A Logic-Based Benders Approach to Home Healthcare Delivery

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Outline

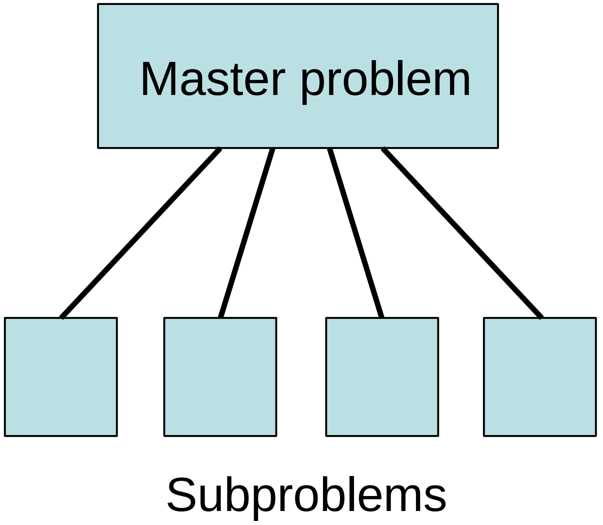
- Logic-based Benders tutorial
  - The algorithm
  - Inference duality
  - Machine scheduling
- Home health care
  - The problem
  - Logic-based Benders model
  - Computational results
  - References

Decomposition

- **Decomposition** breaks a large problem into subproblems that can be solved separately.
  - But with some kind of **communication** among the subproblems.
  - Decomposition is an **essential strategy** for solving today's ever larger and more interconnected models.

Benders Decomposition

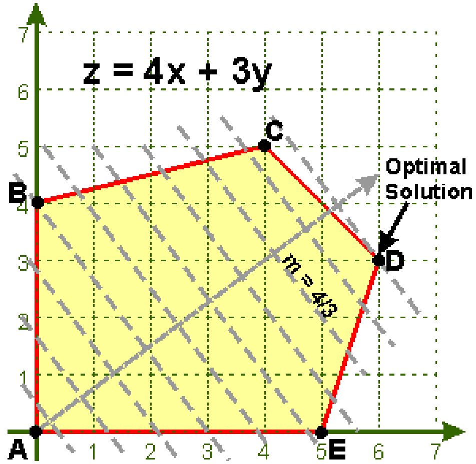
- **Benders decomposition** is a classical strategy that does not sacrifice overall optimality.
  - Separates the problem into a **master problem** and multiple **subproblems**.
    - Variables are partitioned between master and subproblems.
    - Exploits the fact that the problem may **radically simplify** when the master problem variables are fixed to a set of values.



Benders Decomposition

- But classical Benders decomposition has **a serious limitation.**
  - The subproblems must be **linear programming** problems.
    - Or continuous nonlinear programming problems.
    - The **linear programming dual** provides the Benders cuts.

Benders 1962

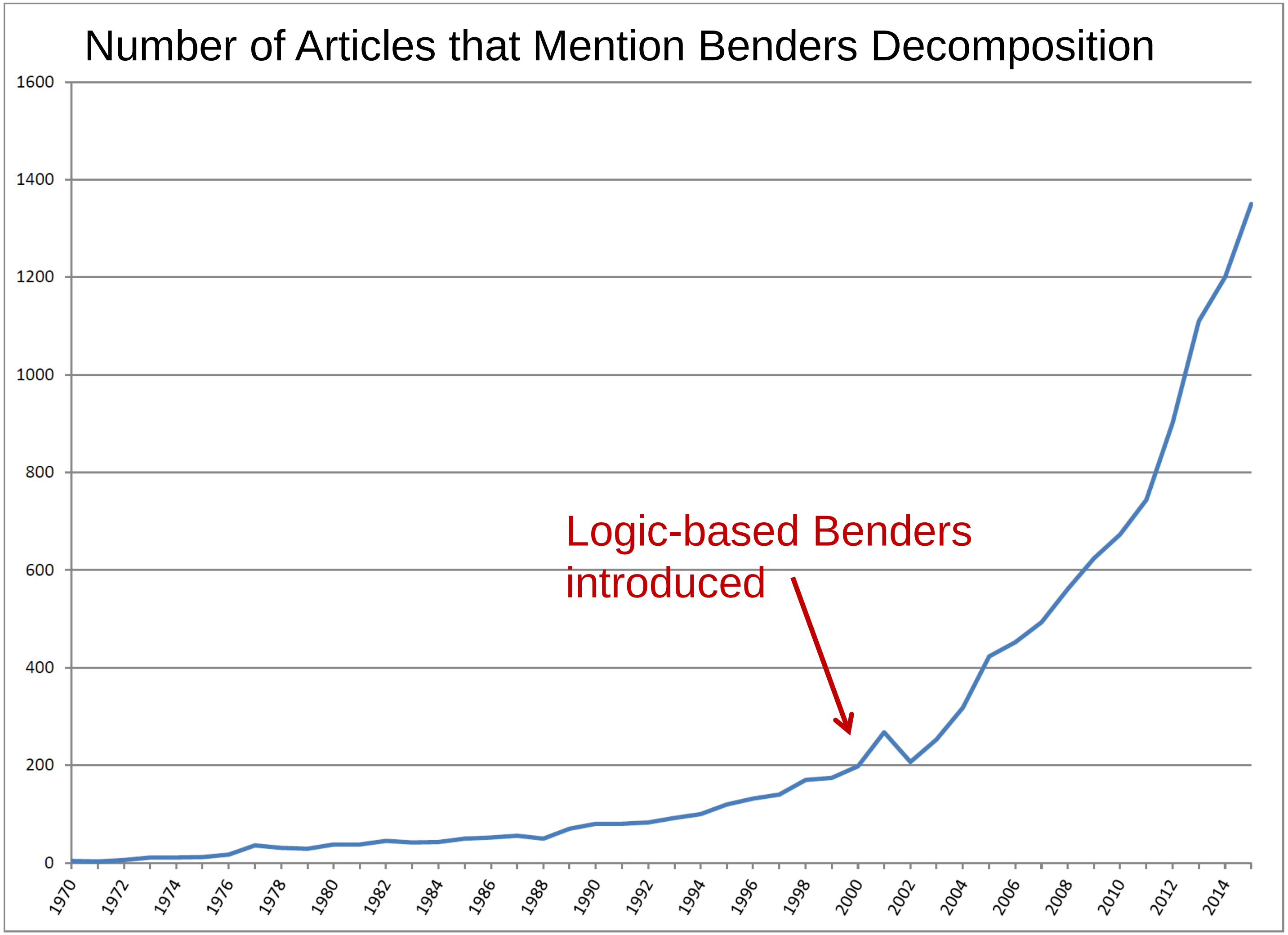


Logic-Based Benders

- **Logic-based Benders decomposition** attempts to overcome this limitation.
  - The subproblems can, in principle, be **any kind of optimization problem**.
    - The Benders cuts are obtained from an **inference dual**.
  - Speedup over state of the art can be **several orders of magnitude**.
  - Yet the Benders cuts must be designed specifically for every class of problems.

JH 1996, 2000
JH & Ottosson 2003

Logic-Based Benders



Source: Google Scholar

Logic-Based Benders

- Logic-based Benders decomposition solves a problem of the form

$$\min f(x,y)$$
$$(x,y) \in S$$
$$x \in D_x, y \in D_y$$

  – Where the problem simplifies when ***x* is fixed** to a specific value.

Logic-Based Benders

- Decompose problem into master and subproblem.
  - Subproblem is obtained by fixing x to solution value in master problem.

Master problem

$$\min z$$
$$z \geq g_k(x) \quad \text{(Benders cuts)}$$
$$x \in D_x$$

Minimize cost z subject to bounds given by Benders cuts, obtained from values of x attempted in previous iterations k.

→ Trial value $\bar{x}$ that solves master

← Benders cut $z \geq g_k(x)$

Subproblem

$$\min f(\bar{x}, y)$$
$$(\bar{x}, y) \in S$$

Obtain proof of optimality (solution of **inference dual**). Use same proof to deduce cost bounds for other assignments, yielding Benders cut.

Logic-Based Benders

- Iterate until master problem value equals best subproblem value so far.
  - This yields optimal solution.

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→ Trial value $\bar{x}$ that solves master

← Benders cut $z \geq g_k(x)$

Subproblem

$$\min f(\bar{x}, y)$$
$$(\bar{x}, y) \in S$$

Obtain proof of optimality (solution of **inference dual**). Use same proof to deduce cost bounds for other assignments, yielding Benders cut.

Logic-Based Benders

- Fundamental concept: inference duality

Primal problem: optimization

$$\min f(x)$$
$$x \in S$$

Find best feasible solution by searching over values of *x*.

Dual problem: Inference

$$\max v$$
$$x \in S \overset{P}{\Longrightarrow} f(x) \geq v$$
$$P \in \mathcal{P}$$

Find a proof of optimal value *v** by searching over proofs *P.*

In classical LP, the proof is a tuple of dual multipliers

Logic-Based Benders

- The proof that solves the dual in iteration k gives a bound $g_k(\bar{x})$ on the optimal value.
  - **The same proof** gives a bound $g_k(x)$ for other values of x.

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$$x \in D_x$$

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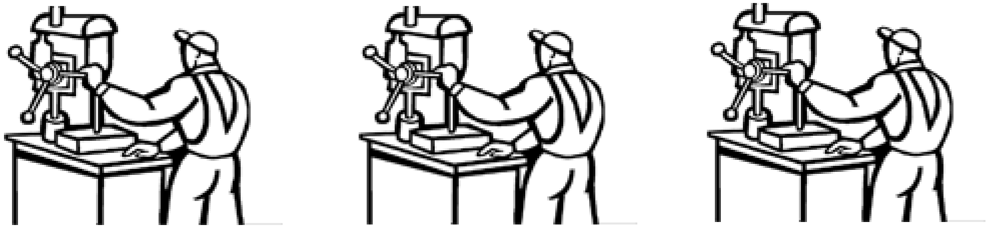
Logic-Based Benders

- Popular optimization duals are special cases of the inference dual.
  - Result from different choices of inference method.
  - For example....
    - Linear programming dual (gives **classical Benders cuts**)
    - Lagrangean dual
    - Surrogate dual
    - Subadditive dual

Machine Scheduling

- Assign tasks to machines.
- Then schedule tasks assigned to each machine.
  - Subject to time windows.
  - Cumulative scheduling: several tasks can run simultaneously, subject to resource limits.
  - Scheduling problem **decouples** into a separate problem for each machine.



Jain & Grossmann 2001

Machine Scheduling

- Assign tasks in master, schedule in subproblem.
  - Combine **mixed integer programming** and **constraint programming**

Master problem

Assign tasks to resources to minimize cost.

Solve by **mixed integer programming**.

Trial assignment $\bar{x}$

Benders cut $z \geq g_k(x)$

Subproblem

Schedule jobs on each machine, subject to time windows.

Constraint programming obtains proof of optimality (dual solution).

Use **same proof** to deduce cost for some other assignments, yielding Benders cut.

Machine Scheduling

- Objective function
  - Cost is based on **task assignment only**.

$$\text{cost} = \sum_{ij} c_{ij} x_{ij}, \quad x_{ij} = 1 \text{ if task } j \text{ assigned to resource } i$$

  - So cost appears only in the **master problem**.
  - Scheduling subproblem is a **feasibility problem**.

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- Benders cuts
  - They have the form $\sum_{j \in J_i} (1 - x_{ij}) \geq 1, \text{ all } i$
    - where J_i is a set of tasks that create infeasibility when assigned to resource i.

Machine Scheduling

- Resulting Benders decomposition:

Master problem

$$\min\ z$$
$$z = \sum_{ij} c_{ij} x_{ij}$$
Benders cuts

Trial assignment $\bar{x}$ →

← Benders cuts

$$\sum_{j \in J_i} (1 - x_{ij}) \geq 1,$$

for infeasible resources *i*

Subproblem

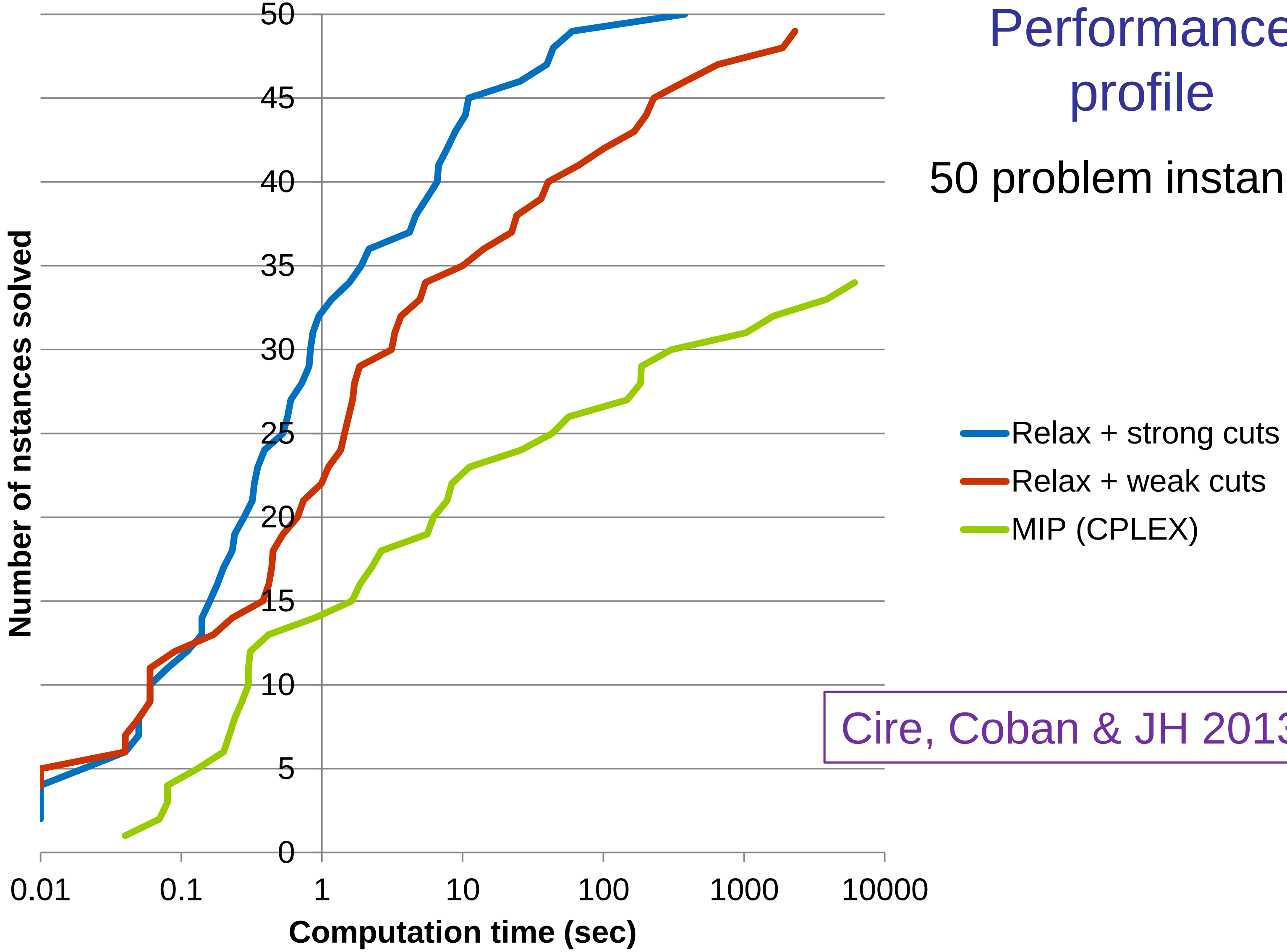
Schedule jobs on each resource.

Constraint programming may obtain proof of infeasibility on some resources (dual solution).

Use **same proof** to deduce infeasibility for some other assignments, yielding Benders cut.

Performance profile

50 problem instances



Cire, Coban & JH 2013

Home Healthcare

- General home health care problem.
  – Assign **aides** to homebound **patients**.
    - …subject to constraints on aide qualifications and patent preferences.
    - One patient may require a team of aides.
  – **Route** each aide through assigned patients, observing **time windows**.
    - …subject to constraints on hours, breaks, etc.

Home Healthcare

- A large industry, and **rapidly growing**.
  - Roughly as large as all courier and delivery services.

Projected Growth
of Home Health Care Industry

	2014	2018
U.S. revenues, $ billions	75	150
World revenues, $ billions	196	306

Increase in U.S. Employment, 2010-2020

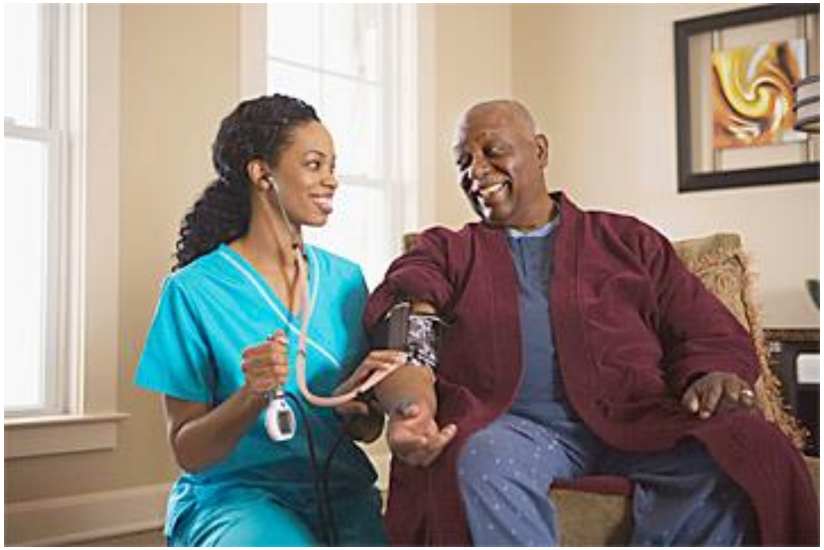
Home health care industry	70%
Entire economy	14%

Home Healthcare

- Advantages of home healthcare
  - Lower cost
    - Hospital & nursing home care is very expensive.
  - No hospital-acquired infections
    - Less exposure to superbugs.
  - Preferred by patients
    - Comfortable, familiar surroundings of home.
    - Sense of control over one's life.
  - Supported by new equipment & technology
    - IT integration with hospital systems.
    - Online consulting with specialists.

Home Healthcare

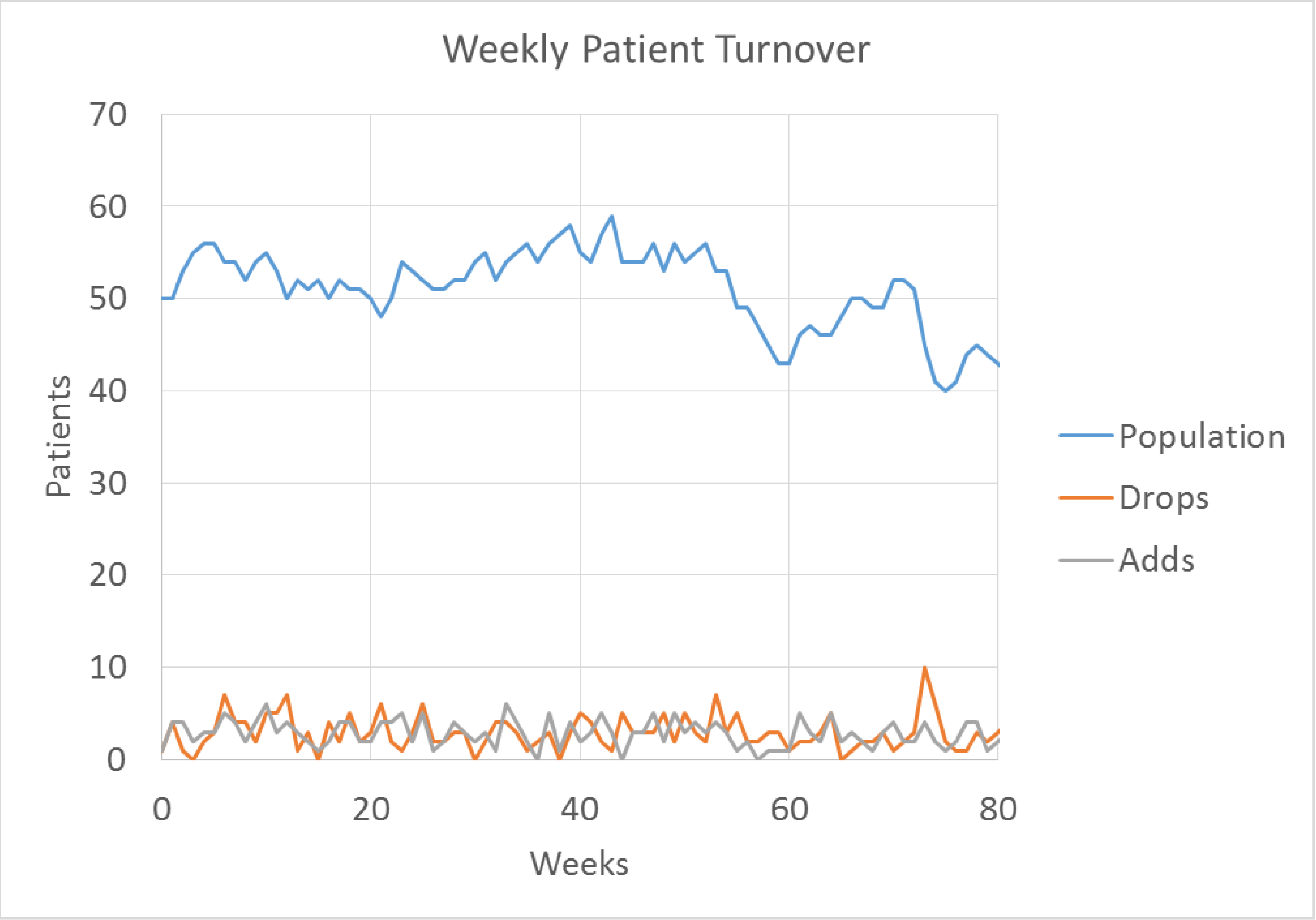
- Critical factor to realize cost savings:
  - Aides must be **efficiently** scheduled.
- This is our task.
  - Focus on home hospice care.



Home Hospice Care

- Distinguishing characteristics
  - Personal & household services
  - Regular weekly schedule
    - For example, Mon-Wed-Fri at 9 am.
  - Same aide each visit
  - Long planning horizon
    - Several weeks
  - Rolling schedule
    - Update schedule as patient population evolves.

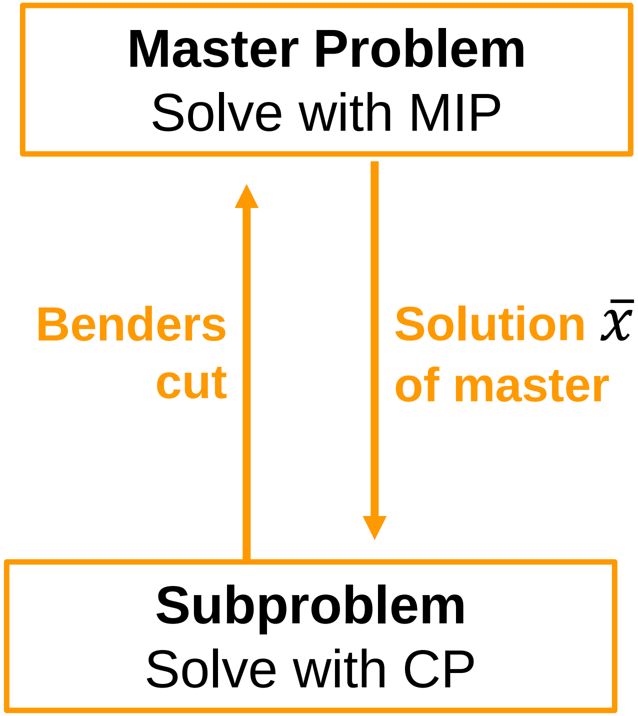
Home Hospice Care



5-8% weekly turnover

Home Hospice Care

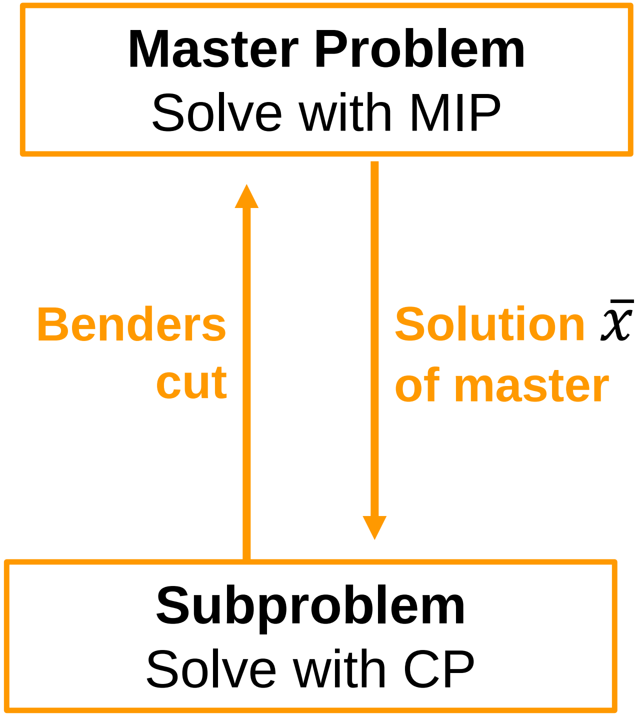
- Solve with Benders decomposition.
  - **Assign aides to patients** in master problem.
    - Maximize number of patients served by a given set of aides.



Heching & JH 2016

Home Hospice Care

- Solve with Benders decomposition.
  - **Assign aides to patients** in master problem.
    - Maximize number of patients served by a given set of aides.
  - **Schedule home visits** in subproblem.
    - Cyclic weekly schedule.
    - Visit each patient same time each day.
    - No visits on weekends.



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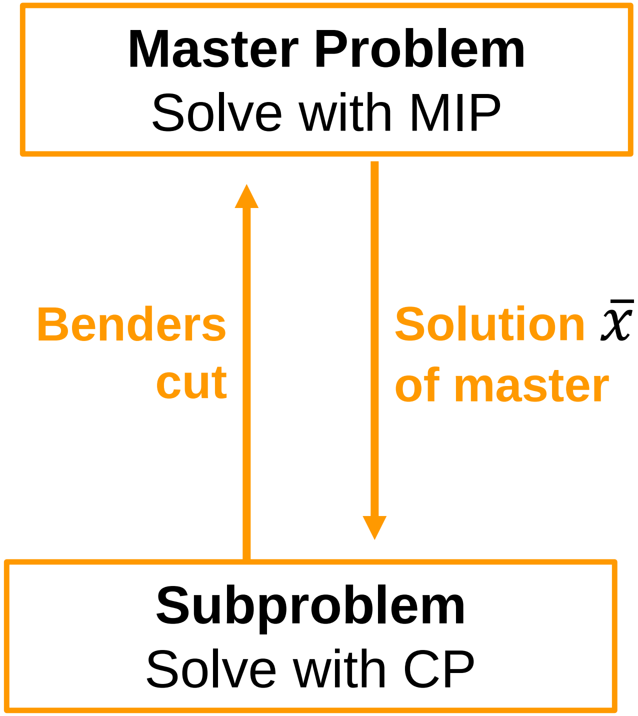
  - Subproblem **decouples** into a scheduling problem for each aide



Master Problem

= 1 if patient *j* scheduled

= 1 if patient *j* assigned to aide *i*

= 1 if patient *j* assigned to aide *i* on day *k*

Required number of visits per week

$$\max \sum_j \delta_j$$

$$\sum_i x_{ij} = \delta_j, \quad \text{all } j$$

$$\sum_{i,k} y_{ijk} = v_j \delta_j, \quad \text{all } j$$

$$y_{ijk} \le x_{ij}, \quad \text{all } i, j, k$$

Spacing constraints on visit days

Benders cuts

Relaxation of subproblem

$$\delta_j, x_{ij}, y_{ijk} \in \{0, 1\}$$

Master Problem

- For a rolling schedule:
  - Schedule **new patients**, drop **departing patients** from schedule.
    - Provide continuity for remaining patients as follows:
  - Old patients served by **same aide** on **same days**.
    - Fix $y_{ijk} = 1$ for the relevant aides, patients, and days.

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    - Fix $y_{ijk} = 1$ for the relevant aides, patients, and days.
  - Alternative: Also served **at same time**.
    - Fix time windows to enforce their current schedule.
  - Alternative: served only by **same aide**.
    - Fix $x_{ij} = 1$ for the relevant aides, patients.

Subproblem

Simplified routing & scheduling problem for aide *i*

*n*th patient in sequence

Patients assigned to aide *i*

$$\text{all-different}\{\pi_{k\nu} \mid \nu = 1, \ldots, |P_i|\}$$

$$[s_j, s_j + p_j] \subseteq [r_j, d_j]$$

$$s_{\pi_{k\nu}} + p_{\pi_{k\nu}} + t_{\pi_{k\nu}\pi_{k,\nu+1}} \leq s_{\pi_{k,\nu+1}}, \quad \text{all } k, \nu$$

start time

Visit duration

Travel time

Modeled with interval variables in CP solver

Benders Cuts

- Generate a cut for each infeasible scheduling problem.
  - Solution of subproblem inference dual is a **proof** of infeasibility.
    - The proof may show **other** patient assignments to be infeasible.
    - Generate **nogood cut** that rules out these assignments.

Benders Cuts

- Generate a cut for each infeasible scheduling problem.
  - Solution of subproblem inference dual is a **proof** of infeasibility.
    - The proof may show **other** patient assignments to be infeasible.
    - Generate **nogood cut** that rules out these assignments.
  - Unfortunately, we **don’t have access** to infeasibility proof in CP solver.

Benders Cuts

- So, strengthen the nogood cuts heuristically.
  - Find a smaller set of patients that create infeasibility…
    - …by re-solving the each infeasible scheduling problem repeatedly.

$$\sum_{j \in \boxed{\bar{P}_i}} (1 - y_{ijk}) \geq 1$$

Reduced set of patients whose assignment to aide *i* creates infeasibility

Subproblem Relaxation

- Include relaxation of subproblem in the master problem.
  - Necessary for good performance.
  - Use **time window relaxation** for each scheduling problem.
  - Simplest relaxation for aide *i* and day *k*:

$$\sum_{j \in J(a,b)} p_j y_{ijk} \leq b - a$$

Set of patients whose time window fits in interval [*a*, *b*].

Can use several intervals.

Subproblem Relaxation

- This relaxation is very weak.
  - Doesn’t take into account travel times.

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  - **Basic idea:** Augment visit duration p_j with travel time to (or from) location j from **closest** patient or aide home base.

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    - As in rolling schedule.

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  - This is **weak** unless most assignments are **fixed**.
    - As in rolling schedule.
  - Find intervals that yield tightest relaxation
    - Short intervals that contain many time windows.

Branch & Check

- A variation of logic-based Benders
  - Solve master problem only once, by branching.
  - At feasible nodes, solve subproblem to obtain Benders cut.
  - **Not the same** as branch & cut.
- Use when master problem is the bottleneck
  - Subproblem solves much faster than master problem.

JH 2000
Thorsteinsson 2003

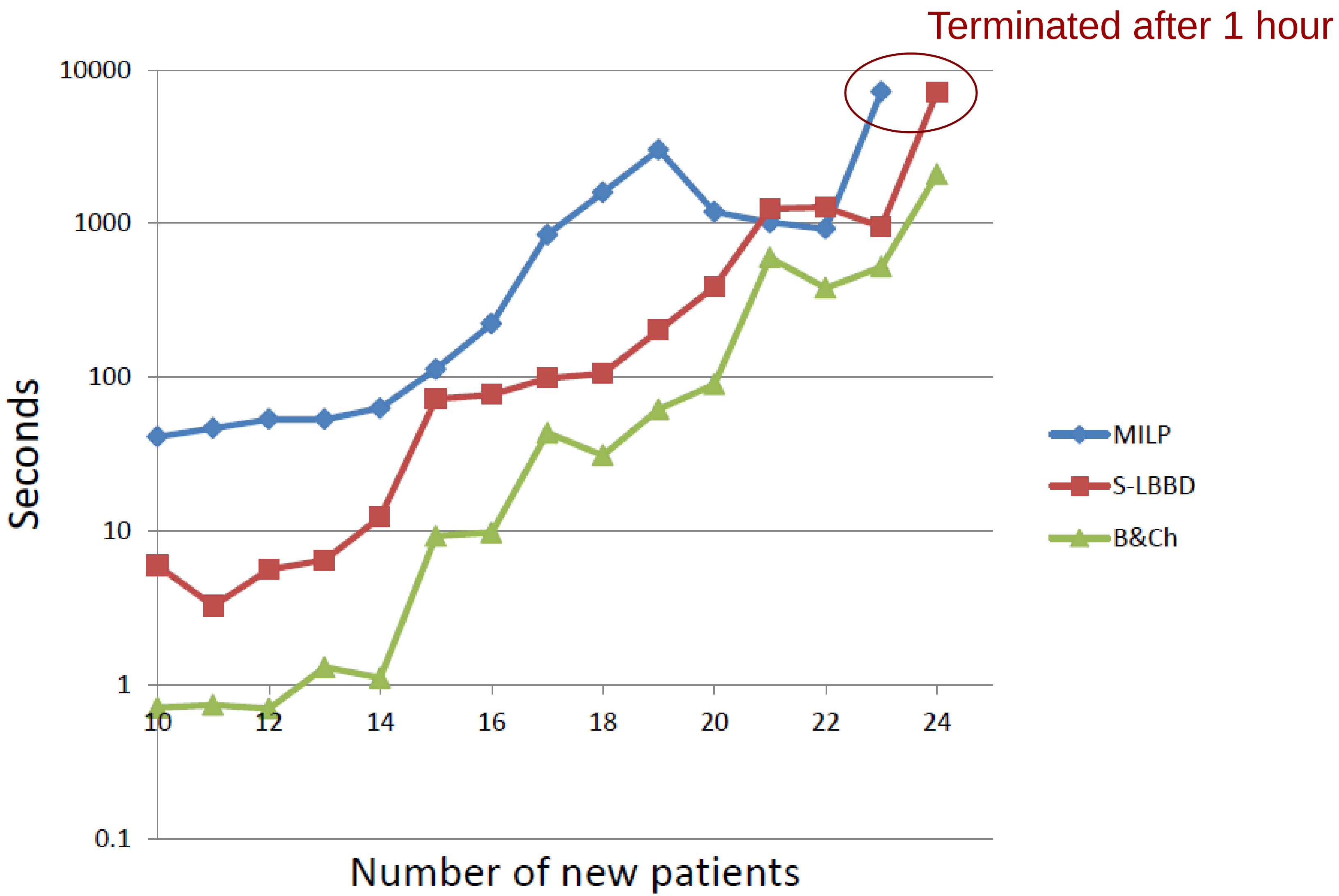
Computational Tests

- Original real-world dataset
  - 60 home hospice patients
    - Mostly 5 visits per week (not on weekends)
  - 18 health care aides with time windows
  - Actual travel distances
- Solver
  - **LBBD:** Hand-written code manages MIP & CP solvers
    - SCIP + Gecode
  - **Branch & check:** Use constraint handler in SCIP
    - SCIP + Gecode
  - **MIP:** SCIP
    - Modified multicommodity flow model of VRPTW

Computational Tests

- Instance generation
  - Start with (suboptimal) solution for the 60 patients, 270 visits
    - Fix this schedule for first *n* patients.
    - Schedule remaining 60 – *n* patients
  - Use 8 of the 18 aides to cover new patients
    - As well as the old patients they already cover.
    - This puts us near the phase transition.

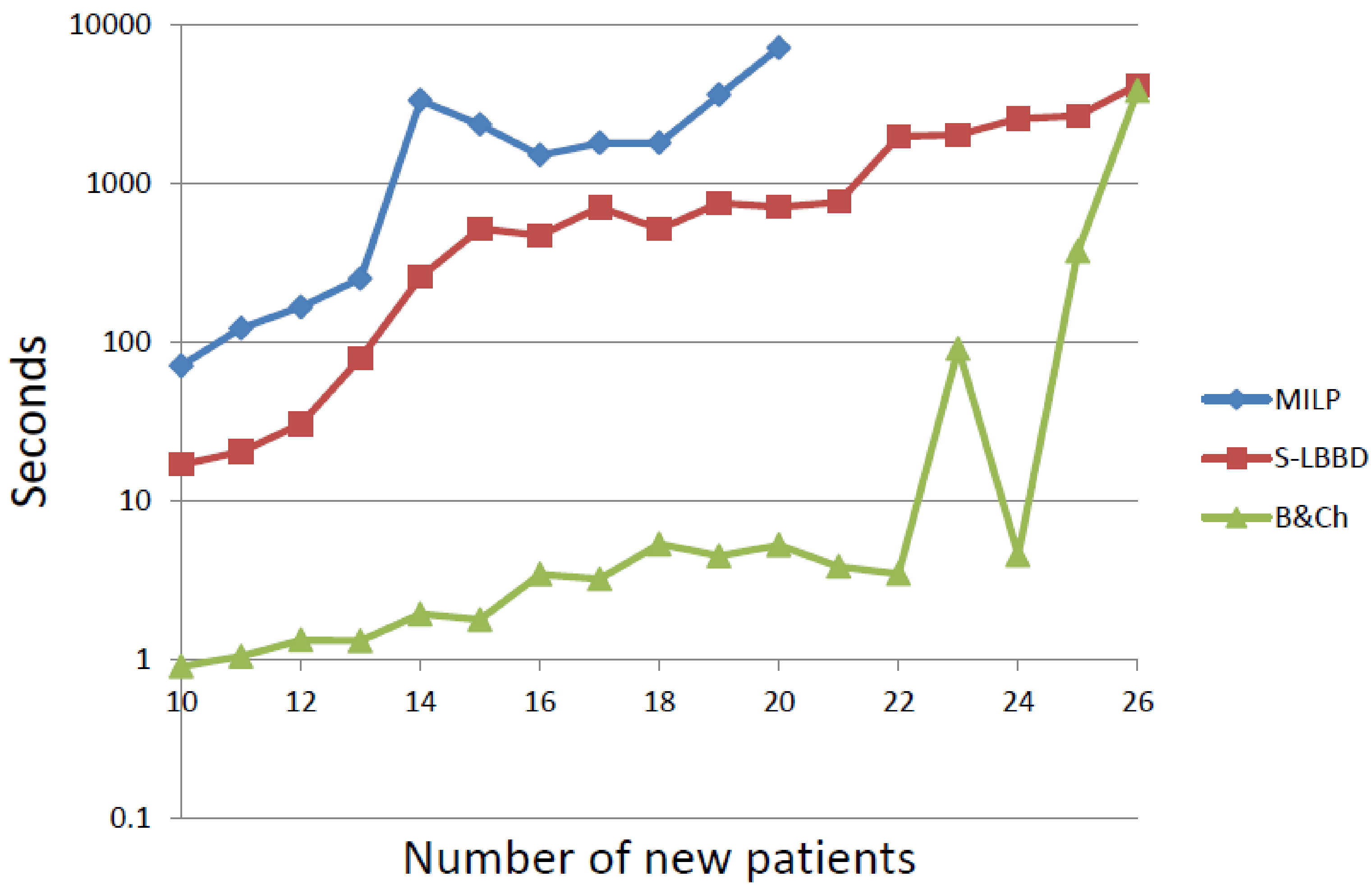
Computation time, original dataset



Computational Tests

- Modified problem
  - Patients receive1-5 visits per week
    - Uniformly distributed
  - Use only of the 18 aides to cover new patients
    - This puts us back near the phase transition.

Computation time, fewer visits per week

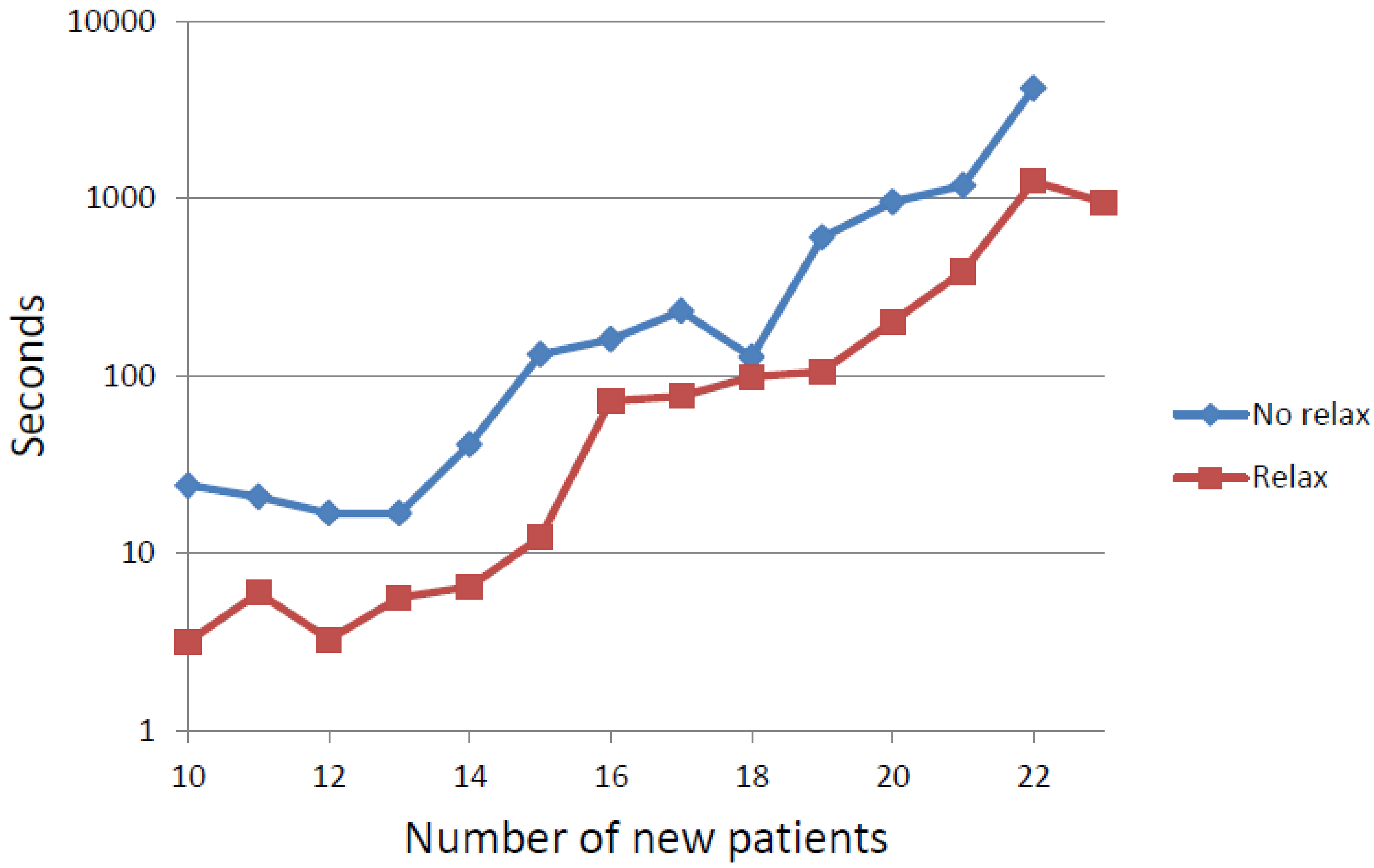


Computational Tests

- Practical implications
  - Branch & check **scales up** to realistic size
    - One month advance planning for original 60-patient dataset
    - Assuming 5-8% weekly turnover
    - Much faster performance for modified dataset
  - Advantage of **exact** solution method
    - We know **for sure** whether existing staff will cover projected demand.

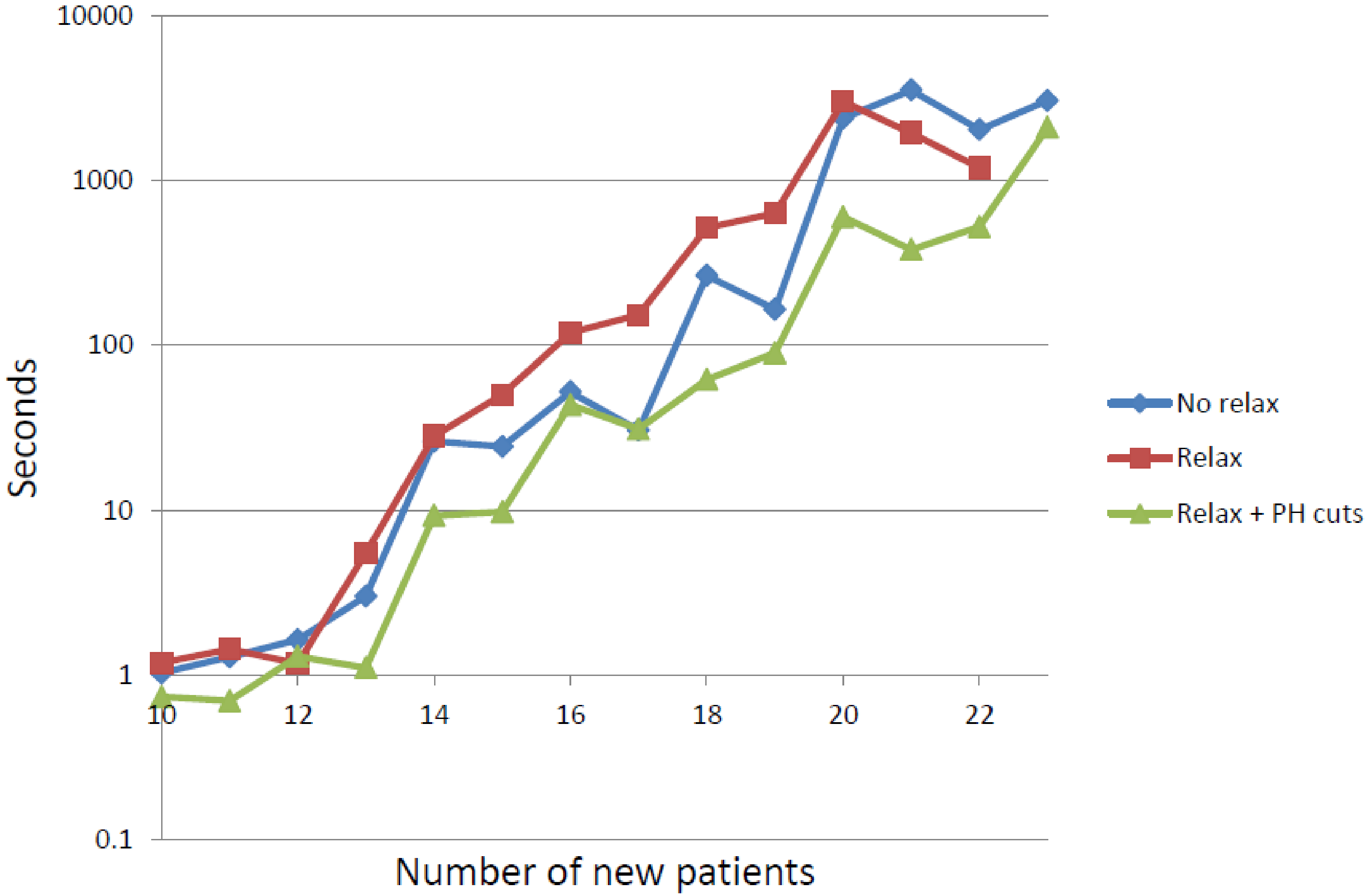
Effect of time window relaxation

Standard LBBD
Original problem data



Effect of time window relaxation and primal heuristic cuts

Branch & check
Original problem data



Computational Tests

- Rasmussen instances
  - From 2 Danish municipalities
    - One-day problem
    - We extended it to 5 days with same schedule each day
    - Reduce number of patients to 30, so MIP has a chance
  - Solve problem from scratch
    - No rolling schedule
  - Two objective functions
    - **Weighted:** Minimize weighted average of travel cost, matching cost (undesirability of assignment), uncovered patients.
    - **Covering:** Minimize number of uncovered patients (same as ours)

Table 6 Solution time (s) for modified Rasmussen instances

Instance	Patients	Crews	Weighted objective			Covering objective		
			MILP	LBBD	B&Ch	MILP	LBBD	B&Ch
hh	30	15	*	3.16	**1.41**	*	**23.3**	441
ll1	30	8	*	1.74	**0.43**	*	108	**1.41**
ll2	30	7	2868	1.56	**0.32**	*	**1.38**	6.45
ll3	30	6	1398	2.16	**0.30**	*	**3.07**	5.98

*Computation time exceeded one hour.

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Standard LBBD tends to be better when subproblem consumes most of the solution time in branch & check

Table 2 Percent of solution time devoted to subproblem

Instances	S-LBBD Avg	S-LBBD Max	B&Ch Avg	B&Ch Max
Original 60-patient instances	0.1	0.2	1.4	3.9
Narrow time windows	0.1	0.1	2.8	6.0
Fewer visits per patient	0.0	0.1	1.7	3.5
Rasmussen, weighted objective	0.4	0.8	6.3	13.6
Rasmussen, covering objective	1.2	1.5	85.6	99.7

Conclusions

- LBBD can scale up despite sequence-dependent costs…
  - …especially when computing a **rolling** schedule
    - Time window relaxation is tight enough in this case
  - Routing & scheduling problems remain small as patient population increases
    - The 4-index MIP variables explode as the population grows
    - …even for a rolling schedule

Conclusions

- LBBD can scale up despite sequence-dependent costs…
  - …especially when computing a **rolling** schedule
    - Time window relaxation is tight enough in this case
  - Routing & scheduling problems remain small as patient population increases
    - The 4-index MIP variables explode as the population grows
    - …even for a rolling schedule
- However…
  - LBBD not designed for temporal dependencies
    - As when multiple aides must visit a patient simultaneously.
    - Unclear how much performance degrades in this case.

References

Applications of Logic-Based Benders Decomposition

Benders decomposition [7] was introduced in 1962 to solve applications that become linear programming (LP) problems when certain *search variables* are fixed. "Generalized" Benders decomposition, proposed by Geoffrion in 1972 [25], extended the method to nonlinear programming subproblems.

Logic-based Benders decomposition (LBBD) allows the subproblem to be any optimization problem. LBBD was introduced in [32], formally developed in 2000 [33], and tested computationally in [39]. *Branch and check* is introduced in [33] and tested computationally in [69]. *Combinatorial Benders cuts* for mixed integer programming are proposed in [18].

One of the first applications [43] was a planning and scheduling problem. Updated experiments [17] show that LBBD is orders of magnitude faster than state-of-the-art MIP, with the advantage over CP even greater). Similar results have been obtained for various planning and scheduling problems [15, 21, 30, 34, 35, 37, 71].

Other successful applications of LBBD include steel production scheduling [29], inventory management [74], concrete delivery [44], shop scheduling [3, 13, 27, 28, 59], hospital scheduling [57], batch scheduling in chemical plants [49, 70], computer processor scheduling [8, 9, 12, 22, 31, 46, 47, 48, 58, 62], logic circuit verification [40], shift scheduling [5, 60], lock scheduling [73], facility location [23, 66], space packing [20, 50], vehicle routing [19, 51, 53, 56, 61, 75], bicycle sharing [45], network design [24, 52, 63, 65], home health care [16], service restoration [26], supply chain management [68], food distribution [64], queuing design and control [67], optimal control of dynamical systems [11], propositional satisfiability [1], quadratic programming [2, 41, 42], chordal completion [10], and sports scheduling [14, 54, 55, 72]. LBBD is compared with branch and check in [6]. It is implemented in the general-purpose solver SIMPL [76].

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