
  - Advantages and disadvantages
- Initial Example: Integrated Methods
  - Freight Transfer
  - Bounds Propagation
  - Cutting Planes
  - Branch-infer-and-relax Tree

## Detailed Outline

- CP Concepts
  - Consistency
  - Hyperarc Consistency
  - Modeling Examples
- CP Filtering Algorithms
  - Element
  - Alldiff
  - Disjunctive Scheduling
  - Cumulative Scheduling
- Linear Relaxation and CP
  - Why relax?
  - Algebraic Analysis of LP
  - Linear Programming Duality
  - LP-Based Domain Filtering
  - Example: Single-Vehicle Routing
  - Disjunctions of Linear Systems

## Detailed Outline

- Mixed Integer/Linear Modeling
  - MILP Representability
  - 4.2 Disjunctive Modeling
  - 4.3 Knapsack Modeling
- Cutting Planes
  - 0-1 Knapsack Cuts
  - Gomory Cuts
  - Mixed Integer Rounding Cuts
  - Example: Product Configuration
- Lagrangean Relaxation and CP
  - Lagrangean Duality
  - Properties of the Lagrangean Dual
  - Example: Fast Linear Programming
  - Domain Filtering
  - Example: Continuous Global Optimization

## Detailed Outline

- Dynamic Programming in CP
  - Example: Capital Budgeting
  - Domain Filtering
  - Recursive Optimization
- CP-based Branch and Price
  - Basic Idea
  - Example: Airline Crew Scheduling
- CP-based Benders Decomposition
  - Benders Decomposition in the Abstract
  - Classical Benders Decomposition
  - Example: Machine Scheduling

# Background Reading

This tutorial is based on:

• J. N. Hooker, *Integrated Methods for Optimization*, Springer (2007). Contains 295 exercises.

• J. N. Hooker, Operations research methods in constraint programming, in F. Rossi, P. van Beek and T. Walsh, eds., *Handbook of Constraint Programming*, Elsevier (2006), pp. 527-570.

# A Glimpse at Constraint Programming

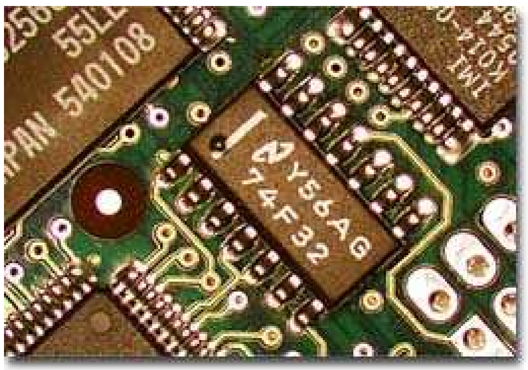
## Early Successes
## Advantages and Disadvantages

## What is constraint programming?

- It is a relatively new technology developed in the computer science and artificial intelligence communities.
- It has found an important role in scheduling, logistics and supply chain management.

# Early commercial successes

- Circuit design (Siemens)




- Container port scheduling (Hong Kong and Singapore)

- Real-time control (Siemens, Xerox)

# Applications


- Job shop scheduling
- Assembly line smoothing and balancing
- Cellular frequency assignment
- Nurse scheduling
- Shift planning
- Maintenance planning
- Airline crew rostering and scheduling
- Airport gate allocation and stand planning

## Applications

- Production scheduling
  - chemicals
  - aviation
  - oil refining
  - steel
  - lumber
  - photographic plates
  - tires
- Transport scheduling (food, nuclear fuel)
- Warehouse management
- Course timetabling

# Advantages and Disadvantages

## CP vs. Mathematical Programming

| MP | CP |
|---|---|
| Numerical calculation | Logic processing |
| Relaxation | Inference (filtering, constraint propagation) |
| Atomistic modeling (linear inequalities) | High-level modeling (global constraints) |
| Branching | Branching |
| Independence of model and algorithm | Constraint-based processing |

# Programming $\neq$ programming

- In **constraint programming**:
    - *programming* = a form of computer programming (constraint-based processing)

- In **mathematical programming**:
    - *programming* = logistics planning (historically)

# CP vs. MP

- In **mathematical programming**, equations (constraints) describe the problem but don't tell how to solve it.

- In **constraint programming**, each constraint invokes a procedure that screens out unacceptable solutions.
    - Much as each line of a computer program invokes an operation.

## Advantages of CP

- Better at sequencing and scheduling
    - …where MP methods have weak relaxations.
- Adding messy constraints makes the problem easier.
    - The more constraints, the better.
- More powerful modeling language.
    - Global constraints lead to succinct models.
    - Constraints convey problem structure to the solver.
- “Better at highly-constrained problems”
    - Misleading – better when constraints propagate well, or when constraints have few variables.

## Disdvantages of CP

- Weaker for continuous variables.
  - Due to lack of numerical techniques
- May fail when constraints contain many variables.
  - These constraints don't propagate well.
- Often not good for funding optimal solutions.
  - Due to lack of relaxation technology.
- May not scale up
  - Discrete combinatorial methods
- Software is not robust
  - Younger field

## Obvious solution…

- Integrate CP and MP.
    - More on this later.

## Trends

- CP is better known in continental Europe, Asia.
  - Less known in North America, seen as threat to OR.
- CP/MP integration is growing
  - Eclipse, Mozart, OPL Studio, SIMPL, SCIP, BARON
- Heuristic methods increasingly important in CP
  - Discrete combinatorial methods
- MP/CP/heuristics may become a single technology.

# Initial Example: Integrated Methods

Freight Transfer

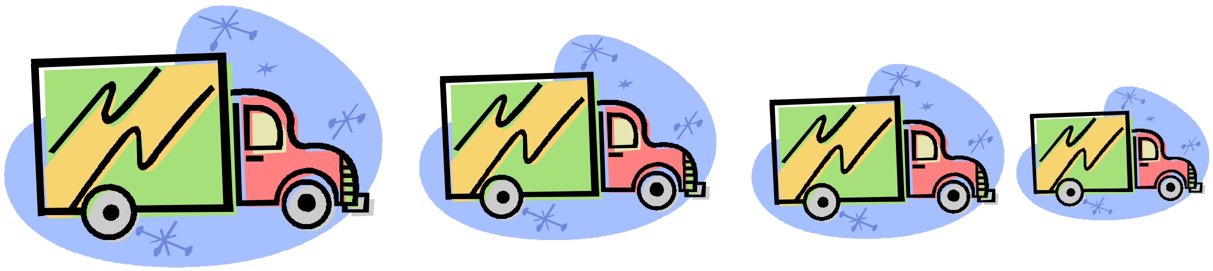
Bounds Propagation

Cutting Planes

Branch-infer-and-relax Tree

# Example: Freight Transfer

- Transport 42 tons of freight using 8 trucks, which come in 4 sizes…



| Truck size | Number available | Capacity (tons) | Cost per truck |
|---|---|---|---|
| 1 | 3 | 7 | 90 |
| 2 | 3 | 5 | 60 |
| 3 | 3 | 4 | 50 |
| 4 | 3 | 3 | 40 |

Number of trucks of type 1

$$\min\ 90x_1 + 60x_2 + 50x_3 + 40x_4$$
$$7x_1 + 5x_2 + 4x_3 + 3x_4 \geq 42$$
$$x_1 + x_2 + x_3 + x_4 \leq 8$$
$$x_i \in \{0,1,2,3\}$$

Knapsack covering constraint

Knapsack packing constraint

| Truck type | Number available | Capacity (tons) | Cost per truck |
|---|---|---|---|
| 1 | 3 | 7 | 90 |
| 2 | 3 | 5 | 60 |
| 3 | 3 | 4 | 50 |
| 4 | 3 | 3 | 40 |

# Bounds propagation

$$\min\ 90x_1 + 60x_2 + 50x_3 + 40x_4$$
$$7x_1 + 5x_2 + 4x_3 + 3x_4 \geq 42$$
$$x_1 + x_2 + x_3 + x_4 \leq 8$$
$$x_i \in \{0,1,2,3\}$$

$$x_1 \geq \left\lceil \frac{42 - 5\cdot 3 - 4\cdot 3 - 3\cdot 3}{7} \right\rceil = 1$$

## Bounds propagation

$$\min\ 90x_1 + 60x_2 + 50x_3 + 40x_4$$
$$7x_1 + 5x_2 + 4x_3 + 3x_4 \geq 42$$
$$x_1 + x_2 + x_3 + x_4 \leq 8$$
$$x_1 \in \{1,2,3\}, \quad x_2, x_3, x_4 \in \{0,1,2,3\}$$

Reduced domain

$$x_1 \geq \left\lceil \frac{42 - 5\cdot 3 - 4\cdot 3 - 3\cdot 3}{7} \right\rceil = 1$$

## Bounds consistency

- Let $\{L_j, \ldots, U_j\}$ be the domain of $x_j$
- A constraint set is **bounds consistent** if for each $j$ :
  - $x_j = L_j$ in some feasible solution and
  - $x_j = U_j$ in some feasible solution.
- Bounds consistency $\Rightarrow$ we will not set $x_j$ to any infeasible values during branching.
- Bounds propagation achieves bounds consistency for a **single inequality**.
  - $7x_1 + 5x_2 + 4x_3 + 3x_4 \geq 42$ is bounds consistent when the domains are $x_1 \in \{1,2,3\}$ and $x_2, x_3, x_4 \in \{0,1,2,3\}$.
- But not necessarily for a **set** of inequalities.

## Bounds consistency

- Bounds propagation may not achieve bounds consistency for a set of constraints.
- Consider set of inequalities

$$x_1 + x_2 \geq 1$$
$$x_1 - x_2 \geq 0$$

with domains $x_1$, $x_2 \in \{0,1\}$, solutions $(x_1,x_2) = (1,0), (1,1)$.

- Bounds propagation has no effect on the domains.
- But constraint set is not bounds consistent because $x_1 = 0$ in no feasible solution.

# Cutting Planes

## Begin with continuous relaxation

$$
\begin{aligned}
&\min\ 90x_1 + 60x_2 + 50x_3 + 40x_4 \\
&7x_1 + 5x_2 + 4x_3 + 3x_4 \geq 42 \\
&x_1 + x_2 + x_3 + x_4 \leq 8 \\
&0 \leq x_i \leq 3, \quad x_1 \geq 1
\end{aligned}
$$

Replace domains with bounds

This is a linear programming problem, which is easy to solve.

Its optimal value provides a lower bound on optimal value of original problem.

## Cutting planes (valid inequalities)

$$
\begin{aligned}
&\min\ 90x_1 + 60x_2 + 50x_3 + 40x_4 \\
&7x_1 + 5x_2 + 4x_3 + 3x_4 \geq 42 \\
&x_1 + x_2 + x_3 + x_4 \leq 8 \\
&0 \leq x_i \leq 3, \quad x_1 \geq 1
\end{aligned}
$$

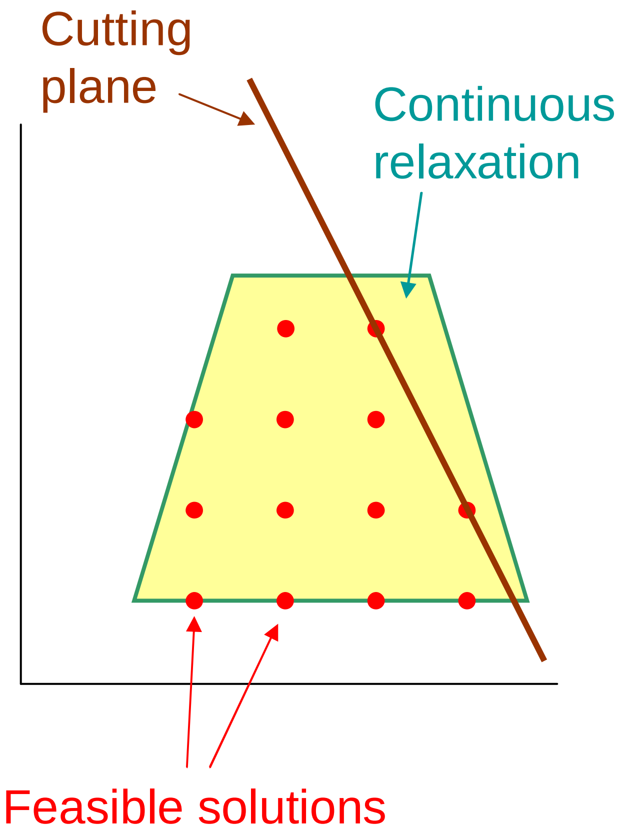
We can create a **tighter** relaxation (larger minimum value) with the addition of **cutting planes**.

# Cutting planes (valid inequalities)

$$\min\ 90x_1 + 60x_2 + 50x_3 + 40x_4$$
$$7x_1 + 5x_2 + 4x_3 + 3x_4 \geq 42$$
$$x_1 + x_2 + x_3 + x_4 \leq 8$$
$$0 \leq x_i \leq 3, \quad x_1 \geq 1$$

All feasible solutions of the original problem satisfy a cutting plane (i.e., it is **valid**).

But a cutting plane may exclude ("**cut off**") solutions of the continuous relaxation.



# Cutting planes (valid inequalities)

$$\min\ 90x_1 + 60x_2 + 50x_3 + 40x_4$$
$$7x_1 + 5x_2 + 4x_3 + 3x_4 \geq 42$$
$$x_1 + x_2 + x_3 + x_4 \leq 8$$
$$0 \leq x_i \leq 3, \quad x_1 \geq 1$$

{1,2} is a **packing**

…because $7x_1 + 5x_2$ alone cannot satisfy the inequality, even with $x_1 = x_2 = 3$.

# Cutting planes (valid inequalities)

$$\min\ 90x_1 + 60x_2 + 50x_3 + 40x_4$$
$$7x_1 + 5x_2 + 4x_3 + 3x_4 \geq 42$$
$$x_1 + x_2 + x_3 + x_4 \leq 8$$
$$0 \leq x_i \leq 3, \quad x_1 \geq 1$$

{1,2} is a **packing**

So, $$4x_3 + 3x_4 \geq 42 - (7\cdot 3 + 5\cdot 3)$$

**Knapsack cut**

which implies

$$x_3 + x_4 \geq \left\lceil \frac{42 - (7\cdot 3 + 5\cdot 3)}{\max\{4,3\}} \right\rceil = 2$$

# Cutting planes (valid inequalities)

Let $x_i$ have domain $[L_i,U_i]$ and let $a \geq 0$.

In general, a **packing** $P$ for $ax \geq a_0$ satisfies

$$\sum_{i \notin P} a_i x_i \geq a_0 - \sum_{i \in P} a_i U_i$$

and generates a **knapsack cut**

$$\sum_{i \notin P} x_i \geq \left\lceil \frac{a_0 - \sum_{i \in P} a_i U_i}{\max_{i \notin P}\{a_i\}} \right\rceil$$

## Cutting planes (valid inequalities)

$$
\begin{aligned}
&\min\ 90x_1 + 60x_2 + 50x_3 + 40x_4 \\
&7x_1 + 5x_2 + 4x_3 + 3x_4 \geq 42 \\
&x_1 + x_2 + x_3 + x_4 \leq 8 \\
&0 \leq x_i \leq 3, \quad x_1 \geq 1
\end{aligned}
$$

| Maximal Packings | **Knapsack cuts** |
|---|---|
| {1,2} | $x_3 + x_4 \geq 2$ |
| {1,3} | $x_2 + x_4 \geq 2$ |
| {1,4} | $x_2 + x_3 \geq 3$ |

Knapsack cuts corresponding to nonmaximal packings can be nonredundant.

## Continuous relaxation with cuts

$$\min\ 90x_1 + 60x_2 + 50x_3 + 40x_4$$
$$7x_1 + 5x_2 + 4x_3 + 3x_4 \geq 42$$
$$x_1 + x_2 + x_3 + x_4 \leq 8$$
$$0 \leq x_i \leq 3, \quad x_1 \geq 1$$
$$\boxed{\begin{aligned} x_3 + x_4 &\geq 2 \\ x_2 + x_4 &\geq 2 \\ x_2 + x_3 &\geq 3 \end{aligned}}$$

Knapsack cuts

Optimal value of 523.3 is a lower bound on optimal value of original problem.

# Branch-infer-and-relax tree

Propagate bounds and solve relaxation of original problem.

$x_1 \in \{\ \ 123\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (2⅓,3,2⅔,0)$
value = 523⅓

# Branch-infer-and-relax tree

Branch on a variable with nonintegral value in the relaxation.

$x_1 \in \{\ \ 123\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (2\tfrac{1}{3}, 3, 2\tfrac{2}{3}, 0)$
value = $523\tfrac{1}{3}$

$x_1 \in \{1,2\}$ $\quad$ $x_1 = 3$

# Branch-infer-and-relax tree

Propagate bounds and solve relaxation.

Since relaxation is infeasible, backtrack.

$x_1 \in \{\ \ 123\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (2⅓,3,2⅔,0)$
value = 523⅓

$x_1 \in \{1,2\}$

$x_1 \in \{\ 12\ \}$
$x_2 \in \{\ \ \ 23\}$
$x_3 \in \{\ 123\}$
$x_4 \in \{\ 123\}$
infeasible relaxation

$x_1 = 3$

# Branch-infer-and-relax tree

Propagate bounds and solve relaxation.

Branch on nonintegral variable.

$x_1 \in \{\ \ 123\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (2⅓,3,2⅔,0)$
value = 523⅓

$x_1 \in \{1,2\}$ | $x_1 = 3$

$x_1 \in \{\ \ 12\ \ \}$
$x_2 \in \{\ \ \ 23\}$
$x_3 \in \{\ \ 123\}$
$x_4 \in \{\ \ 123\}$
infeasible relaxation

$x_1 \in \{\ \ \ \ \ 3\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (3,2.6,2,0)$
value = 526

$x_2 \in \{0,1,2\}$ | $x_2 = 3$

# Branch-infer-and-relax tree

Branch again.

$x_1 \in \{\ \ 123\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (2\frac{1}{3},3,2\frac{2}{3},0)$
value = $523\frac{1}{3}$

$x_1 \in \{1,2\}$ | $x_1 = 3$

$x_1 \in \{\ \ 12\ \ \}$
$x_2 \in \{\ \ \ \ 23\}$
$x_3 \in \{\ \ 123\}$
$x_4 \in \{\ \ 123\}$
infeasible relaxation

$x_1 \in \{\ \ \ \ \ \ 3\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (3,2.6,2,0)$
value = 526

$x_2 \in \{0,1,2\}$ | $x_2 = 3$

$x_1 \in \{\ \ \ \ \ \ 3\}$
$x_2 \in \{012\ \ \}$
$x_3 \in \{\ \ 123\}$
$x_4 \in \{0123\}$
$x = (3,2,2\frac{3}{4},0)$
value = $527\frac{1}{2}$

$x_3 \in \{1,2\}$ | $x_3 = 3$

# Branch-infer-and-relax tree

Solution of relaxation is integral and therefore feasible in the original problem.

This becomes the **incumbent solution**.

$x_1 \in \{\ 123\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (2⅓,3,2⅔,0)$
value = 523⅓

$x_1 \in \{1,2\}$

$x_1 \in \{\ 12\ \}$
$x_2 \in \{\ \ 23\}$
$x_3 \in \{\ 123\}$
$x_4 \in \{\ 123\}$
infeasible relaxation

$x_1 = 3$

$x_1 \in \{\ \ \ 3\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (3,2.6,2,0)$
value = 526

$x_2 \in \{0,1,2\}$

$x_1 \in \{\ \ \ 3\}$
$x_2 \in \{012\ \}$
$x_3 \in \{\ 123\}$
$x_4 \in \{0123\}$
$x = (3,2,2¾,0)$
value = 527½

$x_3 \in \{1,2\}$

$x_1 \in \{\ \ \ 3\}$
$x_2 \in \{\ 12\ \}$
$x_3 \in \{\ 12\ \}$
$x_4 \in \{\ 123\}$
$x = (3,2,2,1)$
value = 530
feasible solution

$x_3 = 3$

$x_2 = 3$

# Branch-infer-and-relax tree

Solution is nonintegral, but we can backtrack because value of relaxation is no better than incumbent solution.

$x_1 \in \{\ \ 123\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (2\tfrac{1}{3},3,2\tfrac{2}{3},0)$
value = $523\tfrac{1}{3}$

$x_1 \in \{1,2\}$

$x_1 \in \{\ 12\ \}$
$x_2 \in \{\ \ 23\}$
$x_3 \in \{\ 123\}$
$x_4 \in \{\ 123\}$
infeasible relaxation

$x_1 = 3$

$x_1 \in \{\ \ \ \ 3\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (3,2.6,2,0)$
value = 526

$x_2 \in \{0,1,2\}$

$x_1 \in \{\ \ \ \ 3\}$
$x_2 \in \{012\ \}$
$x_3 \in \{\ 123\}$
$x_4 \in \{0123\}$
$x = (3,2,2\tfrac{3}{4},0)$
value = $527\tfrac{1}{2}$

$x_2 = 3$

$x_3 \in \{1,2\}$

$x_1 \in \{\ \ \ \ 3\}$
$x_2 \in \{\ 12\ \}$
$x_3 \in \{\ 12\ \}$
$x_4 \in \{\ 123\}$
$x = (3,2,2,1)$
value = 530
feasible solution

$x_3 = 3$

$x_1 \in \{\ \ \ \ 3\}$
$x_2 \in \{012\ \}$
$x_3 \in \{\ \ \ \ 3\}$
$x_4 \in \{012\ \}$
$x = (3,1\tfrac{1}{2},3,\tfrac{1}{2})$
value = 530
backtrack due to bound

# Branch-infer-and-relax tree

Another feasible solution found.

No better than incumbent solution, which is optimal because search has finished.

$x_1 \in \{\ 123\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (2⅓,3,2⅔,0)$
value = 523⅓

$x_1 \in \{1,2\}$

$x_1 \in \{\ 12\ \}$
$x_2 \in \{\ \ 23\}$
$x_3 \in \{\ 123\}$
$x_4 \in \{\ 123\}$
infeasible relaxation

$x_1 = 3$

$x_1 \in \{\ \ \ 3\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (3,2.6,2,0)$
value = 526

$x_2 \in \{0,1,2\}$

$x_1 \in \{\ \ \ 3\}$
$x_2 \in \{012\ \}$
$x_3 \in \{\ 123\}$
$x_4 \in \{0123\}$
$x = (3,2,2¾,0)$
value = 527½

$x_3 \in \{1,2\}$

$x_1 \in \{\ \ \ 3\}$
$x_2 \in \{\ 12\ \}$
$x_3 \in \{\ 12\ \}$
$x_4 \in \{\ 123\}$
$x = (3,2,2,1)$
value = 530
feasible solution

$x_3 = 3$

$x_1 \in \{\ \ \ 3\}$
$x_2 \in \{012\ \}$
$x_3 \in \{\ \ \ 3\}$
$x_4 \in \{012\ \}$
$x = (3,1½,3,½)$
value = 530
backtrack due to bound

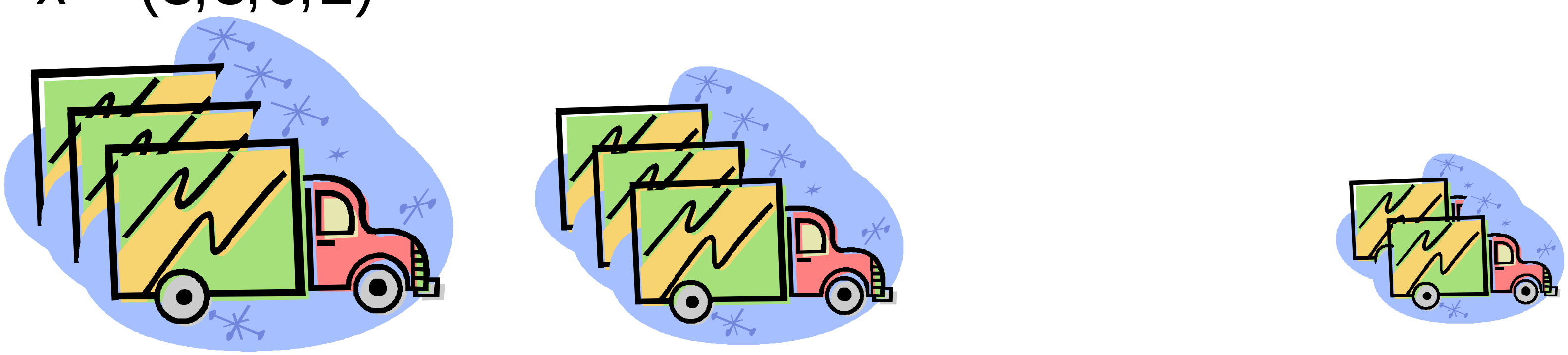
$x_2 = 3$

$x_1 \in \{\ \ \ 3\}$
$x_2 \in \{\ \ \ 3\}$
$x_3 \in \{012\ \}$
$x_4 \in \{012\ \}$
$x = (3,3,0,2)$
value = 530
feasible solution

## Two optimal solutions…

$$x = (3,2,2,1)$$

$$x = (3,3,0,2)$$



# Constraint Programming Concepts

Consistency

Hyperarc Consistency

Modeling Examples

# Consistency

- A constraint set is **consistent** if every partial assignment to the variables that violates no constraint is feasible.
    - i.e., can be extended to a feasible solution.
- Consistency $\neq$ feasibility
    - Consistency means that any infeasible partial assignment is explicitly ruled out by a constraint.
- Fully consistent constraint sets can be solved **without backtracking**.

## Consistency

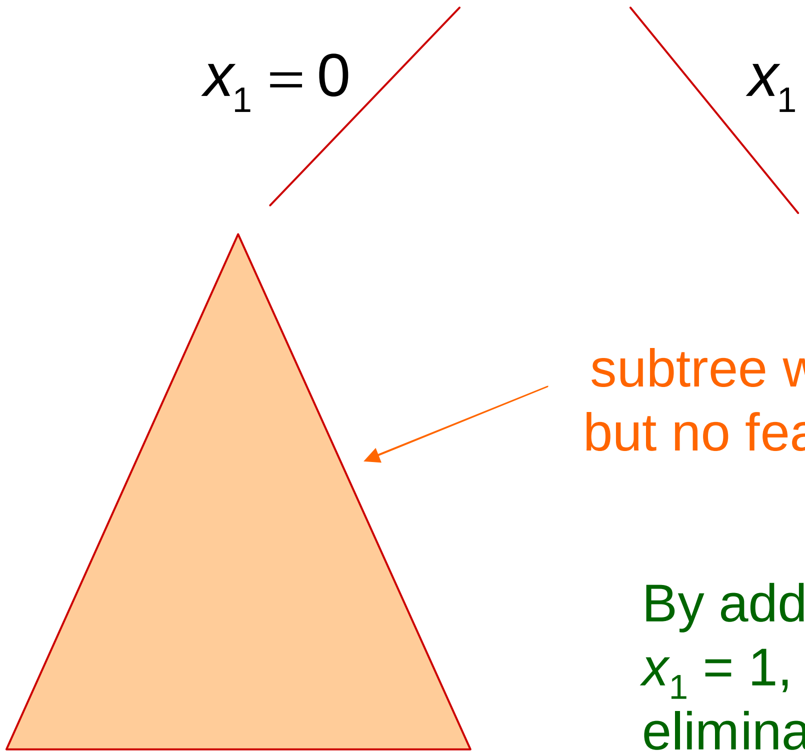
Consider the constraint set

$$
\begin{aligned}
& x_1 + x_{100} \geq 1 \\
& x_1 - x_{100} \geq 0 \\
& x_j \in \{0,1\}
\end{aligned}
$$

It is not consistent, because $x_1 = 0$ violates no constraint and yet is infeasible (no solution has $x_1 = 0$).

Adding the constraint $x_1 = 1$ makes the set consistent.

$$x_1 + x_{100} \geq 1$$
$$x_1 - x_{100} \geq 1$$
other constraints
$$x_j \in \{0,1\}$$



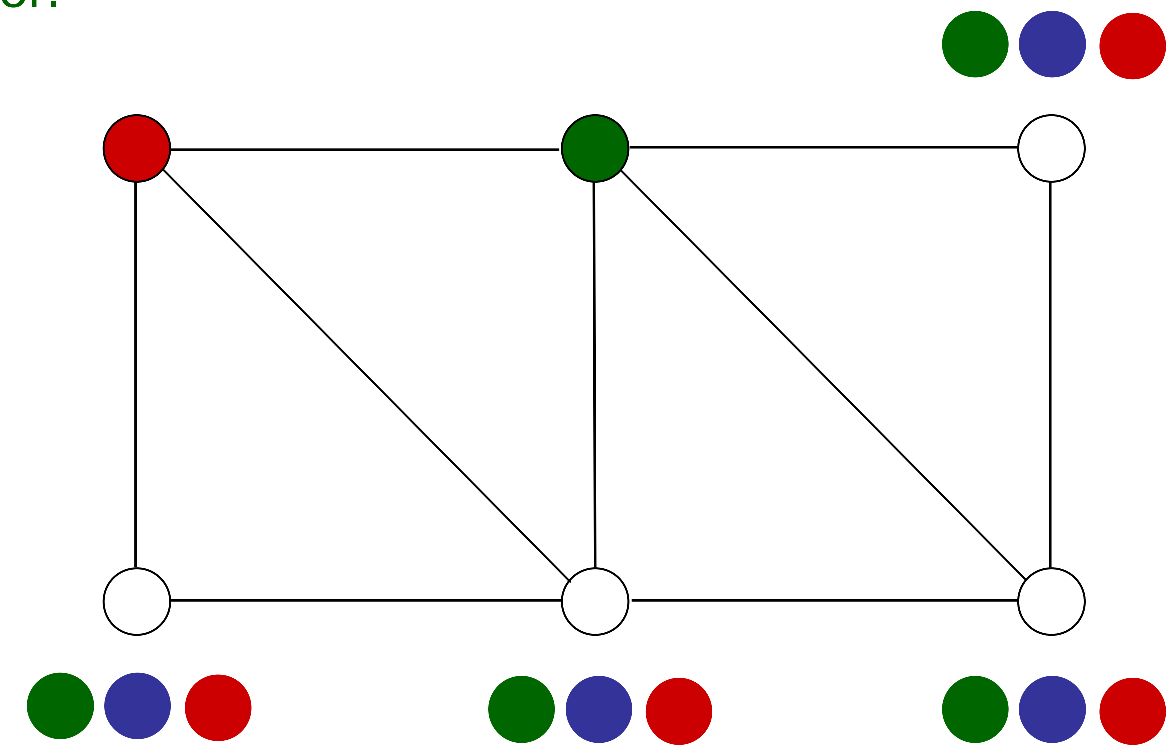

subtree with $2^{99}$ nodes
but no feasible solution

By adding the constraint
$x_1$ = 1, the left subtree is
eliminated

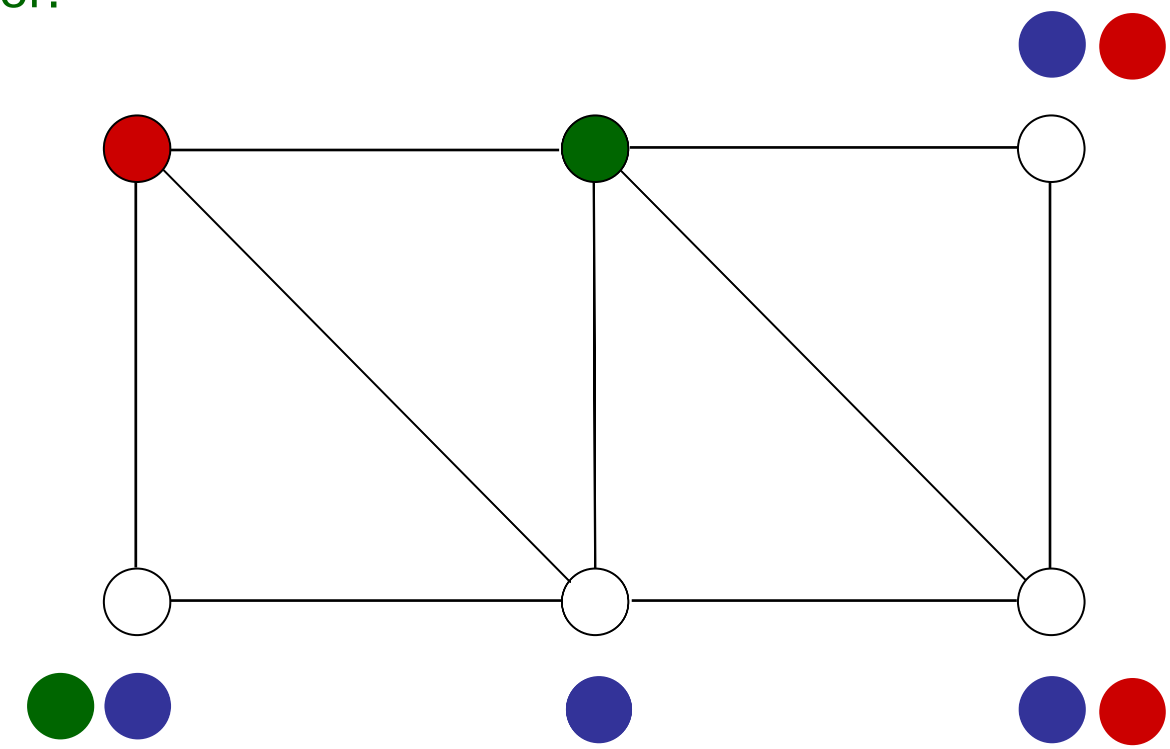
# Hyperarc Consistency

- Also known as **generalized arc consistency**.
- A constraint set is **hyperarc consistent** if every value in every variable domain is part of some feasible solution.
    - That is, the domains are reduced as much as possible.
    - If all constraints are "binary" (contain 2 variables), hyperarc consistent = arc consistent.
    - Domain reduction is CP's biggest engine.

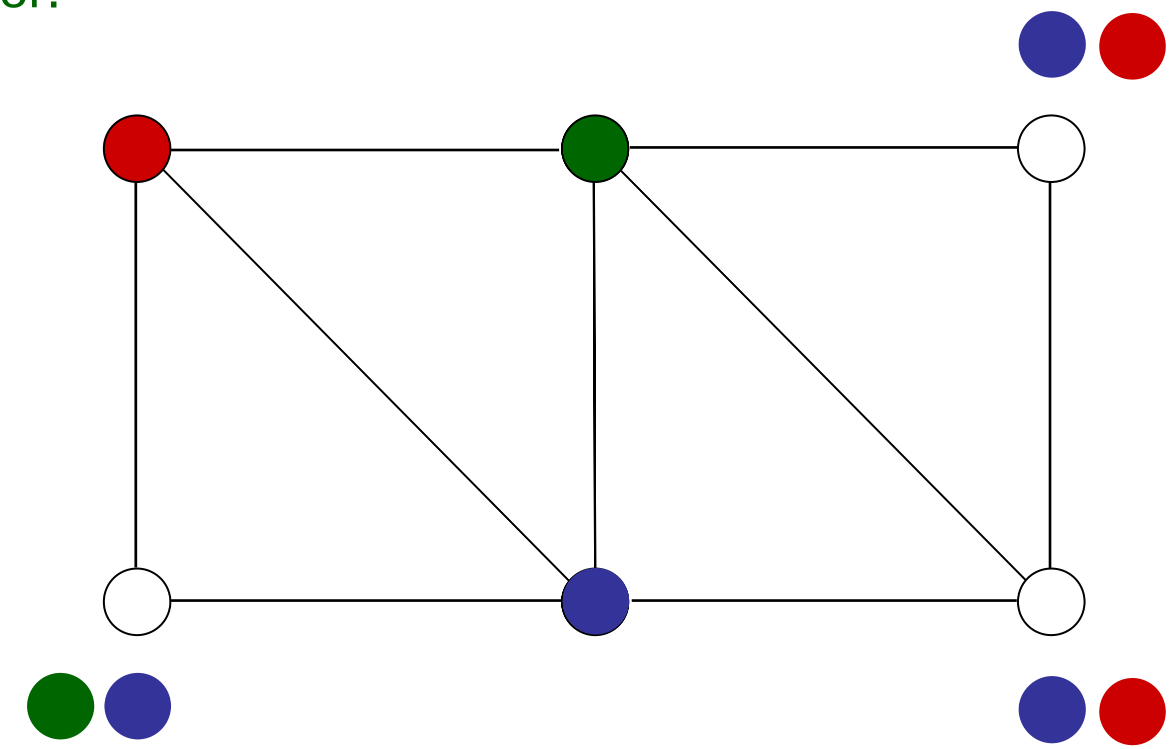
Graph coloring problem that can be solved by arc consistency maintenance alone. Color nodes with red, green, blue with no two adjacent nodes having the same color.



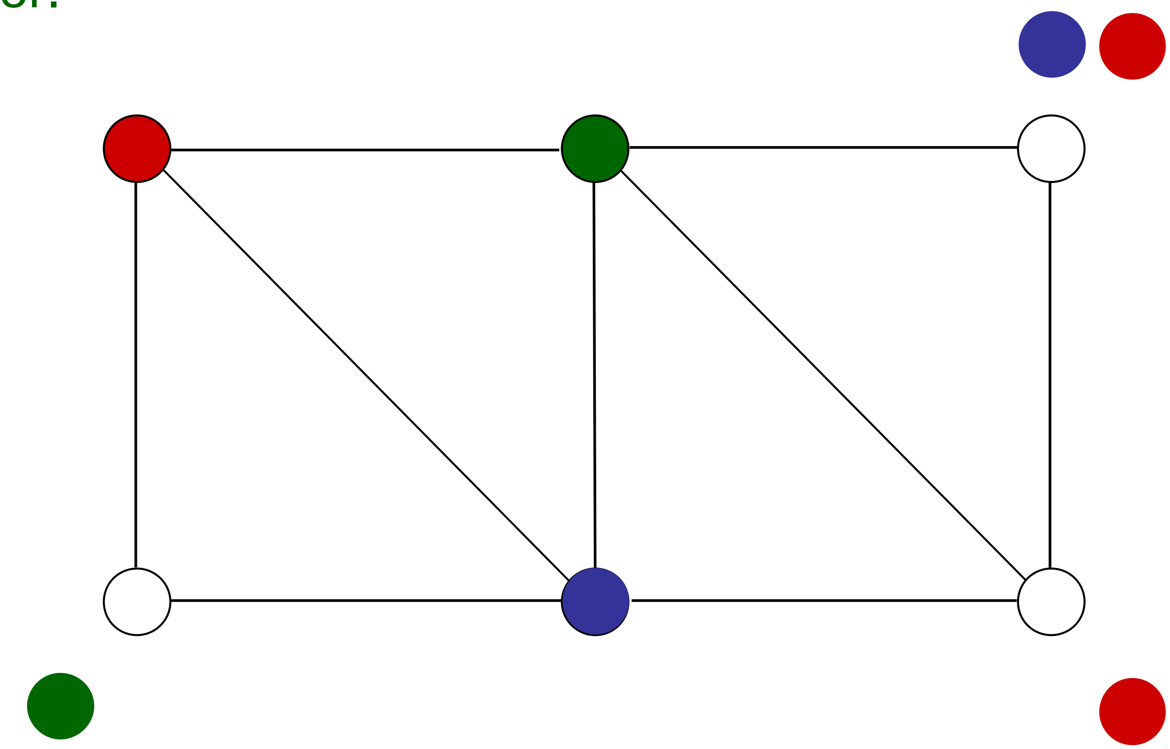
Graph coloring problem that can be solved by arc consistency maintenance alone. Color nodes with red, green, blue with no two adjacent nodes having the same color.



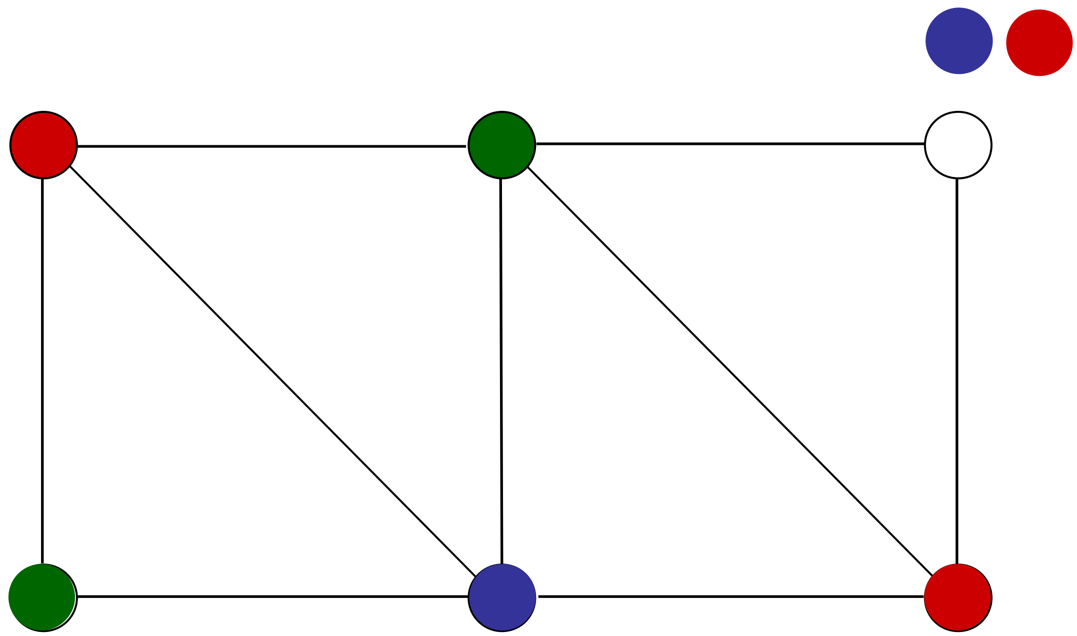
Graph coloring problem that can be solved by arc consistency maintenance alone. Color nodes with red, green, blue with no two adjacent nodes having the same color.



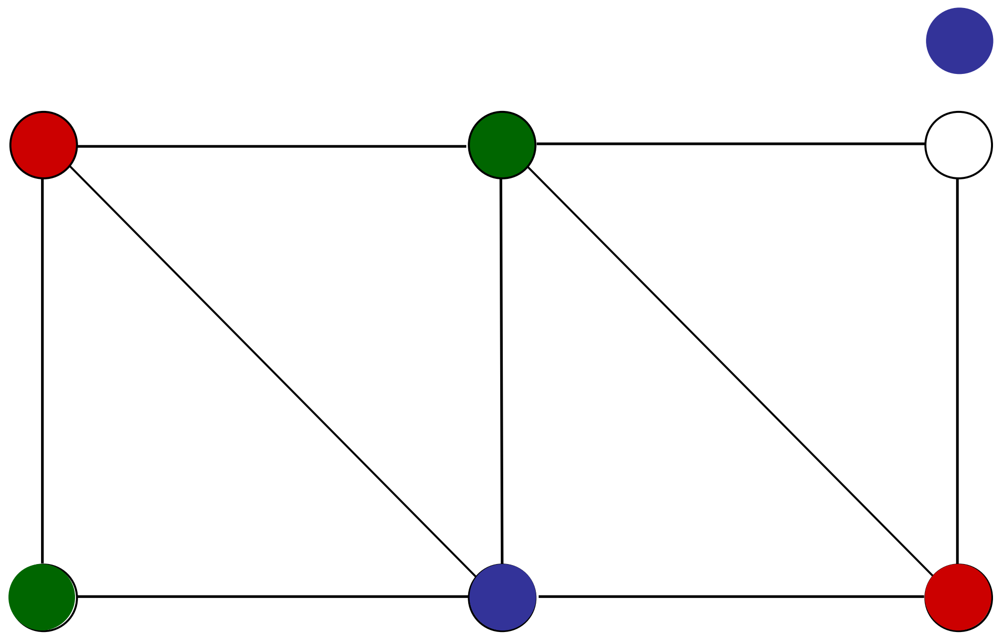
Graph coloring problem that can be solved by arc consistency maintenance alone. Color nodes with red, green, blue with no two adjacent nodes having the same color.



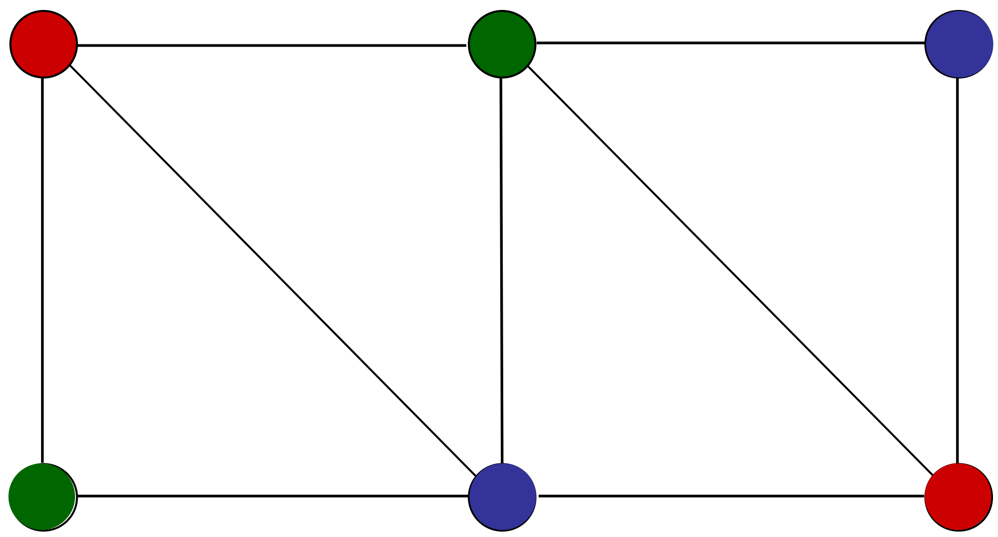
Graph coloring problem that can be solved by arc consistency maintenance alone. Color nodes with red, green, blue with no two adjacent nodes having the same color.



Graph coloring problem that can be solved by arc consistency maintenance alone. Color nodes with red, green, blue with no two adjacent nodes having the same color.



Graph coloring problem that can be solved by arc consistency maintenance alone. Color nodes with red, green, blue with no two adjacent nodes having the same color.



# Modeling Examples with Global Constraints

## Traveling Salesman

Traveling salesman problem:

Let $c_{ij}$ = distance from city $i$ to city $j$.

Find the shortest route that visits each of $n$ cities exactly once.

## Popular 0-1 model

Let $x_{ij}$ = 1 if city $i$ immediately precedes city $j$, 0 otherwise

$$
\begin{aligned}
\min \quad & \sum_{ij} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{i} x_{ij} = 1, \quad \text{all } j \\
& \sum_{j} x_{ij} = 1, \quad \text{all } i \\
& \sum_{i \in V} \sum_{j \in W} x_{ij} \geq 1, \quad \text{all disjoint } V, W \subset \{1, \ldots, n\} \\
& x_{ij} \in \{0, 1\}
\end{aligned}
$$

Subtour elimination constraints

# A CP model

Let $y_k$ = the $k$th city visited.

The model would be written in a specific constraint programming language but would essentially say:

Variable indices

$$\begin{aligned} \min \quad & \sum_k c_{y_k y_{k+1}} \\ \text{s.t.} \quad & \text{alldiff}(y_1,\ldots,y_n) \\ & y_k \in \{1,\ldots,n\} \end{aligned}$$

"Global" constraint

# An alternate CP model

Let $y_k$ = the city visited after city $k$.

$$\begin{array}{ll} \min & \sum_k c_{k y_k} \\ \text{s.t.} & \text{circuit}(y_1,\ldots,y_n) \\ & y_k \in \{1,\ldots,n\} \end{array}$$

Hamiltonian circuit constraint

# Element constraint

The constraint $c_y \leq 5$ can be implemented:

$$z \leq 5$$
$$\text{element}\big(y,(c_1,\ldots,c_n),z\big)$$

Assign $z$ the $y$th value in the list

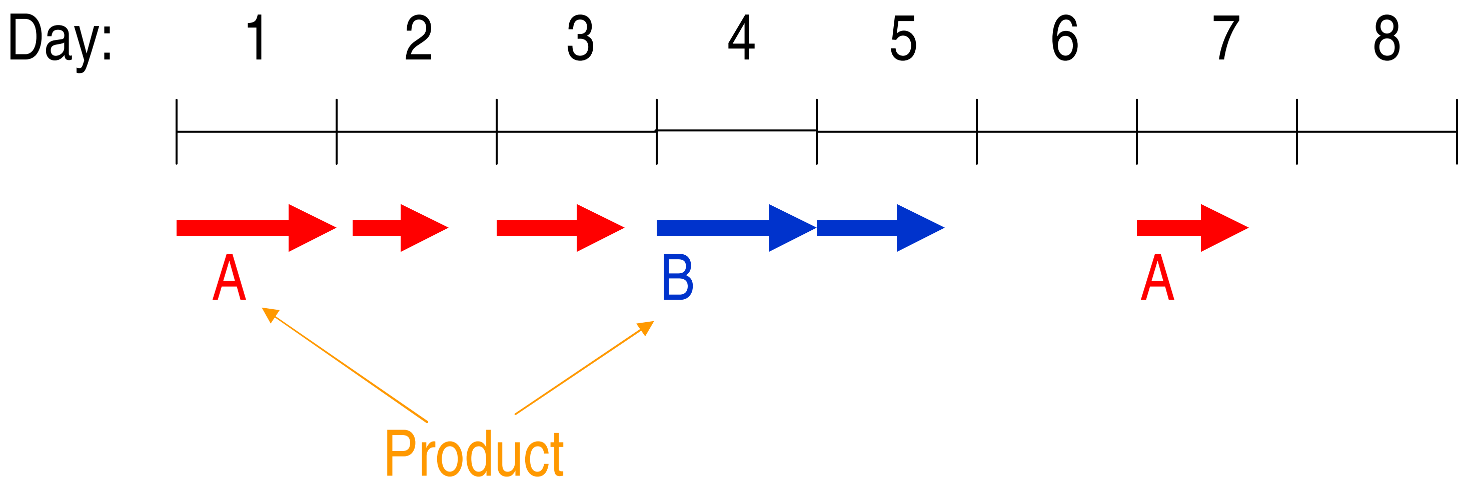
The constraint $x_y \leq 5$ can be implemented

$$z \leq 5$$
$$\text{element}\big(y,(x_1,\ldots,x_n),z\big)$$

Add the constraint $z = x_y$

(this is a slightly different constraint)

## Modeling example: Lot sizing and scheduling




- At most one product manufactured on each day.
- Demands for each product on each day.
- Minimize setup + holding cost.

Integer programming model

*(Wolsey)*

$$\min \sum_{t,i} \left( h_{it} s_{it} + \sum_{j \neq t} q_{ij} \delta_{ijt} \right)$$

Many variables

$$\begin{aligned}
\text{s.t.} \quad & s_{i,t-1} + x_{it} = d_{it} + s_{it}, \quad \text{all } i,t \\
& z_{it} \geq y_{it} - y_{i,t-1}, \quad \text{all } i,t \\
& z_{it} \leq y_{it}, \quad \text{all } i,t \\
& z_{it} \leq 1 - y_{i,t-1}, \quad \text{all } i,t \\
& \delta_{ijt} \geq y_{i,t-1} + y_{jt} - 1, \quad \text{all } i,j,t \\
& \delta_{ijt} \geq y_{i,t-1}, \quad \text{all } i,j,t \\
& \delta_{ijt} \geq y_{jt}, \quad \text{all } i,j,t \\
& x_{it} \leq C y_{it}, \quad \text{all } i,t \\
& \sum_i y_{it} = 1, \quad \text{all } t \\
& y_{it}, z_{it}, \delta_{ijt} \in \{0,1\} \\
& x_{it}, s_{it} \geq 0
\end{aligned}$$

## CP model

Minimize holding and setup costs

$$\min \quad \sum_t \left( q_{y_{t-1}y_t} + \sum_i h_i s_{it} \right)$$

Inventory balance

Production capacity

$$\text{s.t.} \quad s_{i,t-1} + x_{it} = d_{it} + s_{it}, \quad \text{all } i,t$$

$$0 \le x_{it} \le C, \quad s_{it} \ge 0, \quad \text{all } i,t$$

$$(y_t \ne i) \rightarrow (x_{it} = 0), \quad \text{all } i,t$$

## CP model

Minimize holding and setup costs

Variable indices

$$\min \quad \sum_t \left( q_{y_{t-1}y_t} + \sum_i h_i s_{it} \right)$$

Inventory balance

$$\text{s.t.} \quad s_{i,t-1} + x_{it} = d_{it} + s_{it}, \quad \text{all } i,t$$

Production capacity

$$0 \le x_{it} \le C, \quad s_{it} \ge 0, \quad \text{all } i,t$$

$$(y_t \ne i) \rightarrow (x_{it} = 0), \quad \text{all } i,t$$

Production level of product *i* in period *t*

Product manufactured in period *t*

## Cumulative scheduling constraint

- Used for resource-constrained scheduling.
- Total resources consumed by jobs at any one time must not exceed $L$.

$$\text{cumulative}\big((t_1,\ldots,t_n),(p_1,\ldots,p_n),(c_1,\ldots,c_n),L\big)$$

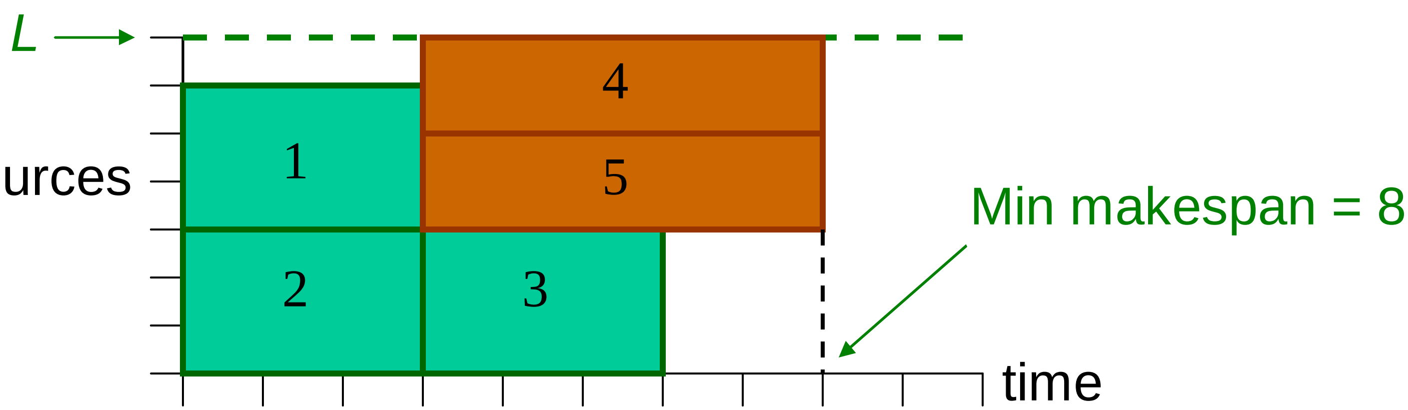
Job start times (variables)

Job processing times

Job resource requirements

## Cumulative scheduling constraint

Minimize makespan (no deadlines, all release times = 0):




L

$$
\begin{aligned}
\text{min} \quad & z \\
\text{s.t.} \quad & \text{cumulative}\big((t_1,\ldots,t_5),(3,3,3,5,5),(3,3,3,2,2),7\big) \\
& z \geq t_1 + 3 \\
& \vdots \\
& z \geq t_5 + 2
\end{aligned}
$$

Resources used

Processing times

Job start times

## Modeling example: Ship loading

- Will use ILOG's OPL Studio modeling language.
  - Example is from OPL manual.
- The problem
  - Load 34 items on the ship in minimum time (min makespan)
  - Each item requires a certain time and certain number of workers.
  - Total of 8 workers available.

## Problem data

| Item | Duration | Labor |
|---|---|---|
| 1 | 3 | 4 |
| 2 | 4 | 4 |
| 3 | 4 | 3 |
| 4 | 6 | 4 |
| 5 | 5 | 5 |
| 6 | 2 | 5 |
| 7 | 3 | 4 |
| 8 | 4 | 3 |
| 9 | 3 | 4 |
| 10 | 2 | 8 |
| 11 | 3 | 4 |
| 12 | 2 | 5 |
| 13 | 1 | 4 |
| 14 | 5 | 3 |
| 15 | 2 | 3 |
| 16 | 3 | 3 |
| 17 | 2 | 6 |

| Item | Duration | Labor |
|---|---|---|
| 18 | 2 | 7 |
| 19 | 1 | 4 |
| 20 | 1 | 4 |
| 21 | 1 | 4 |
| 22 | 2 | 4 |
| 23 | 4 | 7 |
| 24 | 5 | 8 |
| 25 | 2 | 8 |
| 26 | 1 | 3 |
| 27 | 1 | 3 |
| 28 | 2 | 6 |
| 29 | 1 | 8 |
| 30 | 3 | 3 |
| 31 | 2 | 3 |
| 32 | 1 | 3 |
| 33 | 2 | 3 |
| 34 | 2 | 3 |

## Precedence constraints

1 → 2,4
2 →3
3 →5,7
4 →5
5 →6
6 →8
7 →8
8 →9
9 →10
9 →14
10 →11
10 →12

11 →13
12 →13
13 →15,16
14 →15
15 →18
16 →17
17 →18
18 →19
18 →20,21
19 →23
20 → 23
21 → 22

22 →23
23 →24
24 →25
25 →26,30,31,32
26 → 27
27 → 28
28 → 29
30 → 28
31 → 28
32 → 33
33 → 34

Use the cumulative scheduling constraint.

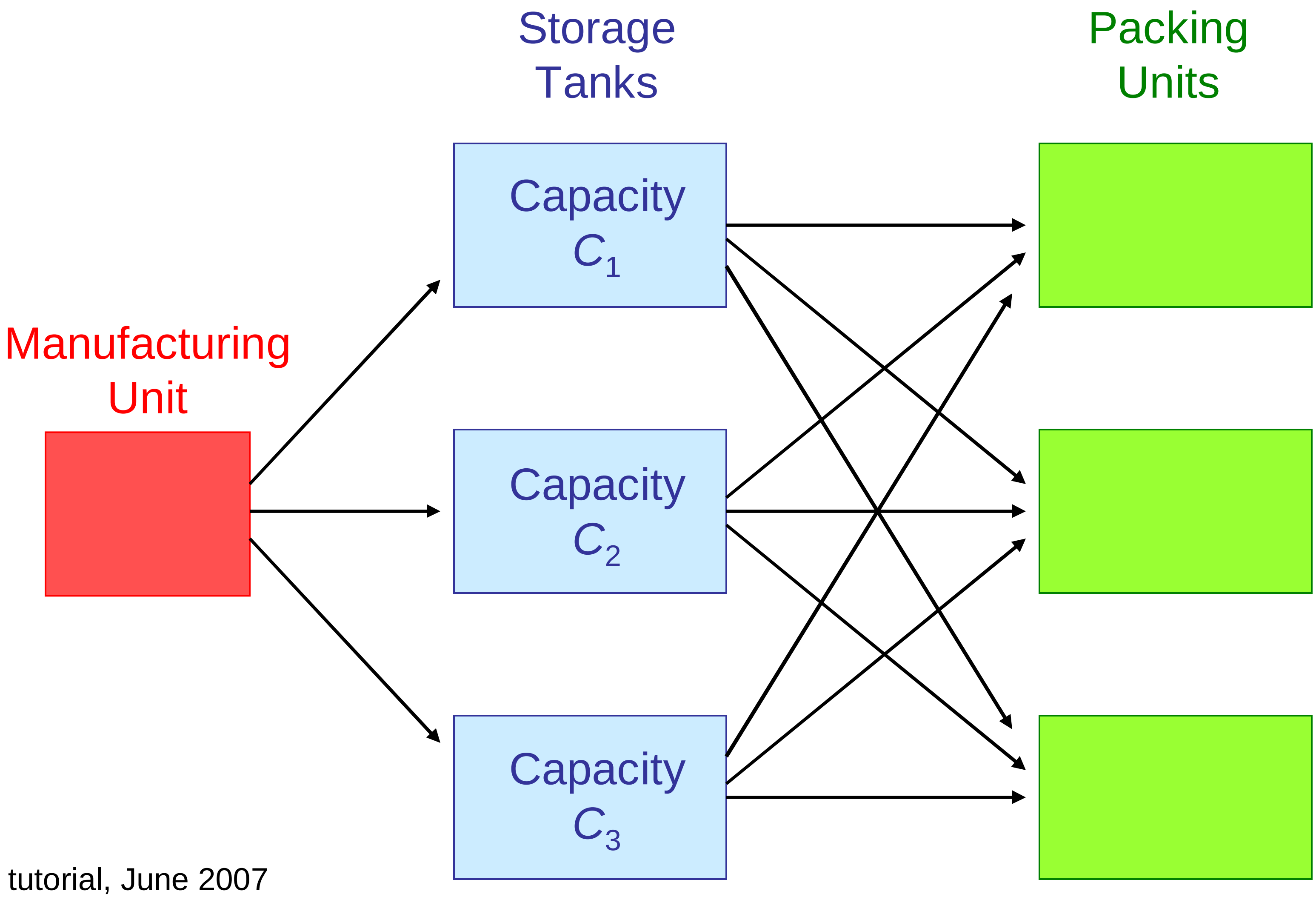
$$\begin{array}{ll} \min & z \\ \text{s.t.} & z \geq t_1 + 3, \quad z \geq t_2 + 4, \quad \text{etc.} \\ & \text{cumulative}\big((t_1,\ldots,t_{34}),(3,4,\ldots,2),(4,4,\ldots,3),8\big) \\ & t_2 \geq t_1 + 3, \quad t_4 \geq t_1 + 3, \quad \text{etc.} \end{array}$$

## OPL model

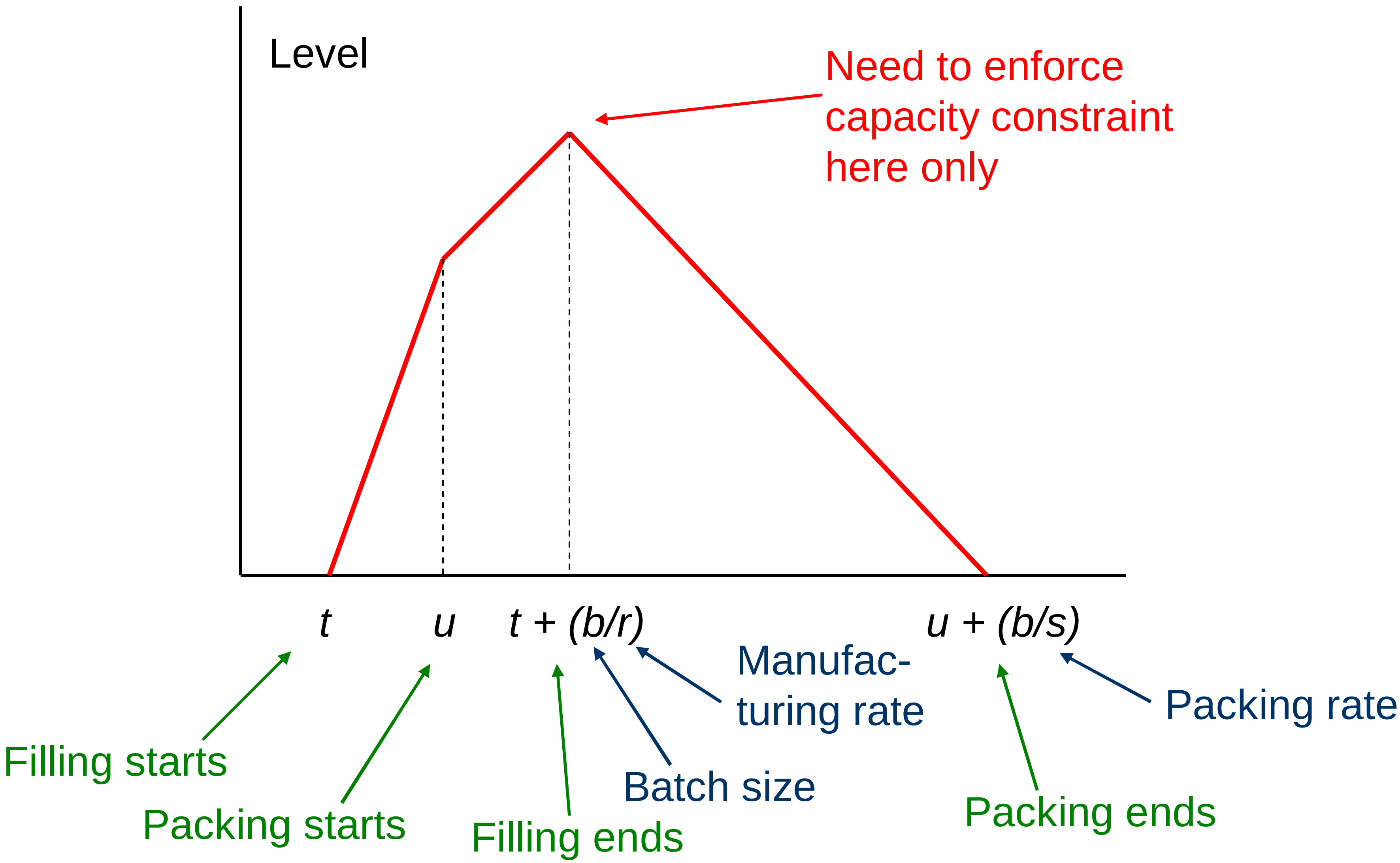
```
int capacity = 8;
int nbTasks = 34;
range Tasks 1..nbTasks;
int duration[Tasks] = [3,4,4,6,…,2];
int totalDuration =
      sum(t in Tasks) duration[t];
int demand[Tasks] = [4,4,3,4,…,3];
struct Precedences {
   int before;
   int after;

}
{Precedences} setOfPrecedences = {
      <1,2>, <1,4>, …, <33,34> };
```

```
scheduleHorizon = totalDuration;
Activity a[t in Tasks](duration[t]);
DiscreteResource res(8);
Activity makespan(0);
minimize
   makespan.end
subject to
   forall(t in Tasks)
      a[t] precedes makespan;
   forall(p in setOfPrecedences)
      a[p.before] precedes a[p.after];
   forall(t in Tasks)
      a[t] requires(demand[t]) res;
};
```

# Modeling example: Production scheduling with intermediate storage



## Filling of storage tank



$$\min \quad T$$ Makespan

$$\text{s.t.} \quad T \ge u_j + \frac{b_j}{s_j}, \quad \text{all } j$$

$$t_j \ge R_j, \quad \text{all } j$$ Job release time

$$\text{cumulative}(t, v, e, m)$$ $m$ storage tanks

$$v_i = u_i + \frac{b_i}{s_i} - t_i, \quad \text{all } i$$ Job duration

$$b_i\left(1 - \frac{s_i}{r_i}\right) + s_i u_i \le C_i, \quad \text{all } i$$ Tank capacity

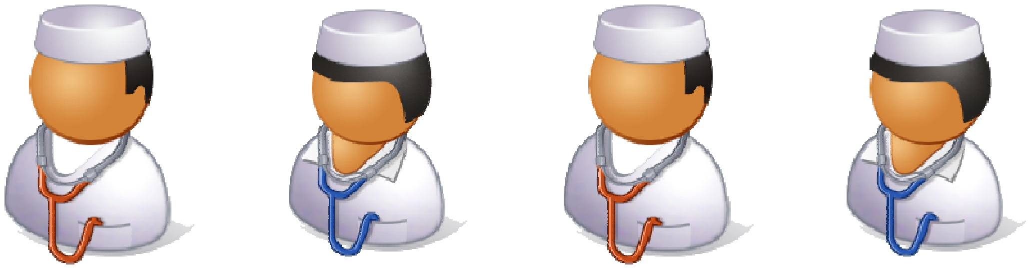
$$\text{cumulative}\left(u, \left(\frac{b_1}{s_1}, \ldots, \frac{b_n}{s_n}\right), e, p\right)$$ $p$ packing units

$$u_j \ge t_j \ge 0$$

$e = (1,\ldots,1)$

## Modeling example: Employee scheduling

- Schedule four nurses in 8-hour shifts.
- A nurse works at most one shift a day, at least 5 days a week.
- Same schedule every week.
- No shift staffed by more than two different nurses in a week.
- A nurse cannot work different shifts on two consecutive days.
- A nurse who works shift 2 or 3 must do so at least two days in a row.



# Two ways to view the problem

## Assign nurses to shifts

| | Sun | Mon | Tue | Wed | Thu | Fri | Sat |
|---|---|---|---|---|---|---|---|
| Shift 1 | A | B | A | A | A | A | A |
| Shift 2 | C | C | C | B | B | B | B |
| Shift 3 | D | D | D | D | C | C | D |

## Assign shifts to nurses

| | Sun | Mon | Tue | Wed | Thu | Fri | Sat |
|---|---|---|---|---|---|---|---|
| Nurse A | 1 | 0 | 1 | 1 | 1 | 1 | 1 |
| Nurse B | 0 | 1 | 0 | 2 | 2 | 2 | 2 |
| Nurse C | 2 | 2 | 2 | 0 | 3 | 3 | 0 |
| Nurse D | 3 | 3 | 3 | 3 | 0 | 0 | 3 |

0 = day off

Use **both** formulations in the same model!

First, assign nurses to shifts.

Let $w_{sd}$ = nurse assigned to shift $s$ on day $d$

$$\text{alldiff}(w_{1d}, w_{2d}, w_{3d}), \quad \text{all } d$$

The variables $w_{1d}$, $w_{2d}$, $w_{3d}$ take different values

That is, schedule 3 different nurses on each day

Use **both** formulations in the same model!

First, assign nurses to shifts.

Let $w_{sd}$ = nurse assigned to shift $s$ on day $d$

$$\text{alldiff}(w_{1d}, w_{2d}, w_{3d}), \quad \text{all } d$$

$$\text{cardinality}\big(w \mid (A, B, C, D), (5,5,5,5), (6,6,6,6)\big)$$

$A$ occurs at least 5 and at most 6 times in the array $w$, and similarly for B, C, D.

That is, each nurse works at least 5 and at most 6 days a week

Use **both** formulations in the same model!

First, assign nurses to shifts.

Let $w_{sd}$ = nurse assigned to shift $s$ on day $d$

$$\text{alldiff}\left(w_{1d}, w_{2d}, w_{3d}\right), \quad \text{all } d$$

$$\text{cardinality}\left(w \mid (A,B,C,D),(5,5,5,5),(6,6,6,6)\right)$$

$$\text{nvalues}\left(w_{s,\text{Sun}},\ldots,w_{s,\text{Sat}} \mid 1,2\right), \quad \text{all } s$$

The variables $w_{s,\text{Sun}}$, …, $w_{s,\text{Sat}}$ take at least 1 and at most 2 different values.

That is, at least 1 and at most 2 nurses work any given shift.

Remaining constraints are not easily expressed in this notation.

So, assign shifts to nurses.

Let $y_{id}$ = shift assigned to nurse *i* on day *d*

$$\text{alldiff}\left(y_{1d}, y_{2d}, y_{3d}\right),\ \ \text{all } d$$

Assign a different nurse to each shift on each day.

This constraint is redundant of previous constraints, but redundant constraints speed solution.

Remaining constraints are not easily expressed in this notation.

So, assign shifts to nurses.

Let $y_{id}$ = shift assigned to nurse $i$ on day $d$

$$\text{alldiff}\left(y_{1d}, y_{2d}, y_{3d}\right),\ \ \text{all } d$$
$$\text{stretch}\left(y_{i,\text{Sun}}, \ldots, y_{i,\text{Sat}} \mid (2,3),(2,2),(6,6),P\right),\ \ \text{all } i$$

Every stretch of 2's has length between 2 and 6.
Every stretch of 3's has length between 2 and 6.

So a nurse who works shift 2 or 3 must do so at least two days in a row.

Remaining constraints are not easily expressed in this notation.

So, assign shifts to nurses.

Let $y_{id}$ = shift assigned to nurse *i* on day *d*

$$\text{alldiff}\left(y_{1d}, y_{2d}, y_{3d}\right), \text{ all } d$$
$$\text{stretch}\left(y_{i,\text{Sun}}, \ldots, y_{i,\text{Sat}} \mid (2,3),(2,2),(6,6),P\right), \text{ all } i$$

Here $P = \{(s,0),(0,s) \mid s = 1,2,3\}$

Whenever a stretch of *a*'s immediately precedes a stretch of *b*'s, (*a*,*b*) must be one of the pairs in *P*.

So a nurse cannot switch shifts without taking at least one day off.

Now we must connect the $w_{sd}$ variables to the $y_{id}$ variables.

Use **channeling constraints**:

$$w_{y_{id}d} = i, \quad \text{all } i,d$$

$$y_{w_{sd}d} = s, \quad \text{all } s,d$$

Channeling constraints increase propagation and make the problem easier to solve.

The complete model is:

$$\text{alldiff}\left(w_{1d}, w_{2d}, w_{3d}\right), \ \ \text{all } d$$
$$\text{cardinality}\left(w \mid (A,B,C,D),(5,5,5,5),(6,6,6,6)\right)$$
$$\text{nvalues}\left(w_{s,\text{Sun}},\ldots,w_{s,\text{Sat}} \mid 1,2\right), \ \ \text{all } s$$

$$\text{alldiff}\left(y_{1d}, y_{2d}, y_{3d}\right), \ \ \text{all } d$$
$$\text{stretch}\left(y_{i,\text{Sun}},\ldots,y_{i,\text{Sat}} \mid (2,3),(2,2),(6,6),P\right), \ \ \text{all } i$$

$$w_{y_{id}d} = i, \ \ \text{all } i,d$$
$$y_{w_{sd}d} = s, \ \ \text{all } s,d$$

# CP Filtering Algorithms

Element

Alldiff

Disjunctive Scheduling

Cumulative Scheduling

# Filtering for element

$$\mathrm{element}\left(y,(x_1,\ldots,x_n),z\right)$$

Variable domains can be easily filtered to maintain hvperarc consistency.

Domain of $z$

$$D_z \leftarrow D_z \cap \bigcup_{j \in D_y} D_{x_j}$$

$$D_y \leftarrow D_y \cap \left\{ j \mid D_z \cap D_{x_j} \neq \varnothing \right\}$$

$$D_{x_j} \leftarrow \left\{ \begin{matrix} D_z & \text{if } D_y = \{j\} \\ D_{x_j} & \text{otherwise} \end{matrix} \right\}$$

## Filtering for element

Example... $\text{element}\left(y,(x_1,x_2,x_3,x_4),z\right)$

The initial domains are:

$$D_z = \{20,30,60,80,90\}$$
$$D_y = \{1,3,4\}$$
$$D_{x_1} = \{10,50\}$$
$$D_{x_2} = \{10,20\}$$
$$D_{x_3} = \{40,50,80,90\}$$
$$D_{x_4} = \{40,50,70\}$$

The reduced domains are:

$$D_z = \{80,90\}$$
$$D_y = \{3\}$$
$$D_{x_1} = \{10,50\}$$
$$D_{x_2} = \{10,20\}$$
$$D_{x_3} = \{80,90\}$$
$$D_{x_4} = \{40,50,70\}$$

# Filtering for alldiff

$$\text{alldiff}\left(y_1,\ldots,y_n\right)$$

Domains can be filtered with an algorithm based on maximum cardinality bipartite matching and a theorem of Berge.

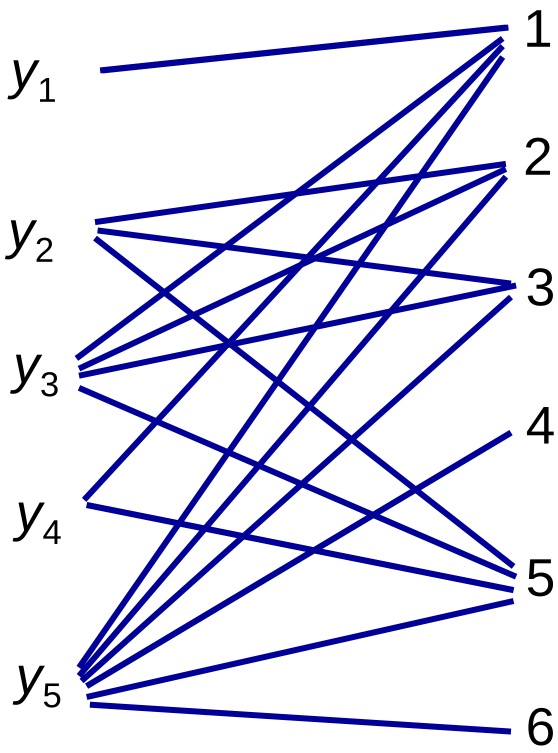
It is a special case of optimality conditions for max flow.

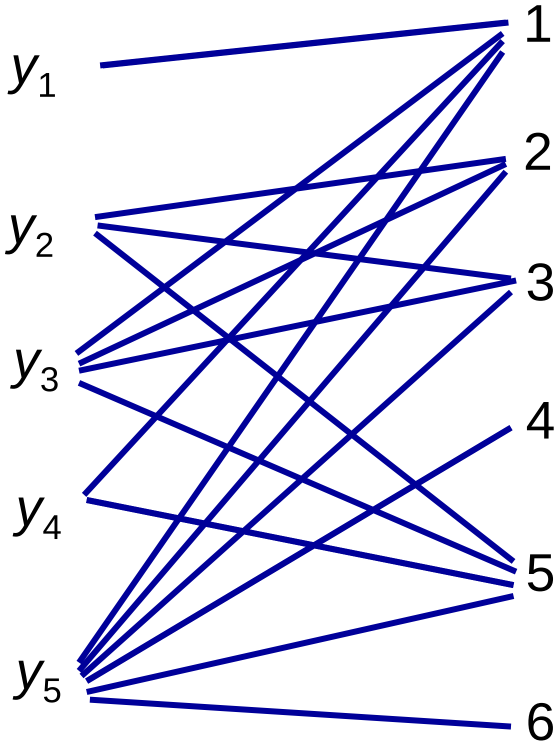
## Filtering for alldiff

Consider the domains

$$y_1 \in \{1\}$$
$$y_2 \in \{2,3,5\}$$
$$y_3 \in \{1,2,3,5\}$$
$$y_4 \in \{1,5\}$$
$$y_5 \in \{1,2,3,4,5,6\}$$

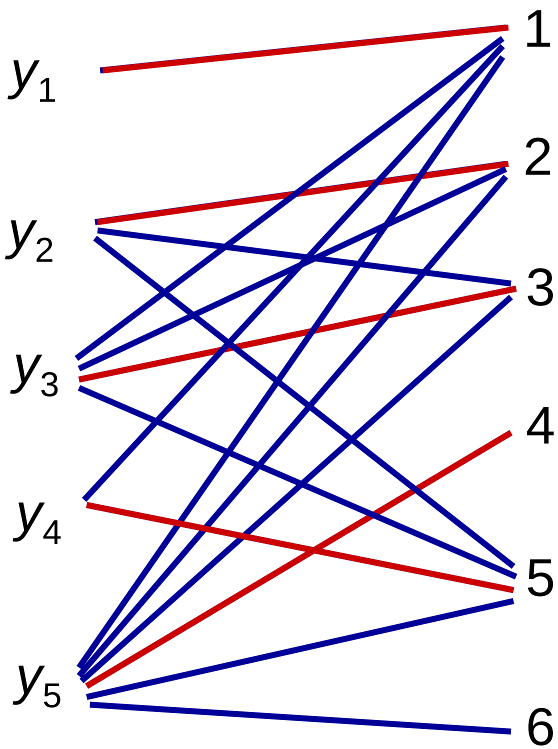
Indicate domains with edges





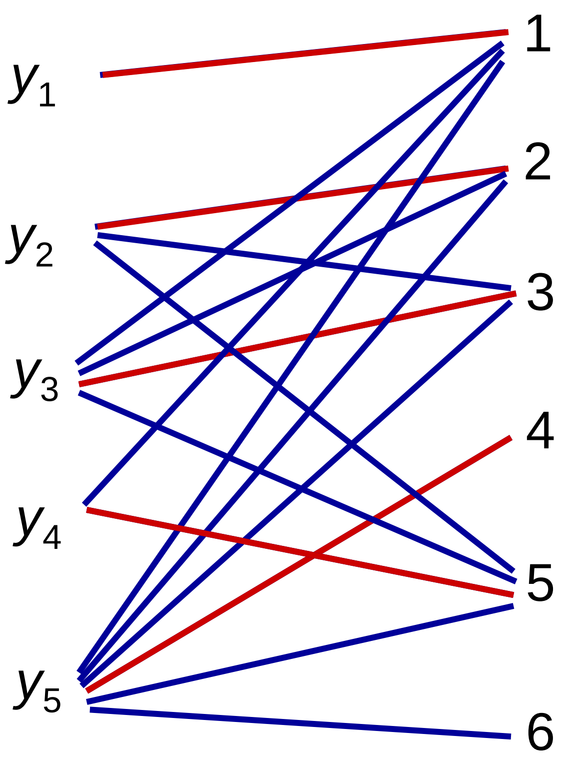

Indicate domains with edges

Find maximum cardinality bipartite matching.




Indicate domains with edges

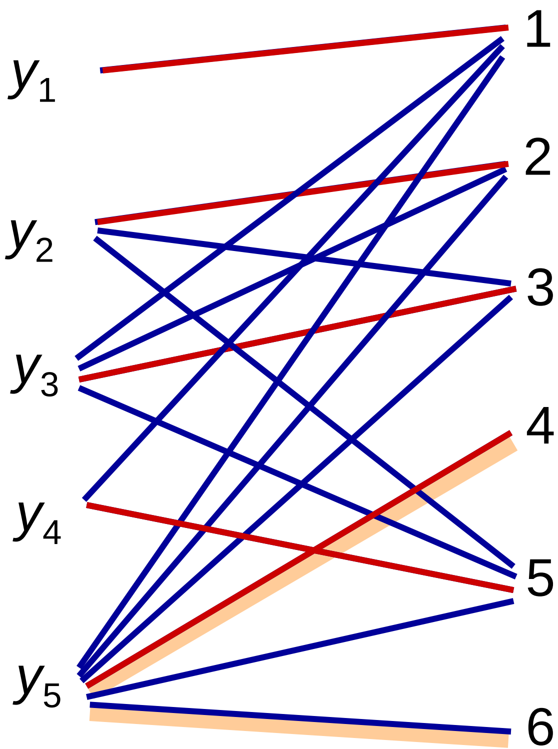
Find maximum cardinality bipartite matching.




Indicate domains with edges

Find maximum cardinality bipartite matching.

Mark edges in alternating paths that start at an uncovered vertex.




Indicate domains with edges

Find maximum cardinality bipartite matching.

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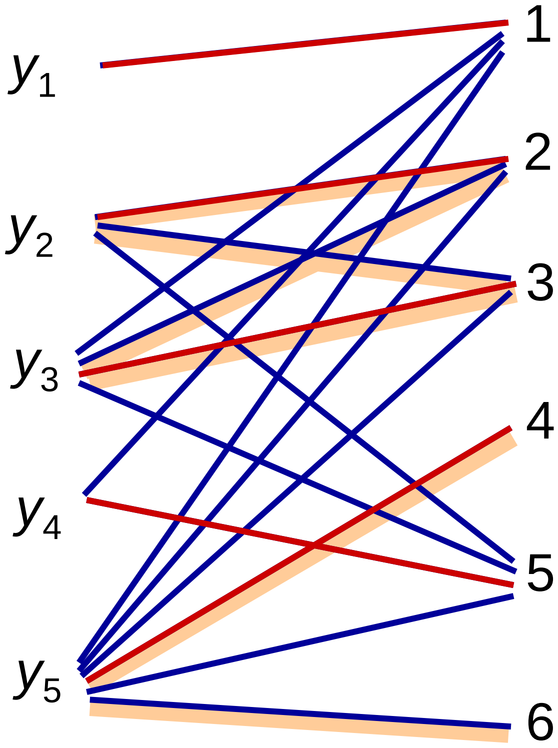
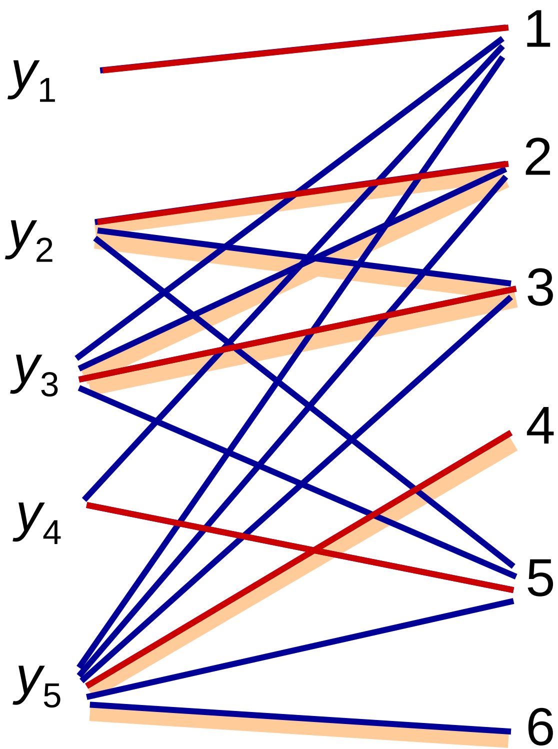



Indicate domains with edges

Find maximum cardinality bipartite matching.

Mark edges in alternating paths that start at an uncovered vertex.

Mark edges in alternating cycles.



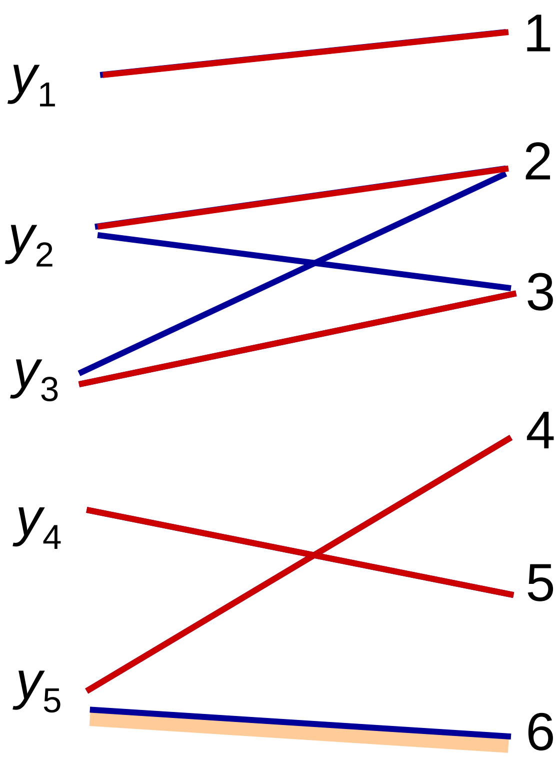

Indicate domains with edges

Find maximum cardinality bipartite matching.

Mark edges in alternating paths that start at an uncovered vertex.

Mark edges in alternating cycles.

Remove unmarked edges not in matching.




Indicate domains with edges

Find maximum cardinality bipartite matching.

Mark edges in alternating paths that start at an uncovered vertex.

Mark edges in alternating cycles.

Remove unmarked edges not in matching.

## Filtering for alldiff

Domains have been filtered:

$$
\begin{aligned}
&y_1 \in \{1\} & & & y_1 \in \{1\} \\
&y_2 \in \{2,3,5\} & & & y_2 \in \{2,3\} \\
&y_3 \in \{1,2,3,5\} & \longrightarrow & & y_3 \in \{2,3\} \\
&y_4 \in \{1,5\} & & & y_4 \in \{5\} \\
&y_5 \in \{1,2,3,4,5,6\} & & & y_5 \in \{4,6\}
\end{aligned}
$$

Hyperarc consistency achieved.

# Disjunctive scheduling

Consider a disjunctive scheduling constraint:

$$\text{disjunctive}\left(\boxed{(s_1,s_2,s_3,s_5}),(p_1,p_2,p_3,p_5)\right)$$

Start time variables

| *Job* $j$ | *Release time* $r_j$ | *Dead-line* $d_j$ | *Processing time* $p_{Aj}$ | $p_{Bj}$ |
|---|---|---|---|---|
| 1 | 0 | 10 | 1 | 5 |
| 2 | 0 | 10 | 3 | 6 |
| 3 | 2 | 7 | 3 | 7 |
| 4 | 2 | 10 | 4 | 6 |
| 5 | 4 | 7 | 2 | 5 |

# Edge finding for disjunctive scheduling

Consider a disjunctive scheduling constraint:

$$\text{disjunctive}\big((s_1,s_2,s_3,s_5),\boxed{(p_1,p_2,p_3,p_5)}\big)$$

Processing times

| Job $j$ | Release time $r_j$ | Dead-line $d_j$ | Processing time $p_{Aj}$ | $p_{Bj}$ |
|---|---|---|---|---|
| 1 | 0 | 10 | 1 | 5 |
| 2 | 0 | 10 | 3 | 6 |
| 3 | 2 | 7 | 3 | 7 |
| 4 | 2 | 10 | 4 | 6 |
| 5 | 4 | 7 | 2 | 5 |

# Edge finding for disjunctive scheduling

Consider a disjunctive scheduling constraint:

$$\text{disjunctive}\big((s_1,s_2,s_3,s_5),(p_1,p_2,p_3,p_5)\big)$$

| Job $j$ | Release time $r_j$ | Dead-line $d_j$ | Processing time $p_{Aj}$ | $p_{Bj}$ |
|---|---|---|---|---|
| 1 | 0 | 10 | 1 | 5 |
| 2 | 0 | 10 | 3 | 6 |
| 3 | 2 | 7 | 3 | 7 |
| 4 | 2 | 10 | 4 | 6 |
| 5 | 4 | 7 | 2 | 5 |

Variable domains defined by time windows and processing times

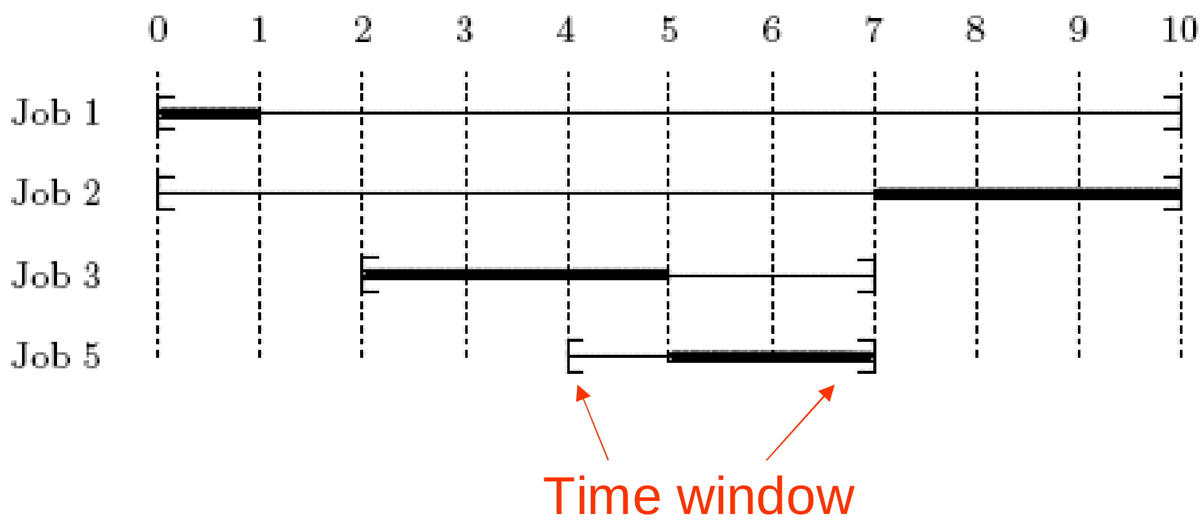
$$s_1 \in [0, 10-1]$$
$$s_2 \in [0, 10-3]$$
$$s_3 \in [2, 7-3]$$
$$s_5 \in [4, 7-2]$$

# Edge finding for disjunctive scheduling

Consider a disjunctive scheduling constraint:

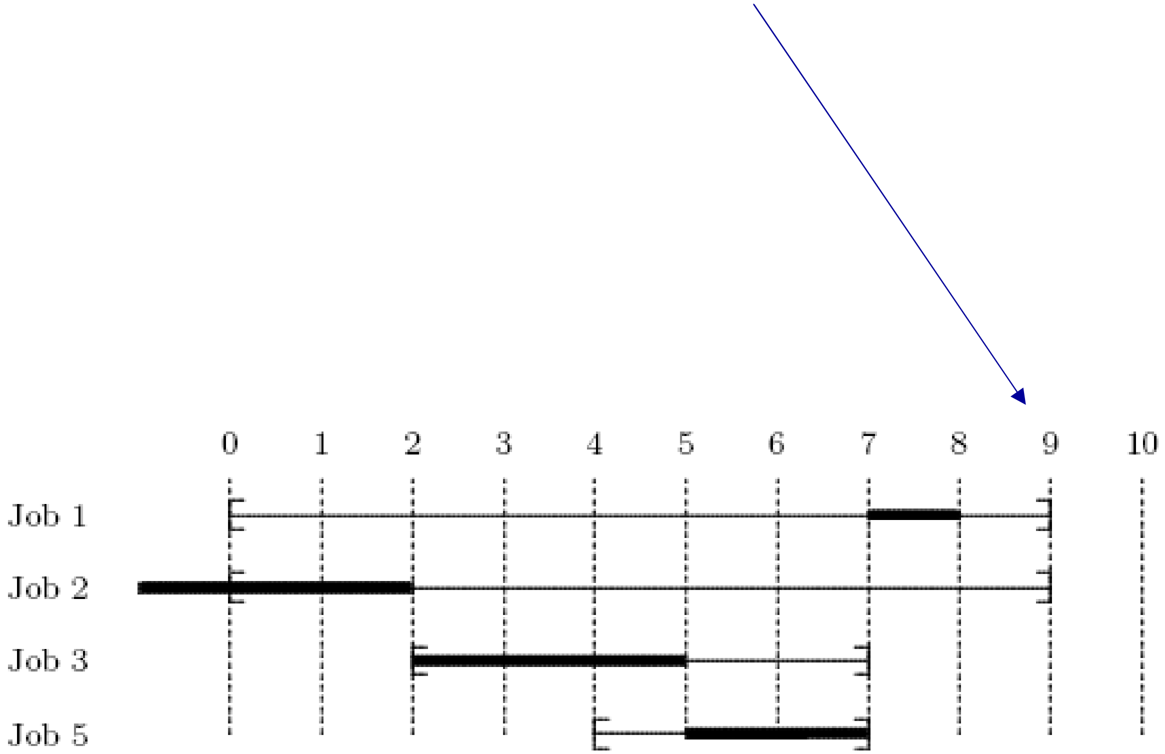
$$\text{disjunctive}\big((s_1,s_2,s_3,s_5),(p_1,p_2,p_3,p_5)\big)$$

A feasible (min makespan) solution:



# Edge finding for disjunctive scheduling

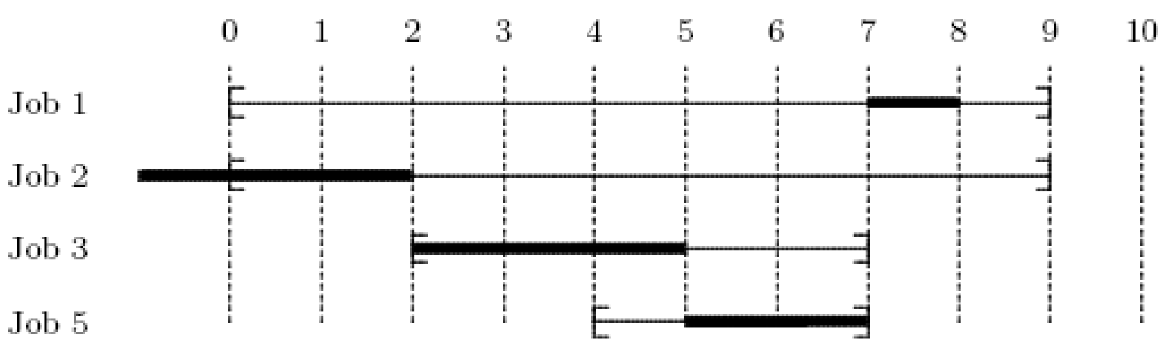
But let's reduce 2 of the deadlines to 9:



# Edge finding for disjunctive scheduling

But let's reduce 2 of the deadlines to 9:

We will use edge finding to prove that there is no feasible schedule.

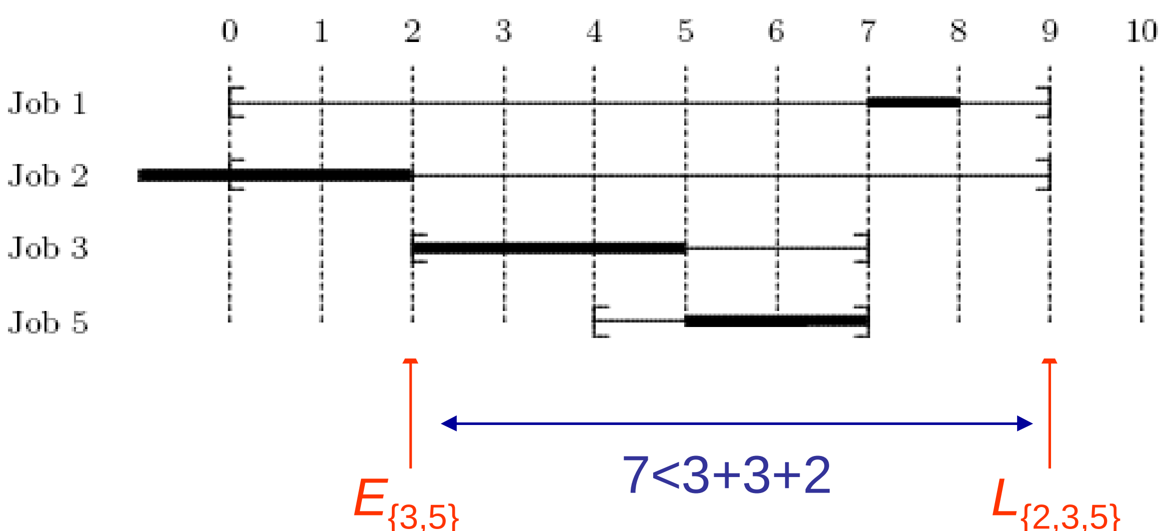


# Edge finding for disjunctive scheduling

We can deduce that job 2 must precede jobs 3 and 4: $2 \ll \{3,5\}$

Because if job 2 is not first, there is not enough time for all 3 jobs within the time windows:

$$L_{\{2,3,5\}} - E_{\{3,5\}} < p_{\{2,3,5\}}$$



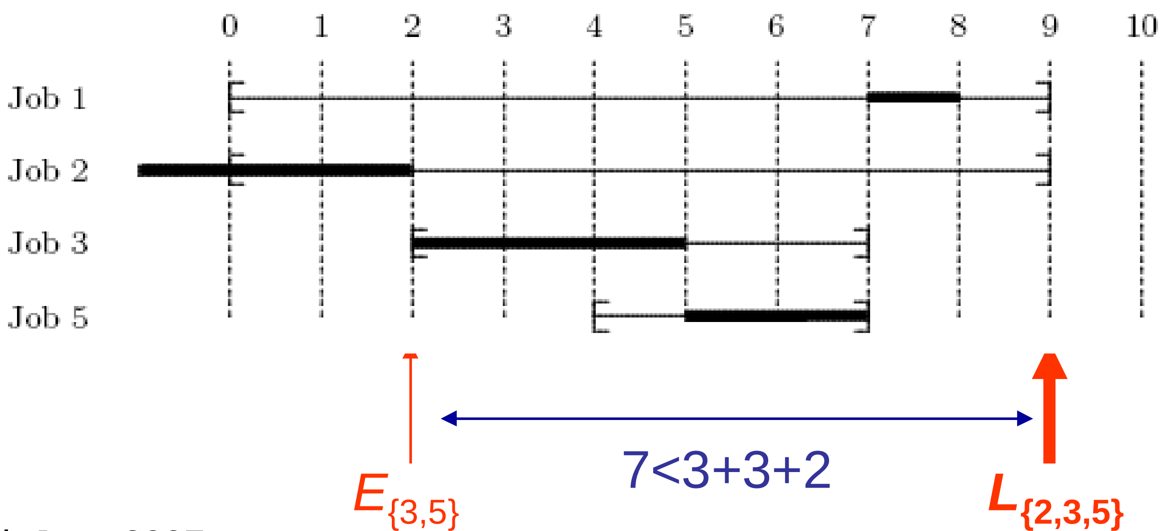
# Edge finding for disjunctive scheduling

We can deduce that job 2 must precede jobs 3 and 4: $2 \ll \{3,5\}$

Because if job 2 is not first, there is not enough time for all 3 jobs within the time windows:

$$\boxed{L_{\{2,3,5\}}} - E_{\{3,5\}} < p_{\{2,3,5\}}$$

Latest deadline



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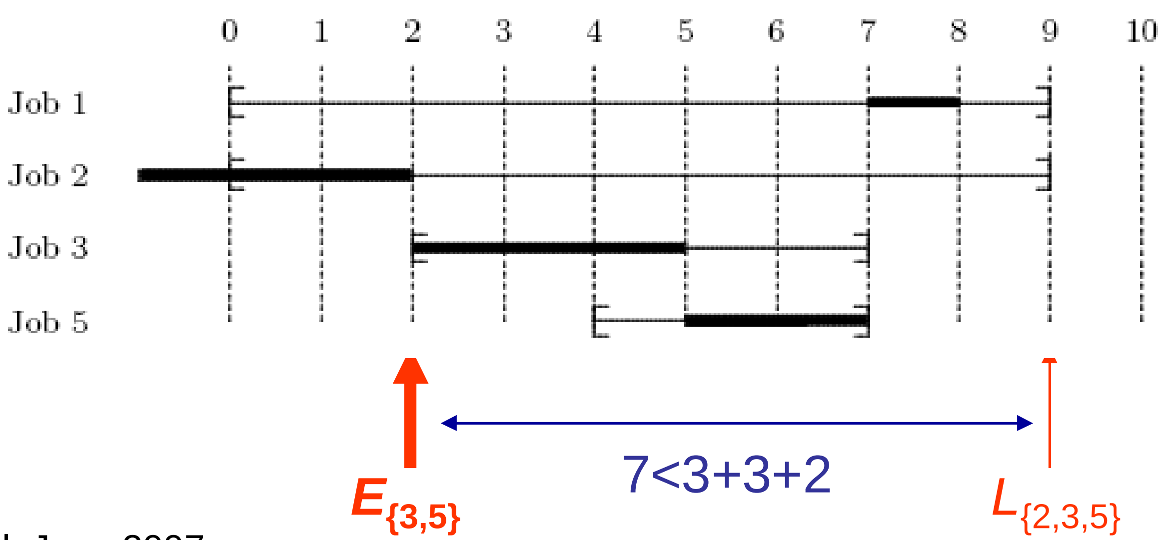
Earliest release time



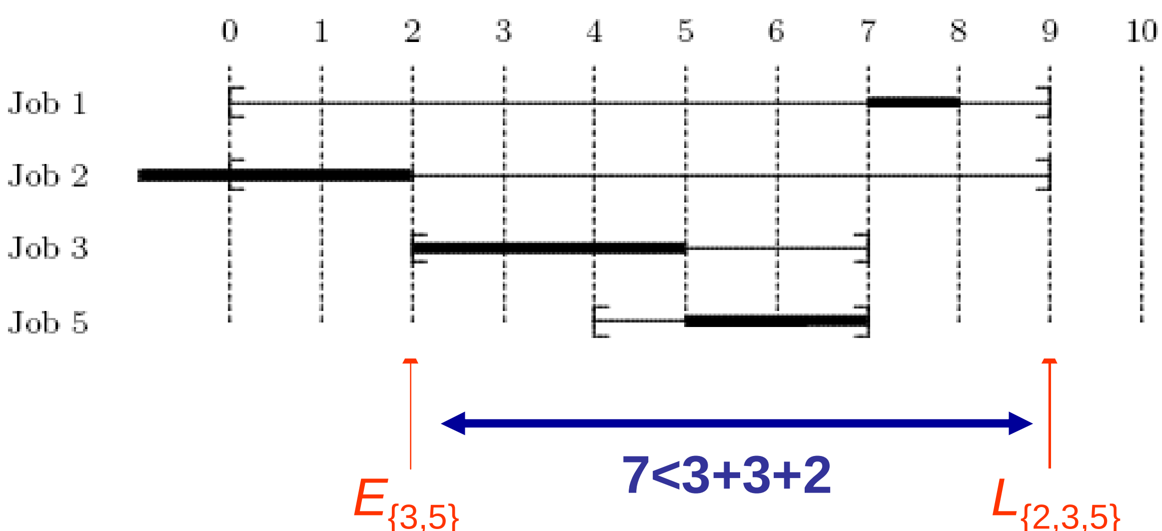
# Edge finding for disjunctive scheduling

We can deduce that job 2 must precede jobs 3 and 4: $2 \ll \{3,5\}$

Because if job 2 is not first, there is not enough time for all 3 jobs within the time windows:

$$L_{\{2,3,5\}} - E_{\{3,5\}} < \boxed{p_{\{2,3,5\}}}$$

Total processing time



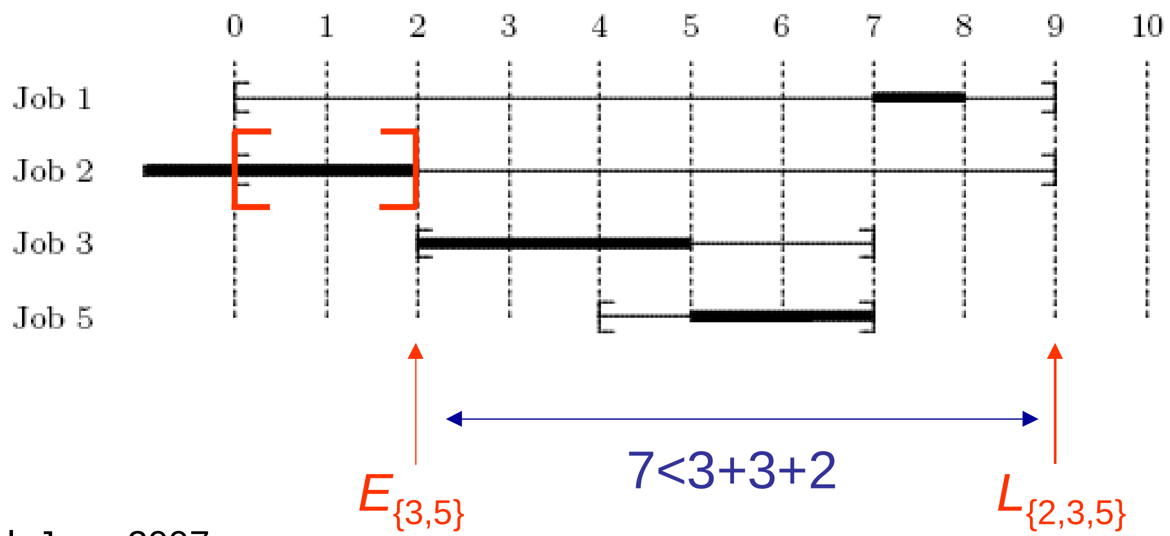
# Edge finding for disjunctive scheduling

We can deduce that job 2 must precede jobs 3 and 4: $2 \ll \{3,5\}$

So we can tighten deadline of job 2 to minimum of

$$L_{\{3\}} - p_{\{3\}} = 4 \qquad L_{\{5\}} - p_{\{5\}} = 5 \qquad L_{\{3,5\}} - p_{\{3,5\}} = 2$$

Since time window of job 2 is now too narrow, there is no feasible schedule.



# Edge finding for disjunctive scheduling

In general, we can deduce that job *k* must precede all the jobs in set *J*: $k \ll J$

If there is not enough time for all the jobs after the earliest release time of the jobs in *J*

$$L_{J\cup\{k\}} - E_J < p_{J\cup\{k\}} \qquad L_{\{2,3,5\}} - E_{\{3,5\}} < p_{\{2,3,5\}}$$

# Edge finding for disjunctive scheduling

In general, we can deduce that job *k* must precede all the jobs in set *J*: $k \ll J$

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$$L_{J\cup\{k\}} - E_J < p_{J\cup\{k\}} \qquad L_{\{2,3,5\}} - E_{\{3,5\}} < p_{\{2,3,5\}}$$

Now we can tighten the deadline for job *k* to:

$$\min_{J'\subset J}\left\{L_{J'} - p_{J'}\right\} \qquad L_{\{3,5\}} - p_{\{3,5\}} = 2$$

## Edge finding for disjunctive scheduling

There is a symmetric rule: $k \gg J$

If there is not enough time for all the jobs before the latest deadline of the jobs in *J:*

$$L_J - E_{J \cup \{k\}} < p_{J \cup \{k\}}$$

Now we can tighten the release date for job *k* to*:*

$$\max_{J' \subset J} \left\{ E_{J'} + p_{J'} \right\}$$

# Edge finding for disjunctive scheduling

**Problem:** how can we avoid enumerating all subsets *J* of jobs to find edges?

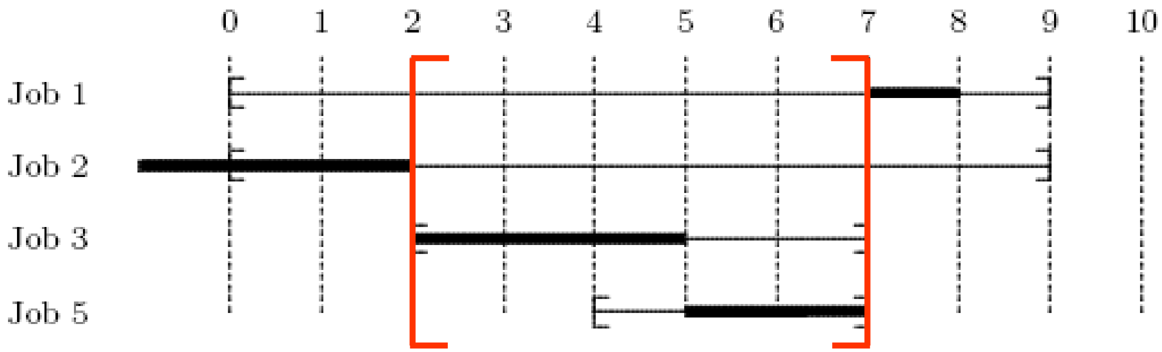
$$L_{J \cup \{k\}} - E_J < p_{J \cup \{k\}}$$

…and all subsets *J´* of *J* to tighten the bounds?

$$\min_{J' \subset J} \left\{ L_{J'} - p_{J'} \right\}$$

# Edge finding for disjunctive scheduling

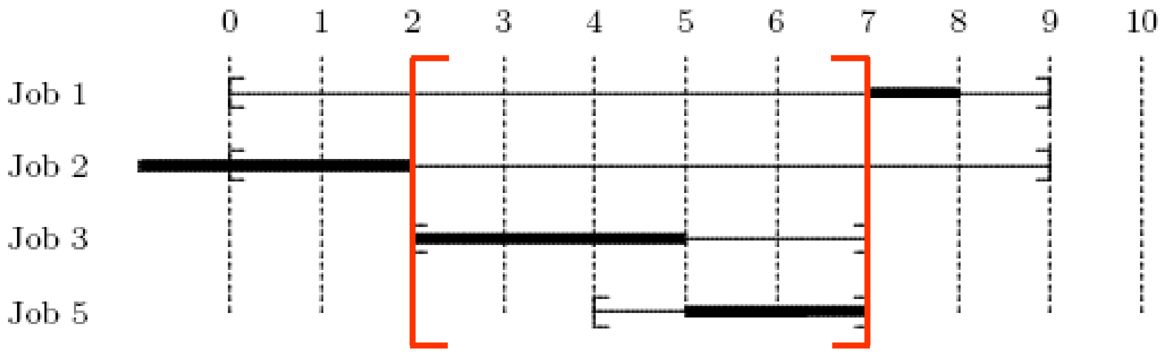
**Key result:** We only have to consider sets $J$ whose time windows lie within some interval.




e.g., $J = \{3,5\}$

# Edge finding for disjunctive scheduling

**Key result:** We only have to consider sets *J* whose time windows lie within some interval.




e.g., $J = \{3,5\}$

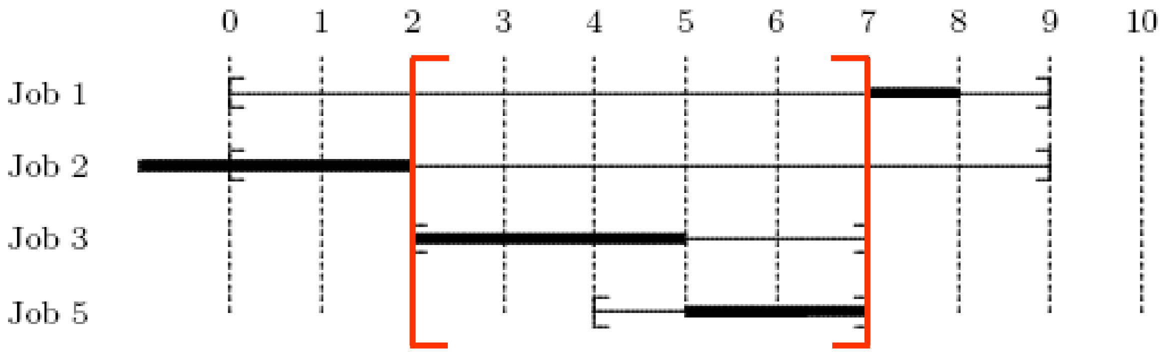
Removing a job from those within an interval only weakens the test

$$L_{J\cup\{k\}} - E_J < p_{J\cup\{k\}}$$

There are a polynomial number of intervals defined by release times and deadlines.

# Edge finding for disjunctive scheduling

**Key result:** We only have to consider sets *J* whose time windows lie within some interval.

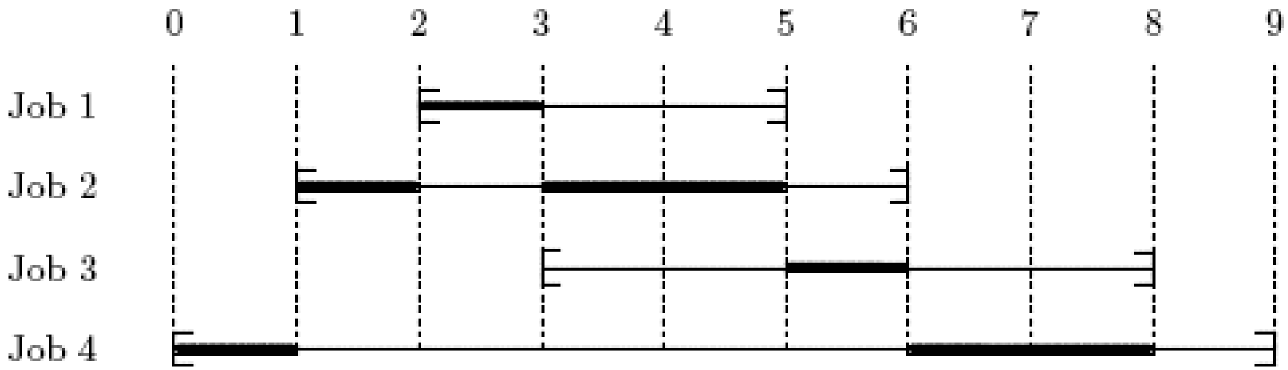



e.g., $J = \{3,5\}$

**Note:** Edge finding does not achieve bounds consistency, which is an NP-hard problem.

# Edge finding for disjunctive scheduling

One $O(n^2)$ algorithm is based on the Jackson pre-emptive schedule (JPS). Using a different example, the JPS is:



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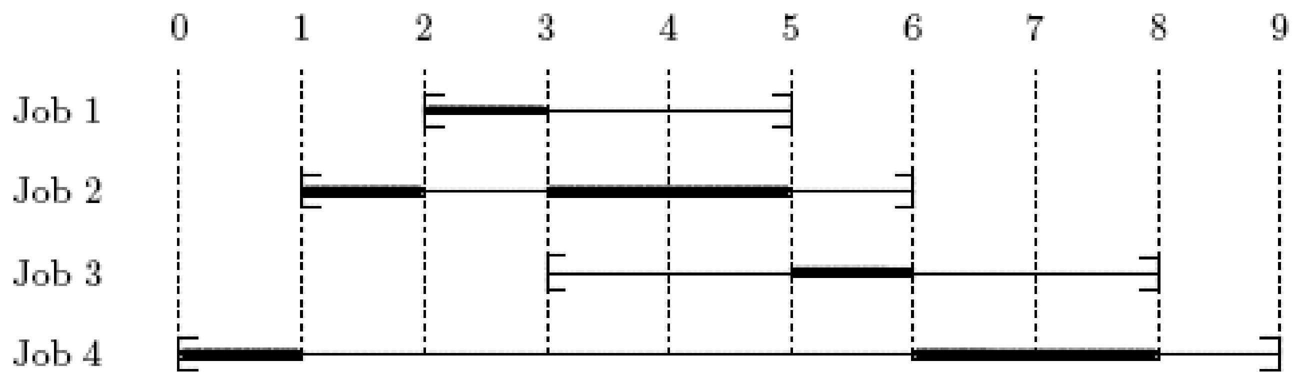
One $O(n^2)$ algorithm is based on the Jackson pre-emptive schedule (JPS). Using a different example, the JPS is:




Jobs unfinished at time $E_i$ in JPS

For each job $i$

Scan jobs $k \in \boxed{J_i}$ in decreasing order of $L_k$

Select first $k$ for which $L_k - E_i < p_i + \bar{p}_{J_{ik}}$

Conclude that $i \gg \boxed{J_{ik}}$ Jobs $j \neq i$ in $J_i$ with $L_j \leq L_k$

Update $E_i$ to $\boxed{\text{JPS}(i,k)}$

Latest completion time in JPS of jobs in $J_{ik}$

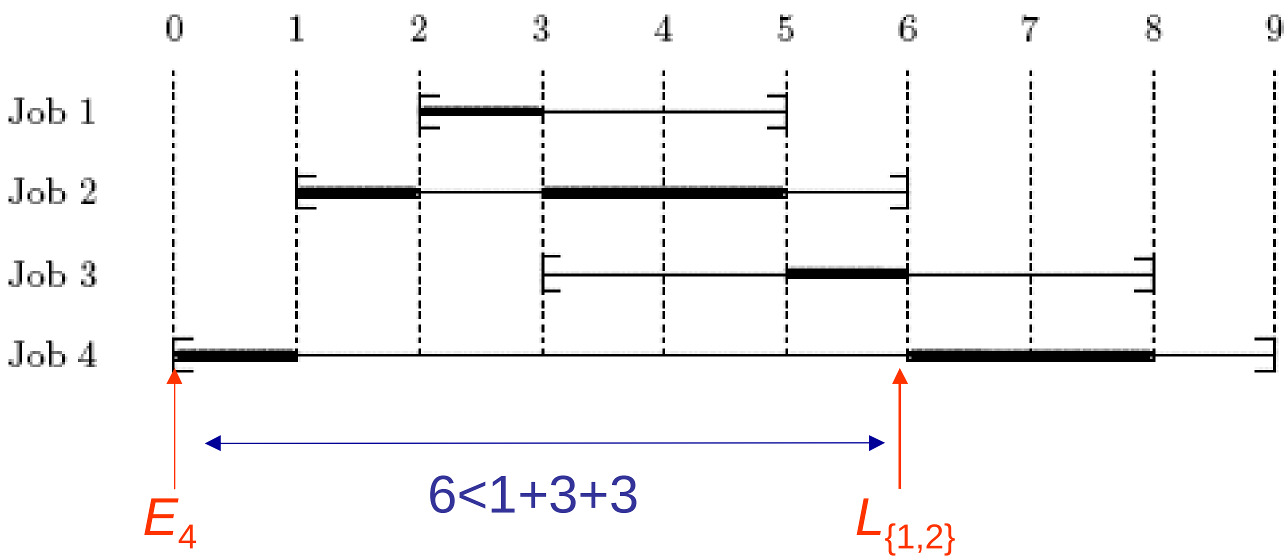
## Not-first/not-last rules

We can deduce that job 4 cannot precede jobs 1 and 2:

$$\neg\left(4 \ll \{1,2\}\right)$$

Because if job 4 is first, there is too little time to complete the jobs before the later deadline of jobs 1 and 2:

$$L_{\{1,2\}} - E_4 < p_1 + p_2 + p_4$$



## Not-first/not-last rules

We can deduce that job 4 cannot precede jobs 1 and 2:

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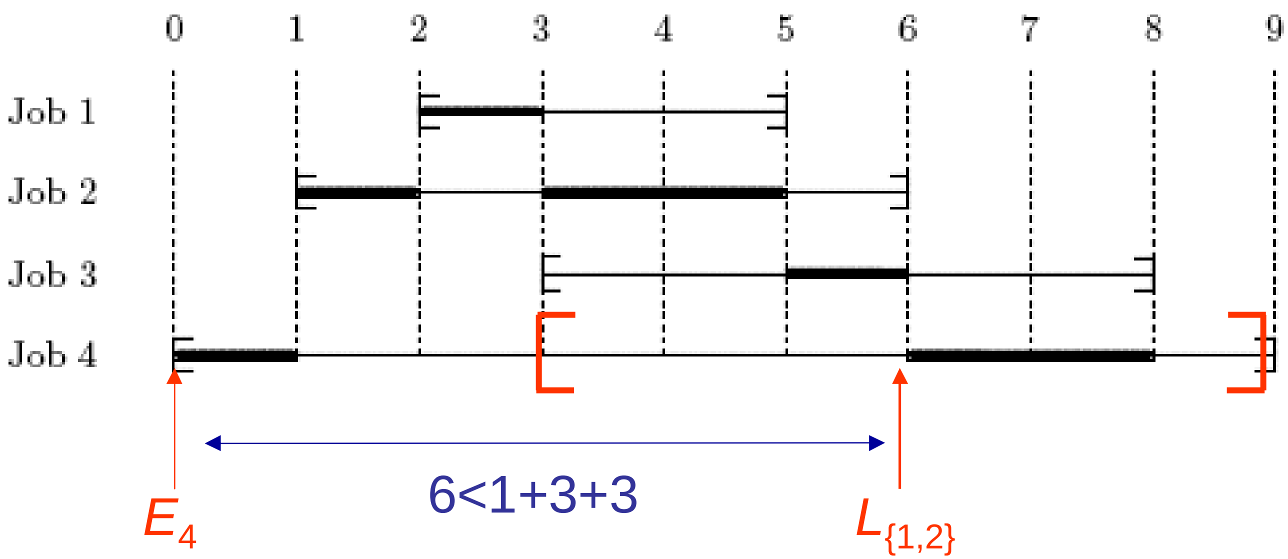
Now we can tighten the release time of job 4 to minimum of:

$$E_1 + p_1 = 3 \qquad E_2 + p_2 = 4$$



## Not-first/not-last rules

In general, we can deduce that job *k* cannot precede all the jobs in *J*:

$$\neg\left(k \ll J\right)$$

if there is too little time after release time of job *k* to complete all jobs before the latest deadline in *J*:

$$L_J - E_k < p_J$$

Now we can update $E_i$ to

$$\min_{j \in J}\left\{E_j + p_j\right\}$$

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In general, we can deduce that job *k* cannot precede all the jobs in *J*:

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Now we can update $E_i$ to

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There is a symmetric not-last rule.

The rules can be applied in polynomial time, although an efficient algorithm is quite complicated.

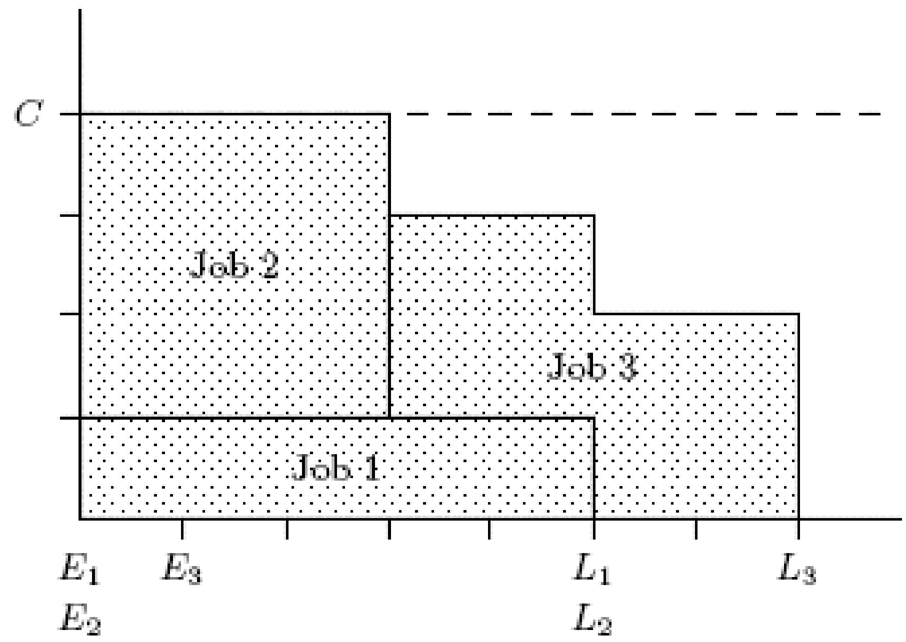
# Cumulative scheduling

Consider a cumulative scheduling constraint:

$$\text{cumulative}\big((s_1,s_2,s_3),(p_1,p_2,p_3),(c_1,c_2,c_3),C\big)$$

| $j$ | $p_j$ | $c_j$ | $E_j$ | $L_j$ |
|---|---|---|---|---|
| 1 | 5 | 1 | 0 | 5 |
| 2 | 3 | 3 | 0 | 5 |
| 3 | 4 | 2 | 1 | 7 |

A feasible solution:

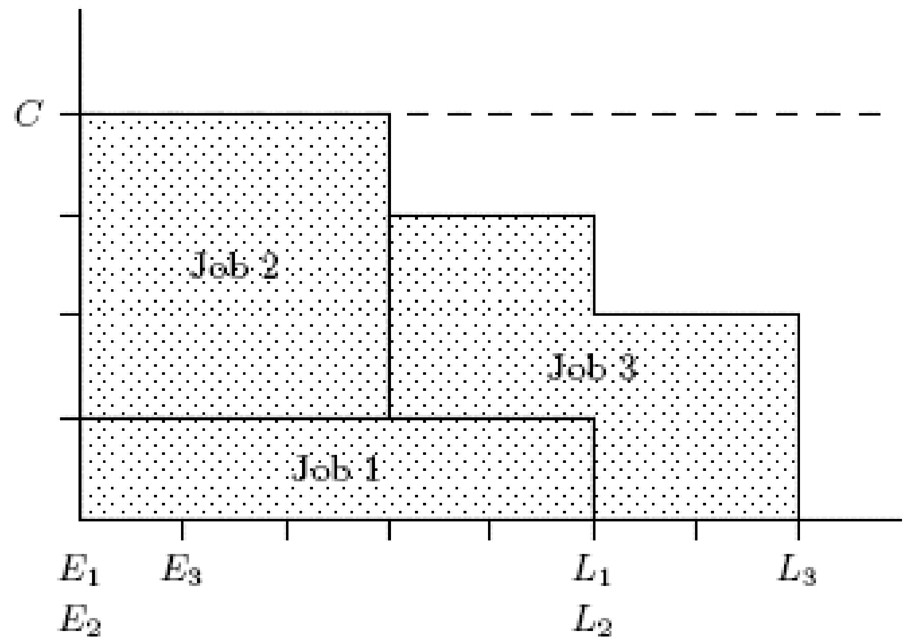


## Edge finding for cumulative scheduling

We can deduce that job 3 must finish after the others finish: $3 > \{1,2\}$

Because the total **energy** required exceeds the area between the earliest release time and the later deadline of jobs 1,2:

$$e_3 + e_{\{1,2\}} > C \cdot \left( L_{\{1,2\}} - E_{\{1,2,3\}} \right)$$



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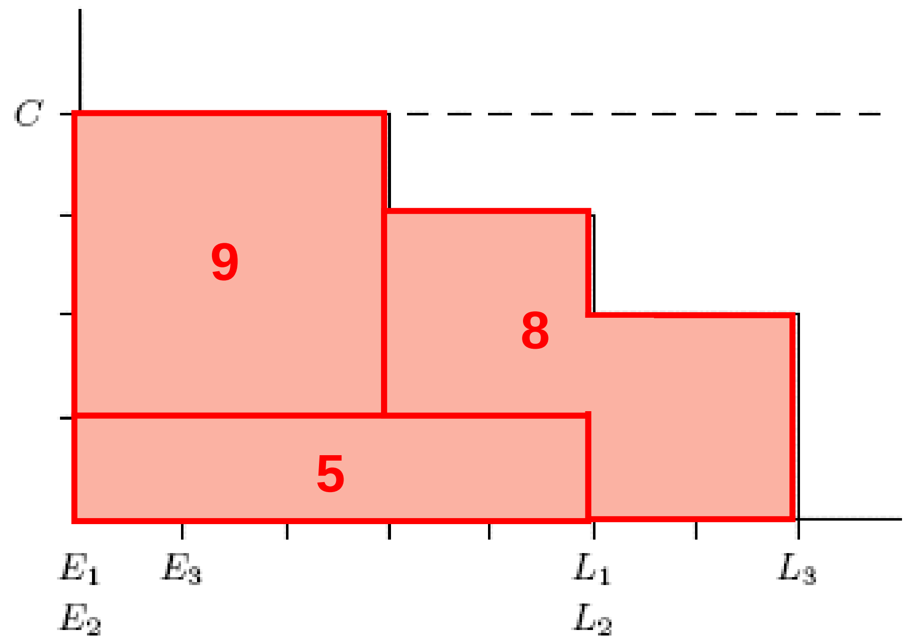
Total energy required = 22



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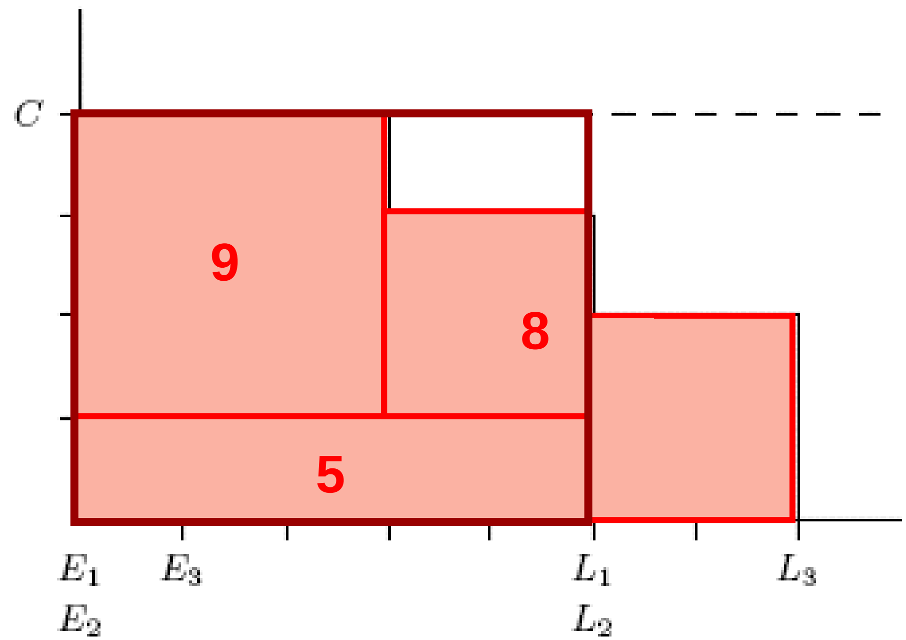
$$\boxed{e_3 + e_{\{1,2\}}} > \boxed{C \cdot \left(L_{\{1,2\}} - E_{\{1,2,3\}}\right)}$$

Total energy required = 22

Area available = 20



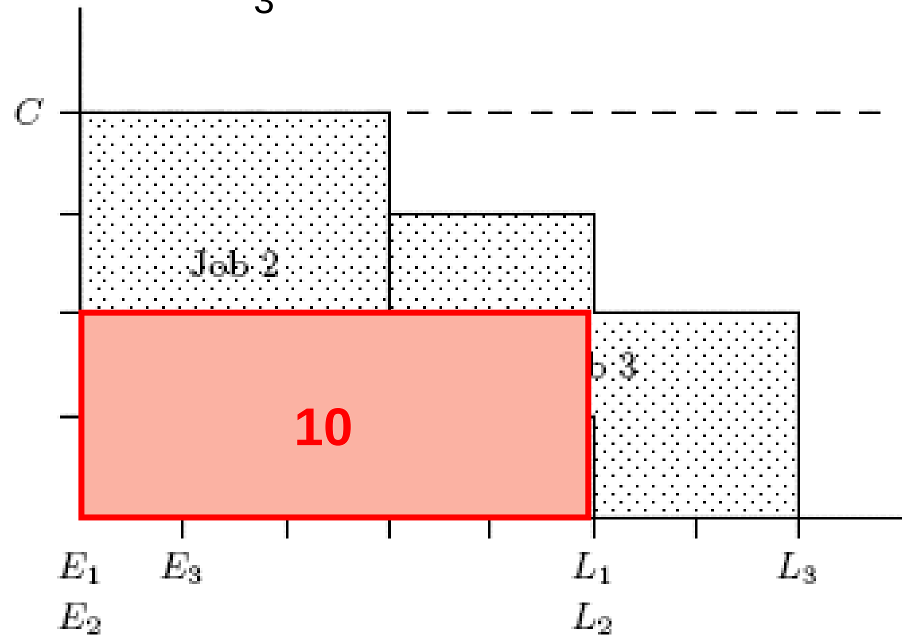
## Edge finding for cumulative scheduling

We can deduce that job 3 must finish after the others finish: $3 > \{1,2\}$

We can update the release time of job 3 to

$$E_{\{1,2\}} + \frac{e_J - \boxed{(C - c_3)(L_{\{1,2\}} - E_{\{1,2\}})}}{c_3}$$

Energy available for jobs 1,2 if space is left for job 3 to start anytime = 10



# Edge finding for cumulative scheduling

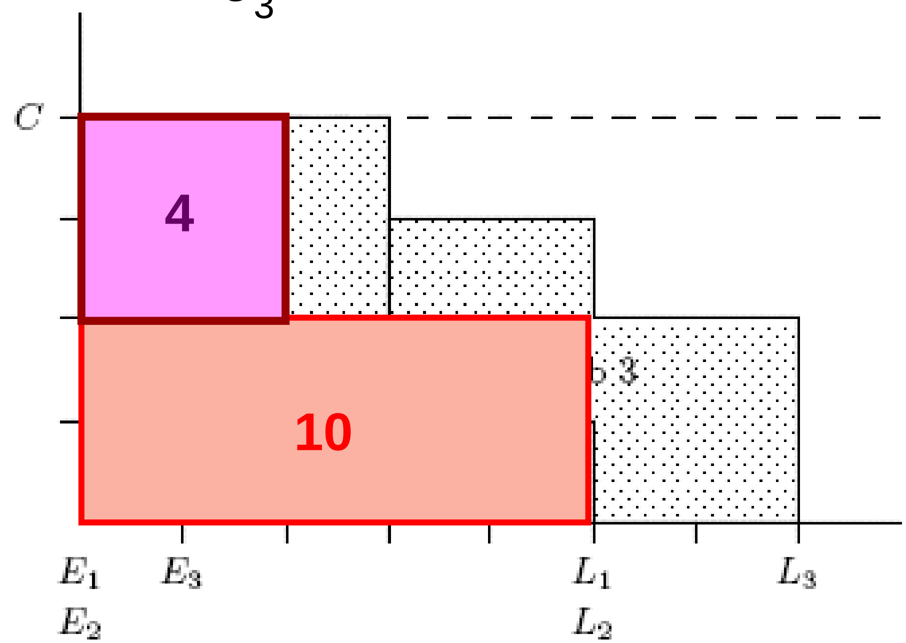
We can deduce that job 3 must finish after the others finish: $3 > \{1,2\}$

We can update the release time of job 3 to

$$E_{\{1,2\}} + \frac{\boxed{e_J - (C - c_3)(L_{\{1,2\}} - E_{\{1,2\}})}}{c_3}$$

Energy available for jobs 1,2 if space is left for job 3 to start anytime = 10

Excess energy required by jobs 1,2 = 4



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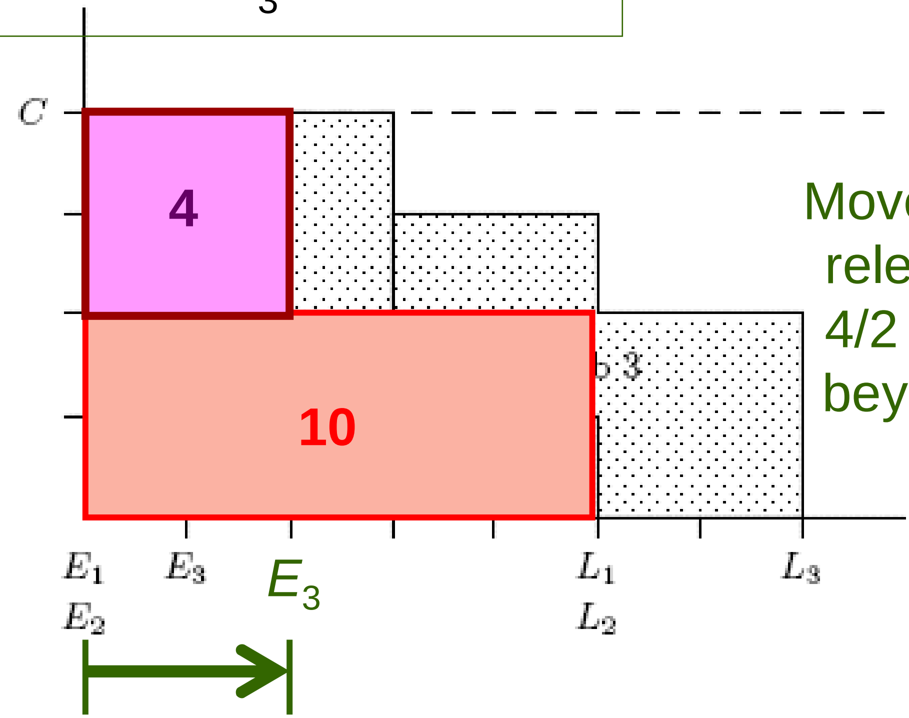



Move up job 3 release time 4/2 = 2 units beyond $E_{\{1,2\}}$

# Edge finding for cumulative scheduling

In general, if $e_{J\cup\{k\}} > C\cdot\left(L_J - E_{J\cup\{k\}}\right)$

then $k > J$, and update $E_k$ to

$$\max_{\substack{J'\subset J \\ e_{J'}-(C-c_k)(L_{J'}-E_{J'})>0}} \left\{ E_{J'} + \frac{e_{J'}-(C-c_k)(L_{J'}-E_{J'})}{c_k} \right\}$$

In general, if $e_{J\cup\{k\}} > C\cdot\left(L_{J\cup\{k\}} - E_J\right)$

then $k < J$, and update $L_k$ to

$$\min_{\substack{J'\subset J \\ e_{J'}-(C-c_k)(L_{J'}-E_{J'})>0}} \left\{ L_{J'} - \frac{e_{J'}-(C-c_k)(L_{J'}-E_{J'})}{c_k} \right\}$$

# Edge finding for cumulative scheduling

There is an $O(n^2)$ algorithm that finds all applications of the edge finding rules.

# Other propagation rules for cumulative scheduling

- Extended edge finding.
- Timetabling.
- Not-first/not-last rules.
- Energetic reasoning.

# Linear Relaxation

Why Relax?

Algebraic Analysis of LP

Linear Programming Duality

LP-Based Domain Filtering

Example: Single-Vehicle Routing

Disjunctions of Linear Systems

# Why Relax?

## Solving a relaxation of a problem can:

- Tighten variable bounds.
- Possibly solve original problem.
- Guide the search in a promising direction.
- Filter domains using reduced costs or Lagrange multipliers.
- Prune the search tree using a bound on the optimal value.
- Provide a more global view, because a single OR relaxation can pool relaxations of several constraints.

## Some OR models that can provide relaxations:

- Linear programming (LP).
- Mixed integer linear programming (MILP)
  - Can itself be relaxed as an LP.
  - LP relaxation can be strengthened with cutting planes.
- Lagrangean relaxation.
- Specialized relaxations.
  - For particular problem classes.
  - For global constraints.

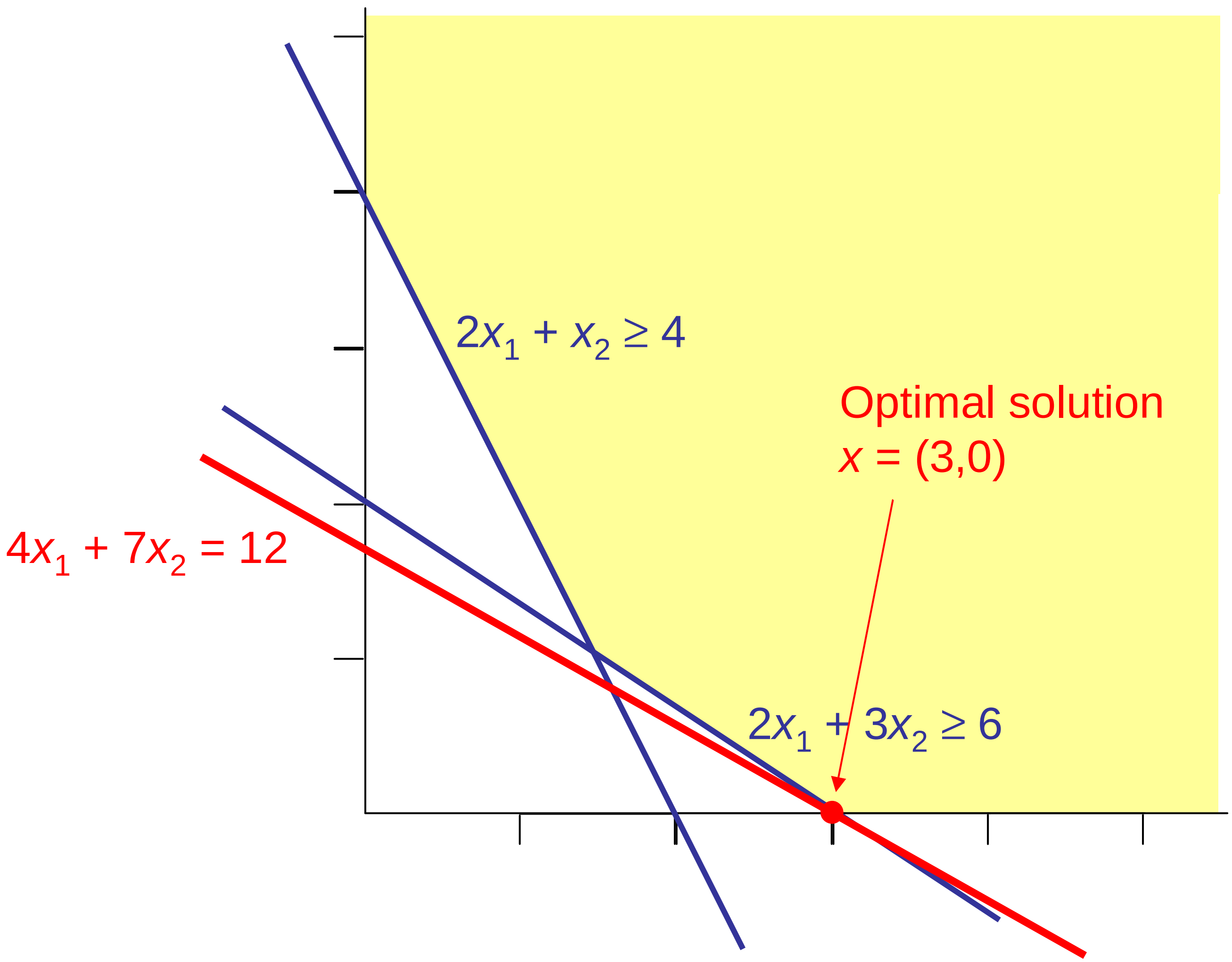
# Motivation

- **Linear programming** is remarkably versatile for representing real-world problems.
- LP is by far the most widely used tool for **relaxation**.
- LP relaxations can be strengthened by **cutting planes.**
  - Based on polyhedral analysis.
- LP has an elegant and powerful **duality theory**.
  - Useful for domain filtering, and much else.
- The LP problem is **extremely well solved**.

# Algebraic Analysis of LP

An example…

$$\min\ 4x_1 + 7x_2$$
$$2x_1 + 3x_2 \geq 6$$
$$2x_1 + x_2 \geq 4$$
$$x_1, x_2 \geq 0$$



## Algebraic Analysis of LP

Rewrite

$$\min\ 4x_1 + 7x_2$$
$$2x_1 + 3x_2 \geq 6$$
$$2x_1 + x_2 \geq 4$$
$$x_1, x_2 \geq 0$$

as

$$\min\ 4x_1 + 7x_2$$
$$2x_1 + 3x_2 - x_3 = 6$$
$$2x_1 + x_2 - x_4 = 4$$
$$x_1, x_2, x_3, x_4 \geq 0$$

In general an LP has the form

$$\min\ cx$$
$$Ax = b$$
$$x \geq 0$$

# Algebraic analysis of LP

Write
$$\min cx$$
$$\boxed{A}x = b$$
$$x \geq 0$$

$m \times n$ matrix

as
$$\min c_B x_B + c_N x_N$$
$$B x_B + N x_N = b$$
$$\boxed{x_B}, \boxed{x_N} \geq 0$$

**Basic** variables

**Nonbasic** variables

where
$$A = [\boxed{B}\ N]$$

**Any** set of $m$ linearly independent columns of A.

These form a **basis** for the space spanned by the columns.

## Algebraic analysis of LP

Write $\begin{aligned} &\min cx \\ &Ax = b \\ &x \geq 0 \end{aligned}$ as $\begin{aligned} &\min c_B x_B + c_N x_N \\ &\boxed{Bx_B + Nx_N = b} \\ &x_B, x_N \geq 0 \end{aligned}$ where $A = [B\ N]$

Solve constraint equation for $x_B$: $x_B = B^{-1}b - B^{-1}N\boxed{x_N}$

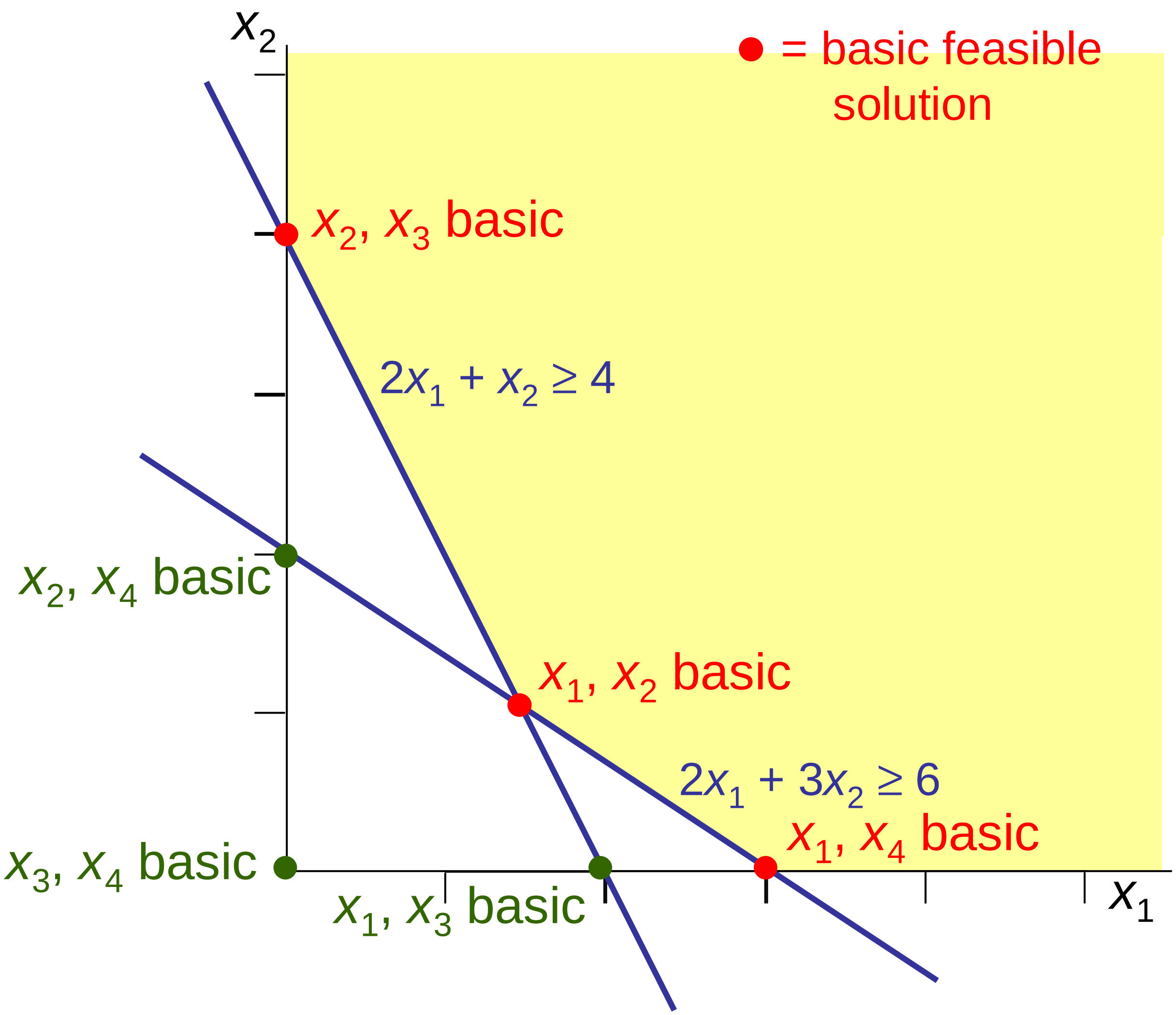
All solutions can be obtained by setting $x_N$ to some value.

The solution is **basic** if $x_N = 0$.

It is a **basic feasible solution** if $x_N = 0$ and $x_B \geq 0$.

## Example…

$$\min \ 4x_1 + 7x_2$$
$$2x_1 + 3x_2 - x_3 = 6$$
$$2x_1 + x_2 - x_4 = 4$$
$$x_1, x_2, x_3, x_4 \geq 0$$



# Algebraic analysis of LP

Write
$$\begin{aligned} &\min cx \\ &Ax = b \\ &x \geq 0 \end{aligned}$$
as
$$\begin{aligned} &\min \boxed{c_B x_B + c_N x_N} \\ &B x_B + N x_N = b \\ &x_B, x_N \geq 0 \end{aligned}$$
where
$$A = [B\ N]$$

Solve constraint equation for $x_B$: $x_B = B^{-1}b - B^{-1}N x_N$

Express cost in terms of nonbasic variables:

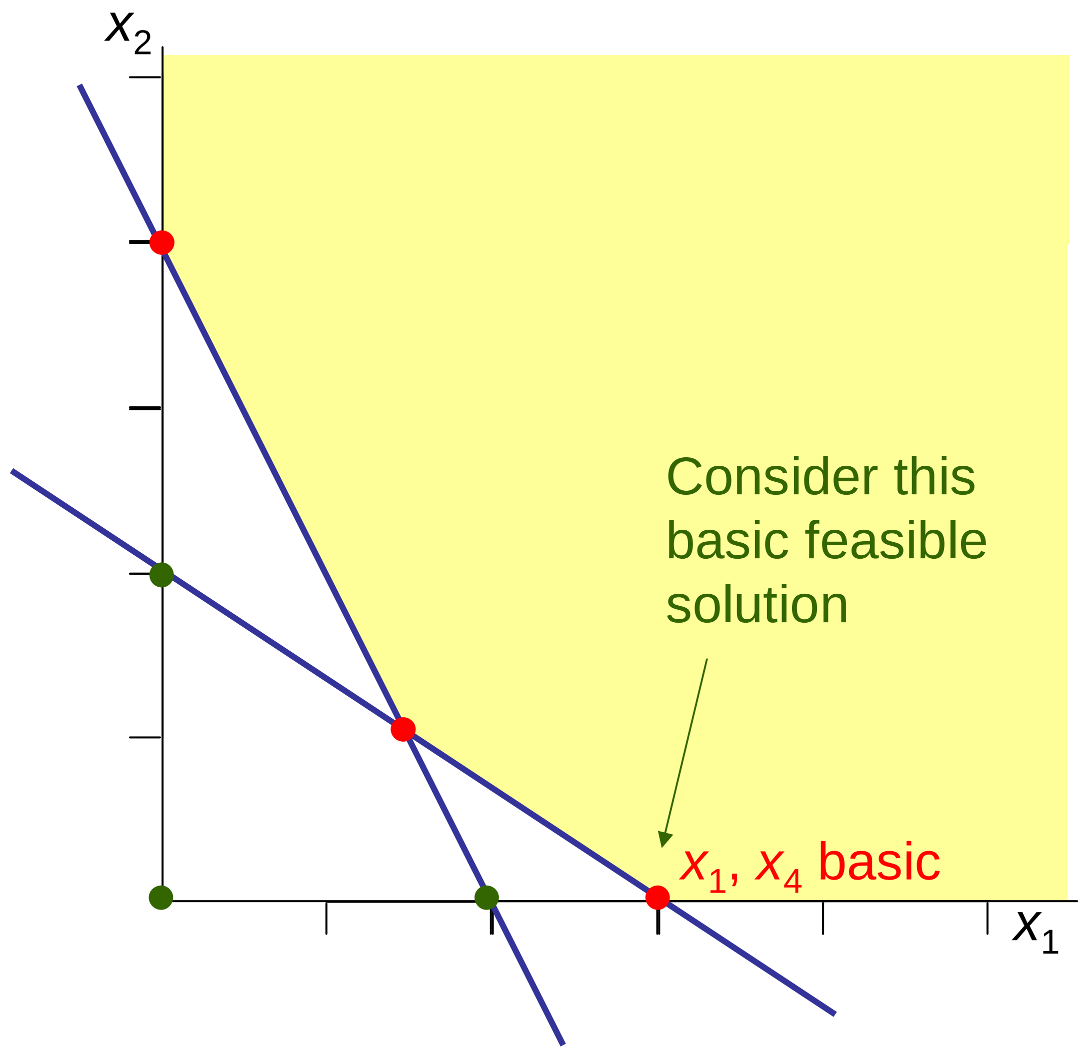
$$c_B B^{-1} b - (\boxed{c_N - c_B B^{-1} N}) x_N$$

Vector of reduced costs

Since $x_N \geq 0$, basic solution $(x_B, 0)$ is optimal if reduced costs are nonnegative.

## Example…

$$\min \ 4x_1 + 7x_2$$
$$2x_1 + 3x_2 - x_3 = 6$$
$$2x_1 + x_2 - x_4 = 4$$
$$x_1, x_2, x_3, x_4 \geq 0$$



# Example…

Write…

$$\min\ \ 4x_1 + 7x_2$$
$$2x_1 + 3x_2 - x_3 = 6$$
$$2x_1 + x_2 - x_4 = 4$$
$$x_1, x_2, x_3, x_4 \geq 0$$

as…

$$\min\ \overbrace{\boxed{[4\ \ 0]\begin{bmatrix}x_1\\x_4\end{bmatrix}}}^{c_Bx_B} + \overbrace{\boxed{[7\ \ 0]\begin{bmatrix}x_2\\x_3\end{bmatrix}}}^{c_Nx_N}$$

$$\underbrace{\boxed{\begin{bmatrix}2 & 0\\2 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_4\end{bmatrix}}}_{Bx_B} + \underbrace{\boxed{\begin{bmatrix}3 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_4\end{bmatrix}}}_{Nx_N} = \underbrace{\boxed{\begin{bmatrix}6\\4\end{bmatrix}}}_{b}$$

$$\begin{bmatrix}x_1\\x_4\end{bmatrix}, \begin{bmatrix}x_1\\x_4\end{bmatrix} \geq \begin{bmatrix}0\\0\end{bmatrix}$$

## Example…

$$\min \overbrace{\boxed{\begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \end{bmatrix}}}^{c_B x_B} + \overbrace{\boxed{\begin{bmatrix} 7 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}}}^{c_N x_N}$$

$$Bx_B \; \boxed{\begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \end{bmatrix}} + \underbrace{\boxed{\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \end{bmatrix}}}_{Nx_N} = \underbrace{\boxed{\begin{bmatrix} 6 \\ 4 \end{bmatrix}}}_{b}$$

$$\begin{bmatrix} x_1 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

# Example…

$$\min \; \overbrace{\begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \end{bmatrix}}^{c_B x_B} + \overbrace{\begin{bmatrix} 7 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}}^{c_N x_N}$$

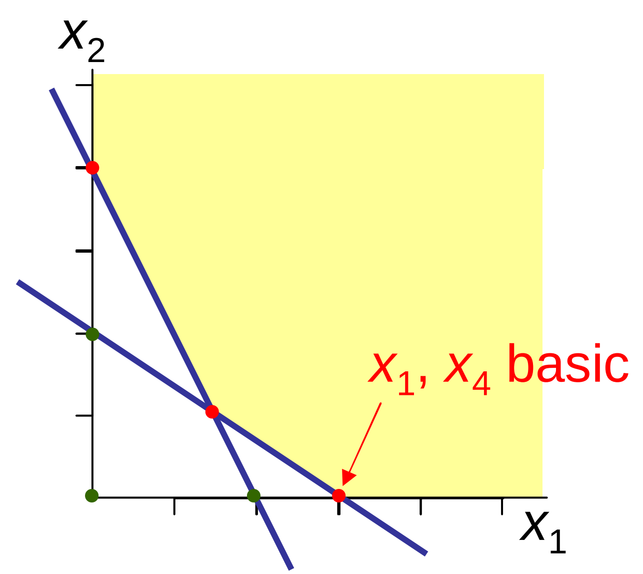
$$Bx_B \; \begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \end{bmatrix}}_{Nx_N} = \underbrace{\begin{bmatrix} 6 \\ 4 \end{bmatrix}}_{b}$$

$$\begin{bmatrix} x_1 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Basic solution is

$$x_B = B^{-1}b - B^{-1}Nx_N = B^{-1}b$$

$$= \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$



# Example…

$$\min \; \overbrace{\boxed{[4 \;\; 0]\begin{bmatrix} x_1 \\ x_4 \end{bmatrix}}}^{c_B x_B} + \overbrace{\boxed{[7 \;\; 0]\begin{bmatrix} x_2 \\ x_3 \end{bmatrix}}}^{c_N x_N}$$

$$\overbrace{\boxed{\begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_4 \end{bmatrix}}}^{Bx_B} + \underbrace{\boxed{\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_4 \end{bmatrix}}}_{Nx_N} = \boxed{\begin{bmatrix} 6 \\ 4 \end{bmatrix}}$$

$$\begin{bmatrix} x_1 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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Reduced costs are

$$c_N - c_B B^{-1} N$$

$$= [7 \quad 0] - [4 \quad 0]\begin{bmatrix} 1/2 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= [1 \quad 2] \geq [0 \quad 0]$$

Solution is optimal

# Linear Programming Duality

An LP can be viewed as an inference problem…

$$\begin{aligned} \min\ & cx \\ & Ax \geq b \\ & x \geq 0 \end{aligned} \quad = \quad \begin{aligned} \max\ & v \\ & Ax \geq b \ \boxed{\overset{x\geq 0}{\Longrightarrow}}\ cx \geq v \end{aligned}$$

implies

**Dual** problem: Find the tightest lower bound on the objective function that is implied by the constraints.

An LP can be viewed as an inference problem…

$$\begin{array}{lcl} \min\ cx & = & \max\ v \\ Ax \geq b & & Ax \geq b \overset{x\geq 0}{\Rightarrow} cx \geq v \\ x \geq 0 & & \end{array}$$

That is, some **surrogate** (nonnegative linear combination) of $Ax \geq b$ dominates $cx \geq v$

From Farkas Lemma: If $Ax \geq b$, $x \geq 0$ is feasible,

$$Ax \geq b \overset{x\geq 0}{\Rightarrow} cx \geq v \quad \text{iff} \quad \lambda Ax \geq \lambda b \ \boxed{\text{dominates}}\ cx \geq v \ \text{ for some } \lambda \geq 0$$

$\lambda A \leq c$ and $\lambda b \geq v$

An LP can be viewed as an inference problem…

$$\begin{array}{l} \min\ cx \\ Ax \geq b \\ x \geq 0 \end{array} \quad = \quad \begin{array}{l} \max\ v \\ Ax \geq b \overset{x\geq 0}{\Rightarrow} cx \geq v \end{array} \quad = \quad \begin{array}{l} \max\ \lambda b \\ \lambda A \leq c \\ \lambda \geq 0 \end{array}$$

This is the **classical LP dual**

From Farkas Lemma: If $Ax \geq b$, $x \geq 0$ is feasible,

$Ax \geq b \overset{x\geq 0}{\Rightarrow} cx \geq v$ iff $\lambda Ax \geq \lambda b$ dominates $cx \geq v$ for some $\lambda \geq 0$

$\lambda A \leq c$ and $\lambda b \geq v$

This equality is called **strong duality.**

$$\min \; cx = \max \; \lambda b$$
$$Ax \geq b \qquad \lambda A \leq c$$
$$x \geq 0 \qquad \lambda \geq 0$$

This is the **classical LP dual**

If $Ax \geq b$, $x \geq 0$ is feasible

Note that the dual of the dual is the **primal** (i.e., the original LP).

## Example

*Primal*

$$\min\ 4x_1 + 7x_2 \quad =$$

$$2x_1 + 3x_2 \ge 6 \quad (\lambda_1)$$

$$2x_1 + x_2 \ge 4 \quad (\lambda_1)$$

$$x_1, x_2 \ge 0$$

*Dual*

$$\max\ 6\lambda_1 + 4\lambda_2 \quad = 12$$

$$2\lambda_1 + 2\lambda_2 \le 4 \quad (x_1)$$

$$3\lambda_1 + \lambda_2 \le 7 \quad (x_2)$$

$$\lambda_1, \lambda_2 \ge 0$$

A dual solution is $(\lambda_1, \lambda_2) = (2,0)$

$$2x_1 + 3x_2 \ge 6 \quad \cdot(\lambda_1 = 2)$$

$$2x_1 + x_2 \ge 4 \quad \cdot(\lambda_2 = 0)$$

Dual multipliers

$$4x_1 + 6x_2 \ge 12$$

Surrogate

dominates

$$4x_1 + 7x_2 \ge 12$$

Tightest bound on cost

## Weak Duality

If x* is feasible in the primal problem

$$\min\ cx$$
$$Ax \geq b$$
$$x \geq 0$$

and $\lambda$* is feasible in the dual problem

$$\max\ \lambda b$$
$$\lambda A \leq c$$
$$\lambda \geq 0$$

then $cx^* \geq \lambda^* b$.

This is because
$cx^* \geq \lambda^* Ax^* \geq \lambda^* b$

$\lambda^*$ is dual feasible and $x^* \geq 0$

$x^*$ is primal feasible and $\lambda^* \geq 0$

# Dual multipliers as marginal costs

Suppose we perturb the RHS of an LP (i.e., change the requirement levels):

$$\begin{aligned}&\min \ cx\\ &Ax \geq b + \Delta b\\ &x \geq 0\end{aligned}$$

The dual of the perturbed LP has the same constraints at the original LP:

$$\begin{aligned}&\max \ \lambda(b + \Delta b)\\ &\lambda A \leq c\\ &\lambda \geq 0\end{aligned}$$

So an optimal solution $\lambda^*$ of the original dual is feasible in the perturbed dual.

## Dual multipliers as marginal costs

Suppose we perturb the RHS of an LP (i.e., change the requirement levels):

$$\min \; cx$$
$$Ax \geq b + \Delta b$$
$$x \geq 0$$

By weak duality, the optimal value of the perturbed LP is at least $\lambda^*(b + \Delta b) = \boxed{\lambda^* b} + \lambda^* \Delta b.$

Optimal value of original LP, by strong duality.

So $\lambda_i^*$ is a lower bound on the marginal cost of increasing the $i$-th requirement by one unit ($\Delta b_i = 1$).

If $\lambda_i^* > 0$, the $i$-th constraint must be tight **(complementary slackness).**

## Dual of an LP in equality form

*Primal*

$$\min c_B x_B + c_N x_N$$
$$B x_B + N x_N = b \qquad (\lambda)$$
$$x_B, x_N \geq 0$$

*Dual*

$$\max \lambda b$$
$$\lambda B \leq c_B \qquad (x_B)$$
$$\lambda N \leq c_N \qquad (x_B)$$
$$\lambda \text{ unrestricted}$$

# Dual of an LP in equality form

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$$\max \lambda b$$
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$$\lambda N \leq c_N \qquad (x_B)$$
$$\lambda \text{ unrestricted}$$

Recall that reduced cost vector is $c_N - \underbrace{\boxed{c_B B^{-1}}}_{\lambda} N = c_N - \lambda N$

this solves the dual
if ($x_B$,0) solves the primal

# Dual of an LP in equality form

*Primal*

$$\min c_B x_B + c_N x_N$$
$$B x_B + N x_N = b \qquad (\lambda)$$
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Recall that reduced cost vector is $c_N - \underbrace{\boxed{c_B B^{-1}}}_{\lambda} N = c_N - \lambda N$

this solves the dual
if $(x_B,0)$ solves the primal

Check: $$\lambda B = c_B B^{-1} B = c_B$$
$$\lambda N = c_B B^{-1} N \leq c_N$$

Because reduced cost is nonnegative
at optimal solution $(x_B,0)$.

# Dual of an LP in equality form

*Primal*

$$\min c_B x_B + c_N x_N$$
$$B x_B + N x_N = b \qquad (\lambda)$$
$$x_B, x_N \geq 0$$

*Dual*

$$\max \lambda b$$
$$\lambda B \leq c_B \qquad (x_B)$$
$$\lambda N \leq c_N \qquad (x_B)$$
$$\lambda \text{ unrestricted}$$

Recall that reduced cost vector is $c_N - \underbrace{\boxed{c_B B^{-1}}}_{\lambda} N = c_N - \lambda N$

this solves the dual
if $(x_B,0)$ solves the primal

In the example,

$$\lambda = c_B B^{-1} = [4 \quad 0] \begin{bmatrix} 1/2 & 0 \\ 1 & -1 \end{bmatrix} = [2 \quad 0]$$

# Dual of an LP in equality form

| *Primal* | | *Dual* | |
|---|---|---|---|
| $\min c_B x_B + c_N x_N$ | | $\max \lambda b$ | |
| $Bx_B + Nx_N = b$ | $(\lambda)$ | $\lambda B \le c_B$ | $(x_B)$ |
| $x_B, x_N \ge 0$ | | $\lambda N \le c_N$ | $(x_B)$ |
| | | $\lambda$ unrestricted | |

Recall that reduced cost vector is $c_N - \underbrace{\boxed{c_B B^{-1}}}_{\lambda} N = c_N - \lambda N$

Note that the reduced cost of an individual variable $x_j$ is $r_j = c_j - \lambda \boxed{A_j}$

Column $j$ of $A$

# LP-based Domain Filtering

Let $\begin{array}{l} \min \ cx \\ Ax \geq b \\ x \geq 0 \end{array}$ be an LP relaxation of a CP problem.

- One way to filter the domain of $x_j$ is to minimize and maximize $x_j$ subject to $Ax \geq b$, $x \geq 0$.
  - This is time consuming.
- A faster method is to use **dual multipliers** to derive valid inequalities.
  - A special case of this method uses **reduced costs** to bound or fix variables.
  - **Reduced-cost variable fixing** is a widely used technique in OR.

## Suppose:

$$\min\ cx$$
$$Ax \geq b$$
$$x \geq 0$$

has optimal solution $x^*$, optimal value $v^*$, and optimal dual solution $\lambda^*$.

…and $\lambda_i^* > 0$, which means the $i$-th constraint is tight (complementary slackness);

…and the LP is a relaxation of a CP problem;

…and we have a feasible solution of the CP problem with value $U$, so that $U$ is an upper bound on the optimal value.

Supposing $\begin{array}{l} \min\ cx \\ Ax \geq b \\ x \geq 0 \end{array}$ has optimal solution $x^*$, optimal value $v^*$, and optimal dual solution $\lambda^*$:

If $x$ were to change to a value other than $x^*$, the LHS of $i$-th constraint $A^i x \geq b_i$ would change by some amount $\Delta b_i$.

Since the constraint is tight, this would increase the optimal value as much as changing the constraint to $A^i x \geq b_i + \Delta b_i$.

So it would increase the optimal value at least $\lambda_i^* \Delta b_i$.

Supposing $\begin{array}{l} \min\ cx \\ Ax \geq b \\ x \geq 0 \end{array}$ has optimal solution $x^*$, optimal value $v^*$, and optimal dual solution $\lambda^*$:

We have found: a change in $x$ that changes $A^i x$ by $\Delta b_i$ increases the optimal value of LP at least $\lambda_i^* \Delta b_i$.

Since optimal value of the LP $\leq$ optimal value of the CP $\leq U$,
we have $\lambda_i^* \Delta b_i \leq U - v^*$, or

$$\Delta b_i \leq \frac{U - v^*}{\lambda_i^*}$$

Supposing $\begin{array}{l} \min\ cx \\ Ax \geq b \\ x \geq 0 \end{array}$ has optimal solution $x^*$, optimal value $v^*$, and optimal dual solution $\lambda^*$:

We have found: a change in $x$ that changes $A^i x$ by $\Delta b_i$ increases the optimal value of LP at least $\lambda_i^* \Delta b_i$.

Since optimal value of the LP ≤ optimal value of the CP ≤ $U$,
we have $\lambda_i^* \Delta b_i \leq U - v^*$, or

$$\Delta b_i \leq \frac{U - v^*}{\lambda_i^*}$$

Since $\Delta b_i = A^i x - A^i x^* = A^i x - b_i$, this implies the inequality

$$A^i x \leq b_i + \frac{U - v^*}{\lambda_i^*}$$

…which can be propagated.

## Example

$$\min\ 4x_1 + 7x_2$$
$$2x_1 + 3x_2 \geq 6 \qquad (\lambda_1 = 2)$$
$$2x_1 + x_2 \geq 4 \qquad (\lambda_1 = 0)$$
$$x_1, x_2 \geq 0$$

Suppose we have a feasible solution of the original CP with value $U$ = 13.

Since the first constraint is tight, we can propagate the inequality

$$A^1 x \leq b_1 + \frac{U - v^*}{\lambda_1^*}$$

or $$2x_1 + 3x_2 \leq 6 + \frac{13-12}{2} = 6.5$$

## Reduced-cost domain filtering

Suppose $x_j^* = 0$, which means the constraint $x_j \geq 0$ is tight.

The inequality $A^i x \leq b_i + \dfrac{U - v^*}{\lambda_i^*}$ becomes $x_j \leq \dfrac{U - v^*}{\boxed{r_j}}$

The dual multiplier for $x_j \geq 0$ is the reduced cost $r_j$ of $x_j$, because increasing $x_j$ (currently 0) by 1 increases optimal cost by $r_j$.

Similar reasoning can bound a variable below when it is at its upper bound.

## Example

$$\min\ 4x_1 + 7x_2$$
$$2x_1 + 3x_2 \geq 6 \qquad (\lambda_1 = 2)$$
$$2x_1 + x_2 \geq 4 \qquad (\lambda_1 = 0)$$
$$x_1, x_2 \geq 0$$

Suppose we have a feasible solution of the original CP with value $U$ = 13.

Since $x_2^*$ = 0, we have $x_2 \leq \dfrac{U - v^*}{r_2}$

or $x_2 \leq \dfrac{13-12}{2} = 0.5$

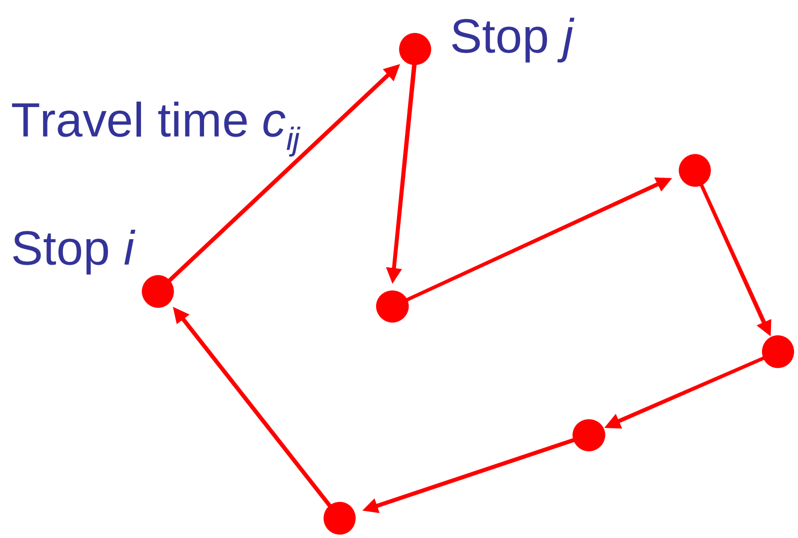
If $x_2$ is required to be integer, we can fix it to zero. This is **reduced-cost variable fixing.**

# Example: Single-Vehicle Routing

A vehicle must make several stops and return home, perhaps subject to time windows.

The objective is to find the order of stops that minimizes travel time.

This is also known as the **traveling salesman problem** (with time windows).



## Assignment Relaxation

$$\min \sum_{ij} c_{ij} x_{ij}$$

= 1 if stop *i* immediately precedes stop *j*

$$\sum_j x_{ij} = \sum_j x_{ji} = 1, \ \text{all } i$$

Stop *i* is preceded and followed by exactly one stop.

$$x_{ij} \in \{0,1\}, \ \text{all } i, j$$

## Assignment Relaxation

$$\min \sum_{ij} c_{ij} x_{ij}$$

= 1 if stop *i* immediately precedes stop *j*

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Stop *i* is preceded and followed by exactly one stop.

$$0 \le x_{ij} \le 1, \text{ all } i, j$$

Because this problem is **totally unimodular**, it can be solved as an LP.

The relaxation provides a very weak lower bound on the optimal value.

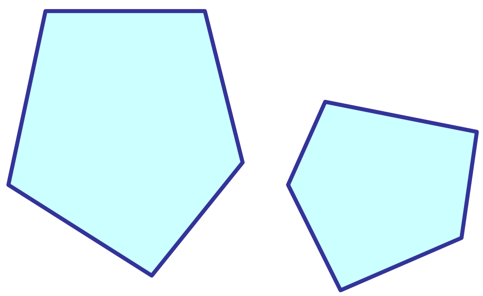
But **reduced-cost variable fixing** can be very useful in a CP context.

# Disjunctions of linear systems

Disjunctions of linear systems often occur naturally in problems and can be given a convex hull relaxation.

A disjunction of linear systems represents a union of polyhedra.

$$\min \; cx$$
$$\bigvee_k \left( A^k x \ge b^k \right)$$



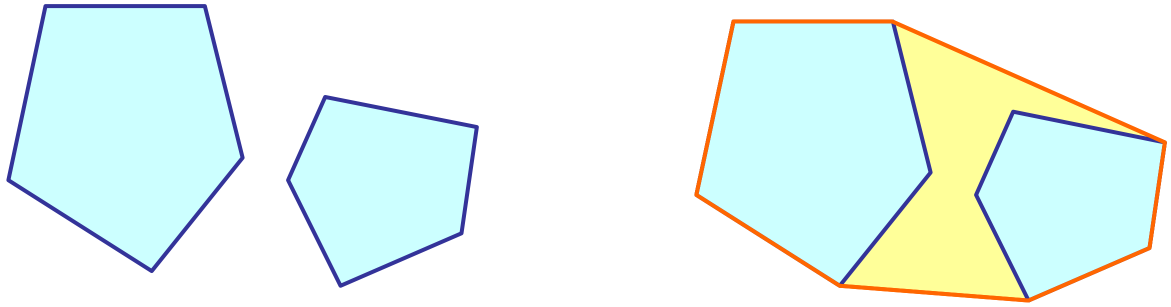
# Relaxing a disjunction of linear systems

Disjunctions of linear systems often occur naturally in problems and can be given a convex hull relaxation.

A disjunction of linear systems represents a union of polyhedra.

We want a convex hull relaxation (tightest linear relaxation).

$$\min \; cx$$
$$\bigvee_k \left( A^k x \geq b^k \right)$$



# Relaxing a disjunction of linear systems

Disjunctions of linear systems often occur naturally in problems and can be given a convex hull relaxation.

The closure of the convex hull of

$$\begin{aligned}&\min\ cx\\&\bigvee_k \left(A^k x \ge b^k\right)\end{aligned}$$

…is described by

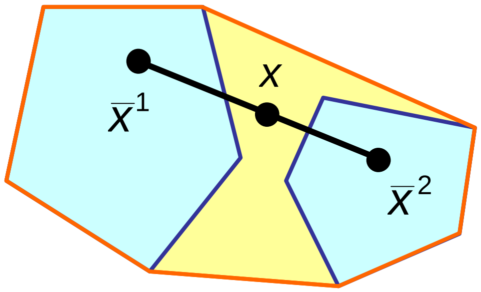
$$\begin{aligned}&\min\ cx\\&A^k x^k \ge b^k y_k,\ \text{all } k\\&\sum_k y_k = 1\\&x = \sum_k x^k\\&0 \le y_k \le 1\end{aligned}$$

## Why?

To derive convex hull relaxation of a disjunction…

Write each solution as a convex combination of points in the polyhedron

$$\begin{aligned}
&\min\ cx\\
&A^k \bar{x}^k \geq b^k,\ \text{all } k\\
&\sum_k y_k = 1\\
&x = \sum_k y_k \bar{x}^k\\
&0 \leq y_k \leq 1
\end{aligned}$$




Convex hull relaxation (tightest linear relaxation)

## Why?

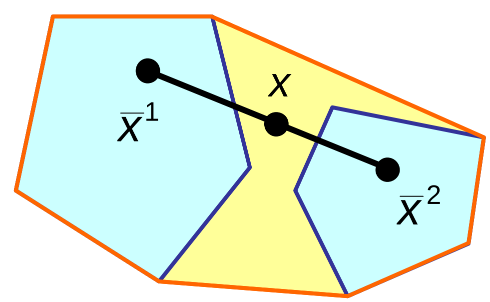
To derive convex hull relaxation of a disjunction…

Write each solution as a convex combination of points in the polyhedron

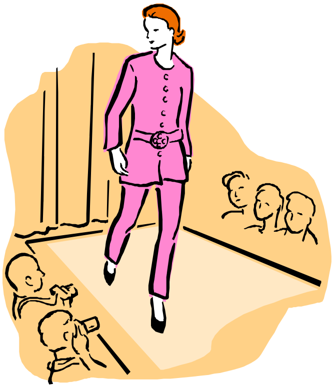
$$\begin{aligned} &\min\ cx \\ &A^k \bar{x}^k \geq b^k,\ \text{all } k \\ &\sum_k y_k = 1 \\ &x = \sum_k y_k \bar{x}^k \\ &0 \leq y_k \leq 1 \end{aligned}$$

Change of variable $x = y_k \bar{x}^k$

$$\begin{aligned} &\min\ cx \\ &A^k x^k \geq b^k y_k,\ \text{all } k \\ &\sum_k y_k = 1 \\ &x = \sum_k x^k \\ &0 \leq y_k \leq 1 \end{aligned}$$




Convex hull relaxation (tightest linear relaxation)



# Mixed Integer/Linear Modeling

MILP Representability

Disjunctive Modeling

Knapsack Modeling

## Motivation

A **mixed integer/linear programming** (MILP) problem has the form

$$
\begin{aligned}
&\min\ cx + dy \\
&Ax + by \geq b \\
&x, y \geq 0 \\
&y \text{ integer}
\end{aligned}
$$

- We can **relax** a CP problem by modeling some constraints with an MILP.
- If desired, we can then **relax the MILP** by dropping the integrality constraint, to obtain an LP.
- The LP relaxation can be strengthened with **cutting planes**.
- The first step is to learn **how to write** MILP models.

# MILP Representability

A subset $S$ of $\mathbb{R}^n$ is **MILP representable** if it is the projection onto $x$ of some MILP constraint set of the form

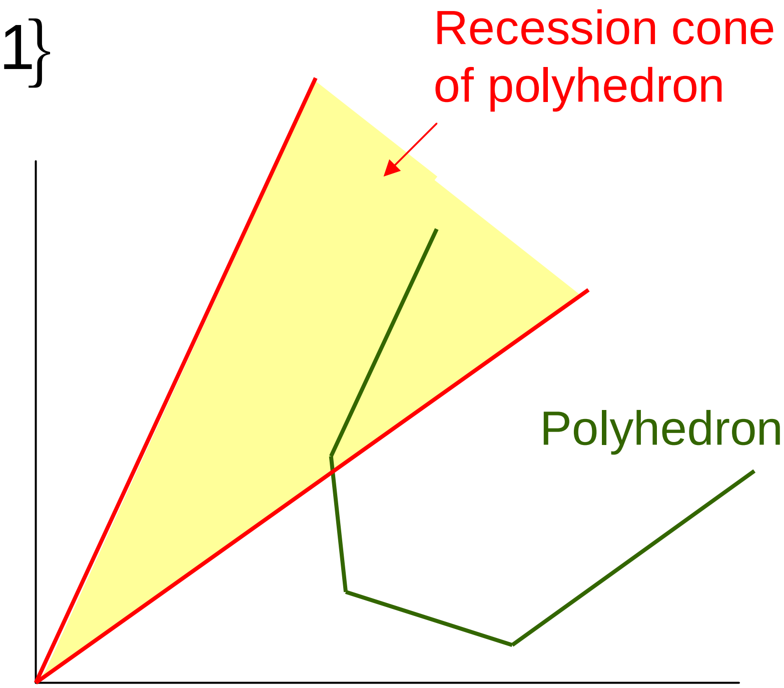
$$
\begin{aligned}
& Ax + Bu + Dy \geq b \\
& x, y \geq 0 \\
& x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ y_k \in \{0,1\}
\end{aligned}
$$

# MILP Representability

A subset $S$ of $\mathbb{R}^n$ is **MILP representable** if it is the projection onto $x$ of some MILP constraint set of the form

$$Ax + Bu + Dy \geq b$$
$$x, y \geq 0$$
$$x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ y_k \in \{0,1\}$$

**Theorem**. $S \subset \mathbb{R}^n$ is MILP representable if and only if $S$ is the union of finitely many polyhedra having the same recession cone.



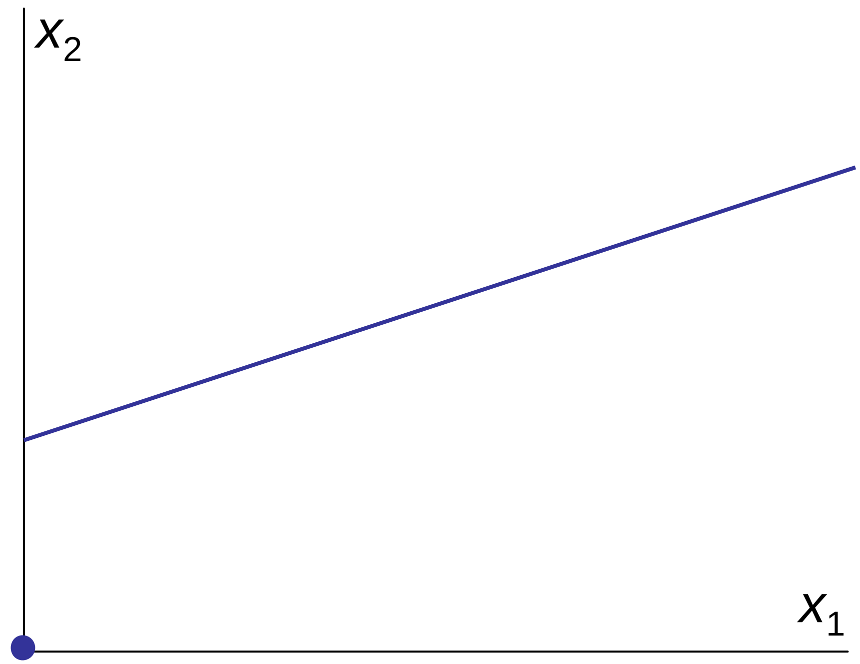
## Example: Fixed charge function

Minimize a fixed charge function:

$$\min \ x_2$$

$$x_2 \geq \begin{Bmatrix} 0 & \text{if } x_1 = 0 \\ f + cx_1 & \text{if } x_1 > 0 \end{Bmatrix}$$

$$x_1 \geq 0$$



## Example

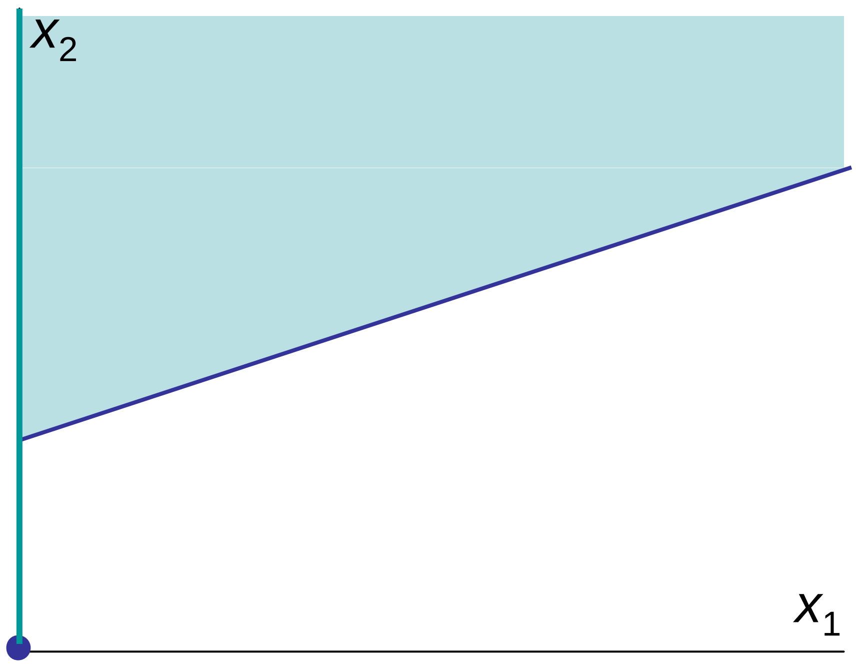
Minimize a fixed charge function:

$$\min \ x_2$$

$$x_2 \geq \begin{cases} 0 & \text{if } x_1 = 0 \\ f + cx_1 & \text{if } x_1 > 0 \end{cases}$$

$$x_1 \geq 0$$

Feasible set



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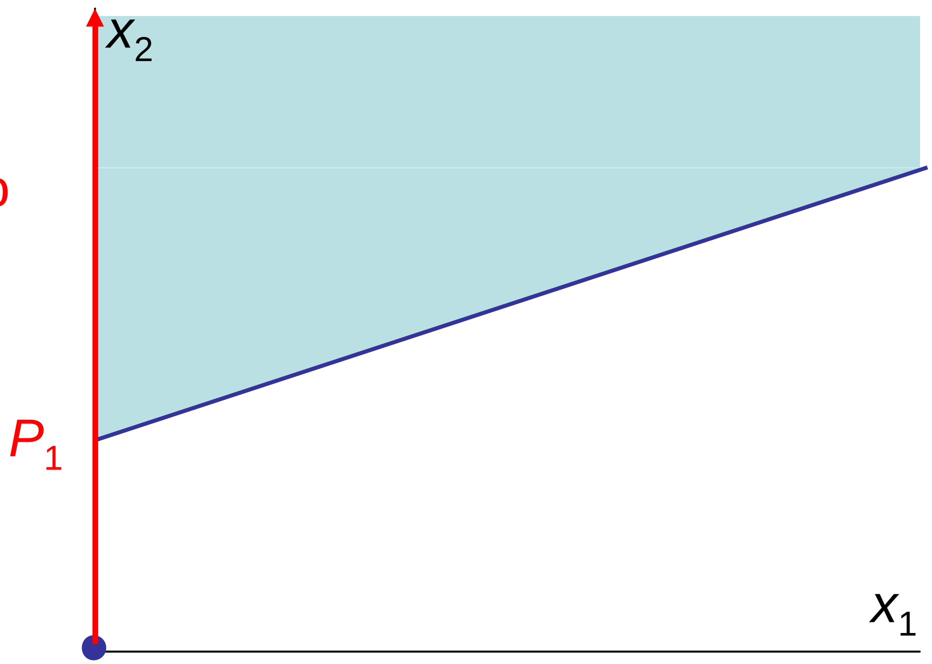
$$x_1 \geq 0$$

Union of two polyhedra $P_1$, $P_2$



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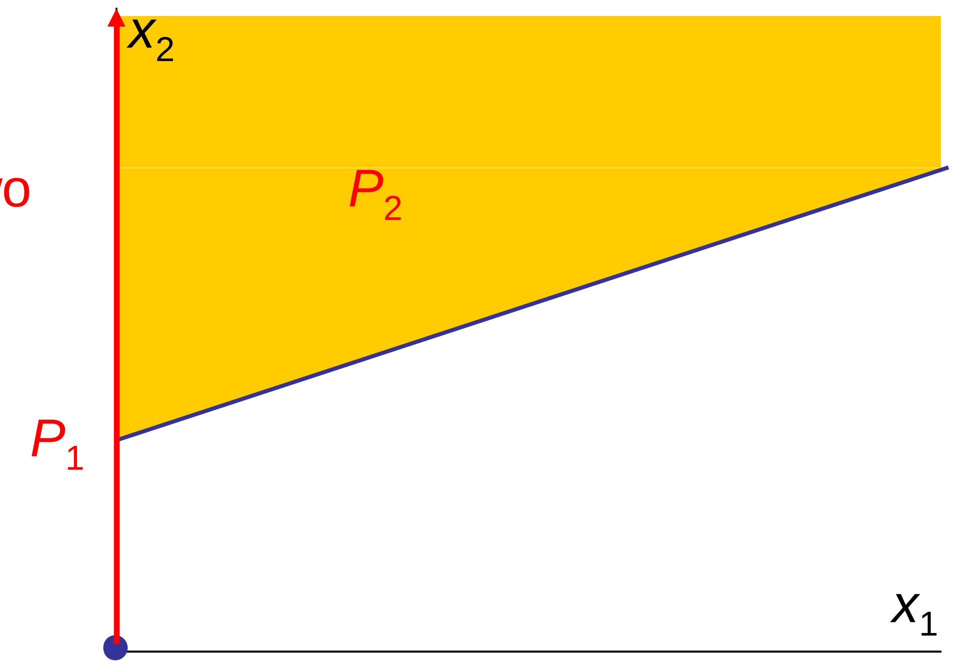

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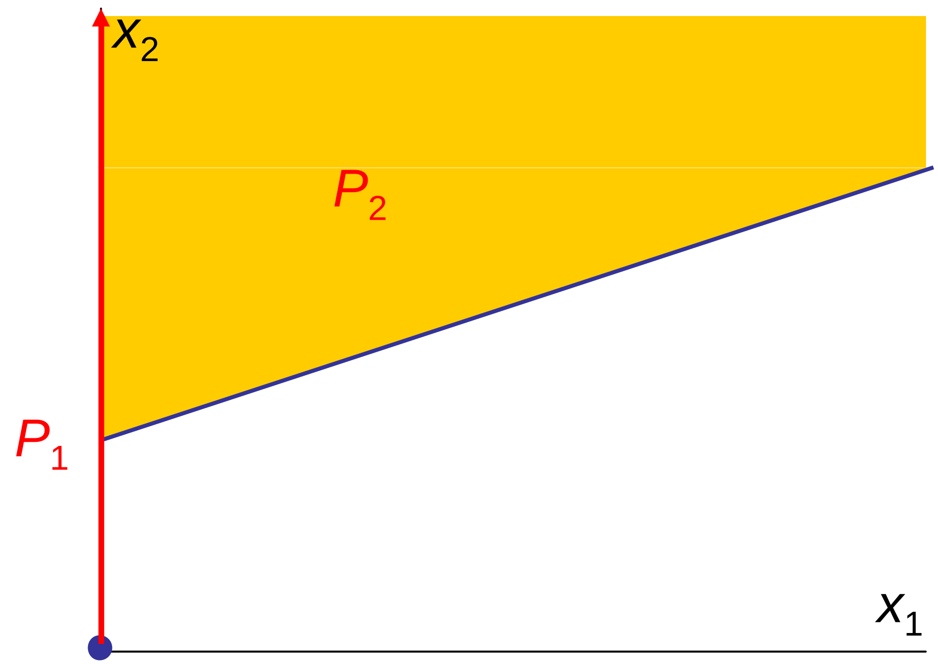
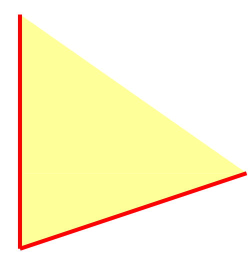
$$x_2 \geq \begin{cases} 0 & \text{if } x_1 = 0 \\ f + cx_1 & \text{if } x_1 > 0 \end{cases}$$

$$x_1 \geq 0$$

The polyhedra have different recession cones.




$P_1$ recession cone



$P_2$ recession cone

## Example

Minimize a fixed charge function:

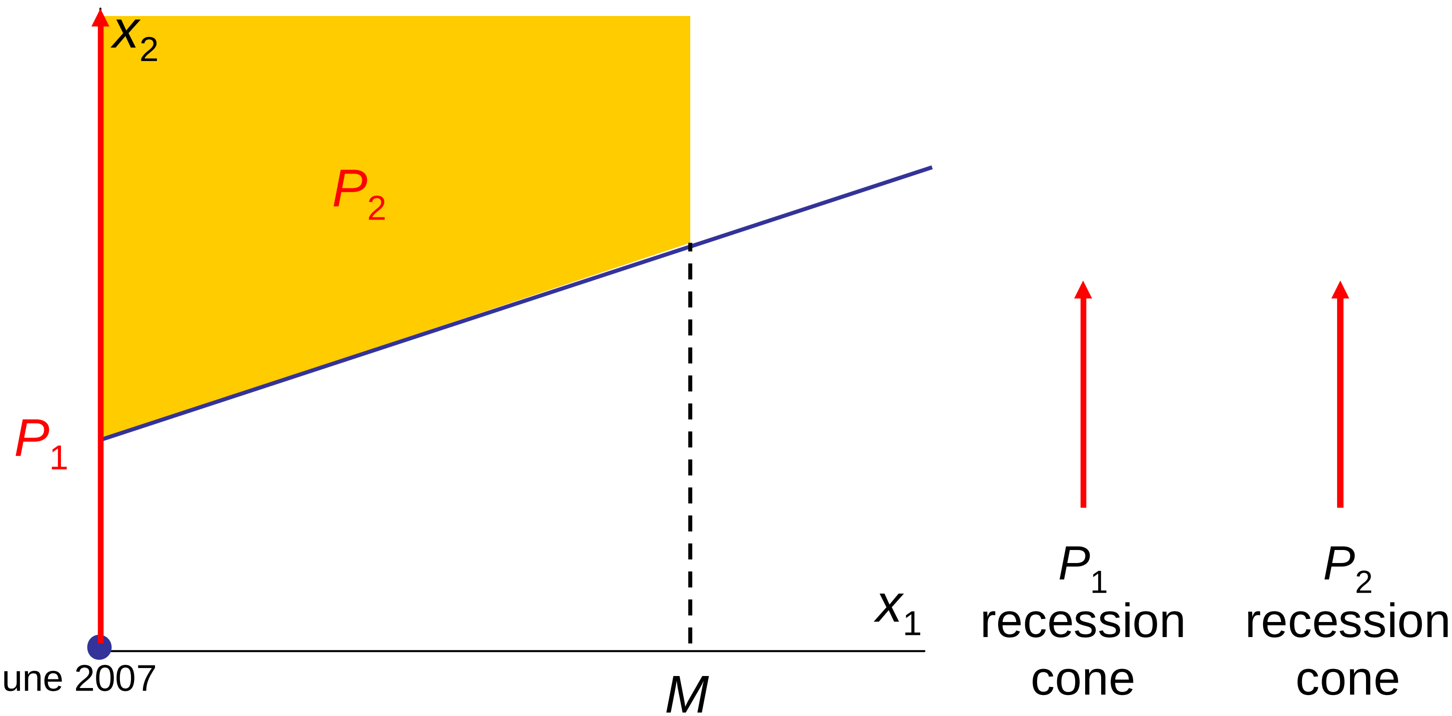
$$\min \; x_2$$

$$x_2 \geq \begin{Bmatrix} 0 & \text{if } x_1 = 0 \\ f + cx_1 & \text{if } x_1 > 0 \end{Bmatrix}$$

Add an upper bound on $x_1$

$$0 \leq x_1 \leq M$$

The polyhedra have the same recession cone.



# Modeling a union of polyhedra

Start with a disjunction of linear systems to represent the union of polyhedra.

The *k*th polyhedron is $\{x \mid A^k x \geq b\}$

$$\min\ cx$$
$$\bigvee_k \left( A^k x \geq b^k \right)$$

Introduce a 0-1 variable $y_k$ that is 1 when $x$ is in polyhedron <u>*k*</u>.

Disaggregate $x$ to create an $x^k$ for each $k$.

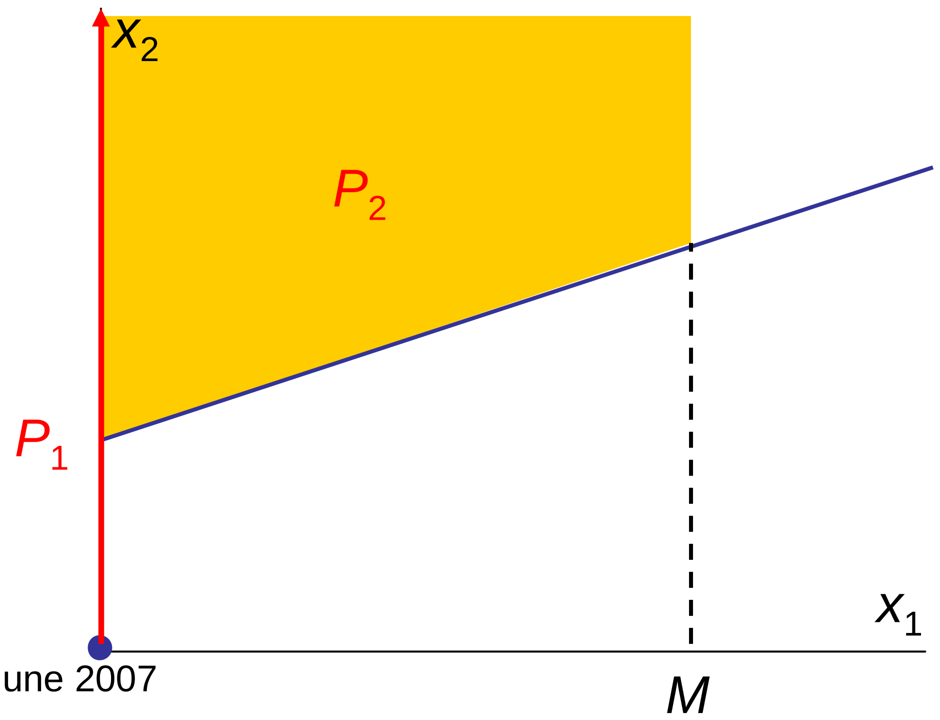
$$\min\ cx$$
$$A^k x^k \geq b^k y_k, \text{ all } k$$
$$\sum_k y_k = 1$$
$$x = \sum_k x^k$$
$$y_k \in \{0,1\}$$

## Example

Start with a disjunction of linear systems to represent the union of polyhedra

$$\min \ x_2$$

$$\begin{pmatrix} x_1 = 0 \\ x_2 \geq 0 \end{pmatrix} \vee \begin{pmatrix} 0 \leq x_1 \leq M \\ x_2 \geq f + cx_1 \end{pmatrix}$$



# Example

Start with a disjunction of linear systems to represent the union of polyhedra

$$\min \; x_2$$
$$\begin{pmatrix} x_1 = 0 \\ x_2 \ge 0 \end{pmatrix} \vee \begin{pmatrix} 0 \le x_1 \le M \\ x_2 \ge f + cx_1 \end{pmatrix}$$

Introduce a 0-1 variable $y_k$ that is 1 when $x$ is in polyhedron <u>$k$</u>.

Disaggregate $x$ to create an $x^k$ for each $k$.

$$\min \; cx$$
$$x_1^1 = 0, \;\; x_2^1 \ge 0$$
$$0 \le x_1^2 \le My_2, \quad -cx_1^2 + x_2^2 \ge fy_2$$
$$y_1 + y_2 = 1, \;\; y_k \in \{0,1\}$$
$$x = x^1 + x^2$$

# Example

To simplify:

Replace $x_1^2$ with $x_1$.

Replace $x_2^2$ with $x_2$.

Replace $y_2$ with $y$.

$$
\begin{aligned}
&\min\ x_2 \\
&x_1^1 = 0,\ \ x_2^1 \ge 0 \\
&0 \le x_1^2 \le My_2, \quad -cx_1^2 + x_2^2 \ge fy_2 \\
&y_1 + y_2 = 1,\ \ y_k \in \{0,1\} \\
&x = x^1 + x^2
\end{aligned}
$$

This yields

$$
\begin{aligned}
&\min\ x_2 \\
&0 \le x_1 \le My \\
&x_2 \ge fy + cx_1 \\
&y \in \{0,1\}
\end{aligned}
$$

or

$$
\begin{aligned}
&\min\ fy + cx \\
&0 \le x \le \boxed{M}y \\
&y \in \{0,1\}
\end{aligned}
$$

"Big $M$ "

# Disjunctive Modeling

Disjunctions often occur naturally in problems and can be given an MILP model.

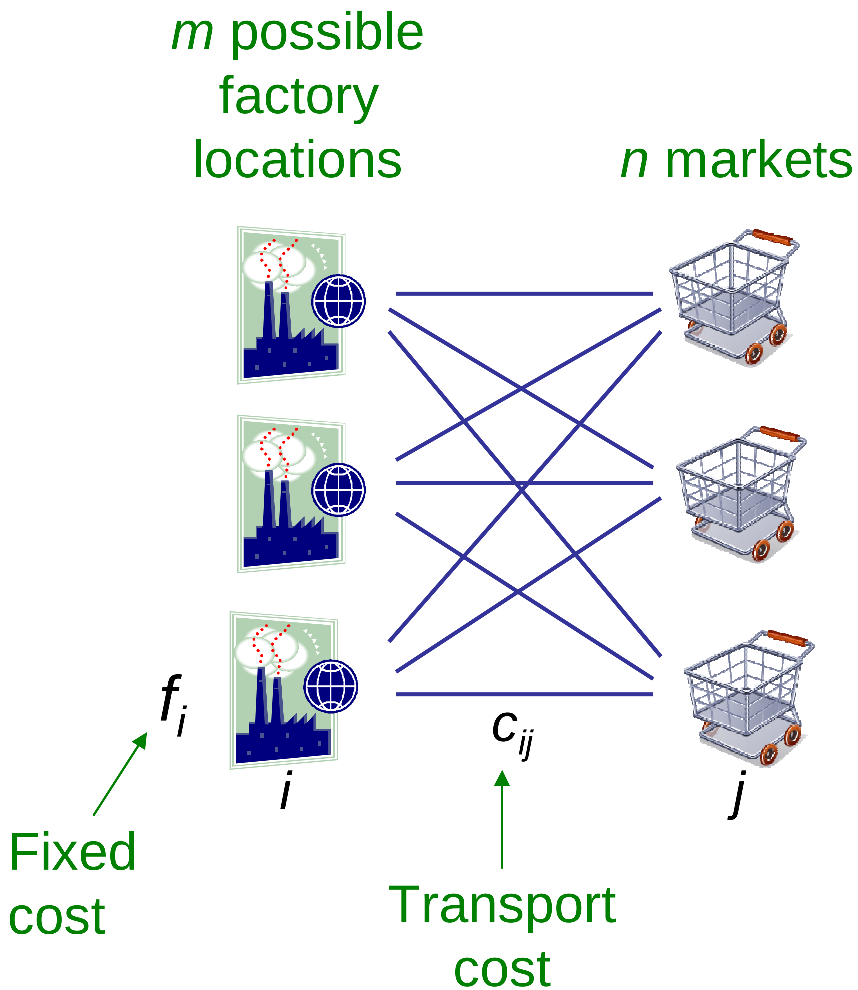
Recall that a disjunction of linear systems (representing polyhedra with the same recession cone)

$$\min \; cx$$
$$\bigvee_k \left( A^k x \geq b^k \right)$$

…has the MILP model

$$\begin{aligned}
&\min \; cx \\
&A^k x^k \geq b^k y_k, \text{ all } k \\
&\sum_k y_k = 1 \\
&x = \sum_k x^k \\
&y_k \in \{0,1\}
\end{aligned}$$

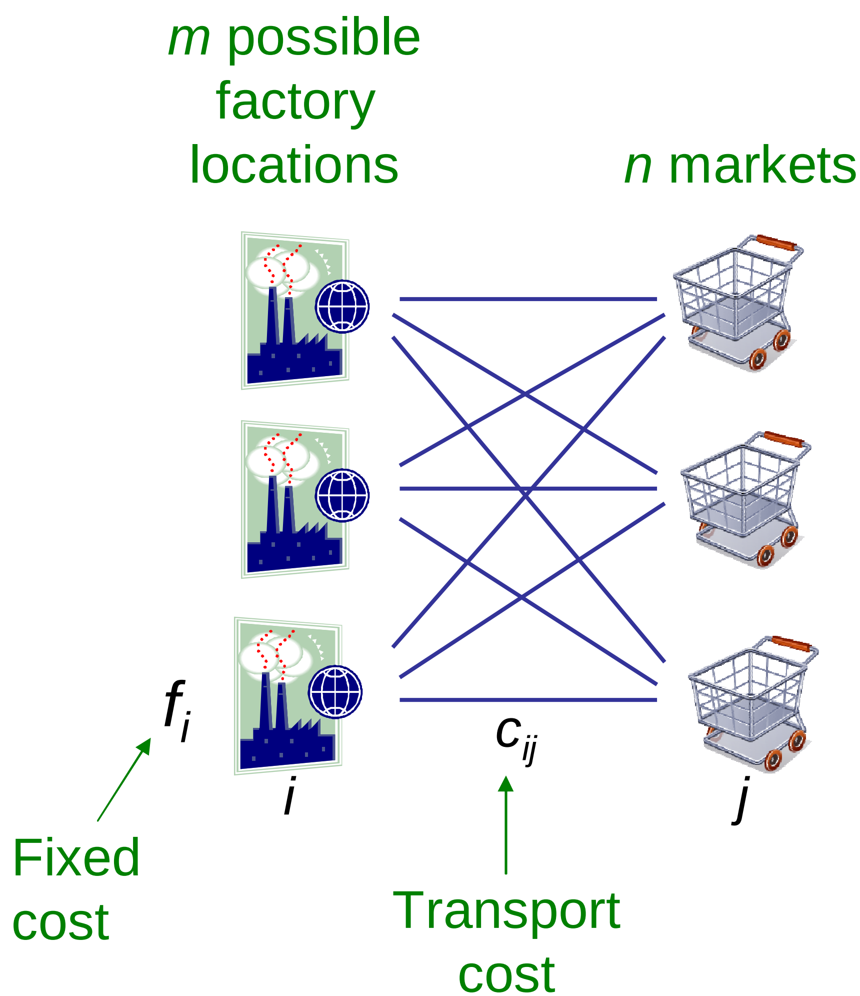
# Example:  Uncapacitated facility location




Locate factories to serve markets so as to minimize total fixed cost and transport cost.

No limit on production capacity of each factory.

## Uncapacitated facility location




Disjunctive model:

Fraction of market *j*'s demand satisfied from location *i*

$$\min \sum_i z_i + \sum_{ij} c_{ij} \boxed{x_{ij}}$$

$$\begin{pmatrix} x_{ij} = 0, \text{ all } j \\ z_i = 0 \end{pmatrix} \vee \begin{pmatrix} 0 \le x_{ij} \le 1, \text{ all } j \\ z_i \ge f_i \end{pmatrix}, \quad \text{all } i$$

$$\sum_i x_{ij} = 1, \quad \text{all } j$$

No factory at location *i*

Factory at location *i*

# Uncapacitated facility location

**MILP formulation:**

$$\min \sum_i f_i y_i + \sum_{ij} c_{ij} x_{ij}$$

$$0 \le x_{ij} \le y_i, \quad \text{all } i, j$$

$$y_i \in \{0,1\}$$

**Disjunctive model:**

$$\min \sum_i z_i + \sum_{ij} c_{ij} x_{ij}$$

$$\begin{pmatrix} x_{ij} = 0, \ \text{all } j \\ z_i = 0 \end{pmatrix} \vee \begin{pmatrix} 0 \le x_{ij} \le 1, \ \text{all } j \\ z_i \ge f_i \end{pmatrix}, \quad \text{all } i$$

$$\sum_i x_{ij} = 1, \quad \text{all } j$$

No factory at location *i*

Factory at location *i*

## Uncapacitated facility location

Maximum output from location i

MILP formulation:

$$\min \sum_i f_i y_i + \sum_{ij} c_{ij} x_{ij}$$
$$0 \le x_{ij} \le y_i, \quad \text{all } i, j$$
$$y_i \in \{0,1\}$$

Beginner's model:

$$\min \sum_i f_i y_i + \sum_{ij} c_{ij} x_{ij}$$
$$\sum_j x_{ij} \le \boxed{n y_i}, \quad \text{all } i, j$$
$$y_i \in \{0,1\}$$

Based on capacitated location model.

It has a **weaker continuous relaxation** (obtained by replacing $y_i \in \{0,1\}$ with $0 \le y_i \le 1$).

This beginner's mistake can be avoided by starting with disjunctive formulation.

# Knapsack Modeling

- Knapsack models consist of **knapsack covering** and **knapsack packing** constraints.
- The freight transfer model presented earlier is an example.
- We will consider a similar example that combines disjunctive and knapsack modeling.
- Most OR professionals are unlikely to write a model as good as the one presented here.

## Note on tightness of knapsack models

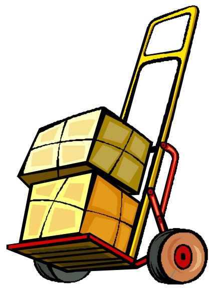
- The continuous relaxation of a knapsack model is not in general a convex hull relaxation.

    - A disjunctive formulation would provide a convex hull relaxation, but there are exponentially many disjuncts.

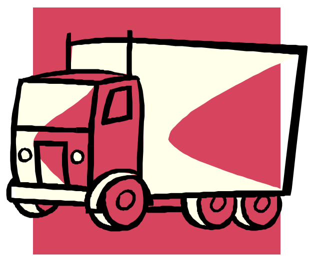
- Knapsack cuts can significantly tighten the relaxation.

# Example: Package transport

Each package $j$ has size $a_j$



Each truck $i$ has capacity $Q_i$ and costs $c_i$ to operate



Disjunctive model

Knapsack constraints

$$\min \sum_i z_i$$

$$\sum_i Q_i y_i \geq \sum_j a_j; \quad \sum_i x_{ij} = 1, \text{ all } j$$

$$\begin{pmatrix} y_i = 1 \\ z_i = c_i \\ \sum_j a_j x_{ij} \leq Q_i \\ 0 \leq x_{ij} \leq 1, \text{ all } j \end{pmatrix} \vee \begin{pmatrix} y_i = 0 \\ z_i = 0 \\ x_{ij} = 0 \end{pmatrix}, \quad \text{all } i$$

$$\boxed{x_{ij}}, \boxed{y_i} \in \{0,1\}$$

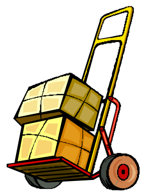
Truck $i$ used

Truck $i$ not used

1 if truck $i$ carries package $j$

1 if truck $i$ is used

# Example: Package transport



## MILP model

$$\min \sum_i c_i y_i$$

$$\sum_i Q_i y_i \ge \sum_j a_j; \quad \sum_i x_{ij} = 1, \text{ all } j$$

$$\sum_j a_j x_{ij} \le Q_i y_i, \quad \text{all } i$$

$$x_{ij} \le y_i, \quad \text{all } i, j$$

$$x_{ij}, y_i \in \{0,1\}$$

## Disjunctive model

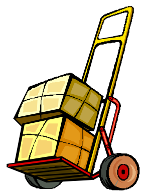
$$\min \sum_i z_i$$

$$\sum_i Q_i y_i \ge \sum_j a_j; \quad \sum_i x_{ij} = 1, \text{ all } j$$

$$\begin{pmatrix} y_i = 1 \\ z_i = c_i \\ \sum_j a_j x_{ij} \le Q_i \\ 0 \le x_{ij} \le 1, \text{ all } j \end{pmatrix} \vee \begin{pmatrix} y_i = 0 \\ z_i = 0 \\ x_{ij} = 0 \end{pmatrix}, \quad \text{all } i$$

$$x_{ij}, y_i \in \{0,1\}$$

## Example: Package transport



Most OR professionals would omit this constraint, since it is the sum over *i* of the next constraint. But it generates very effective knapsack cuts.

### MILP model

$$\min \sum_i c_i y_i$$

$$\boxed{\sum_i Q_i y_i \ge \sum_j a_j}; \quad \sum_i x_{ij} = 1, \text{ all } j$$

$$\sum_j a_j x_{ij} \le Q_i \boxed{y_i}, \quad \text{all } i$$

$$x_{ij} \le y_i, \quad \text{all } i, j$$

$$x_{ij}, y_i \in \{0,1\}$$

Modeling trick; unobvious without disjunctive approach

# Cutting Planes

0-1 Knapsack Cuts

Gomory Cuts

Mixed Integer Rounding Cuts

Example: Product Configuration

# To review…

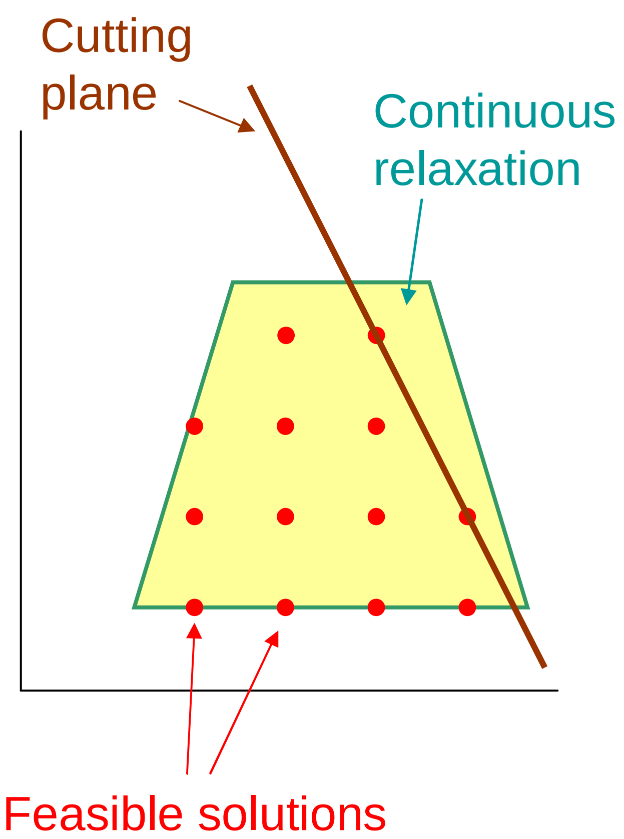
A **cutting plane** (cut, valid inequality) for an MILP model:

- …is **valid**

    - It is satisfied by all feasible solutions of the model.

- …**cuts off** solutions of the continuous relaxation.

    - This makes the relaxation tighter.



## Motivation

- **Cutting planes** (cuts) tighten the continuous relaxation of an MILP model.
- **Knapsack cuts**
  - Generated for individual knapsack constraints.
  - We saw **general integer knapsack cuts** earlier.
  - **0-1 knapsack cuts** and **lifting** techniques are well studied and widely used.
- **Rounding cuts**
  - Generated for the entire MILP, they are widely used.
  - **Gomory cuts** for integer variables only.
  - **Mixed integer rounding cuts** for any MILP.

# 0-1 Knapsack Cuts

0-1 knapsack cuts are designed for knapsack constraints with 0-1 variables.

The analysis is different from that of general knapsack constraints, to exploit the special structure of 0-1 inequalities.

# 0-1 Knapsack Cuts

0-1 knapsack cuts are designed for knapsack constraints with 0-1 variables.

The analysis is different from that of general knapsack constraints, to exploit the special structure of 0-1 inequalities.

Consider a 0-1 knapsack packing constraint $ax \le a_0$. (Knapsack covering constraints are similarly analyzed.)

Index set $J$ is a **cover** if $\sum_{j \in J} a_j > a_0$

The **cover inequality** $\sum_{j \in J} x_j \le |J| - 1$ is a **0-1 knapsack cut** for $ax \le a_0$

Only **minimal** covers need be considered.

## Example

$J$ = {1,2,3,4} is a cover for

$$6x_1 + 5x_2 + 5x_3 + 5x_4 + 8x_5 + 3x_6 \le 17$$

This gives rise to the cover inequality

$$x_1 + x_2 + x_3 + x_4 \le 3$$

Index set $J$ is a **cover** if $\sum_{j \in J} a_j > a_0$

The **cover inequality** $\sum_{j \in J} x_j \le |J| - 1$ is a **0-1 knapsack cut** for $ax \le a_0$

Only **minimal** covers need be considered.

## Sequential lifting

- A cover inequality can often be strengthened by **lifting** it into a higher dimensional space.
    - That is, by adding variables.
- **Sequential lifting** adds one variable at a time.
- **Sequence-independent lifting** adds several variables at once.

## Sequential lifting

To lift a cover inequality $\sum_{j \in J} x_j \le |J| - 1$

add a term to the left-hand side $\sum_{j \in J} x_j + \pi_k x_k \le |J| - 1$

where $\pi_k$ is the largest coefficient for which the inequality is still valid.

So, $\pi_k = |J| - 1 - \max_{\substack{x_j \in \{0,1\} \\ \text{for } j \in J}} \left\{ \sum_{j \in J} x_j \,\middle|\, \sum_{j \in J} a_j x_j \le a_0 - a_k \right\}$

This can be done repeatedly (by dynamic programming).

## Example

Given $6x_1 + 5x_2 + 5x_3 + 5x_4 + 8x_5 + 3x_6 \leq 17$

To lift $x_1 + x_2 + x_3 + x_4 \leq 3$

add a term to the left-hand side $x_1 + x_2 + x_3 + x_4 + \pi_5 x_5 \leq 3$

where

$$\pi_5 = 3 - \max_{\substack{x_j \in \{0,1\} \\ \text{for } j \in \{1,2,3,4\}}} \left\{ x_1 + x_2 + x_3 + x_4 \,\middle|\, 6x_1 + 5x_2 + 5x_3 + 5x_4 \leq 17 - 8 \right\}$$

This yields $x_1 + x_2 + x_3 + x_4 + 2x_5 \leq 3$

Further lifting leaves the cut unchanged.

But if the variables are added in the order $x_6$, $x_5$, the result is different:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$$

## Sequence-independent lifting

- Sequence-independent lifting usually yields a weaker cut than sequential lifting.
  - But it adds all the variables at once and is much faster.
  - Commonly used in commercial MILP solvers.

## Sequence-independent lifting

To lift a cover inequality $\sum_{j \in J} x_j \le |J| - 1$

add terms to the left-hand side $\sum_{j \in J} x_j + \sum_{j \notin J} \rho(a_j) x_k \le |J| - 1$

where
$$\rho(u) = \begin{cases} j & \text{if } A_j \le u \le A_{j+1} - \Delta \ \text{ and } \ j \in \{0, \ldots, p-1\} \\ j + (u - A_j)/\Delta & \text{if } A_j - \Delta \le u < A_j - \Delta \ \text{ and } \ j \in \{1, \ldots, p-1\} \\ p + (u - A_p)/\Delta & \text{if } A_p - \Delta \le u \end{cases}$$

with
$$\Delta = \sum_{j \in J} a_j - a_0 \qquad A_j = \sum_{k=1}^{j} a_k$$
$$J = \{1, \ldots, p\} \qquad A_0 = 0$$

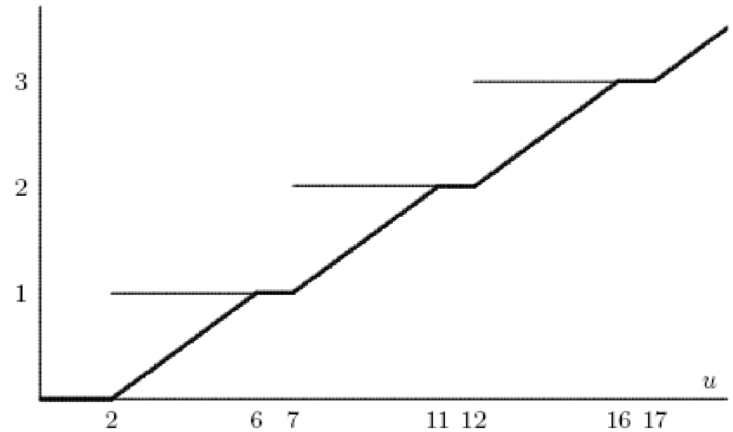
# Example

Given $6x_1 + 5x_2 + 5x_3 + 5x_4 + 8x_5 + 3x_6 \leq 17$

To lift $x_1 + x_2 + x_3 + x_4 \leq 3$

Add terms $x_1 + x_2 + x_3 + x_4 + \rho(8)x_5 + \rho(3)x_6 \leq 3$

where ρ($u$) is given by




This yields the lifted cut

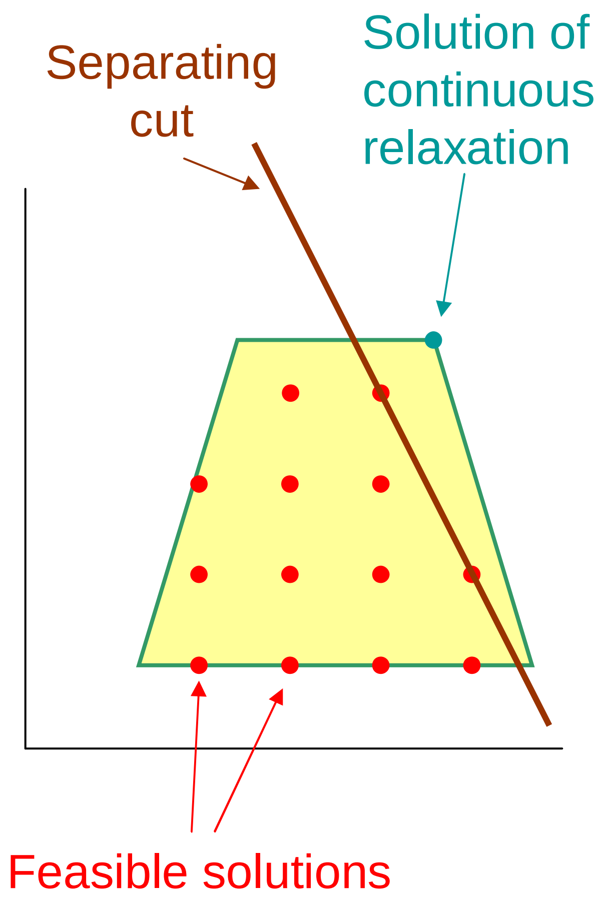
$$x_1 + x_2 + x_3 + x_4 + (5/4)x_5 + (1/4)x_6 \leq 3$$

# Gomory Cuts

• When an integer programming problem has a nonintegral solution, we can generate at least one **Gomory cut** to cut off that solution.

- This is a special case of a **separating cut**, because it separates the current solution of the relaxation from the feasible set.

• Gomory cuts are widely used and very effective in MILP solvers.



# Gomory cuts

Given an integer programming problem

$$\min cx$$
$$Ax = b$$
$$x \geq 0 \text{ and integral}$$

Let $(x_B, 0)$ be an optimal solution of the continuous relaxation, where

$$x_B = \hat{b} - \hat{N} x_N$$
$$\hat{b} = B^{-1} b, \quad \hat{N} = B^{-1} N$$

Then if $x_i$ is nonintegral in this solution, the following **Gomory cut** is violated by $(x_B, 0)$:

$$x_i + \lfloor \hat{N}_i \rfloor x_N \leq \lfloor \hat{b}_i \rfloor$$

## Example

$$\min\ 2x_1 + 3x_2$$
$$x_1 + 3x_2 \ge 3$$
$$4x_1 + 3x_2 \ge 6$$
$$x_1, x_2 \ge 0 \text{ and integral}$$

or

$$\min\ 2x_1 + 3x_2$$
$$x_1 + 3x_2 - x_3 = 3$$
$$4x_1 + 3x_2 - x_4 = 6$$
$$x_j \ge 0 \text{ and integral}$$

Optimal solution of the continuous relaxation has

$$x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$$

$$\hat{N} = \begin{bmatrix} 1/3 & -1/3 \\ -4/9 & 1/9 \end{bmatrix}$$

$$\hat{b} = \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$$

# Example

$$\min\ 2x_1 + 3x_2$$
$$x_1 + 3x_2 \geq 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1, x_2 \geq 0 \text{ and integral}$$

or

$$\min\ 2x_1 + 3x_2$$
$$x_1 + 3x_2 - x_3 = 3$$
$$4x_1 + 3x_2 - x_4 = 6$$
$$x_j \geq 0 \text{ and integral}$$

Optimal solution of the continuous relaxation has

$$x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$$

$$\hat{N} = \begin{bmatrix} 1/3 & -1/3 \\ -4/9 & 1/9 \end{bmatrix}$$

$$\hat{b} = \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$$

The Gomory cut $x_i + \lfloor \hat{N}_i \rfloor x_N \leq \lfloor \hat{b}_i \rfloor$

is $x_2 + \lfloor [-4/9 \quad 1/9] \rfloor \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \leq \lfloor 2/3 \rfloor$

or $x_2 - x_3 \leq 0$

In $x_1, x_2$ space this is $x_1 + 2x_2 \geq 3$

# Example

$$\min\ 2x_1 + 3x_2$$
$$x_1 + 3x_2 \geq 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1, x_2 \geq 0 \text{ and integral}$$

or

$$\min\ 2x_1 + 3x_2$$
$$x_1 + 3x_2 - x_3 = 3$$
$$4x_1 + 3x_2 - x_4 = 6$$
$$x_j \geq 0 \text{ and integral}$$

Optimal solution of the continuous relaxation has

$$x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$$

$$\hat{N} = \begin{bmatrix} 1/3 & -1/3 \\ -4/9 & 1/9 \end{bmatrix}$$

$$\hat{b} = \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$$

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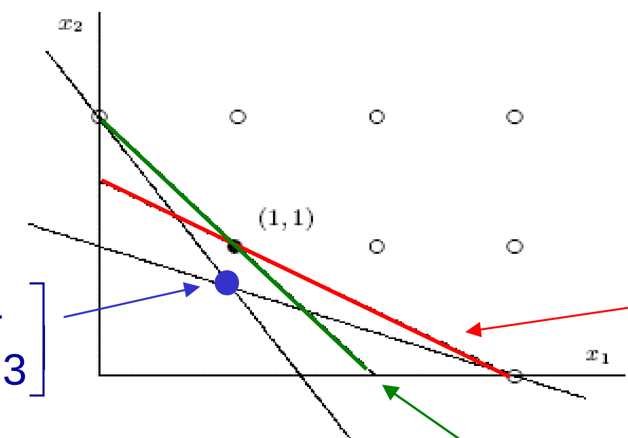



Gomory cut $x_1 + 2x_2 \geq 3$

Gomory cut after re-solving LP with previous cut.

# Mixed Integer Rounding Cuts

- **Mixed integer rounding** (MIR) **cuts** can be generated for solutions of any relaxed MILP in which one or more integer variables has a fractional value.
  - Like Gomory cuts, they are separating cuts.
  - MIR cuts are widely used in commercial solvers.

# MIR cuts

Given an MILP problem

$$\min cx + dy$$
$$Ax + Dy = b$$
$$x, y \geq 0 \text{ and } y \text{ integral}$$

In an optimal solution of the continuous relaxation, let

$J = \{ j \mid y_j \text{ is nonbasic}\}$

$K = \{ j \mid x_j \text{ is nonbasic}\}$

$N$ = nonbasic cols of $[A\ D]$

Then if $y_i$ is nonintegral in this solution, the following **MIR cut** is violated by the solution of the relaxation:

$$y_i + \sum_{j \in J_1} \lceil \hat{N}_{ij} \rceil y_j + \sum_{j \in J_2} \left( \lfloor \hat{N}_{ij} \rfloor + \frac{\text{frac}(\hat{N}_{ij})}{\text{frac}(\hat{b}_i)} \right) + \frac{1}{\text{frac}(\hat{b}_i)} \sum_{j \in K} \hat{N}^+_{ij} x_j \geq \hat{N}_{ij} \lceil \hat{b}_i \rceil$$

where $J_1 = \left\{ j \in J \,\middle|\, \text{frac}(\hat{N}_{ij}) \geq \text{frac}(\hat{b}_j) \right\}$ $\quad J_2 = J \setminus J_1$

## Example

$$3x_1 + 4x_2 - 6y_1 - 4y_2 = 1$$
$$x_1 + 2x_2 - y_1 - y_2 = 3$$
$$x_j, y_j \geq 0, \quad y_j \text{ integer}$$

Take basic solution $(x_1, y_1) = (8/3, 17/3)$.

Then $\hat{N} = \begin{bmatrix} 1/3 & 2/3 \\ -2/3 & 8/3 \end{bmatrix}$ $\hat{b} = \begin{bmatrix} 8/3 \\ 17/3 \end{bmatrix}$

$J = \{2\}$, $K = \{2\}$, $J_1 = \varnothing$, $J_2 = \{2\}$

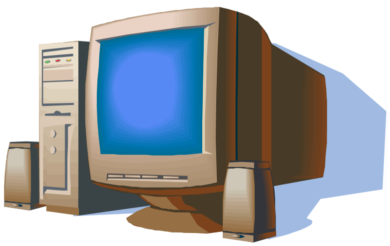
The MIR cut is $y_1 + \left( \lfloor 1/3 \rfloor + \dfrac{1/3}{2/3} \right) y_2 + \dfrac{1}{2/3} (2/3)^+ x_2 \geq \lceil 8/3 \rceil$

or $y_1 + (1/2) y_2 + x_2 \geq 3$

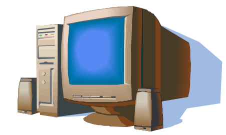
# Example: Product Configuration

This example illustrates:

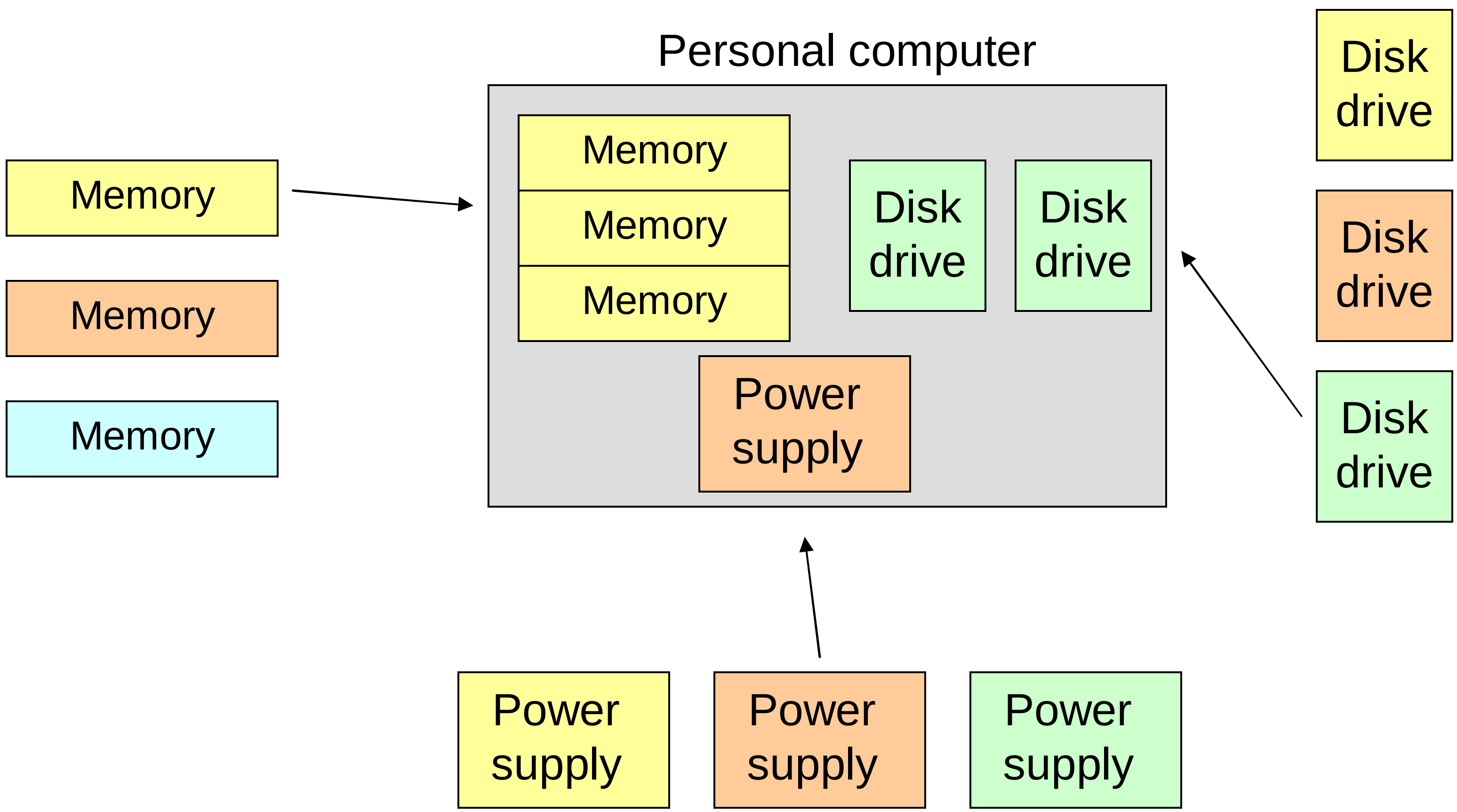
- Combination of propagation and relaxation.
- Processing of variable indices.
- Continuous relaxation of element constraint.



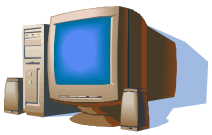
# The problem



Choose what type of each component, and how many



# Model of the problem



Unit cost of producing attribute *j*

Amount of attribute *j* produced by type $t_i$ of component *i*

Amount of attribute *j* produced
(< 0 if consumed):
*memory, heat, power, weight, etc.*

$$\min \sum_j c_j v_j$$

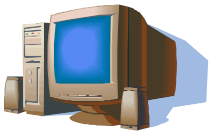
$$v_j = \sum_{ik} q_i A_{ijt_i}, \text{ all } j$$

$$L_j \le v_j \le U_j, \text{ all } j$$

$t_i$ is a **variable index**

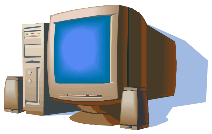
Quantity of component *i* installed

## To solve it:



- **Branch** on domains of $t_i$ and $q_i$.
- **Propagate** *element* constraints and bounds on $v_j$.
  - **Variable index** is converted to specially structured *element* constraint.
  - Valid **knapsack** cuts are derived and propagated.
- Use linear continuous **relaxations**.
  - Special purpose **MILP** relaxation for *element.*

# Propagation



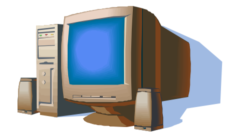
$$\min \sum_j c_j v_j$$

$$v_j = \sum_{ik} q_i A_{ijt_i}, \text{ all } j$$

$$\boxed{L_j \le v_j \le U_j, \text{ all } j}$$

*This* is propagated
in the usual way

## Propagation



$$v_j = \sum_i z_i, \text{ all } j$$

$$\text{element}\big(t_i, (q_i A_{ij1}, \ldots, q_i A_{ijn}), z_i\big), \text{ all } i, j$$

$$\min \sum_j c_j v_j$$

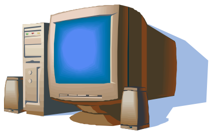
$$\boxed{v_j = \sum_{ik} q_i A_{ijt_i}, \text{ all } j}$$

*This* is rewritten as

$$\boxed{L_j \le v_j \le U_j, \text{ all } j}$$

*This* is propagated in the usual way

# Propagation

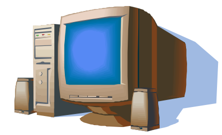


$$v_j = \sum_i z_i, \text{ all } j$$

$$\text{element}\left(t_i, (q_i, A_{ij1}, \ldots, q_i A_{ijn}), z_i\right), \text{ all } i, j$$

*This* can be propagated by
(a) using specialized **filters** for *element* constraints of this form…

## Propagation



$$v_j = \sum_i z_i, \text{ all } j$$

$$\text{element}\left(t_i, (q_i, A_{ij1}, \ldots, q_i A_{ijn}), z_i\right), \text{ all } i, j$$

*This* is propagated by
(a) using specialized **filters** for *element* constraints of this form,
(b) adding **knapsack cuts** for the valid inequalities:

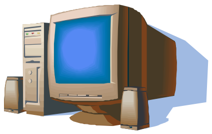
$$\sum_i \max_{k \in D_{t_i}} \left\{A_{ijk}\right\} q_i \geq \underline{v}_j, \text{ all } j$$

$$\sum_i \min_{k \in D_{t_i}} \left\{A_{ijk}\right\} q_i \leq \bar{v}_j, \text{ all } j$$

and (c) propagating the knapsack cuts.

$[\underline{v}_j, \bar{v}_j]$ is current domain of $v_j$

## Relaxation



$$\min \sum_j c_j v_j$$

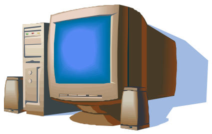
$$v_j = \sum_{ik} q_i A_{ijt_i}, \text{ all } j$$

$$\boxed{L_j \le v_j \le U_j, \text{ all } j}$$

*This* is relaxed as

$$\underline{v}_j \le v_j \le \bar{v}_j$$

## Relaxation



$$v_j = \sum_i z_i, \text{ all } j$$

$$\text{element}\left(t_i, (q_i, A_{ij1}, \ldots, q_i A_{ijn}), z_i\right), \text{ all } i, j$$

$$\min \sum_j c_j v_j$$

$$\boxed{v_j = \sum_{ik} q_i A_{ijt_i}, \text{ all } j}$$

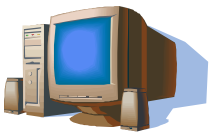
$$\boxed{L_j \le v_j \le U_j, \text{ all } j}$$

*This* is relaxed by relaxing *this* and adding the knapsack cuts.

*This* is relaxed as

$$\underline{v}_j \le v_j \le \overline{v}_j$$

## Relaxation



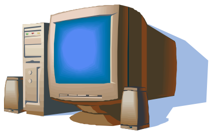
$$v_j = \sum_i z_i, \text{ all } j$$

$$\text{element}\left(t_i, (q_i A_{ij1}, \ldots, q_i A_{ijn}), z_i\right), \text{ all } i, j$$

*This* is relaxed by replacing each *element* constraint with a disjunctive **convex hull** relaxation:

$$z_i = \sum_{k \in D_{t_i}} A_{ijk} q_{ik}, \quad q_i = \sum_{k \in D_{t_i}} q_{ik}$$

## Relaxation



So the following LP relaxation is solved at each node of the search tree to obtain a lower bound:

$$\min \sum_j c_j v_j$$

$$v_j = \sum_i \sum_{k \in D_{t_i}} A_{ijk} q_{ik}, \text{ all } j$$

$$q_j = \sum_{k \in D_{t_i}} q_{ik}, \text{ all } i$$

$$\underline{v}_j \le v_j \le \bar{v}_j, \text{ all } j$$

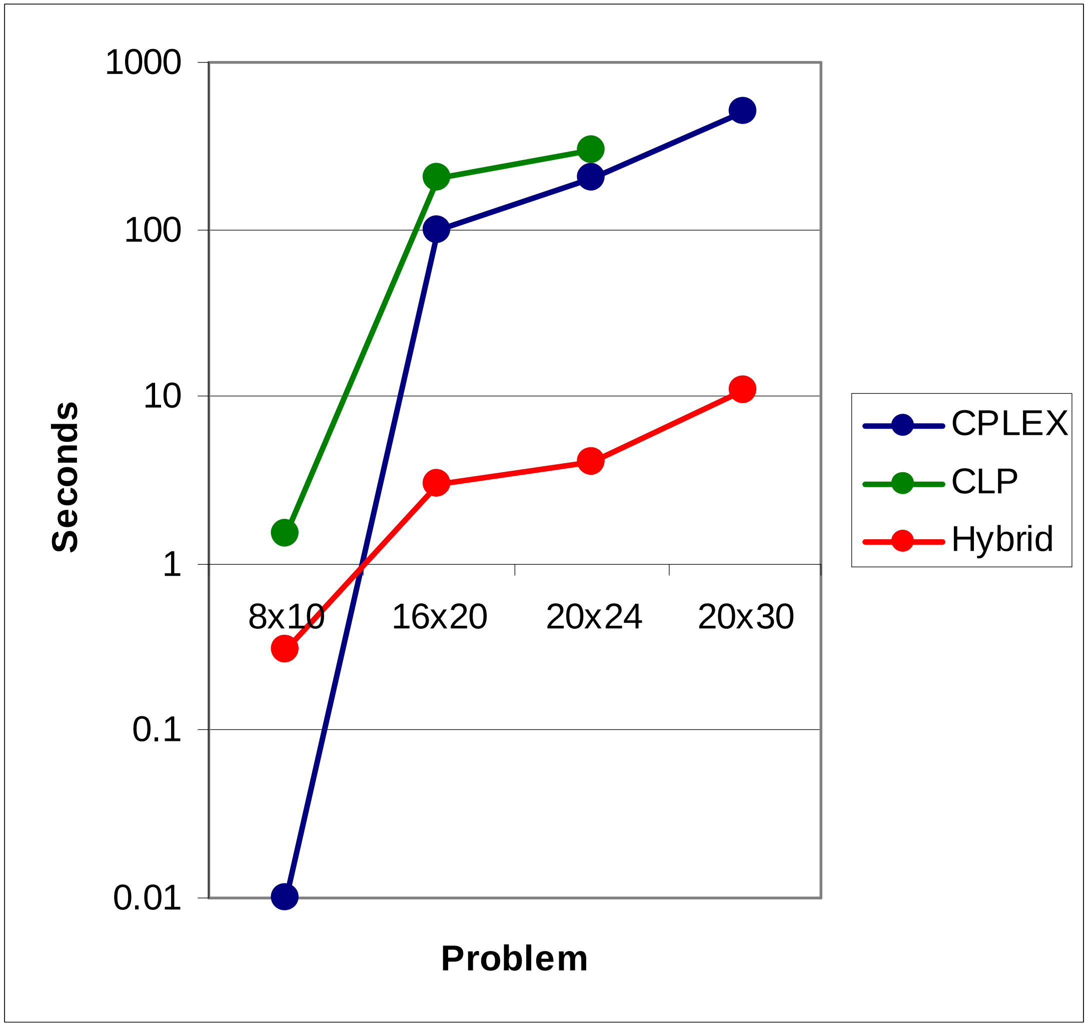
$$\underline{q}_i \le q_i \le \bar{q}_i, \text{ all } i$$

$$\text{knapsack cuts for } \sum_i \max_{k \in D_{t_i}} \{A_{ijk}\} q_i \ge \underline{v}_j, \text{ all } j$$

$$\text{knapsack cuts for } \sum_i \min_{k \in D_{t_i}} \{A_{ijk}\} q_i \le \bar{v}_j, \text{ all } j$$

$$q_{ik} \ge 0, \text{ all } i,k$$

# Computational Results



# Lagrangean Relaxation

Lagrangean Duality

Properties of the Lagrangean Dual

Example: Fast Linear Programming

Domain Filtering

Example: Continuous Global Optimization

# Motivation

• **Lagrangean relaxation** can provide better bounds than LP relaxation.

• The **Lagrangean dual** generalizes LP duality.

• It provides **domain filtering** analogous to that based on LP duality.

- This is a key technique in **continuous global optimization**.

• Lagrangean relaxation gets rid of troublesome constraints by **dualizing** them.

- That is, moving them into the objective function.

- The Lagrangean relaxation may **decouple**.

# Lagrangean Duality

Consider an inequality-constrained problem

$$\min \; f(x)$$
$$g(x) \geq 0$$
$$x \in S$$

Hard constraints

Easy constraints

The object is to get rid of (**dualize**) the hard constraints by moving them into the objective function.

## Lagrangean Duality

Consider an inequality-constrained problem

$$\min \; f(x)$$
$$g(x) \geq 0$$
$$x \in S$$

It is related to an inference problem

$$\max \; v$$
$$g(x) \geq b \overset{s \in S}{\boxed{\Rightarrow}} f(x) \geq v$$

implies

**Lagrangean Dual** problem: Find the tightest lower bound on the objective function that is implied by the constraints.

Primal

$$\min\ f(x)$$
$$g(x) \geq 0$$
$$x \in S$$

Dual

$$\max\ v$$
$$g(x) \geq b \overset{s \in S}{\Rightarrow} f(x) \geq v$$

Surrogate

Let us say that

$g(x) \geq 0 \overset{x \in S}{\Rightarrow} f(x) \geq v$ iff $\boxed{\lambda g(x) \geq 0}$ $\boxed{\text{dominates}}$ $f(x) - v \geq 0$

for some $\lambda \geq 0$

$\lambda g(x) \leq f(x) - v$ for all $x \in S$

That is, $v \leq f(x) - \lambda g(x)$ for all $x \in S$

Primal

$\min\ f(x)$

$g(x) \geq 0$

$x \in S$

Dual

$\max\ v$

$g(x) \geq b \overset{s \in S}{\Rightarrow} f(x) \geq v$

Surrogate

Let us say that

$g(x) \geq 0 \overset{x \in S}{\Rightarrow} f(x) \geq v$ iff $\boxed{\lambda g(x) \geq 0}$ $\boxed{\text{dominates}}$ $f(x) - v \geq 0$

for some $\lambda \geq 0$

$\lambda g(x) \leq f(x) - v$ for all $x \in S$

That is, $v \leq f(x) - \lambda g(x)$ for all $x \in S$

Or $v \leq \min_{x \in S} \{ f(x) - \lambda g(x) \}$

Primal

$$\min\ f(x)$$
$$g(x) \geq 0$$
$$x \in S$$

Dual

$$\max\ v$$
$$g(x) \geq b \overset{s \in S}{\Rightarrow} f(x) \geq v$$

Surrogate

Let us say that

$g(x) \geq 0 \overset{x \in S}{\Rightarrow} f(x) \geq v$ iff $\boxed{\lambda g(x) \geq 0}$ $\boxed{\text{dominates}}$ $f(x) - v \geq 0$

for some $\lambda \geq 0$

$\lambda g(x) \leq f(x) - v$ for all $x \in S$

That is, $v \leq f(x) - \lambda g(x)$ for all $x \in S$

Or $v \leq \min_{x \in S} \{f(x) - \lambda g(x)\}$

So the dual becomes

$$\max\ v$$
$$v \leq \min_{x \in S} \{f(x) - \lambda g(x)\} \text{ for some } \lambda \geq 0$$

Now we have…

Primal

$$\min \; f(x)$$
$$\boxed{g(x) \ge 0}$$
$$x \in S$$

These constraints are **dualized**

Dual

$$\max \; v$$
$$v \le \min_{x \in S}\{f(x) - \lambda g(x)\} \text{ for some } \lambda \ge 0$$

or

$$\max_{\lambda \ge 0} \theta(\lambda)$$

where

$$\theta(\lambda) = \min_{x \in S}\{f(x) - \lambda g(x)\}$$

**Lagrangean relaxation**

Vector of **Lagrange multipliers**

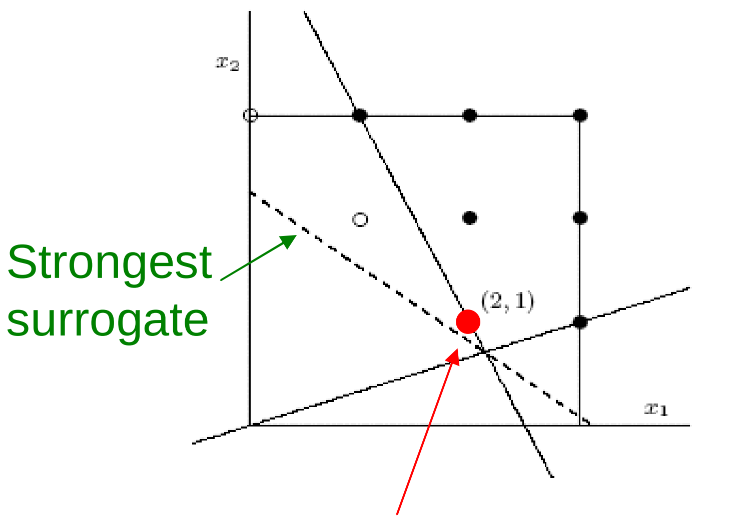
The Lagrangean dual can be viewed as the problem of finding the Lagrangean relaxation that gives the tightest bound.

# Example

$$\min\ 3x_1 + 4x_2$$
$$-x_1 + 3x_2 \geq 0$$
$$2x_1 + x_2 - 5 \geq 0$$
$$x_1, x_2 \in \{0,1,2,3\}$$

The Lagrangean relaxation is

$$\theta(\lambda_1,\lambda_2) = \min_{x_j \in \{0,\ldots,3\}} \{3x_1 + 4x_2 - \lambda_1(-x_1 + 3x_2) - \lambda_2(2x_1 + x_2 - 5)\}$$
$$= \min_{x_j \in \{0,\ldots,3\}} \{(3 + \lambda_1 - 2\lambda_2)x_1 + (4 - 3\lambda_1 - \lambda_2)x_2 + 5\lambda_2\}$$




The Lagrangean relaxation is easy to solve for any given $\lambda_1$, $\lambda_2$:

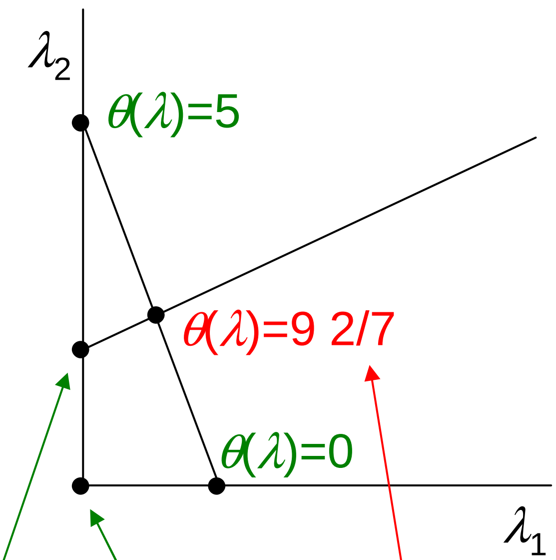
$$x_1 = \begin{cases} 0 & \text{if } 3 + \lambda_1 - 2\lambda_2 \geq 0 \\ 3 & \text{otherwise} \end{cases}$$

$$x_2 = \begin{cases} 0 & \text{if } 4 - 3\lambda_1 - \lambda_2 \geq 0 \\ 3 & \text{otherwise} \end{cases}$$

# Example

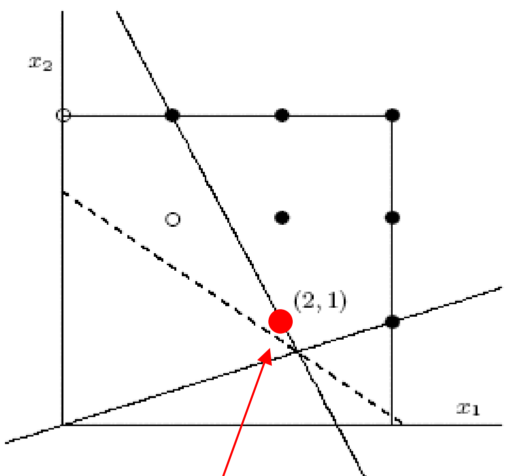
$\theta(\lambda_1,\lambda_2)$ is piecewise linear and concave.

$$\min\ 3x_1 + 4x_2$$
$$-x_1 + 3x_2 \geq 0$$
$$2x_1 + x_2 - 5 \geq 0$$
$$x_1, x_2 \in \{0,1,2,3\}$$




$\theta(\lambda)=0$

$\theta(\lambda)=7.5$




Solution of Lagrangean dual:

$(\lambda_1,\lambda_2)$ = (5/7, 13/7), $\theta(\lambda)$ = 9 2/7

Optimal solution (2,1)
Value = 10

Note **duality gap** between 10 and 9 2/7 (no strong duality).

# Example

$$\min\ 3x_1 + 4x_2$$
$$-x_1 + 3x_2 \geq 0$$
$$2x_1 + x_2 - 5 \geq 0$$
$$x_1, x_2 \in \{0,1,2,3\}$$

Note: in this example, the Lagrangean dual provides the same bound (9 2/7) as the continuous relaxation of the IP.

This is because the Lagrangean relaxation can be solved as an LP:

$$\theta(\lambda_1,\lambda_2) = \min_{x_j \in \{0,\ldots,3\}} \{(3+\lambda_1 - 2\lambda_2)x_1 + (4 - 3\lambda_1 - \lambda_2)x_2 + 5\lambda_2\}$$
$$= \min_{0 \leq x_j \leq 3} \{(3+\lambda_1 - 2\lambda_2)x_1 + (4 - 3\lambda_1 - \lambda_2)x_2 + 5\lambda_2\}$$

Lagrangean duality is useful when the Lagrangean relaxation is tighter than an LP but nonetheless easy to solve.

# Properties of the Lagrangean dual

**Weak duality:** For any feasible $x^*$ and any $\lambda^* \geq 0$, $f(x^*) \geq \theta(\lambda^*)$.

In particular,
$$\begin{array}{l} \min \; f(x) \\ g(x) \geq 0 \\ x \in S \end{array} \quad \geq \quad \max_{\lambda \geq 0} \theta(\lambda)$$

**Concavity:** $\theta(\lambda)$ is concave. It can therefore be maximized by local search methods.

**Complementary slackness**: If $x^*$ and $\lambda^*$ are optimal, and there is no duality gap, then $\lambda^* g(x^*) = 0$.

## Solving the Lagrangean dual

Let $\lambda^k$ be the $k$th iterate, and let $\lambda^{k+1} = \lambda^k + \alpha_k \boxed{\xi^k}$

Subgradient of $\theta(\lambda)$ at $\lambda = \lambda^k$

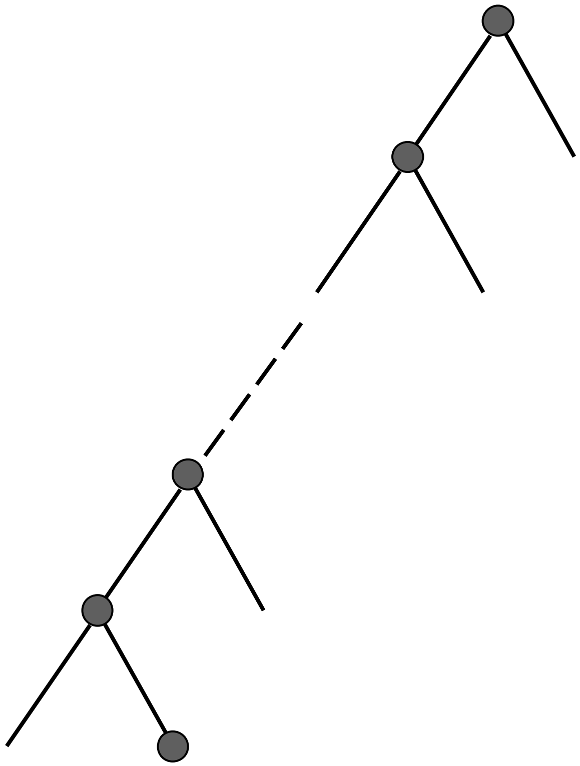
If $x^k$ solves the Lagrangean relaxation for $\lambda = \lambda^k$, then $\xi^k = g(x^k)$.

This is because $\theta(\lambda) = f(x^k) + \lambda g(x^k)$ at $\lambda = \lambda^k$.

The stepsize $\alpha_k$ must be adjusted so that the sequence converges but not before reaching a maximum.

# Example: Fast Linear Programming

- In CP contexts, it is best to process each node of the search tree very rapidly.
- Lagrangean relaxation may allow very fast calculation of a lower bound on the optimal value of the LP relaxation at each node.
- The idea is to solve the Lagrangean dual at the root node (which is an LP) and use the same Lagrange multipliers to get an LP bound at other nodes.



At root node, solve $\min\ cx$

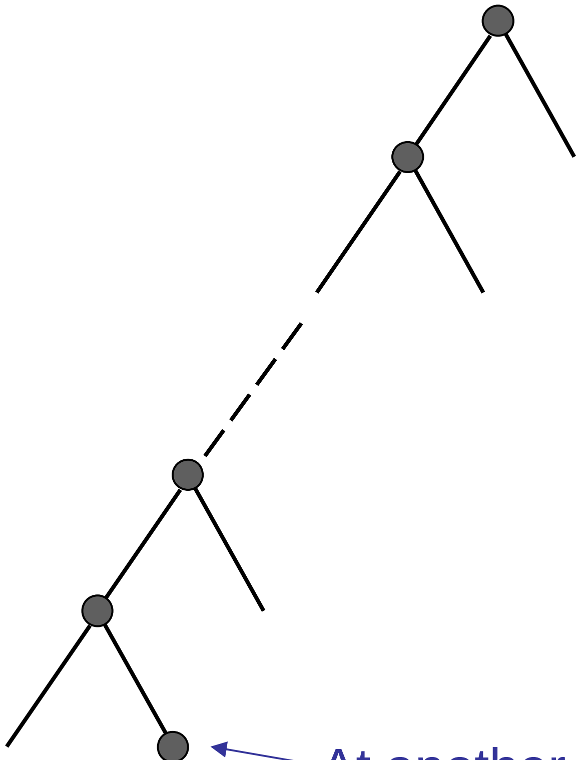
Dualize → $Ax \geq b \quad (\lambda)$

Special structure, e.g. variable bounds → $Dx \geq d$

$x \geq 0$

The (partial) LP dual solution $\lambda^*$ solves the Lagrangean dual in which

$$\theta(\lambda) = \min_{\substack{Dx \geq d \\ x \geq 0}} \{ cx - \lambda(Ax - b) \}$$



At root node, solve

$$\begin{aligned} &\min\ cx \\ &Ax \ge b \quad (\lambda) \\ &Dx \ge d \\ &x \ge 0 \end{aligned}$$

Dualize

Special structure, e.g. variable bounds

The (partial) LP dual solution $\lambda^*$ solves the Lagrangean dual in which

$$\theta(\lambda) = \min_{\substack{Dx \ge d \\ x \ge 0}} \{cx - \lambda(Ax - b)\}$$

At another node, the LP is

$$\begin{aligned} &\min\ cx \\ &Ax \ge b \quad (\lambda) \\ &Dx \ge d \\ &Hx \ge h \\ &x \ge 0 \end{aligned}$$

Branching constraints, etc.

Here $\theta(\lambda^*)$ is still a lower bound on the optimal value of the LP and can be quickly calculated by solving a specially structured LP.

# Domain Filtering

Suppose:

$$\begin{aligned} &\min\ f(x) \\ &g(x) \geq 0 \\ &x \in S \end{aligned}$$

has optimal solution $x^*$, optimal value $v^*$, and optimal Lagrangean dual solution $\lambda^*$.

…and $\lambda_i^* > 0$, which means the *i*-th constraint is tight (complementary slackness);

…and the problem is a relaxation of a CP problem;

…and we have a feasible solution of the CP problem with value $U$, so that $U$ is an upper bound on the optimal value.

Supposing $\begin{array}{l} \min \ f(x) \\ g(x) \geq 0 \\ x \in S \end{array}$ has optimal solution $x^*$, optimal value $v^*$, and optimal Lagrangean dual solution $\lambda^*$:

If $x$ were to change to a value other than $x^*$, the LHS of $i$-th constraint $g_i(x) \geq 0$ would change by some amount $\Delta_i$.

Since the constraint is tight, this would increase the optimal value as much as changing the constraint to $g_i(x) - \Delta_i \geq 0$.

So it would increase the optimal value at least $\lambda_i^* \Delta_i$.

(It is easily shown that Lagrange multipliers are marginal costs. Dual multipliers for LP are a special case of Lagrange multipliers.)

Supposing $\begin{aligned} &\min\ f(x) \\ &g(x) \geq 0 \\ &x \in S \end{aligned}$ has optimal solution $x^*$, optimal value $v^*$, and optimal Lagrangean dual solution $\lambda^*$:

We have found: a change in $x$ that changes $g_i(x)$ by $\Delta_i$ increases the optimal value at least $\lambda_i^* \Delta_i$.

Since optimal value of this problem $\leq$ optimal value of the CP $\leq U$, we have $\lambda_i^* \Delta_i \leq U - v^*$, or

$$\Delta_i \leq \frac{U - v^*}{\lambda_i^*}$$

Supposing $\begin{array}{l} \min\ f(x) \\ g(x) \geq 0 \\ x \in S \end{array}$ has optimal solution $x^*$, optimal value $v^*$, and optimal Lagrangean dual solution $\lambda^*$:

We have found: a change in $x$ that changes $g_i(x)$ by $\Delta_i$ increases the optimal value at least $\lambda_i^*\Delta_i$.

Since optimal value of this problem $\leq$ optimal value of the CP $\leq U$, we have $\lambda_i^*\Delta_i \leq U - v^*$, or

$$\Delta_i \leq \frac{U - v^*}{\lambda_i^*}$$

Since $\Delta_i = g_i(x) - g_i(x^*) = g_i(x)$, this implies the inequality

$$g_i(x) \leq \frac{U - v^*}{\lambda_i^*}$$

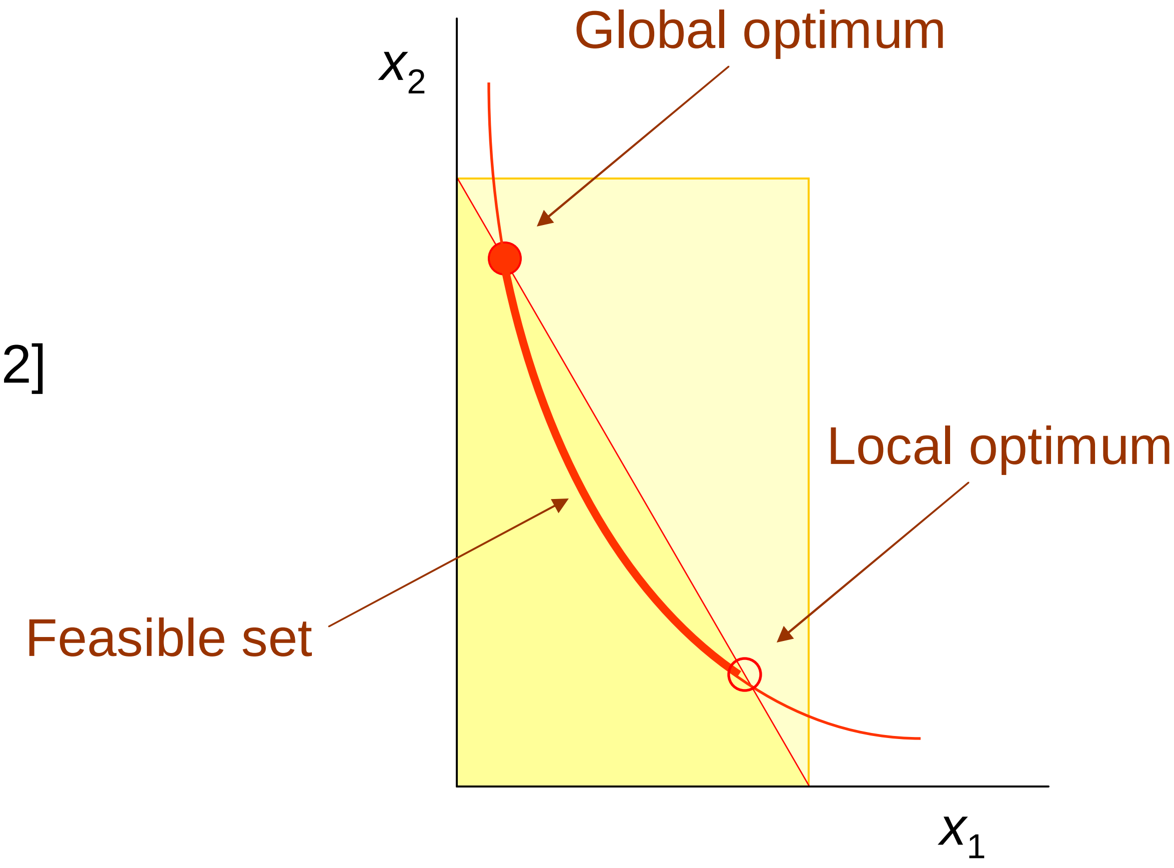
…which can be propagated.

# Example:  Continuous Global Optimization

- Some of the best continuous global solvers (e.g., BARON) combine OR-style relaxation with CP-style interval arithmetic and domain filtering.
- The use of Lagrange multipliers for domain filtering is a key technique in these solvers.

# Continuous Global Optimization

$$\max \; x_1 + x_2$$
$$4x_1x_2 = 1$$
$$2x_1 + x_2 \le 2$$
$$x_1 \in [0,1], \quad x_2 \in [0,2]$$

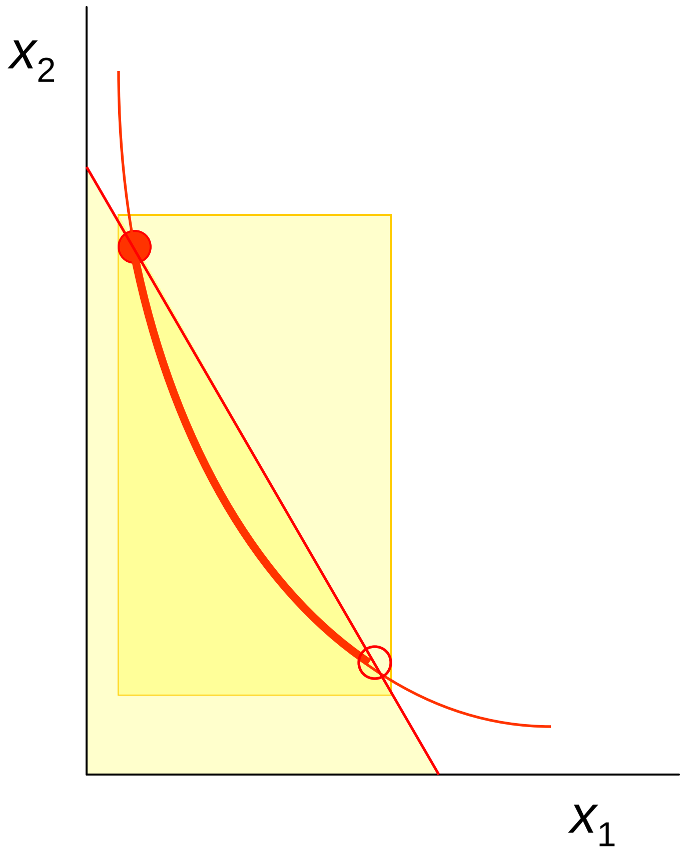


## To solve it:

- **Search**: split interval domains of $x_1, x_2$.
  - Each **node** of search tree is a problem restriction.
- **Propagation:** Interval propagation, domain filtering.
  - Use **Lagrange multipliers** to infer valid inequality for propagation.
  - **Reduced-cost variable** fixing is a special case.
- **Relaxation:** Use function **factorization** to obtain linear continuous relaxation.

# Interval propagation

Propagate intervals
[0,1], [0,2]
through constraints
to obtain
[1/8,7/8], [1/4,7/4]



## Relaxation (function factorization)

Factor complex functions into elementary functions that have known linear relaxations.

Write $4x_1x_2 = 1$ as $4y = 1$ where $y = x_1x_2$.

This factors $4x_1x_2$ into linear function $4y$ and bilinear function $x_1x_2$.

Linear function $4y$ is its own linear relaxation.

## Relaxation (function factorization)

Factor complex functions into elementary functions that have known linear relaxations.

Write $4x_1x_2 = 1$ as $4y = 1$ where $y = x_1x_2$.

This factors $4x_1x_2$ into linear function $4y$ and bilinear function $x_1x_2$.

Linear function $4y$ is its own linear relaxation.

Bilinear function $y = x_1x_2$ has relaxation:

$$\underline{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \underline{x}_2 \le y \le \underline{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1 \underline{x}_2$$
$$\bar{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1 \bar{x}_2 \le y \le \bar{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \bar{x}_2$$

where domain of $x_j$ is $[\underline{x}_j, \bar{x}_j]$

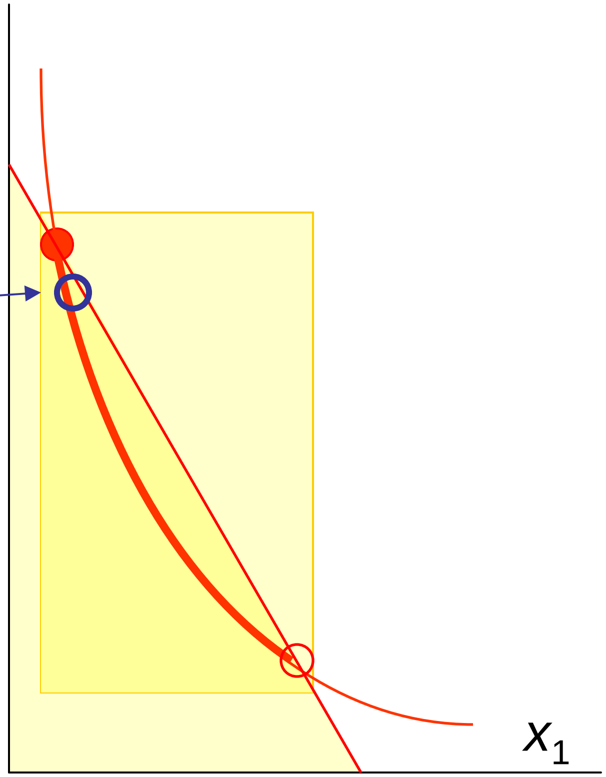
## Relaxation (function factorization)

The linear relaxation becomes:

$$
\begin{aligned}
&\min \;\; x_1 + x_2 \\
&4y = 1 \\
&2x_1 + x_2 \le 2 \\
&\underline{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \underline{x}_2 \le y \le \underline{x}_2 x_1 + \overline{x}_1 x_2 - \overline{x}_1 \underline{x}_2 \\
&\overline{x}_2 x_1 + \overline{x}_1 x_2 - \overline{x}_1 \overline{x}_2 \le y \le \overline{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \overline{x}_2 \\
&\underline{x}_j \le x_j \le \overline{x}_j, \;\; j = 1,2
\end{aligned}
$$

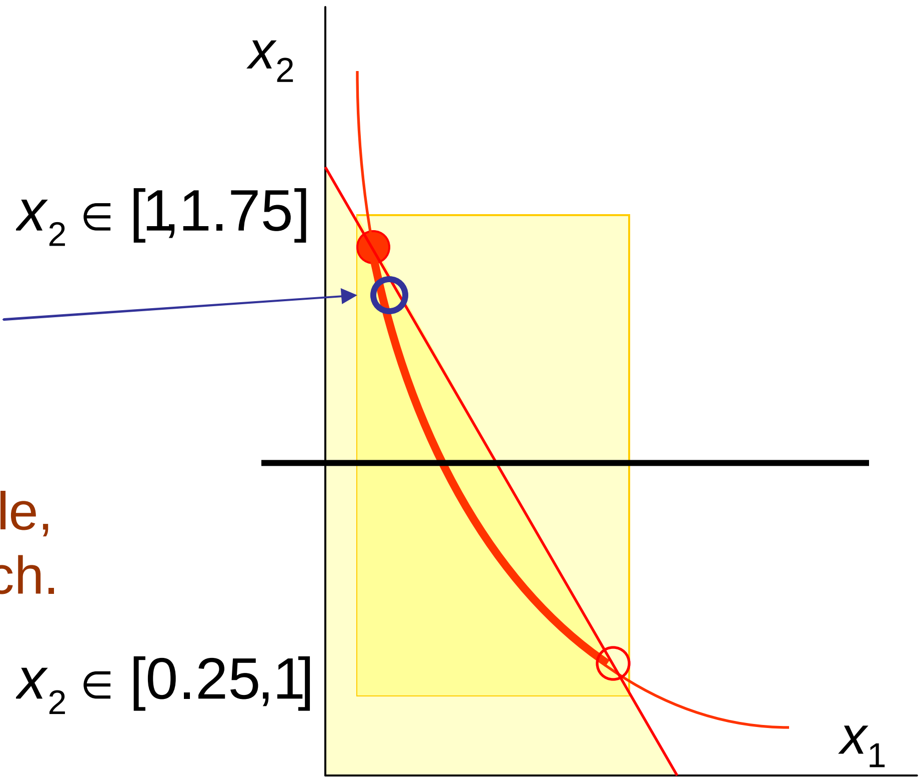
# Relaxation (function factorization)

Solve linear relaxation.

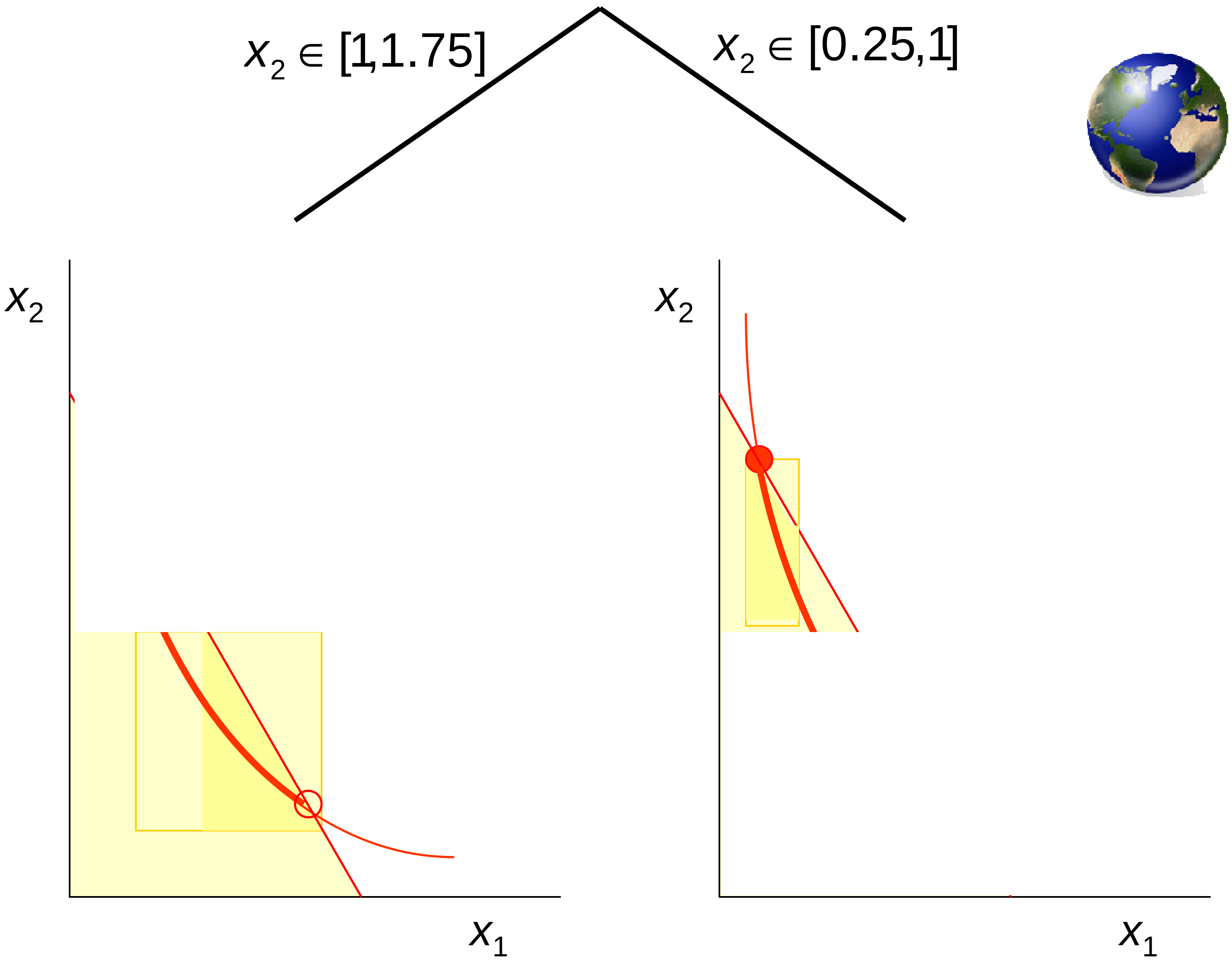


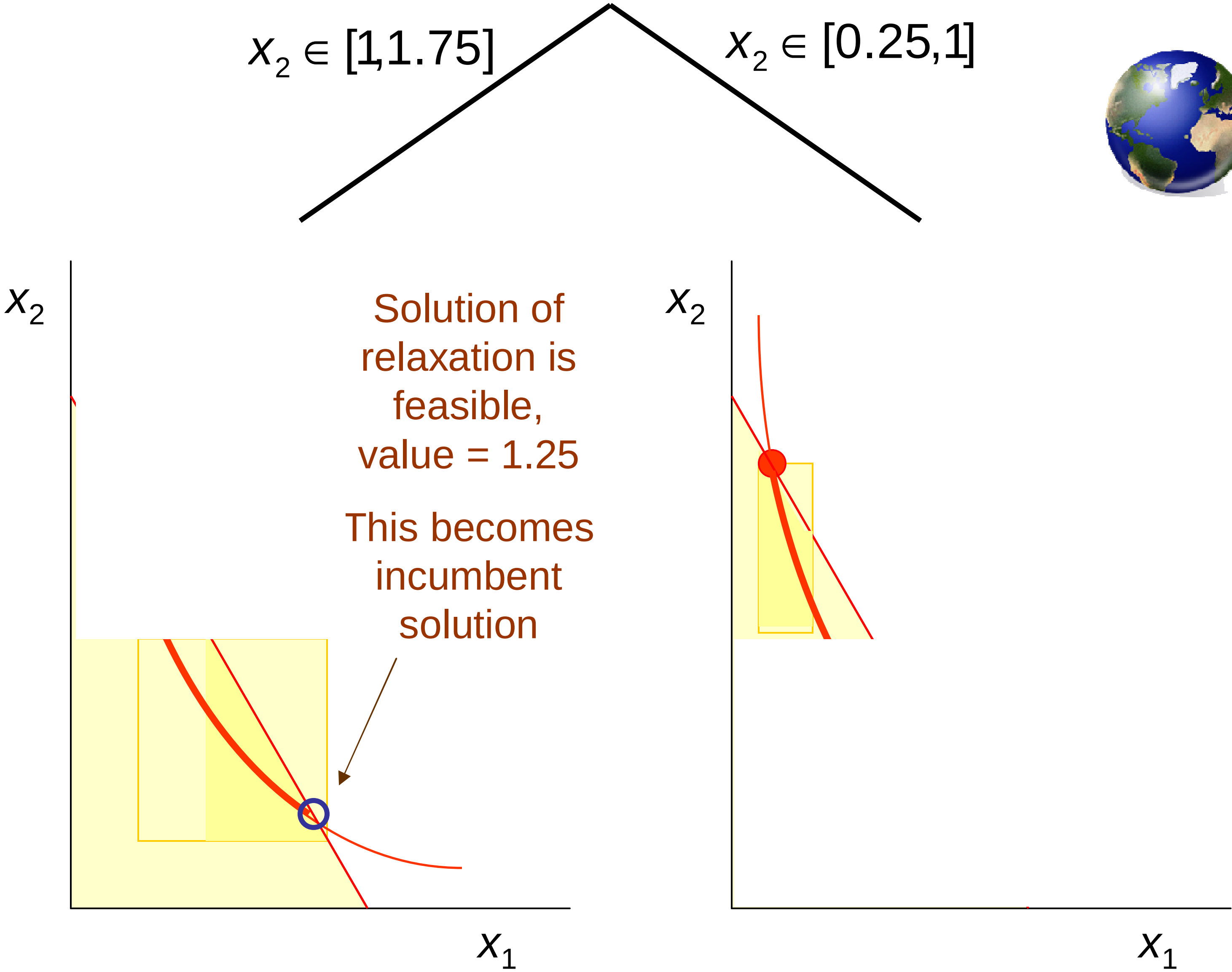
# Relaxation (function factorization)

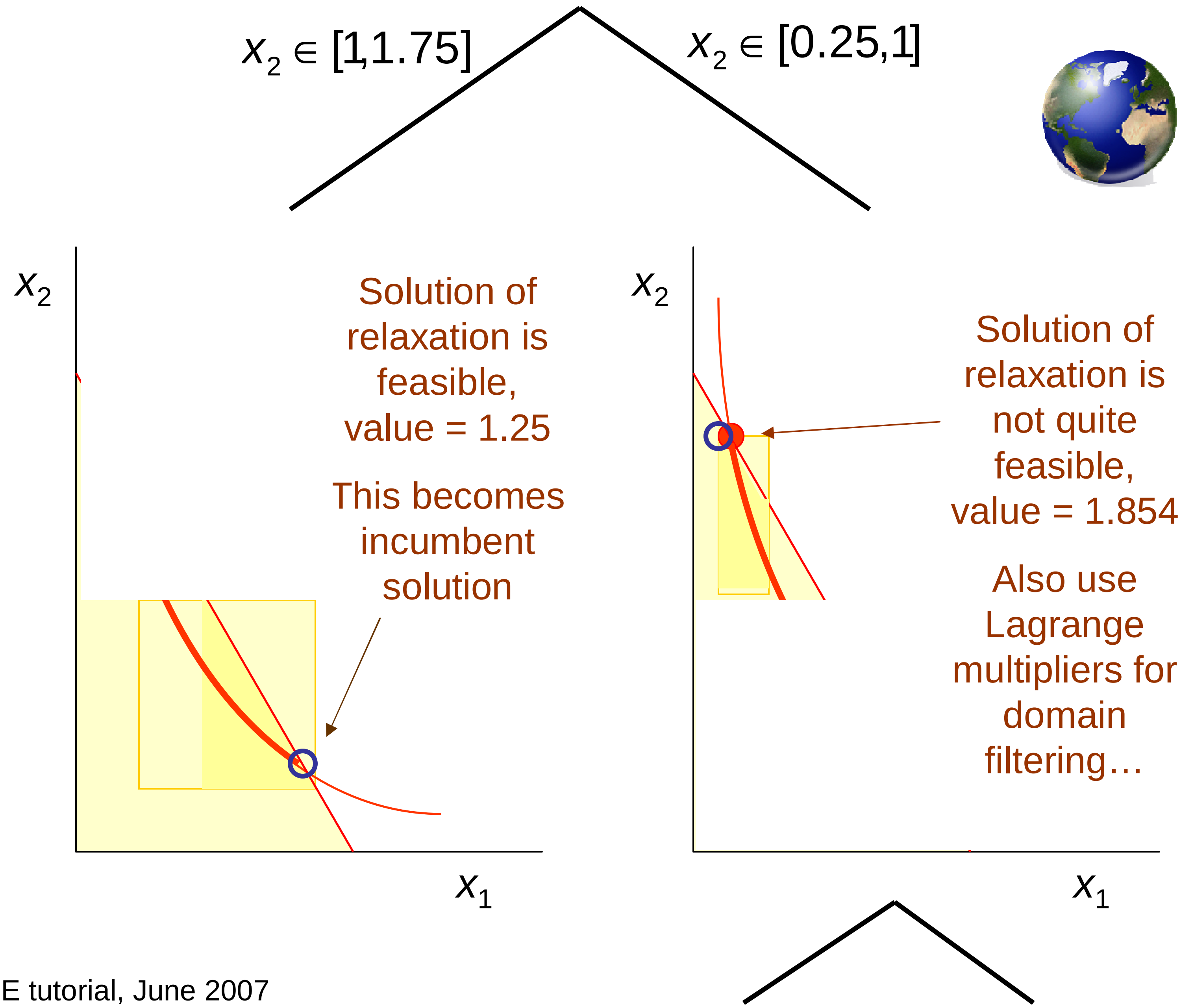
Solve linear relaxation.




Since solution is infeasible,
split an interval and branch.


$x_2 \in [1,1.75]$
$x_2 \in [0.25,1]$
$x_2$
$x_1$
$x_2$
$x_1$


$x_2 \in [1,1.75]$
$x_2 \in [0.25,1]$
$x_2$
$x_1$
Solution of relaxation is feasible, value = 1.25
This becomes incumbent solution
$x_2$
$x_1$


$x_2 \in [1,1.75]$
$x_2 \in [0.25,1]$
$x_2$
$x_1$
Solution of relaxation is feasible, value = 1.25
This becomes incumbent solution
$x_2$
$x_1$
Solution of relaxation is not quite feasible, value = 1.854
Also use Lagrange multipliers for domain filtering…

# Relaxation (function factorization)

$$\min \; x_1 + x_2$$
$$4y = 1$$
$$2x_1 + x_2 \le 2$$
$$\underline{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \underline{x}_2 \le y \le \underline{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1 \underline{x}_2$$
$$\bar{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1 \bar{x}_2 \le y \le \bar{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \bar{x}_2$$
$$\underline{x}_j \le x_j \le \bar{x}_j, \quad j = 1,2$$

Associated Lagrange multiplier in solution of relaxation is $\lambda_2 = 1.1$

# Relaxation (function factorization)

$$
\begin{aligned}
&\min \; x_1 + x_2 \\
&4y = 1 \\
&2x_1 + x_2 \le 2 \\
&\underline{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \underline{x}_2 \le y \le \underline{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1 \underline{x}_2 \\
&\bar{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1 \bar{x}_2 \le y \le \bar{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \bar{x}_2 \\
&\underline{x}_j \le x_j \le \bar{x}_j, \quad j = 1,2
\end{aligned}
$$

Associated Lagrange multiplier in solution of relaxation is $\lambda_2 = 1.1$

This yields a valid inequality for propagation:

$$2x_1 + x_2 \ge 2 - \frac{\boxed{1.854} - \boxed{1.25}}{\boxed{1.1}} = 1.451$$

Value of relaxation

Lagrange multiplier

Value of incumbent solution

# Dynamic Programming in CP

## Example: Capital Budgeting
## Domain Filtering
## Recursive Optimization

## Motivation

- **Dynamic programming** (DP) is a highly versatile technique that can exploit recursive structure in a problem.
- **Domain filtering** is straightforward for problems modeled as a DP.
- DP is also important in designing **filters** for some global constraints, such as the *stretch* constraint (employee scheduling).
- **Nonserial DP** is related to bucket elimination in CP and exploits the structure of the primal graph.
- DP modeling is the **art** of keeping the state space small while maintaining a Markovian property.
- We will examine only **one simple example** of serial DP.

## Example: Capital Budgeting

We wish to built power plants with a total cost of at most 12 million Euros.

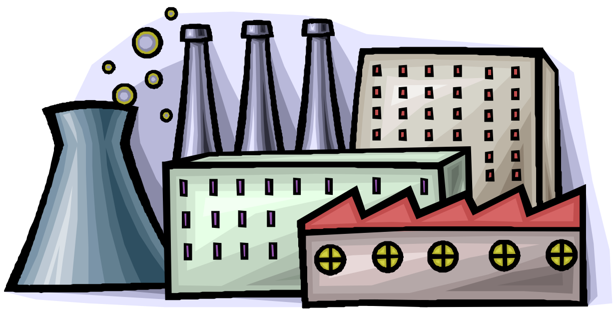
There are three types of plants, costing 4, 2 or 3 million Euros each. We must build one or two of each type.

The problem has a simple knapsack packing model:

$$4x_1 + 2x_2 + 3x_3 \le 12$$

$$\boxed{x_j} \in \{1,2\}$$

Number of factories of type *j*



## Example: Capital Budgeting

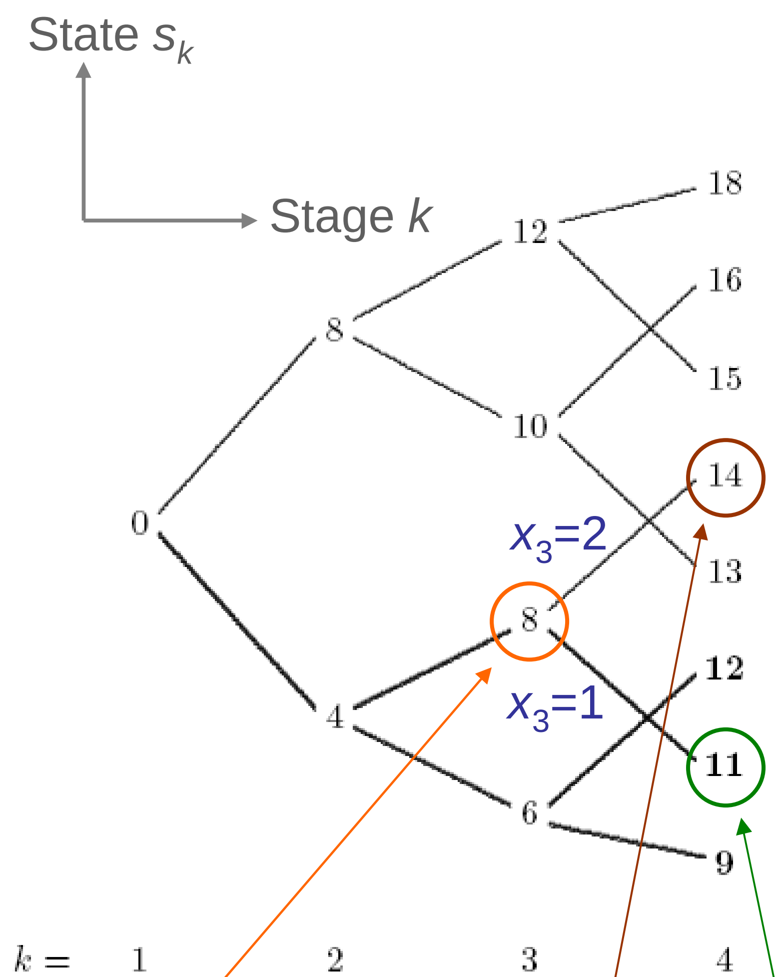
$$4x_1 + 2x_2 + 3x_3 \le 12$$
$$x_j \in \{1, 2\}$$

In general the recursion for $ax \le b$ is

$$f_k(s_k) = \max_{x_k \in D_{x_k}} \{ f_{k+1}(s_k + a_k x_k) \}$$

= 1 if there is a path from state $s_k$ to a feasible solution, 0 otherwise

State is sum of first $k$ terms of $ax$




$f_3(8) = \max\{f_4(8+3\cdot1), f_4(8+3\cdot2)\} = \max\{1,0\} = 1$

$f_4(14)=0$

$f_4(11)=1$

## Example: Capital Budgeting

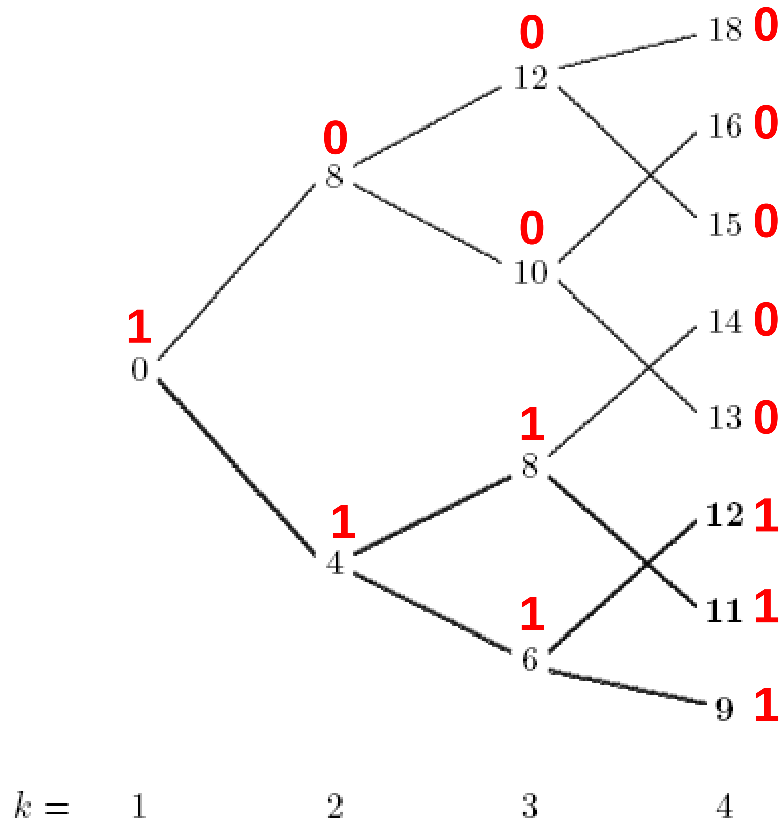
$$4x_1 + 2x_2 + 3x_3 \leq 12$$
$$x_j \in \{1,2\}$$

In general the recursion for $ax \leq b$ is

$$f_k(s_k) = \max_{x_k \in D_{x_k}} \{f_{k+1}(s_k + a_k x_k)\}$$

Boundary condition:

$$f_{n+1}(s_{n+1}) = \begin{cases} 1 & \text{if } s_{n+1} \leq b \\ 0 & \text{otherwise} \end{cases}$$




$f_k(s_k)$ for each state $s_k$

# Example: Capital Budgeting

$$4x_1 + 2x_2 + 3x_3 \le 12$$
$$x_j \in \{1,2\}$$

The problem is feasible.

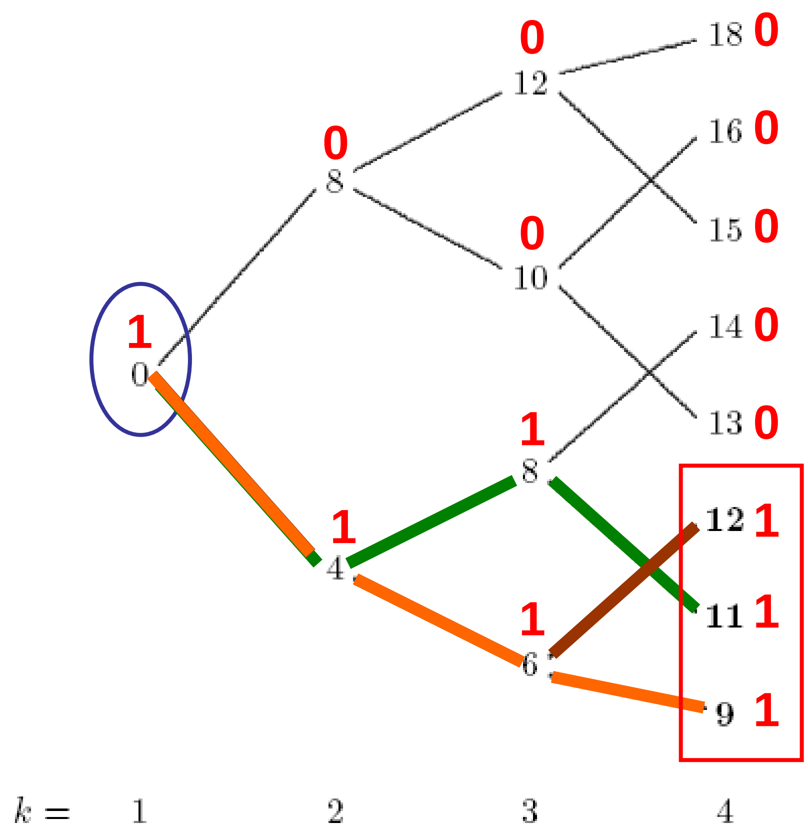
Each path to 0 is a feasible solution.

Path 1: x = (1,2,1)

Path 2: x = (1,1,2)

Path 3: x = (1,1,1)

Possible costs are 9,11,12.



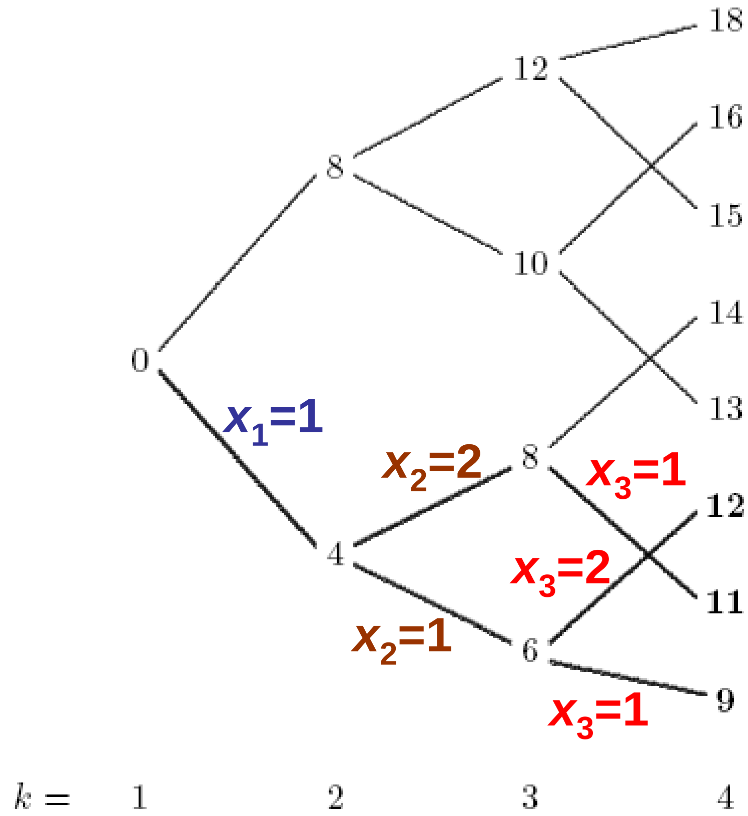

$f_k(s_k)$ for each state $s_k$

# Domain Filtering

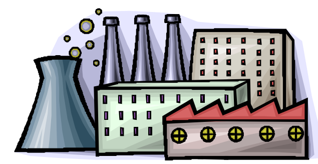
$$4x_1 + 2x_2 + 3x_3 \le 12$$
$$x_j \in \{1, 2\}$$

To filter domains: observe what values of $x_k$ occur on feasible paths.

$$D_{x_3} = \{1, 2\}$$
$$D_{x_2} = \{1, 2\}$$
$$D_{x_1} = \{1\}$$



# Recursive Optimization



$$\max \ 15x_1 + 10x_2 + 12x_3$$
$$4x_1 + 2x_2 + 3x_3 \le 12$$
$$x_j \in \{1, 2\}$$

Maximize revenue

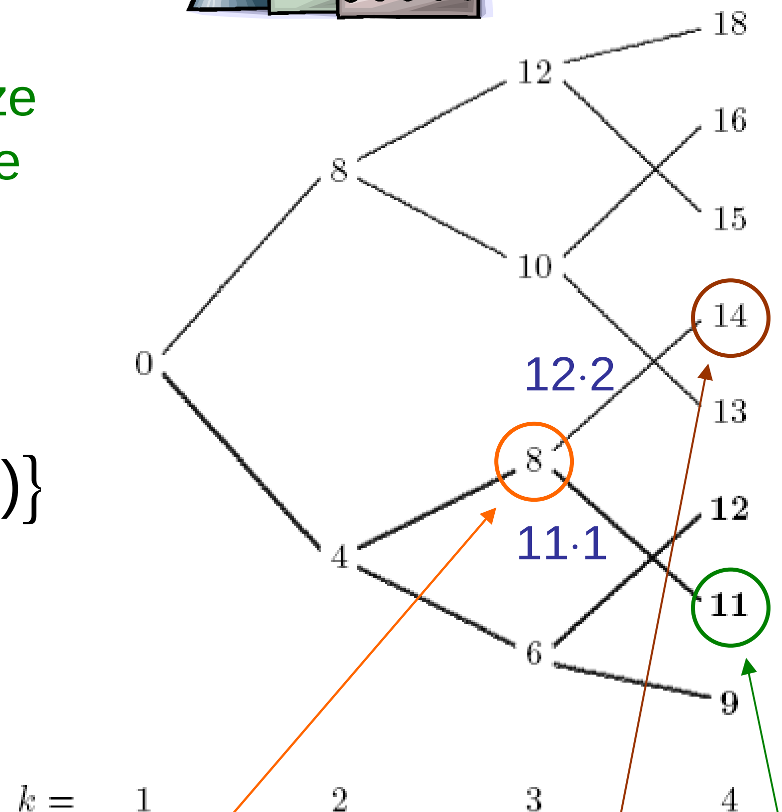
The recursion includes arc values:

$$f_k(s_k) = \max_{x_k \in D_{x_k}} \{ c_k x_k + f_{k+1}(s_k + a_k x_k) \}$$

= value on max value path from $s_k$ to final stage

(value to go)

Arc value




$f_3(8) = \max\{12 \cdot 1 + f_4(8+3 \cdot 1),\ 12 \cdot 2 + f_4(8+3 \cdot 2)\}$
$= \max\{12, -\infty\} = 12$

$f_4(14) = -\infty$

$f_4(11) = 0$

## Recursive optimization

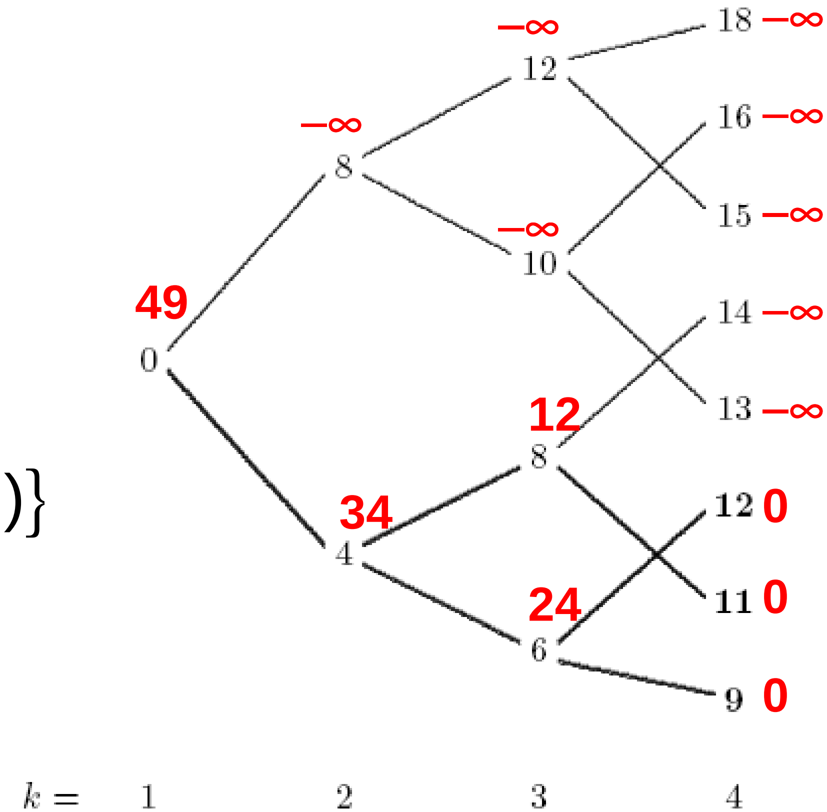
$$\max\ 15x_1 + 10x_2 + 12x_3$$
$$4x_1 + 2x_2 + 3x_3 \le 12$$
$$x_j \in \{1,2\}$$

The recursion includes arc values:

$$f_k(s_k) = \max\{c_k x_k + f_{k+1}(s_k + a_k x_k)\}$$

Boundary condition:

$$f_{n+1}(s_{n+1}) = \begin{cases} 0 & \text{if } s_{n+1} \le b \\ -\infty & \text{otherwise} \end{cases}$$




$f_k(s_k)$ for each state $s_k$

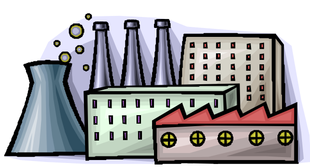
# Recursive optimization

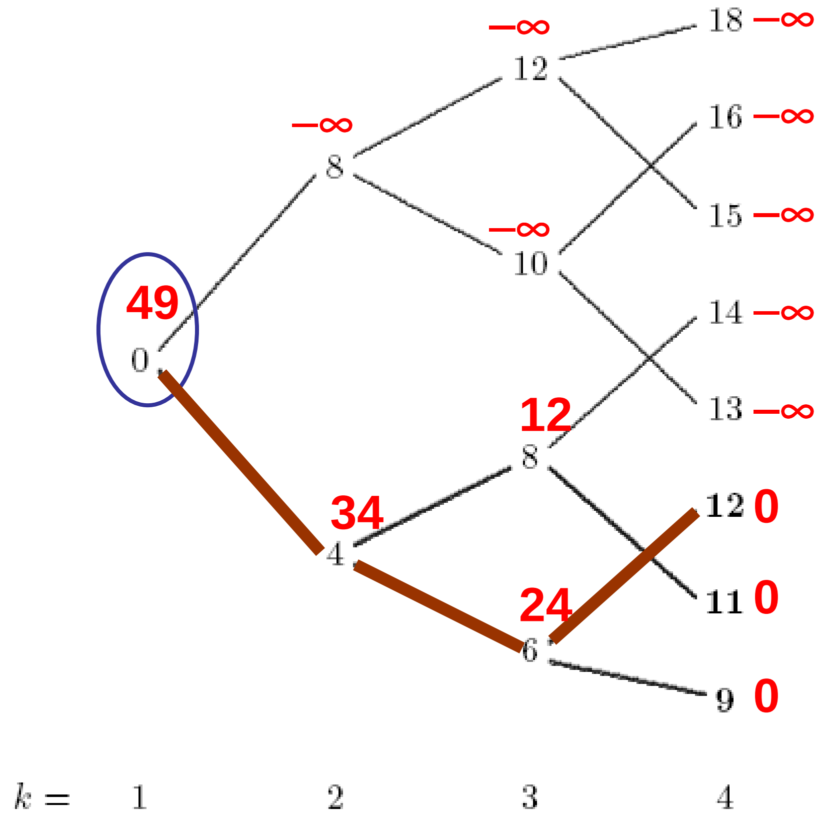
$$\max\ 15x_1 + 10x_2 + 12x_3$$
$$4x_1 + 2x_2 + 3x_3 \le 12$$
$$x_j \in \{1, 2\}$$

The maximum revenue is 49.

The optimal path is easy to retrace.

$(x_1, x_2, x_3) = (1,1,2)$






$f_k(s_k)$ for each state $s_k$

# CP-based Branch and Price

## Basic Idea

## Example: Airline Crew Scheduling

## Motivation

- **Branch and price** allows solution of integer programming problems with a huge number of variables.
- The problem is solved by a branch-and-relax method. The difference lies in how the LP relaxation is solved.
- Variables are added to the LP relaxation only as needed.
- Variables are **priced** to find which ones should be added.
- **CP** is useful for solving the pricing problem, particularly when constraints are complex.
- **CP-based branch and price** has been successfully applied to airline crew scheduling, transit scheduling, and other transportation-related problems.

# Basic Idea

Suppose the LP relaxation of an integer programming problem has a huge number of variables:

$$\begin{aligned} &\min\ cx \\ &Ax = b \\ &x \geq 0 \end{aligned}$$

We will solve a **restricted master problem**, which has a small subset of the variables:

$$\begin{aligned} &\min\ \sum_{j \in J} c_j x_j \\ &\sum_{j \in J} \boxed{A_j} x_j = b \qquad (\lambda) \\ &x_j \geq 0 \end{aligned}$$

Column $j$ of $A$

Adding $x_k$ to the problem would improve the solution if $x_k$ has a negative reduced cost:

$$r_k = c_k - \lambda A_k < 0$$

## Basic Idea

Adding $x_k$ to the problem would improve the solution if $x_k$ has a negative reduced cost:

$$r_k = c_k - \lambda A_k < 0$$

Computing the reduced cost of $x_k$ is known as **pricing** $x_k$.

So we solve the pricing problem:

Cost of column $y$

$$\min \boxed{c_y} - \lambda y$$
$$y \text{ is a column of } A$$

If the solution $y^*$ satisfies $c_{y^*} - \lambda y^* < 0$, then we can add column $y$ to the restricted master problem.

# Basic Idea

The pricing problem $\max \; \lambda y$

$y$ is a column of $A$

need not be solved to optimality, so long as we find a column with negative reduced cost.

However, when we can no longer find an improving column, we solved the pricing problem to optimality to make sure we have the optimal solution of the LP.

If we can state constraints that the columns of $A$ must satisfy, CP may be a good way to solve the pricing problem.

# Example: Airline Crew Scheduling

We want to assign crew members to flights to minimize cost while covering the flights and observing complex work rules.

Flight data

| $j$ | $s_j$ | $f_j$ |
|---|---|---|
| 1 | 0 | 3 |
| 2 | 1 | 3 |
| 3 | 5 | 8 |
| 4 | 6 | 9 |
| 5 | 10 | 12 |
| 6 | 14 | 16 |

Start time    Finish time

A **roster** is the sequence of flights assigned to a single crew member.

The gap between two consecutive flights in a roster must be from 2 to 3 hours. Total flight time for a roster must be between 6 and 10 hours.

For example,

flight 1 cannot immediately precede 6

flight 4 cannot immediately precede 5.

The possible rosters are:

(1,3,5), (1,4,6), (2,3,5), (2,4,6)

# Airline Crew Scheduling

There are 2 crew members, and the possible rosters are:

| 1 | 2 | 3 | 4 |
|---|---|---|---|
| (1,3,5), | (1,4,6), | (2,3,5), | (2,4,6) |

The LP relaxation of the problem is:

Cost of assigning crew member 1 to roster 2

$$\min z$$

$$\begin{bmatrix} 10 & 12 & 7 & 13 & 9 & 11 & 6 & 12 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{24} \end{bmatrix} \begin{matrix} = \\ = \\ = \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \end{matrix} \begin{bmatrix} z \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x_{ik} \geq 0, \text{ all } i, k$$

= 1 if we assign crew member 1 to roster 2, = 0 otherwise.

Each crew member is assigned to exactly 1 roster.

Each flight is assigned at least 1 crew member.

# Airline Crew Scheduling

There are 2 crew members, and the possible rosters are:

| 1 | 2 | 3 | 4 |
|---|---|---|---|
| (1,3,5), | (1,4,6), | (2,3,5), | (2,4,6) |

The LP relaxation of the problem is:

Cost of assigning crew member 1 to roster 2

$$\min z$$

$$\begin{bmatrix} 10 & 12 & 7 & 13 & 9 & 11 & 6 & 12 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{24} \end{bmatrix} \begin{matrix} = \\ = \\ = \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \end{matrix} \begin{bmatrix} z \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x_{ik} \geq 0, \text{ all } i, k$$

= 1 if we assign crew member 1 to roster 2, = 0 otherwise.

Each crew member is assigned to exactly 1 roster.

Each flight is assigned at least 1 crew member.

Rosters that cover flight 1.

# Airline Crew Scheduling

There are 2 crew members, and the possible rosters are:

1 2 3 4
(1,3,5), (1,4,6), (2,3,5), (2,4,6)

The LP relaxation of the problem is:

Cost of assigning crew member 1 to roster 2

$$\min z$$

$$\begin{bmatrix} 10 & 12 & 7 & 13 & 9 & 11 & 6 & 12 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & \boxed{1} & 0 & 0 & \boxed{1} & \boxed{1} \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{24} \end{bmatrix} \begin{matrix} = \\ = \\ = \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \end{matrix} \begin{bmatrix} z \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x_{ik} \geq 0, \text{ all } i, k$$

= 1 if we assign crew member 1 to roster 2, = 0 otherwise.

Each crew member is assigned to exactly 1 roster.

Each flight is assigned at least 1 crew member.

Rosters that cover flight 2.

# Airline Crew Scheduling

There are 2 crew members, and the possible rosters are:

1 2 3 4

(1,3,5), (1,4,6), (2,3,5), (2,4,6)

The LP relaxation of the problem is:

Cost of assigning crew member 1 to roster 2

$$\min z$$

$$\begin{bmatrix} 10 & 12 & 7 & 13 & 9 & 11 & 6 & 12 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ \boxed{1} & 0 & \boxed{1} & 0 & \boxed{1} & 0 & \boxed{1} & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{24} \end{bmatrix} \begin{matrix} = \\ = \\ = \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \end{matrix} \begin{bmatrix} z \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x_{ik} \geq 0, \ \text{all } i, k$$

= 1 if we assign crew member 1 to roster 2, = 0 otherwise.

Each crew member is assigned to exactly 1 roster.

Each flight is assigned at least 1 crew member.

Rosters that cover flight 3.

# Airline Crew Scheduling

There are 2 crew members, and the possible rosters are:

1 2 3 4

(1,3,5), (1,4,6), (2,3,5), (2,4,6)

The LP relaxation of the problem is:

Cost of assigning crew member 1 to roster 2

min $z$

$$\begin{bmatrix} 10 & 12 & 7 & 13 & 9 & 11 & 6 & 12 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & \boxed{1} & 0 & \boxed{1} & 0 & \boxed{1} \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{24} \end{bmatrix} \begin{matrix} = \\ = \\ = \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \end{matrix} \begin{bmatrix} z \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$x_{ik} \geq 0$, all $i, k$

= 1 if we assign crew member 1 to roster 2, = 0 otherwise.

Each crew member is assigned to exactly 1 roster.

Each flight is assigned at least 1 crew member.

Rosters that cover flight 4.

# Airline Crew Scheduling

There are 2 crew members, and the possible rosters are:

| 1 | 2 | 3 | 4 |
|---|---|---|---|
| (1,3,5), | (1,4,6), | (2,3,5), | (2,4,6) |

The LP relaxation of the problem is:

Cost of assigning crew member 1 to roster 2

$$\min z$$

$$\begin{bmatrix} 10 & 12 & 7 & 13 & 9 & 11 & 6 & 12 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \boxed{1} & 0 & \boxed{1} & 0 & \boxed{1} & 0 & \boxed{1} & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{24} \end{bmatrix} \begin{matrix} = \\ = \\ = \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \end{matrix} \begin{bmatrix} z \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x_{ik} \geq 0, \text{ all } i, k$$

= 1 if we assign crew member 1 to roster 2, = 0 otherwise.

Each crew member is assigned to exactly 1 roster.

Each flight is assigned at least 1 crew member.

Rosters that cover flight 5.

# Airline Crew Scheduling

There are 2 crew members, and the possible rosters are:

1 2 3 4

(1,3,5), (1,4,6), (2,3,5), (2,4,6)

The LP relaxation of the problem is:

Cost of assigning crew member 1 to roster 2

$$\min z$$

$$\begin{bmatrix} 10 & 12 & 7 & 13 & 9 & 11 & 6 & 12 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & \boxed{1} & 0 & \boxed{1} & 0 & \boxed{1} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{24} \end{bmatrix} \begin{matrix} = \\ = \\ = \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \end{matrix} \begin{bmatrix} z \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x_{ik} \geq 0, \text{ all } i, k$$

= 1 if we assign crew member 1 to roster 2, = 0 otherwise.

Each crew member is assigned to exactly 1 roster.

Each flight is assigned at least 1 crew member.

Rosters that cover flight 6.

# Airline Crew Scheduling

There are 2 crew members, and the possible rosters are:

1 2 3 4

(1,3,5), (1,4,6), (2,3,5), (2,4,6)

The LP relaxation of the problem is:

Cost $c_{12}$ of assigning crew member 1 to roster 2

$$\min\ z$$

$$\begin{bmatrix} 10 & 12 & 7 & 13 & 9 & 11 & 6 & 12 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & \boxed{1} & 0 & \boxed{1} & 0 & \boxed{1} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{24} \end{bmatrix} \begin{array}{c} = \\ = \\ = \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \end{array} \begin{bmatrix} z \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x_{ik} \geq 0,\ \text{all } i, k$$

= 1 if we assign crew member 1 to roster 2, = 0 otherwise.

Each crew member is assigned to exactly 1 roster.

Each flight is assigned at least 1 crew member.

In a real problem, there can be **millions** of rosters.

# Airline Crew Scheduling

We start by solving the problem with a subset of the columns:

Optimal dual solution

$$\min\ z$$

$$\begin{bmatrix} 10 & 13 & 9 & 12 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{14} \\ x_{21} \\ x_{24} \end{bmatrix} \begin{array}{c} = \\ = \\ = \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \end{array} \begin{bmatrix} z \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \boxed{\begin{array}{c} \\ (10) \\ (9) \\ (0) \\ (0) \\ (0) \\ (0) \\ (0) \\ (3) \end{array}} \begin{array}{c} \\ u_1 \\ u_2 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array}$$

$$x_{ik} \geq 0, \text{ all } i, k$$

# Airline Crew Scheduling

We start by solving the problem with a subset of the columns:

Dual variables

$$\min\ z$$

$$\begin{bmatrix}10 & 13 & 9 & 12\\ 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{bmatrix}\begin{bmatrix}x_{11}\\ x_{14}\\ x_{21}\\ x_{24}\end{bmatrix}\begin{array}{c}=\\ =\\ =\\ \geq\\ \geq\\ \geq\\ \geq\\ \geq\\ \geq\end{array}\begin{bmatrix}z\\ 1\\ 1\\ 1\\ 1\\ 1\\ 1\\ 1\\ 1\end{bmatrix}\qquad\begin{array}{c}\\ (10)\\ (9)\\ (0)\\ (0)\\ (0)\\ (0)\\ (0)\\ (3)\end{array}\ \boxed{\begin{array}{c}u_1\\ u_2\\ v_1\\ v_2\\ v_3\\ v_4\\ v_5\\ v_6\end{array}}$$

$$x_{ik} \geq 0,\ \text{all } i, k$$

# Airline Crew Scheduling

We start by solving the problem with a subset of the columns:

Dual variables

$$\min\ z$$

$$\begin{bmatrix}10&13&9&12\\1&1&0&0\\0&0&1&1\\1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{bmatrix}\begin{bmatrix}x_{11}\\x_{14}\\x_{21}\\x_{24}\end{bmatrix}\begin{matrix}=\\=\\=\\\geq\\\geq\\\geq\\\geq\\\geq\\\geq\end{matrix}\begin{bmatrix}z\\1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}\qquad\begin{matrix}(10)&u_1\\(9)&u_2\\(0)&v_1\\(0)&v_2\\(0)&v_3\\(0)&v_4\\(0)&v_5\\(3)&v_6\end{matrix}$$

$$x_{ik}\geq 0,\ \text{all } i,k$$

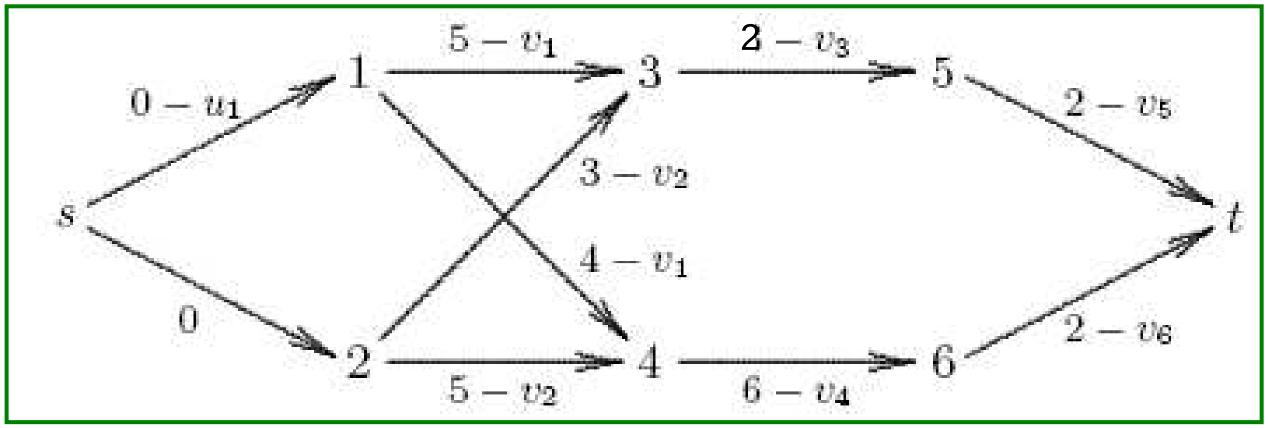
The reduced cost of an excluded roster *k* for crew member *i* is

$$c_{ik} - u_i - \sum_{j \text{ in roster k}} v_j$$

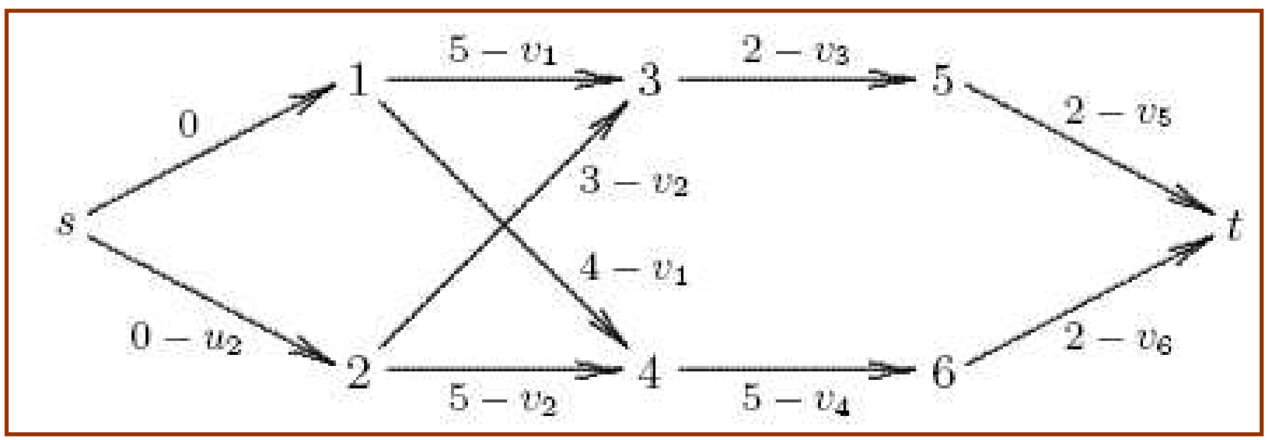
We will formulate the pricing problem as a shortest path problem.

# Pricing problem

Crew
member 1



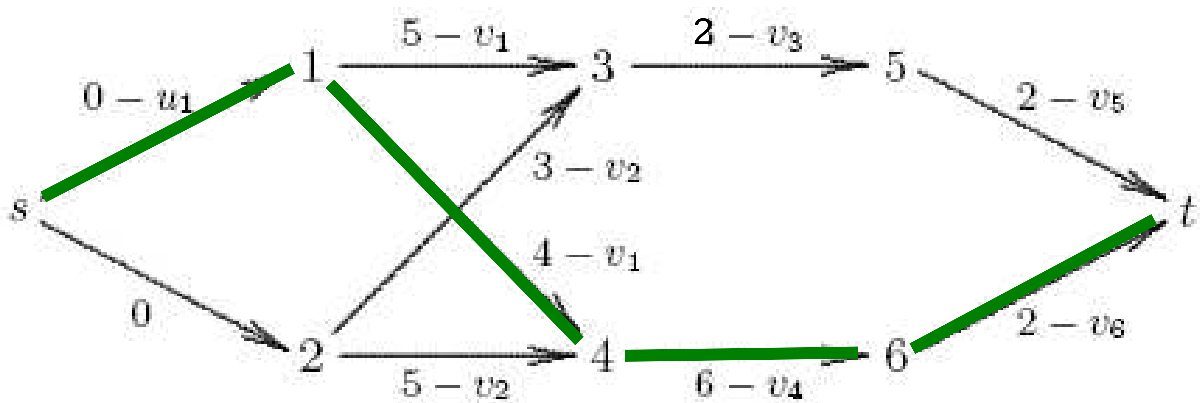

Crew
member 2



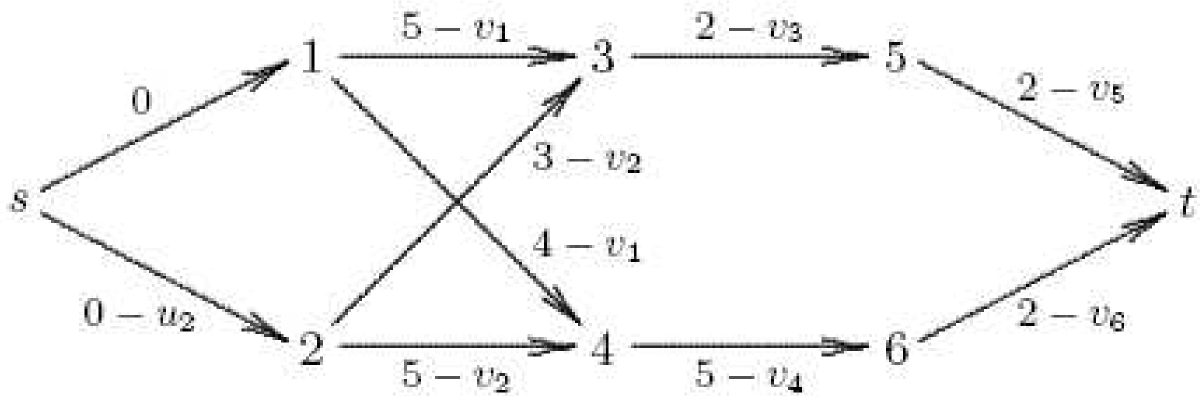
# Pricing problem

Each s-t path corresponds to a roster, provided the flight time is within bounds.

Crew member 1




Crew member 2

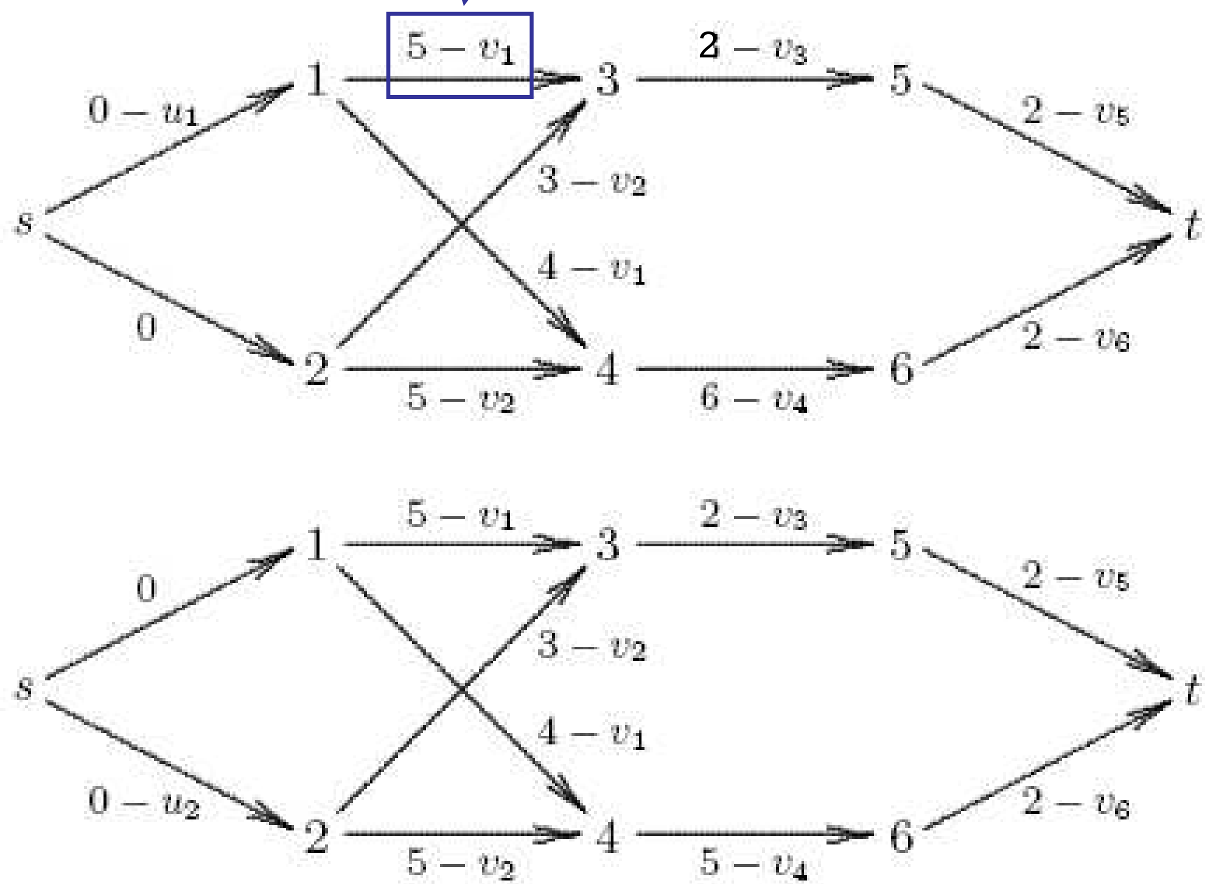


# Pricing problem

Cost of flight 3 if it immediately follows flight 1, offset by dual multiplier for flight 1

Crew member 1

Crew member 2



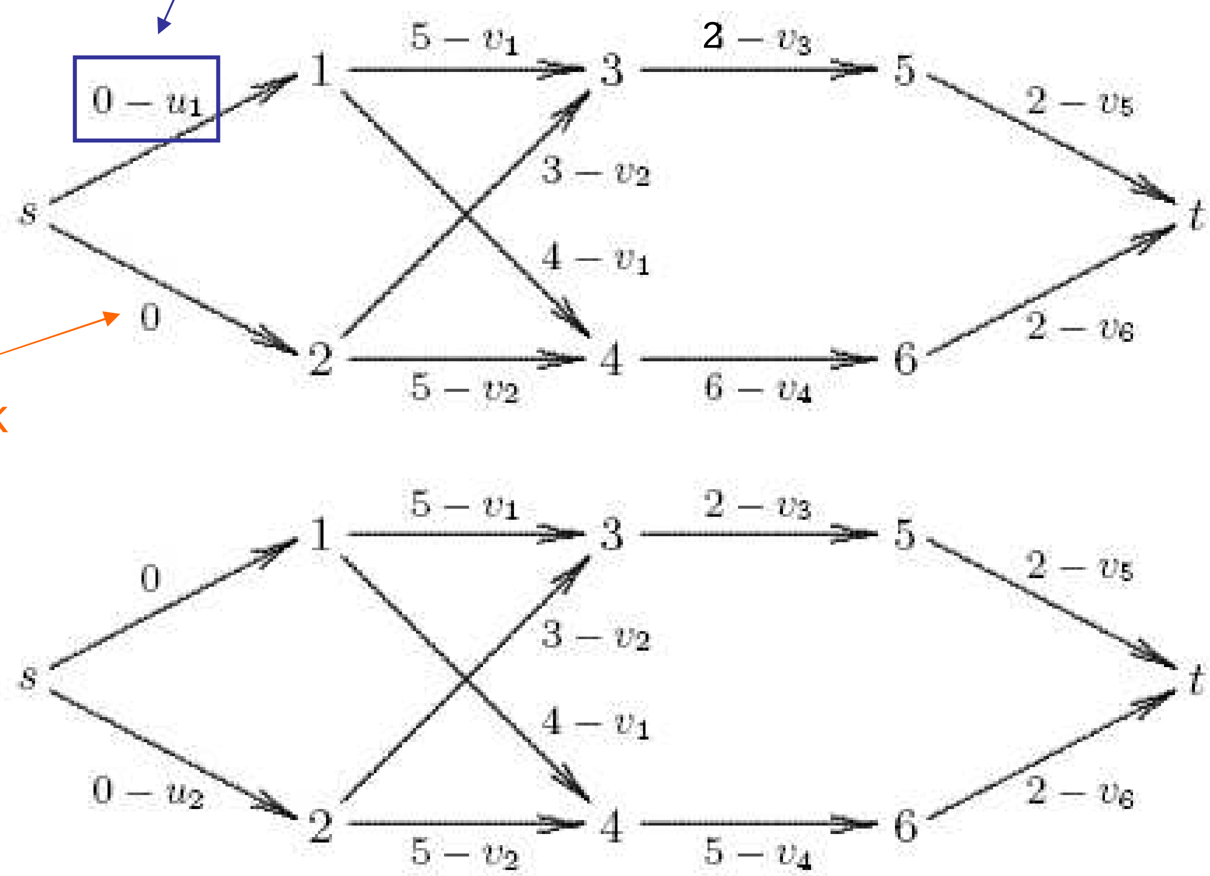
# Pricing problem

Cost of transferring from home to flight 1, offset by dual multiplier for crew member 1

Crew member 1

Dual multiplier omitted to break symmetry

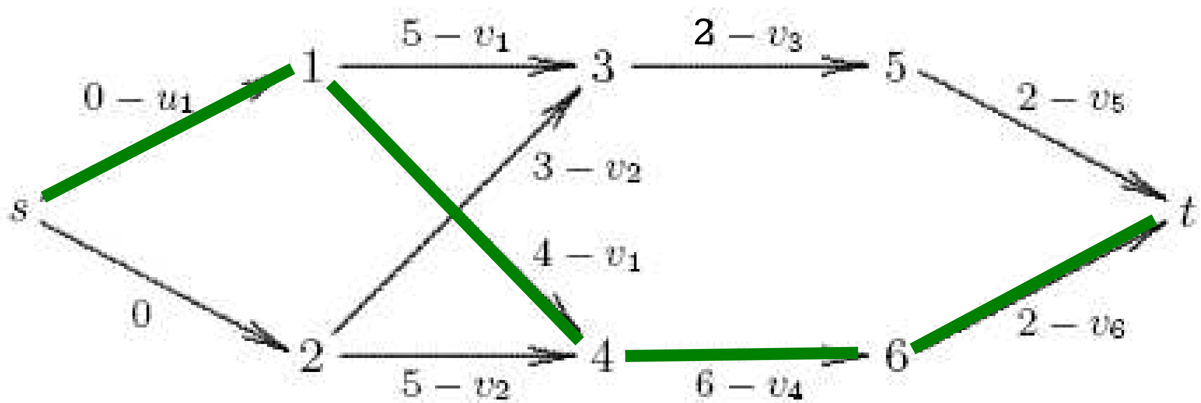
Crew member 2



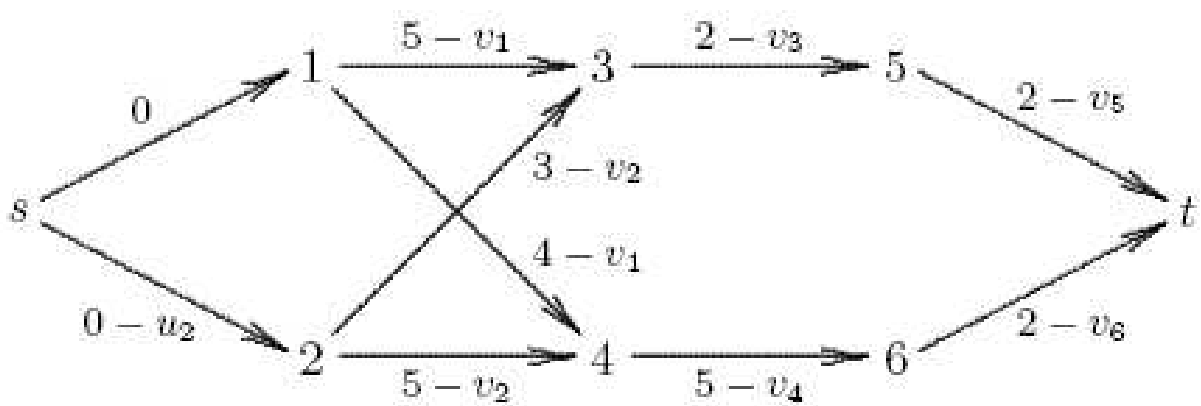
# Pricing problem

Length of a path is reduced cost of the corresponding roster.

Crew member 1



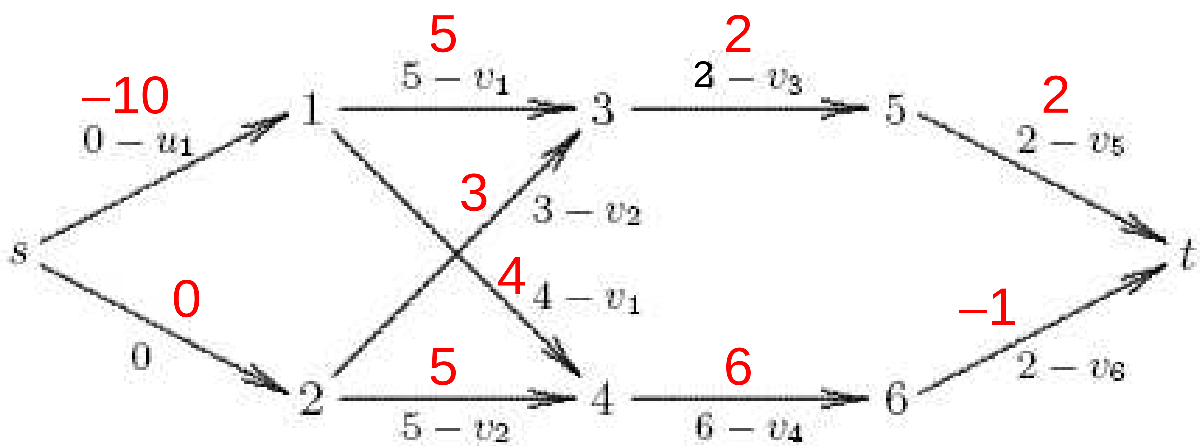

Crew member 2



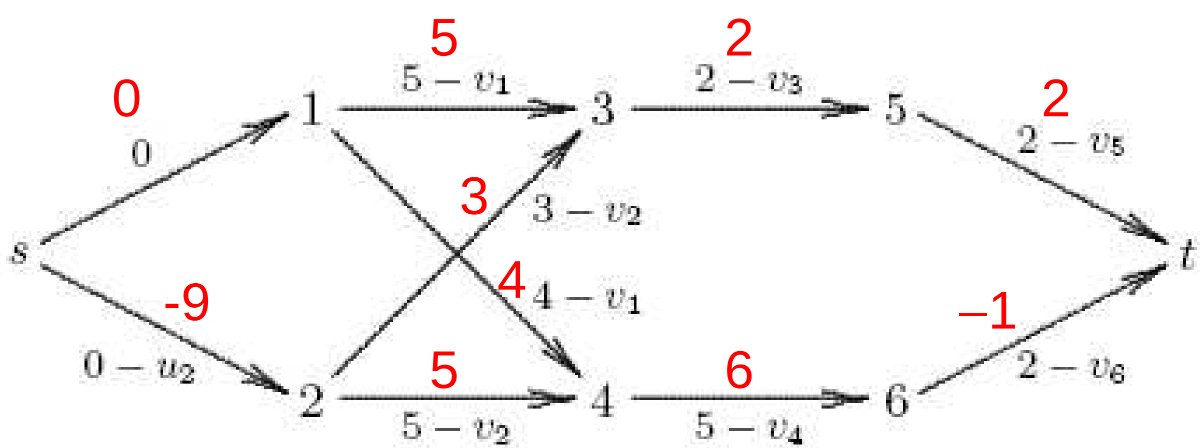
# Pricing problem

Arc lengths using dual solution of LP relaxation

Crew member 1




Crew member 2

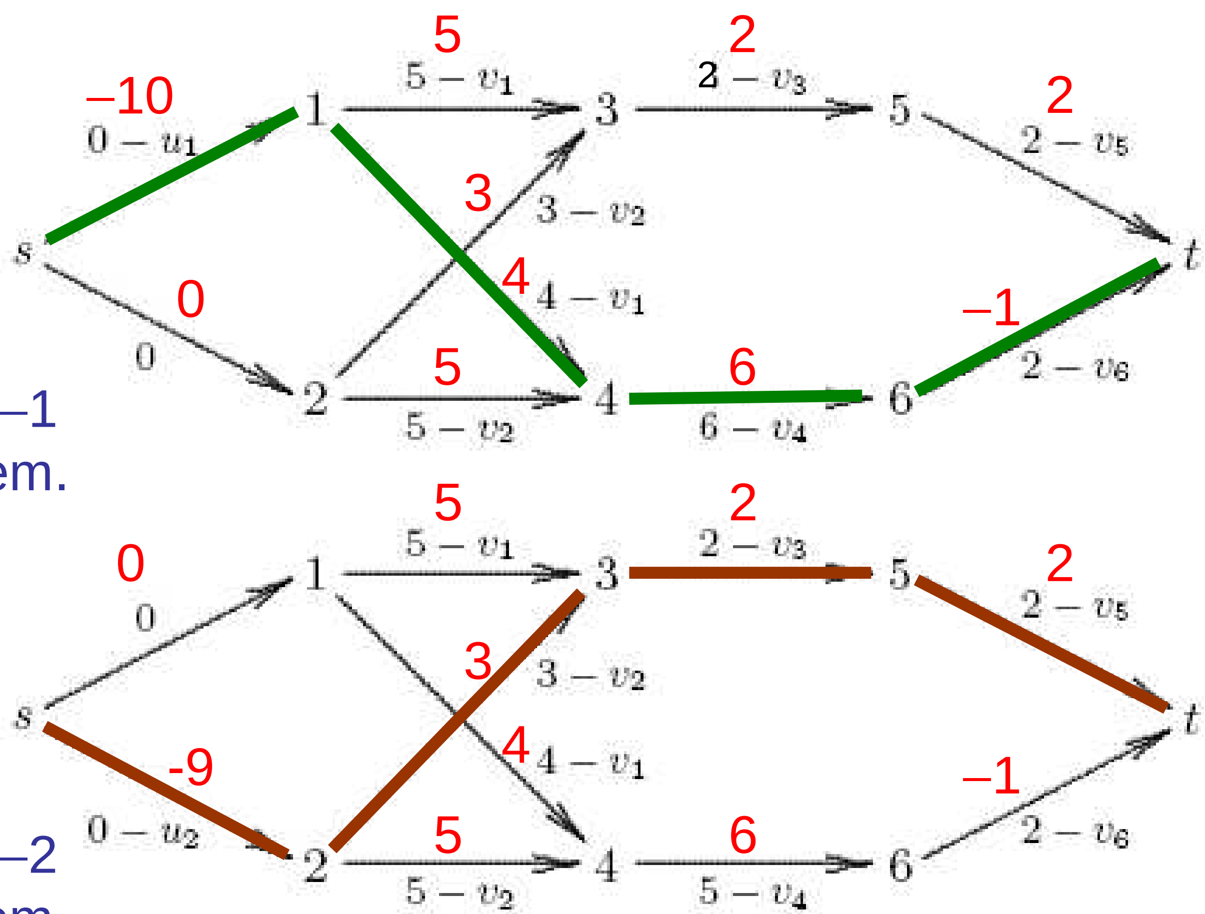


# Pricing problem

## Solution of shortest path problems

Crew member 1

Reduced cost = –1
Add $x_{12}$ to problem.




Crew member 2

Reduced cost = –2
Add $x_{23}$ to problem.

After $x_{12}$ and $x_{23}$ are added to the problem, no remaining variable has negative reduced cost.

# Pricing problem

The shortest path problem cannot be solved by traditional shortest path algorithms, due to the bounds on total path length.

It **can** be solved by CP:

Set of flights assigned to crew member *i*

Path length

Graph

*Path* global constraint → $\text{Path}(\boxed{X_i}, \boxed{z_i}, \boxed{G})$, all flights $i$

*Setsum* global constraint → $T_{\min} \le \sum_{j \in X_i} \left( \boxed{f_j - s_j} \right) \le T_{\max}$

$X_i \subset \{\text{flights}\}, \;\; z_i < 0, \text{ all } i$

Duration of flight *j*


The
END
is
NEAR


# CP-based Benders Decomposition

Benders Decomposition in the Abstract

Classical Benders Decomposition

Example: Machine Scheduling

## Motivation

- **Benders decomposition** allows us to apply CP and OR to different parts of the problem.
- It searches over values of certain variables that, when fixed, result in a much simpler **subproblem**.
- The search learns from past experience by accumulating **Benders cuts** (a form of nogood).
- The technique can be **generalized** far beyond the original OR conception.
- Generalized Benders methods have resulted in the **greatest speedups** achieved by combining CP and OR.

# Benders Decomposition in the Abstract

Benders decomposition can be applied to problems of the form

$$\begin{aligned}&\min\ f(x,y)\\&S(x,y)\\&x\in D_x,\ \ y\in D_y\end{aligned}$$

When x is fixed to some value, the resulting **subproblem** is much easier:

$$\begin{aligned}&\min\ f(\bar{x},y)\\&S(\bar{x},y)\\&y\in D_y\end{aligned}$$

…perhaps because it decouples into smaller problems.

For example, suppose *x* assigns jobs to machines, and *y* schedules the jobs on the machines.

When *x* is fixed, the problem decouples into a separate scheduling subproblem for each machine.

# Benders Decomposition

We will search over assignments to $x$.  This is the **master problem**.

In iteration $k$ we assume $x = x^k$ and solve the subproblem

$$\begin{array}{l} \min \ f(x^k, y) \\ S(x^k, y) \\ y \in D_y \end{array}$$

and get optimal value $v_k$

We generate a **Benders cut** (a type of nogood) $\boxed{v} \geq B_{k+1}(x)$

that satisfies $B_{k+1}(x^k) = v_k$.

Cost in the original problem

The Benders cut says that if we set $x = x^k$ again, the resulting cost $v$ will be at least $v_k$.  To do better than $v_k$, we must try something else.

It also says that any other $x$ will result in a cost of at least $B_{k+1}(x)$, perhaps due to some similarity between $x$ and $x^k$.

# Benders Decomposition

We will search over assignments to $x$. This is the **master problem**.

In iteration $k$ we assume $x = x^k$ and solve the subproblem

$$\begin{aligned} &\min\ f(x^k, y) \\ &S(x^k, y) \\ &y \in D_y \end{aligned}$$

and get optimal value $v_k$

We generate a **Benders cut** (a type of nogood) $\boxed{v} \geq B_{k+1}(x)$

that satisfies $B_{k+1}(x) = v_k$.

Cost in the original problem

We add the Benders cut to the master problem, which becomes

$$\begin{aligned} &\min\ v \\ &v \geq B_i(x),\ i = 1, \ldots, k+1 \\ &x \in D_x \end{aligned}$$

Benders cuts generated so far

# Benders Decomposition

We now solve the master problem

$$\begin{aligned} &\min \ v \\ &v \geq B_i(x), \ i = 1,\ldots,k+1 \\ &x \in D_x \end{aligned}$$

to get the next trial value $x^{k+1}$.

The master problem is a relaxation of the original problem, and its optimal value is a **lower bound** on the optimal value of the original problem.

The subproblem is a restriction, and its optimal value is an **upper bound**.

The process continues until the bounds meet.

The Benders cuts partially define the **projection** of the feasible set onto $x$. We hope not too many cuts are needed to find the optimum.

# Classical Benders Decomposition

The classical method applies to problems of the form

$$\begin{aligned} &\min\ f(x) + cy \\ &g(x) + Ay \geq b \\ &x \in D_x,\ \ y \geq 0 \end{aligned}$$

and the subproblem is an LP

$$\begin{aligned} &\min\ f(x^k) + cy \\ &Ay \geq b - g(x^k) \quad (\lambda) \\ &y \geq 0 \end{aligned}$$

whose dual is

$$\begin{aligned} &\max\ f(x^k) + \lambda\left(b - g(x^k)\right) \\ &\lambda A \leq c \\ &\lambda \geq 0 \end{aligned}$$

Let $\lambda^k$ solve the dual.

By strong duality, $B_{k+1}(x) = f(x) + \lambda^k(b - g(x))$ is the tightest lower bound on the optimal value $v$ of the original problem when $x = x^k$.

Even for other values of $x$, $\lambda^k$ **remains feasible in the dual**. So by weak duality, $B_{k+1}(x)$ remains a lower bound on $v$.

## Classical Benders

So the master problem

$$\begin{aligned} &\min\ v \\ &v \ge B_i(x),\ i = 1,\ldots,k+1 \\ &x \in D_x \end{aligned}$$

becomes

$$\begin{aligned} &\min\ v \\ &v \ge f(x) + \lambda^i (b - g(x)),\ i = 1,\ldots,k+1 \\ &x \in D_x \end{aligned}$$

In most applications the master problem is

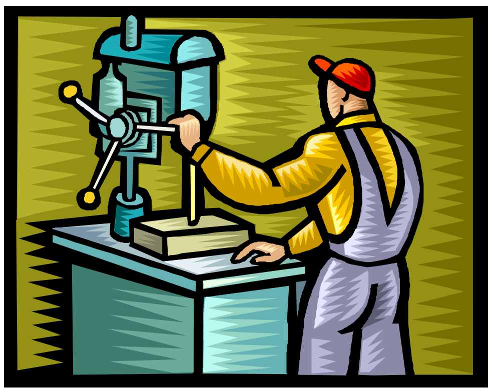
- an MILP
- a nonlinear programming problem (NLP), or
- a mixed integer/nonlinear programming problem (MINLP).

# Example: Machine Scheduling

• Assign 5 jobs to 2 machines (A and B), and schedule the machines assigned to each machine within time windows.

• The objective is to minimize **makespan**.

Time lapse between start of first job and end of last job.



• Assign the jobs in the **master problem**, to be solved by **MILP**.

• Schedule the jobs in the **subproblem**, to be solved by **CP**.

# Machine Scheduling

Job Data

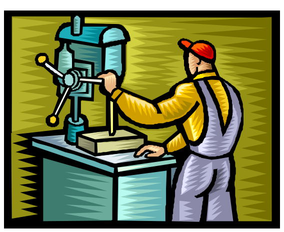
| *Job* $j$ | *Release time* $r_j$ | *Dead-line* $d_j$ | *Processing time* $p_{Aj}$ | $p_{Bj}$ |
|---|---|---|---|---|
| 1 | 0 | 10 | 1 | 5 |
| 2 | 0 | 10 | 3 | 6 |
| 3 | 2 | 7 | 3 | 7 |
| 4 | 2 | 10 | 4 | 6 |
| 5 | 4 | 7 | 2 | 5 |

Machine A

Machine B

Once jobs are assigned, we can minimize overall makespan by minimizing makespan on each machine individually.

So the subproblem decouples.



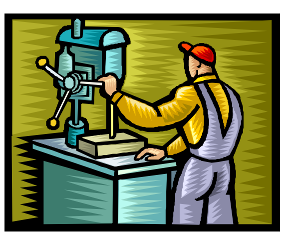
# Machine Scheduling

Job Data

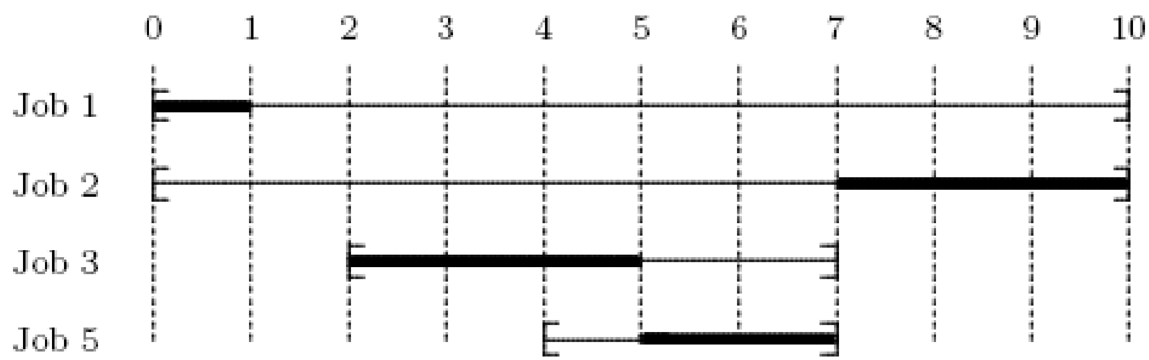
| *Job* $j$ | *Release time* $r_j$ | *Dead-line* $d_j$ | *Processing time* $p_{Aj}$ | $p_{Bj}$ |
|---|---|---|---|---|
| 1 | 0 | 10 | 1 | 5 |
| 2 | 0 | 10 | 3 | 6 |
| 3 | 2 | 7 | 3 | 7 |
| 4 | 2 | 10 | 4 | 6 |
| 5 | 4 | 7 | 2 | 5 |

Once jobs are assigned, we can minimize overall makespan by minimizing makespan on each machine individually.

So the subproblem decouples.



Minimum makespan schedule for jobs 1, 2, 3, 5 on machine A



# Machine Scheduling

The problem is

Start time of job *j*

$$\min \; M$$

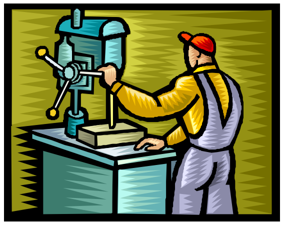
$$M \geq \boxed{s_j} + p_{x_j j}, \text{ all } j$$

Time windows

$$r_j \leq s_j \leq d_j - p_{x_j j}, \text{ all } j$$

Jobs cannot overlap

$$\text{disjunctive}\Big( (s_j \,|\, x_j = i), (p_{ij} \,|\, x_j = i) \Big), \text{ all } i$$



# Machine Scheduling

The problem is

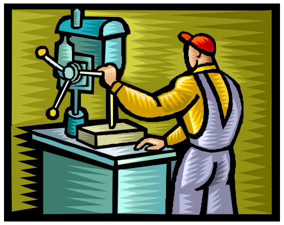
Start time of job *j*

Time windows

Jobs cannot overlap

$$\begin{aligned}&\min\ M\\ &M \ge \boxed{s_j} + p_{x_j j},\ \text{all } j\\ &r_j \le s_j \le d_j - p_{x_j j},\ \text{all } j\\ &\text{disjunctive}\big((s_j \,|\, x_j = i),(p_{ij} \,|\, x_j = i)\big),\ \text{all } i\end{aligned}$$

For a fixed assignment $\bar{x}$ the subproblem on each machine *i* is

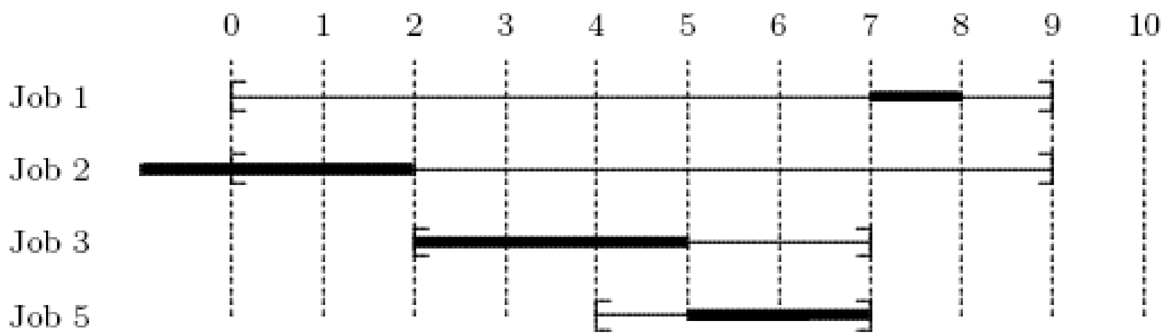


$$\begin{aligned}&\min\ M\\ &M \ge s_j + p_{\bar{x}_j j},\ \text{all } j \text{ with } \bar{x}_j = i\\ &r_j \le s_j \le d_j - p_{\bar{x}_j j},\ \text{all } j \text{ with } \bar{x}_j = i\\ &\text{disjunctive}\big((s_j \,|\, \bar{x}_j = i),(p_{ij} \,|\, \bar{x}_j = i)\big)\end{aligned}$$

# Benders cuts

Suppose we assign jobs 1,2,3,5 to machine A in iteration *k*.

We can prove that 10 is the optimal makespan by proving that the schedule is infeasible with makespan 9.




Edge finding derives infeasibility by reasoning only with jobs 2,3,5. So these jobs alone create a minimum makespan of 10.

So we have a Benders cut

$$v \geq B_{k+1}(x) = \begin{cases} 10 & \text{if } x_2 = x_3 = x_4 = A \\ 0 & \text{otherwise} \end{cases}$$

## Benders cuts

We want the master problem to be an MILP, which is good for assignment problems.

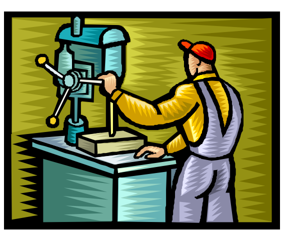
So we write the Benders cut

$$v \ge B_{k+1}(x) = \begin{cases} 10 & \text{if } x_2 = x_3 = x_4 = A \\ 0 & \text{otherwise} \end{cases}$$

Using 0-1 variables:

$$v \ge 10\left(x_{A2} + x_{A3} + \boxed{x_{A5}} - 2\right)$$
$$v \ge 0$$

= 1 if job 5 is assigned to machine A



## Master problem

The master problem is an MILP:

$$\min\ v$$

$$\sum_{j=1}^{5} p_{Aj} x_{Aj} \le 10, \text{ etc.}$$

$$\sum_{j=1}^{5} p_{Bj} x_{Bj} \le 10, \text{ etc.}$$

$$v \ge \sum_{j=1}^{5} p_{ij} x_{ij},\ \ v \ge 2 + \sum_{j=3}^{5} p_{ij} x_{ij}, \text{ etc.},\ \ i = A,B$$

$$\mathrm{v} \ge 10(x_{A2} + x_{A3} + x_{A5} - 2)$$

$$v \ge 8x_{B4}$$

$$x_{ij} \in \{0,1\}$$

Constraints derived from time windows

Constraints derived from release times

Benders cut from machine A

Benders cut from machine B

# Stronger Benders cuts

If all release times are the same, we can strengthen the Benders cuts.

We are now using the cut

$$v \ge M_{ik} \left( \sum_{j \in J_{ik}} x_{ij} - |J_{ik}| + 1 \right)$$

Min makespan on machine *i* in iteration *k*

Set of jobs assigned to machine *i* in iteration *k*

A stronger cut provides a useful bound even if only some of the jobs in $J_{ik}$ are assigned to machine *i*:

$$v \ge M_{ik} - \sum_{j \in J_{ik}} (1 - x_{ij}) p_{ij}$$

These results can be generalized to cumulative scheduling.


THE
END