# Modeling with Metaconstraints and Semantic Typing of Variables

André Ciré
University of Toronto

John Hooker
Carnegie Mellon University

Tallys Yunes
University of Miami

*INFORMS Journal on Computing* (2016) 1-13

# Basic Problem

- **Metaconstraints** (in particular, **global constraints**) in a model help convey **problem structure** to the solver.
    - But they pose a fundamental problem of **variable management**.
    - How to solve it?

# Basic Problem

- **Metaconstraints** (in particular, **global constraints**) in a model help convey **problem structure** to the solver.
    - But they pose a fundamental problem of **variable management**.
    - How to solve it?
- Treat **variable declarations** as **database queries.**
    - In a system of **semantic typing.**

# Exploiting Problem Structure

- You can't solve hard problems without exploiting special structure (**No Free Lunch** Theorem).

# Exploiting Problem Structure

- You can’t solve hard problems without exploiting special structure (**No Free Lunch** Theorem).
- For SAT solvers:
  - Efficient encoding of problem in SAT form

# Exploiting Problem Structure

- You can't solve hard problems without exploiting special structure (**No Free Lunch** Theorem).
- For SAT solvers:
  - Efficient encoding of problem in SAT form
- For CP solvers:
  - Careful choice of global constraints
  - Redundant constraints, search strategy, etc.

# Exploiting Problem Structure

- You can't solve hard problems without exploiting special structure (**No Free Lunch** Theorem).
- For SAT solvers:
  - Efficient encoding of problem in SAT form
- For CP solvers:
  - Careful choice of global constraints
  - Redundant constraints, search strategy, etc.
- For MIP solvers:
  - Careful choice of variables for tight formulation
  - Addition of valid inequalities

# Conveying structure to the solver(s)

- Formulate problem with **global constraints** or **metaconstraints** to reveal structure
- Automatically convert these to **optimal formulation** for the solvers(s)
    - Best choice of **variables**.
    - Reformulation of **constraints**.
        - For **effective propagation** or **tight relaxation**
    - Best choice of **domain filters**.
    - Generation of **valid inequalities**

# Conveying structure to the solver(s)

- Formulate problem with **global constraints** or **metaconstraints** to reveal structure
- Automatically convert these to **optimal formulation** for the solvers(s)
  - Best choice of **variables**.
  - Reformulation of **constraints**.
    - For **effective propagation** or **tight relaxation**
  - Best choice of **domain filters**.
  - Generation of **valid inequalities**
- However, metaconstraints pose a fundamental problem of **variable management**…

# Variable management problem

- Reformulation typically introduces **new variables**
  - Different metaconstraints may introduce variables that are functionally **the same variable**
  - …or related in some other way.
  - Recognizing these relationships is essential to obtaining a good model (e.g., a tight continuous relaxation)
  - How can the solver “understand” what is going on in the model?

# Variable management problem

- Reformulation typically introduces **new variables**
  - Different metaconstraints may introduce variables that are functionally **the same variable**
  - …or related in some other way.
  - Recognizing these relationships is essential to obtaining a good model (e.g., a tight continuous relaxation)
  - How can the solver "understand" what is going on in the model?
- Proposal: Model with **semantic typing of variables**.

# Variable management problem

- **Example**: Let $x_j$ = worker assigned to job $j$

  $c_{ji}$ = cost of assigning worker $i$ to job $j$

  Find **min-cost assignment**:

$$\min \sum_j c_{jx_j}$$

$$\text{alldiff}(x_1, \ldots, x_n)$$

# Variable management problem

- **Example**: Let $x_j$ = worker assigned to job $j$
  $c_{ji}$ = cost of assigning worker $i$ to job $j$
  Find **min-cost assignment**:

$$\min \sum_j c_{jx_j}$$

$$\text{alldiff}(x_1, \ldots, x_n)$$

Objective function is reformulated with 0-1 variables: $\min \sum_{ij} c_{ij} y_{ij}$ where $x_j = \sum_i i y_{ij}$

# Variable management problem

- **Example**: Let $x_j$ = worker assigned to job $j$
  $c_{ji}$ = cost of assigning worker $i$ to job $j$
  Find **min-cost assignment**:

$$\min \sum_j c_{jx_j}$$

$$\text{alldiff}(x_1, \ldots, x_n)$$

Objective function is reformulated with 0-1 variables: $\min \sum_{ij} c_{ij} y_{ij}$ where $x_j = \sum_i i y_{ij}$

Alldiff constraint is reformulated with 0-1 variables: $\sum_i y'_{ij} = 1, \text{ all } j; \quad \sum_j y'_{ij} = 1, \text{ all } i$

# Variable management problem

- **Example**: Let $x_j$ = worker assigned to job $j$
  $c_{ji}$ = cost of assigning worker $i$ to job $j$
  Find **min-cost assignment**:

$$\min \sum_j c_{jx_j}$$

$$\text{alldiff}(x_1, \ldots, x_n)$$

Objective function is reformulated with 0-1 variables: $\min \sum_{ij} c_{ij} y_{ij}$ where $x_j = \sum_i i y_{ij}$

Alldiff constraint is reformulated with 0-1 variables: $\sum_i y'_{ij} = 1, \text{ all } j; \ \sum_j y'_{ij} = 1, \text{ all } i$

How does the solver know that we want $y_{ij} = y'_{ij}$, allowing the problem to be solved rapidly as a **classical assignment problem**?

# Semantic typing

- We assume that variables are **declared**.

# Semantic typing

- We assume that variables are **declared**.
- **Semantic typing** assigns a different meaning to each variable…
  - By associating the variable with a multi-place **predicate** and **keyword**.
  - The keyword "**queries**" the relation denoted by the predicate, as one queries a **relational database**.

# Semantic typing

- We assume that variables are **declared**.
- **Semantic typing** assigns a different meaning to each variable…
  - By associating the variable with a multi-place **predicate** and **keyword**.
  - The keyword "**queries**" the relation denoted by the predicate, as one queries a **relational database**.
- Advantage:
  - This allows the solver to **deduce relationships** between variables, both original or introduced.
  - Can automatically add **channeling constraints**.
  - It is also **good modeling practice**.

# How variables are introduced

- A model may include **two formulations** of the problem that use related variables.
    - Common in CP, because it strengthens **propagation**.

# How variables are introduced

- A model may include **two formulations** of the problem that use related variables.
  - Common in CP, because it strengthens **propagation**.
  - For example,

$$x_i = \text{ job assigned to worker } i$$

$$y_j = \text{ worker assigned to job } j$$

  - Solver should generate **channeling constraints** to relate the variables to each other:

$$j = x_{y_j}, \qquad i = y_{x_i}$$

# How variables are introduced

- The solver may reformulate a **disjunction of linear systems**

$$\bigcup_k A_k x \geq b^k$$

using a convex hull (or big-$M$ ) formulation:

$$A_k x^k \geq b^k y_k, \quad \text{all } k$$
$$x = \sum_k x^k, \quad \sum_k y_k = 1$$
$$y_k \in \{0,1\}, \quad \text{all } k$$

- Other constraints may be based on **same set of alternatives**, and corresponding auxiliary variables ($y_k$ etc.) should be equated.

# How variables are introduced

- A nonlinear or global solver may use **McCormick factorization** to replace nonlinear subexpressions with auxiliary variables
  - … to obtain a linear relaxation.

# How variables are introduced

- A nonlinear or global solver may use **McCormick factorization** to replace nonlinear subexpressions with auxiliary variables
  - … to obtain a linear relaxation.
  - For example, bilinear term *xy* can be linearized by replacing it with new variable *z* and constraints

$$L_y x + L_x y - L_x L_y \le z \le L_y x + U_x y - L_x U_y$$
$$U_y x + U_x y - U_x U_y \le z \le U_y x + L_x y - U_x L_y$$

where $x \in [L_x, U_x],\ \ y \in [L_y, U_y]$

  - Factorization of different constraints may create variables for identical subexpressions.
  - They should be identified to get a tight relaxation.

# How variables are introduced

- The solver may reformulate different **global constraints** from CP by introducing variables that have the same meaning.

# How variables are introduced

- The solver may reformulate different **global constraints** from CP by introducing variables that have the same meaning.

  - For example, **sequence** constraint limits how many jobs of a given type can occur in given time interval:

$$\text{sequence}(x), \quad x_i = \text{ job in position } i$$

and **cardinality** constraint limits how many times a given job appears

$$\text{cardinality}(x), \quad x_j = \text{ job in position } j$$

Both may introduce variables

$$y_{ij} = 1 \text{ when job } j \text{ occurs in position } i$$

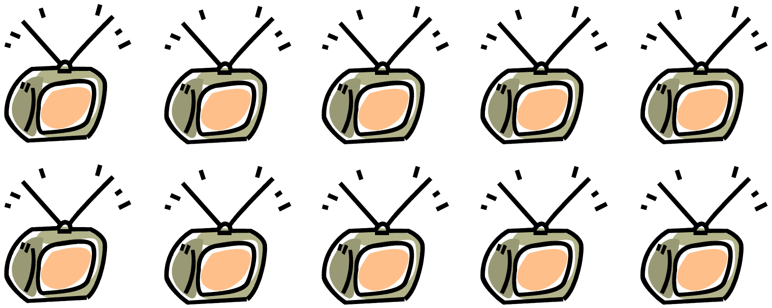
that should be identified.

# How variables are introduced

- The solver may introduce equivalent variables while interpreting metaconstraints designed for **classical MIP modeling situations:**
  - Fixed-charge network flow
  - Facility location
  - Lot sizing
  - Job shop scheduling
  - Assignment (3-dim, quadratic, etc.)
  - Piecewise linear

# Motivating example

- Allocate 10 advertising spots to 5 products



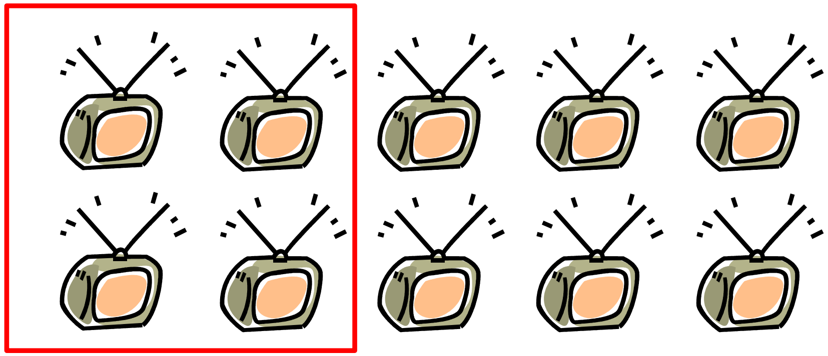
$x_i$ = how many spots allocated to product *i*

$y_{ij}$ = 1 if *j* spots allocated to product *i*

A B C D E

# Motivating example

- Allocate 10 advertising spots to 5 products



≤ 4 spots per product

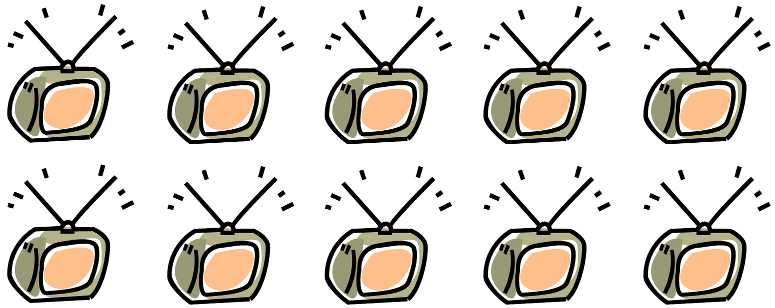
$x_i$ = how many spots allocated to product *i*

$y_{ij}$ = 1 if *j* spots allocated to product *i*

A B C D E

# Motivating example

- Allocate 10 advertising spots to 5 products



$\leq$ 4 spots per product

Advertise $\leq$ 3 products

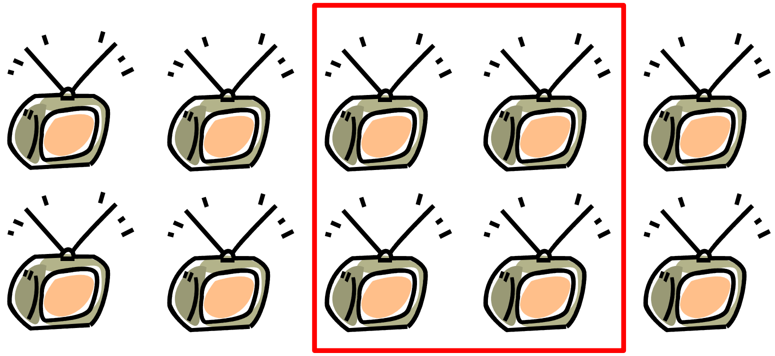
$x_i =$ how many spots allocated to product $i$

$y_{ij} =$ 1 if $j$ spots allocated to product $i$

A B C D E

# Motivating example

- Allocate 10 advertising spots to 5 products



$x_i$ = how many spots allocated to product *i*

$y_{ij}$ = 1 if *j* spots allocated to product *i*

A B C D E

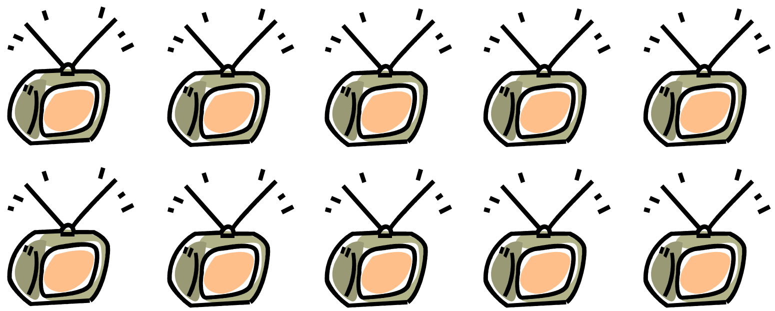
$\leq$ 4 spots per product

Advertise $\leq$ 3 products

$\geq$ 4 spots for at least one product

# Motivating example

- Allocate 10 advertising spots to 5 products



$x_i$ = how many spots allocated to product *i*

$y_{ij}$ = 1 if *j* spots allocated to product *i*

A B C D E

$\leq$ 4 spots per product

Advertise $\leq$ 3 products

$\geq$ 4 spots for at least one product

$P_{ij}$ = profit from allocating *j* spots to product *i*

Objective: maximize profit

# Motivating example

```
spots in {0..4}
product in {A,B,C,D,E})
```

Index sets

# Motivating example

```
spots in {0..4}
product in {A,B,C,D,E}
data P{product,spots}
```

Data input

# Motivating example

```
spots in {0..4}
product in {A,B,C,D,E}
data P{product,spots}
x[i] is howmany spots allocate(product i)
```

Declaration of variable $x_i$

# Motivating example

```
spots in {0..4}
product in {A,B,C,D,E}
data P{product,spots}
x[i] is howmany spots allocate(product i)
```

Declaration of variable $x_i$

This makes it a variable declaration

^

# Motivating example

```
spots in {0..4}
product in {A,B,C,D,E}
data P{product,spots}
x[i] is howmany spots allocate(product i)
```

Declaration of variable $x_i$

This is the semantic type

# Motivating example

```
spots in {0..4}
product in {A,B,C,D,E}
data P{product,spots}
x[i] is howmany spots allocate(product i)
```

Declaration of variable $x_i$

Indicates an integer quantity

Other keywords:
***howmuch***
***whether***

# Motivating example

```
spots in {0..4}
product in {A,B,C,D,E}
data P{product,spots}
x[i] is howmany spots allocate(product i)
```

Declaration of variable $x_i$

How many of what?

# Motivating example

```
spots in {0..4}
product in {A,B,C,D,E}
data P{product,spots}
x[i] is howmany spots allocate(product i)
```

Declaration of variable $x_i$

**2-place predicate** associated with variable *x*

Every variable is associated with a predicate that gives it meaning

# Motivating example

```
spots in {0..4}
product in {A,B,C,D,E}
data P{product,spots}
x[i] is howmany spots allocate(product i)
```

Declaration of variable $x_i$

Other term of the predicate

# Motivating example

```
spots in {0..4}
product in {A,B,C,D,E}
data P{product,spots}
x[i] is howmany spots allocate(product i)
```

Declaration of variable $x_i$

Associates index of **x[i]** with index set **product**

# Motivating example

$$\max \sum_i P_{i x_i}$$

```
spots in {0..4}
product in {A,B,C,D,E}
data P{product,spots}
x[i] is howmany spots allocate(product i)
maximize sum{product i} P[i,x[i]]
```

Objective function

# Motivating example

$$\max \sum_i P_{ix_i}$$

$$\sum_i x_i \leq 10$$

```
spots in {0..4}
product in {A,B,C,D,E}
data P{product,spots}
x[i] is howmany spots allocate(product i)
maximize sum{product i} P[i,x[i]]
sum{product i} x[i] <= 10
```

10 spots available

# Motivating example

$$\max \sum_i P_{ix_i}$$

$$\sum_i x_i \leq 10$$

```
spots in {0..4}
product in {A,B,C,D,E}
data P{product,spots}
x[i] is howmany spots allocate(product i)
maximize sum{product i} P[I,x[i]]
sum{product i} x[i] <= 10
y[i,j] is whether allocate(product i, spots j)
```

Declare $y_{ij}$

Indicates 0-1 variable

# Motivating example

$$\max \sum_i P_{ix_i}$$

$$\sum_i x_i \le 10$$

```
spots in {0..4}
product in {A,B,C,D,E}
data P{product,spots}
x[i] is howmany spots allocate(product i)
maximize sum{product i} P[i,x[i]]
sum{product i} x[i] <= 10
y[i,j] is whether allocate(product i, spots j)
```

Declare $y_{ij}$

Associated with same predicate as **x[i]**

# Motivating example

$$\max \sum_i P_{ix_i}$$

$$\sum_i x_i \le 10, \quad \boxed{\sum_i y_{i0} \ge 2}$$

```
spots in {0..4}
product in {A,B,C,D,E}
data P{product,spots}
x[i] is howmany spots allocate(product i)
maximize sum{product i} P[i,x[i]]
sum{product i} x[i] <= 10
y[i,j] is whether allocate(product i, spots j)
sum{product i} y[i,0] >= 2
```

At least 2 products not advertised

# Motivating example

$$\max \sum_i P_{i x_i}$$

$$\sum_i x_i \leq 10, \quad \sum_i y_{i0} \geq 2, \quad \boxed{\sum_i y_{i4} \geq 1}$$

```
spots in {0..4}
product in {A,B,C,D,E}
data P{product,spots}
x[i] is howmany spots allocate(product i)
maximize sum{product i} P[i,x[i]]
sum{product i} x[i] <= 10
y[i,j] is whether allocate(product i, spots j)
sum{product i} y[i,0] >= 2
sum{product i} y[i,4] >= 1
```

At least 1 product gets ≥4 spots

# Motivating example

$$\max \sum_i P_{ix_i}$$

$$\sum_i x_i \le 10,\ \sum_i y_{i0} \ge 2,\ \sum_i y_{i4} \ge 1$$

$$\sum_j y_{ij} = 1,\ x_i = \sum_j j y_{ij},\ \text{all } i$$

```
spots in {0..4}
product in {A,B,C,D,E}
data P{product,spots}
x[i] is howmany spots allocate(product i)
maximize sum{product i} P[i,x[i]]
sum{product i} x[i] <= 10
y[i,j] is whether allocate(product i, spots j)
sum{product i} y[i,0] >= 2
sum{product i} y[i,4] >= 1
{product i} sum{spots j} y[i,j] = 1
{product i} x[i] = sum{spots j} j*y[i,j]
```

Solver generates linking constraints because
**x[i]** and **y[i,j]** are associated with the same predicate.

# Motivating example

$$\max \sum_i z_i$$

$$\sum_i x_i \le 10,\ \sum_i y_{i0} \ge 2,\ \sum_i y_{i4} \ge 1$$

$$\sum_j y_{ij} = 1,\ x_i = \sum_j j y_{ij},\ \text{all } i$$

```
spots in {0..4}
product in {A,B,C,D,E}
data P{product,spots}
x[i] is howmany spots allocate (product i)
maximize sum{product i} P[i,x[i]]
```

This constraint must be linearized. Solver generates

$$z_i = \sum_{j=0}^{4} P_{ij} y'_{ij},\ \sum_{j=0}^{4} y'_{ij} = 1,\ x_i = \sum_{j=0}^{4} j y'_{ij},\ \text{all } i$$

```
y'[i,j] is whether allocate(product i, spots j)
```

# Motivating example

$$\max \sum_i z_i$$

$$\sum_i x_i \le 10, \ \sum_i y_{i0} \ge 2, \ \sum_i y_{i4} \ge 1$$

$$\sum_j y_{ij} = 1, \ x_i = \sum_j j y_{ij}, \text{ all } i$$

```
spots in {0..4}
product in {A,B,C,D,E}
data P{product,spots}
x[i] is howmany spots allocate (product i)
maximize sum{product i} P[i,x[i]]
```

This constraint must be linearized. Solver generates

$$z_i = \sum_{j=0}^{4} P_{ij} y'_{ij}, \ \sum_{j=0}^{4} y'_{ij} = 1, \ x_i = \sum_{j=0}^{4} j y'_{ij}, \text{ all } i$$

```
y′[i,j] is whether allocate(product i, spots j)
```

**y** and **y′** are identified because they have the same type:

```
y[i,j] is whether allocate(product i, spots j)
```

# Predicates and relations

Predicate **`allocate`** denotes 2-place **relation** (set of tuples).
Schematically indicated by:

| 1 | 2 |
|---|---|
| **`product`** | **`spots`** |
| $i$ | $x_i$ |

# Predicates and relations

Predicate **`allocate`** denotes 2-place **relation** (set of tuples).
Schematically indicated by:

| 1 | 2 |
|---|---|
| **`product`** | **`spots`** |
| $i$ | $x_i$ |

Column corresponding to a variable must be a **function** of other columns.

# Predicates and relations

Predicate **`allocate`** denotes 2-place **relation** (set of tuples).
Schematically indicated by:

| 1 | 2 |
|---|---|
| **`product`** | **`spots`** |
| $i$ | $x_i$ |

Declaration of **`x[i]`** as

```
howmany spots allocate (product i)
```

and **`y[i,j]`** as

```
whether allocate (product i, spots j)
```

**query** the relation for how many and whether.

# Predicates and relations

Predicate **`allocate`** denotes 2-place **relation** (set of tuples).
Schematically indicated by:

| 1 | 2 |
|---|---|
| **`product`** | **`spots`** |
| $i$ | $x_i$ |

Declaration of **`x[i]`** as

**`howmany spots allocate (product i)`**

and **`y[i,j]`** as

**`whether allocate (product i, spots j)`**

**query** the relation for how many and whether.

In general, **keywords** are **queries** (analogous to **relational database**)

# Predicates and relations

Relation table reveals channeling constraints.  For example,

```
x[i] is which job assign(worker i)
y[j] is which worker assign(job i)
```

| 1 | 2 |
|---|---|
| job | worker |
| $j, x_i$ | $i, y_j$ |

We can read off the channeling constraints

$$j = x_i = x_{y_i}$$

$$i = y_j = y_{x_i}$$

# Predicates and relations

If several jobs can be assigned to a worker, we declare

```
z[i] is whichset job assign(worker i)
```

The channeling constraints are

$$j \in z_{y_i}$$

# Previous work

- **Model management** uses semantic typing to help combine models and use inheritance.
  - Originally inspired by object-oriented programming
    Bradley & Clemence (1988)
  - *Quiddity:* a rigorous attempt to analyze conditions for variable identification
    Bhargava, Kimbrough & Krishnan (1991)
  - **SML** uses typing in a structured modeling framework
    Geoffrion (1992)
  - **Ascend** uses strongly-typed, object-oriented modeling
    Bhargava, Krishnan & Piela (1998)

# Previous work

- Our semantic typing differs:
  - **Less ambitious** because it doesn't attempt model management.
    - There is only one model.
  - **More ambitious** because we recognize relationships other than equivalence.
  - We manage variables **introduced by solver**.

# Previous work

- Modeling systems that convey some structure to solver:
  - CP modeling systems use **global constraints**.
  - AIMMS uses **typed index sets**.
  - MiniZinc reformulates **metaconstraints** for specific solvers.
  - Savile Row uses **common subexpression elimination**.
  - OPL, Xpress-Kalis, Comet, etc., use **interval variables**.
  - SAT solver SymChaff uses high-level **AI planning language** PDDL.
  - SIMPL has **full metaconstraint capability**.

# Previous work

- However, **none of these systems** deals systematically with the variable management problem.
  - We address it with semantic typing of variables.

# Assignment problem

$$\min \sum_i c_{ix_i}$$

$$\text{alldiff}\left(x_1,\ldots,x_n\right)$$

```
worker in {1..m}
job in {1..n}
data C{worker,job}
x[i] is which job assign(worker i)
minimize sum{worker i} C[i,x[i]]
alldiff{x[*]}
```

# Assignment problem

$$\min \sum_i c_{i x_i}$$

$$\text{alldiff}\,(x_1,\ldots,x_n)$$

```
worker in {1..m}
job in {1..n}
data C{worker,job}
x[i] is which job assign(worker i)
minimize sum{worker i} C[i,x[i]]
alldiff{x[*]}
```

Objective function
is formulated

$$\max \sum_i c_{ij} y_{ij}, \;\; x_i = \sum_j y_{ij}, \text{ all } i$$

```
y[i,j] is whether assign(worker i, job j)
```

# Assignment problem

$$\min \sum_i c_{ix_i}$$

$$\text{alldiff}\left(x_1,\ldots,x_n\right)$$

```
worker in {1..m}
job in {1..n}
data C{worker,job}
x[i] is which job assign(worker i)
minimize sum{worker i} C[i,x[i]]
alldiff{x[*]}
```

Objective function is formulated

$$\max \sum_i c_{ij} y_{ij}, \;\; x_i = \sum_j y_{ij}, \text{ all } i$$

```
y[i,j] is whether assign(worker i, job j)
```

Alldiff is formulated

$$\sum_j y'_{ij} = 1, \text{ all } i, \;\; \sum_i y'_{ij} = 1, \text{ all } j, \;\; x_i = \sum_j j y'_{ij}, \text{ all } i$$

```
y'[i,j] is whether assign(worker i, job j)
```

Solver identifies *y* and *y*′ to create classical AP.

# Latin squares

|  | *j* |  |  |
|---|---|---|---|
|  | 2 | 3 | 1 |
| *i* | 3 | 1 | 2 |
|  | 1 | 2 | 3 |

Numbers in every row and column are distinct.
We will use **three** formulations to improve propagation.

# Latin squares

$$\text{alldiff}\left(x_{i1},\ldots x_{in}\right), \text{ all } i$$

$$\text{alldiff}\left(x_{1j},\ldots x_{nj}\right), \text{ all } j$$

$$\text{alldiff}\left(y_{i1},\ldots y_{in}\right), \text{ all } i$$

$$\text{alldiff}\left(y_{1k}\ldots y_{nk}\right), \text{ all } k$$

$$\text{alldiff}\left(z_{j1},\ldots x_{jn}\right), \text{ all } j$$

$$\text{alldiff}\left(z_{1k},\ldots x_{nk}\right), \text{ all } k$$

*j* (columns), *i* (rows)

| | | |
|---|---|---|
| 2 | 3 | 1 |
| 3 | 1 | 2 |
| 1 | 2 | 3 |

Numbers in every row and column are distinct.
We will use **three** formulations to improve propagation.

```
row, col, num in {1..n}
x[i,j] is which num assign(row i, col j)
y[i,k] is which col assign(row i, num k)
z[j,k] is which row assign(col j, num k)
```

$$\text{alldiff}\left(x_{i1},\ldots x_{in}\right),\ \text{all } i$$

$$\text{alldiff}\left(x_{1j},\ldots x_{nj}\right),\ \text{all } j$$

$$\text{alldiff}\left(y_{i1},\ldots y_{in}\right),\ \text{all } i$$

$$\text{alldiff}\left(y_{1k}\ldots y_{nk}\right),\ \text{all } k$$

$$\text{alldiff}\left(z_{j1},\ldots x_{jn}\right),\ \text{all } j$$

$$\text{alldiff}\left(z_{1k},\ldots x_{nk}\right),\ \text{all } k$$

# Latin squares

*j*

*i*

| | | |
|---|---|---|
| 2 | 3 | 1 |
| 3 | 1 | 2 |
| 1 | 2 | 3 |

Numbers in every row and column are distinct.
We will use **three** formulations to improve propagation.

```
row, col, num in {1..n}
x[i,j] is which num assign(row i, col j)
y[i,k] is which col assign(row i, num k)
z[j,k] is which row assign(col j, num k)
{row i} alldiff{x[i,*]); {col j} alldiff{x[*,j])
{row i} alldiff{y[i,*]); {num k} alldiff{y[*,j])
{col j} alldiff{z[j,*]); {num k} alldiff{z[*,k])
```

# Latin squares

The predicate **`assign`** denotes the 3-place relation

| 1 | 2 | 3 |
|---|---|---|
| **`num`** | **`col`** | **`row`** |
| $k$, $x_{ij}$ | $j$, $y_{ik}$ | $i$, $z_{jk}$ |

```
row, col, num in {1..n}
x[i,j] is which num assign(row i, col j)
y[i,k] is which col assign(row i, num k)
z[j,k] is which row assign(col j, num k)
{row i} alldiff{x[i,*]); {col j} alldiff{x[*,j])
{row i} alldiff{y[i,*]); {num k} alldiff{y[*,j])
{col j} alldiff{z[j,*]); {num k} alldiff{z[*,k])
```

# Latin squares

The predicate **`assign`** denotes the 3-place relation

| 1 | 2 | 3 |
|---|---|---|
| **`num`** | **`col`** | **`row`** |
| $k, x_{ij}$ | $j, y_{ik}$ | $i, z_{jk}$ |

We can read off the channeling constraints:

$$k = x_{z_{jk} y_{ik}}, \quad j = y_{z_{jk} x_{ij}}, \quad i = z_{y_{ik} x_{ikj}}, \quad \text{all } i, j, k$$

which can be propagated.

# Latin squares

```
{row i} alldiff{x[i,*]); {col j} alldiff{x[*,j])
{row i} alldiff{y[i,*]); {num k} alldiff{y[*,j])
{col j} alldiff{z[j,*]); {num k} alldiff{z[*,k])
```

The 3 formulations generate 3 identical MIP models:

$$x_{ij} = \sum_k k\delta^x_{ijk}; \ \sum_k \delta^x_{ijk} = 1, \text{ all } i,j; \ \sum_j \delta^x_{ijk} = 1, \text{ all } i,k; \ \sum_i \delta^x_{ijk} = 1, \text{ all } j,k$$

$$y_{ik} = \sum_j j\delta^y_{ijk}, \ \sum_j \delta^y_{ijk} = 1, \text{ all } i,k; \ \sum_k \delta^y_{ijk} = 1, \text{ all } i,j; \ \sum_i \delta^y_{ijk} = 1, \text{ all } j,k$$

$$z_{jk} = \sum_i i\delta^z_{ijk}, \ \sum_i \delta^z_{ijk} = 1, \text{ all } j,k; \ \sum_k \delta^z_{ijk} = 1, \text{ all } i,j; \ \sum_j \delta^z_{ijk} = 1, \text{ all } i,k$$

# Latin squares

```
{row i} alldiff{x[i,*]); {col j} alldiff{x[*,j])
{row i} alldiff{y[i,*]); {num k} alldiff{y[*,j])
{col j} alldiff{z[j,*]); {num k} alldiff{z[*,k])
```

The 3 formulations generate 3 identical MIP models:

$$x_{ij} = \sum_k k\delta^x_{ijk};\ \sum_k \delta^x_{ijk} = 1,\ \text{all } i,j;\ \sum_j \delta^x_{ijk} = 1,\ \text{all } i,k;\ \sum_i \delta^x_{ijk} = 1,\ \text{all } j,k$$

$$y_{ik} = \sum_j j\delta^y_{ijk},\ \sum_j \delta^y_{ijk} = 1,\ \text{all } i,k;\ \sum_k \delta^y_{ijk} = 1,\ \text{all } i,j;\ \sum_i \delta^y_{ijk} = 1,\ \text{all } j,k$$

$$z_{jk} = \sum_i i\delta^z_{ijk},\ \sum_i \delta^z_{ijk} = 1,\ \text{all } j,k;\ \sum_k \delta^z_{ijk} = 1,\ \text{all } i,j;\ \sum_j \delta^z_{ijk} = 1,\ \text{all } i,k$$

The solver declares $\delta^x_{ijk},\ \delta^y_{ijk},\ \delta^z_{ijk}$

```
whether assign(row i, col j, num k)
```

So it treats them as the same variable and generates only 1 MIP model.

# Multiple `which` variables

In general, an *n*-place predicate that denotes the relation

| 1 | … | $k$ | $k+1$ | … | $n$ |
|---|---|---|---|---|---|
| $\texttt{term}_1$ | … | $\texttt{term}_k$ | $\texttt{term}_{k+1}$ | … | $\texttt{term}_n$ |
| $i_1,\ x^1_{i(1)}$ | … | $i_k,\ x^k_{i(k)}$ | $i_{k+1}$ | … | $i_n$ |

for **`which`** variables, where $i(j) = i_1 \cdots i_{j-1} i_{j+1} \cdots i_n$

generates the channeling constraints

$$i_j = x^j_{x^1_{i(1)} \cdots \ x^{j-1}_{i(j-1)}\ x^{j+1}_{i(j+1)} \cdots \ x^k_{i(k)}\ i_{k+1} \cdots \ i_n}, \text{ all } i_1,\ldots,i_n\ ,\ j = 1,\ldots,k$$

# Multiple `whether` variables

**`whether`** keywords serve as projection operators on the relation.

```
y[i,j,d] is whether assign(worker i, job j, day d)
```

Project out *d* :

```
y1[i,j] is whether assign(worker i, job j)
```

Project out *j* and *d* :

```
y2[i] is whether assign(worker i)
```

# Short forms

Declare $x_i$ to be cost of activity *i* :

**x[i] is *howmuch* cost(activity i)**

which is short for the formal declaration

**x[i] is *howmuch* cost cost(activity i)**

in which a new term **cost** is generated

Declare x to be cost:

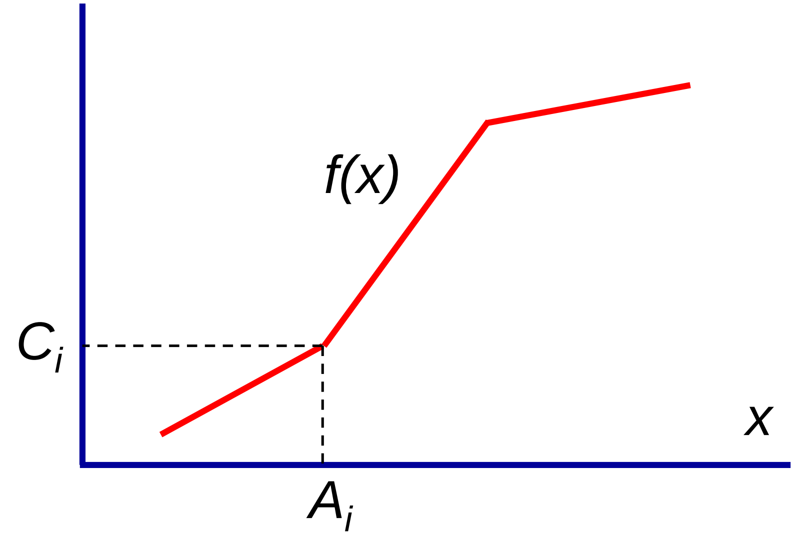
***x* is *howmuch* cost**

which is short for

***x* is *howmuch* cost cost()**

# Piecewise linear

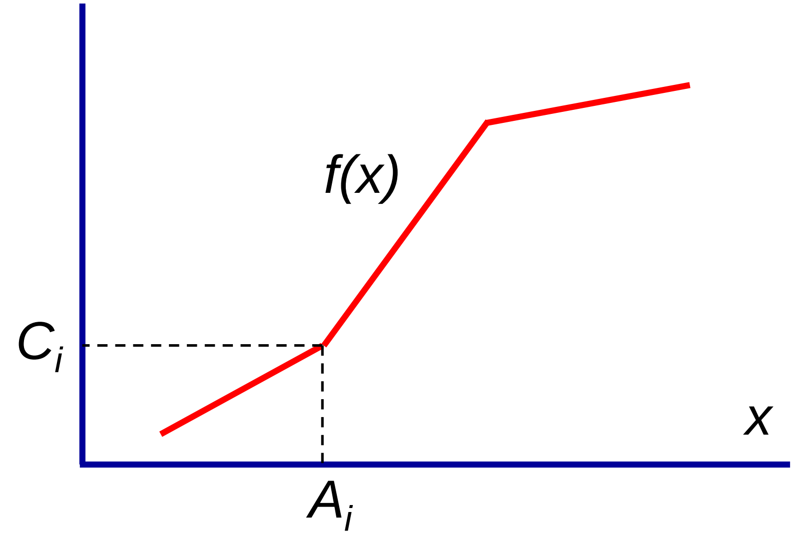
Piecewise linear function $z = f(x)$
Breakpoints in $A$, ordinates in $C$




```
x is howmuch output
index in {1..n}
data A,C{index}
z is howmuch cost
piecewise(x,z,A,C)
```

this metaconstraint defines $z = f(x)$

# Piecewise linear




Piecewise linear function $z = f(x)$
Breakpoints in $A$, ordinates in $C$

```
x is howmuch output
index in {1..n}
data A,C{index}
z is howmuch cost
piecewise(x,z,A,C)
```

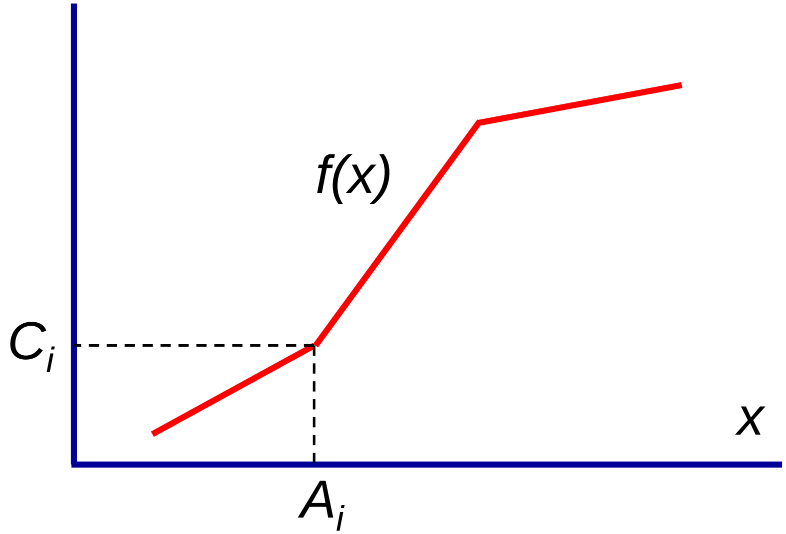
Solver generates the **locally ideal** model

$$x = a_1 + \sum_{i=1}^{n-1} \bar{x}_i, \quad z = c_1 + \sum_{i=1}^{n-1} \frac{c_{i+1} - c_i}{a_{i+1} - a_i} \bar{x}_i$$

$$(a_{i+1} - a_i)\delta_{i+1} \le \bar{x}_i \le (a_{i+1} - a_i)\delta_i, \quad \delta_i \in \{0,1\}, \; i = 1, \ldots, n-1$$

We need to declare auxiliary variables $\delta_i$, $x_i$

# Piecewise linear




Piecewise linear function $z = f(x)$
Breakpoints in $A$, ordinates in $C$

```
x is howmuch output
index in {1..n}
data A,C{index}
z is howmuch cost
piecewise(x,z,A,C)
```

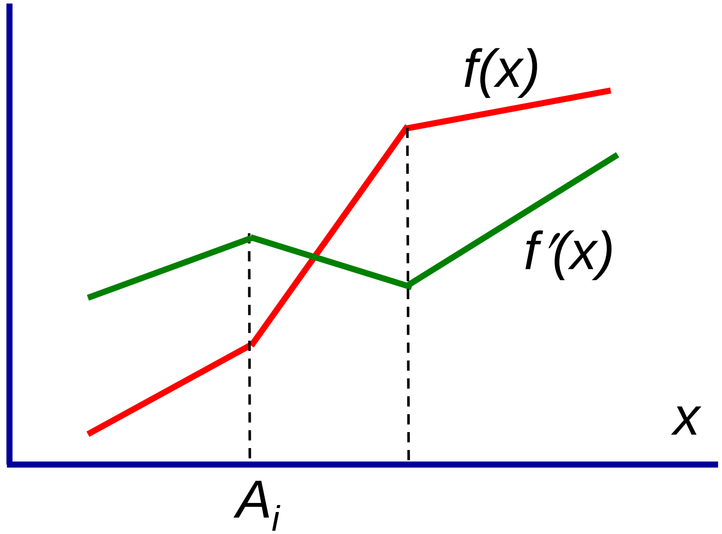
**`piecewise`** constraint induces solver to declare a new index set that associates **`index`** with **`A`**, and use it to declare $\delta_i$, $x_i$

```
xbar[i] is howmuch output.A(index i)
delta[i] is whether lastpositive output.A(index i)
```

Both declarations create predicates inherited from **`output`** and **`A`**

# Piecewise linear

Suppose there is another piecewise function on the same break points

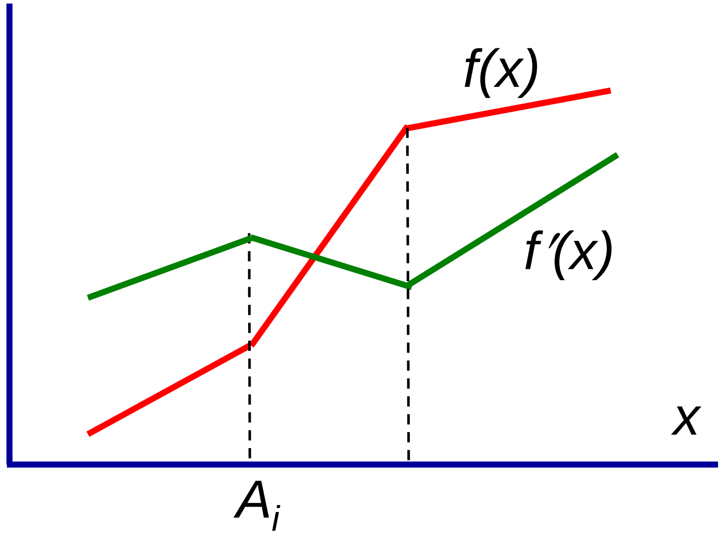



```
x is howmuch output
index in {1..n}
data A,C{index}
z is howmuch cost
piecewise(x,z,A,C)
data C′{index}
z′ is howmuch profit
piecewise(x,z′,A,C′)
x′[i] is howmuch cost output.A(index i)
delta′[i] is whether lastpositive output.A(index)
```

# Piecewise linear

Suppose there is another piecewise function on the same break points

```
x is howmuch output
index in {1..n}
data A,C{index}
z is howmuch cost
piecewise(x,z,A,C)
data C′{index}
z′ is howmuch profit
piecewise(x,z′,A,C′)
x′[i] is howmuch cost output.A(index i)
delta′[i] is whether lastpositive output.A(index)
```




Because new piecewise constraint is associated with the same *x* and *A*, solver again creates **output.A**.

The solver creates variables $\delta_i'$ and $x_i'$ with same types as $\delta_i$ and $x_i$ and so identifies them.

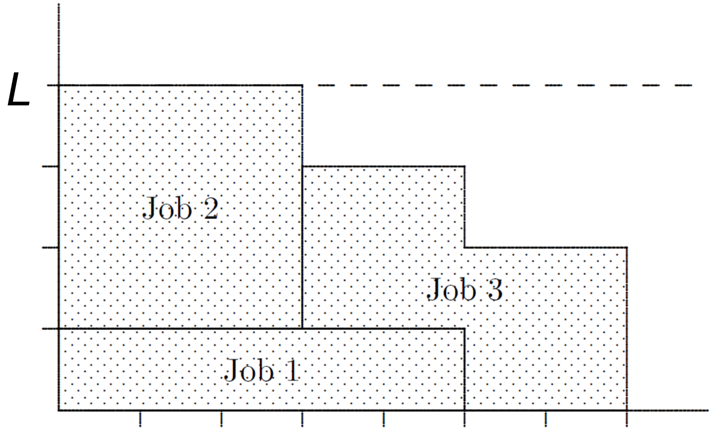
# Interval variables

$$\text{cumulative}(x, D, R, L)$$

$$x_j \subseteq W_j, \text{ all } j$$

Each job *j* runs for a time interval $x_j$.
We wish to schedule jobs so that total resource consumption never exceeds *L*.

**job *in* {1..n}**
**time *in* {t..T}**
***data* W,D,R{job}** window, duration, resource
**running *in* [time,time]** makes **running** an interval variable
**x[j] *is* *when* running sched(job j) *subset* W[j]**
***cumulative*(x,D,R,L)**



# Interval variables

$$\text{cumulative}(x, D, R, L)$$

$$x_j \subseteq W_j, \text{ all } j$$

Each job *j* runs for a time interval $x_j$.
We wish to schedule jobs so that total resource consumption never exceeds *L*.

```
job in {1..n}
time in {t..T}
data W,D,R{job}
running in [time,time]
x[j] is when running sched(job j) subset W[j]
cumulative(x,D,R,L)
```

`data W,D,R{job}` window, duration, resource

`running in [time,time]` makes **running** an interval variable

Solver generates the model

$$\sum_t \delta_{jt} = 1, \text{ all } j; \quad \sum_j R_j \phi_{jt} \le L, \text{ all } t$$

$$\varphi_{jt} \ge \delta_{jt'}, \text{ all } t, t' \text{ with } 0 \le t - t' < D_j, \text{ all } j$$

```
delta[j,t] is whether running.start sched(job j, time t)
phi[j,t] is whether running sched(job j, time t)
```

# Interval variables

Suppose we want finish times to be separated by at least $T_0$

$$\text{cumulative}\left(x, D, R, L\right)$$

$$x_j \subseteq W_j, \text{ all } j$$

$$\left| x_j^{\text{end}} - x_k^{\text{end}} \right| \geq T_0, \text{ all } j, k,\ j \neq k$$

```
job in {1..n}
time in {t..T}
data W,D,R{job}
running in [time,time]
x[j] is when running sched(job j) subset W[j]
cumulative(x,D,R,L)
{job j, job k | j<>k} |x[j].end - x[k].end| >= T0
delta[j,t] is whether running.start sched(job j, time t)
phi[j,t] is whether running sched(job j, time t)
```

# Interval variables

$$\text{cumulative}\left(x, D, R, L\right)$$

$$x_j \subseteq W_j, \text{ all } j$$

$$\left|x_j^{\text{end}} - x_k^{\text{end}}\right| \geq T_0, \text{ all } j,k,\ j \neq k$$

Suppose we want finish times to be separated by at least $T_0$

```
job in {1..n}
time in {t..T}
data W,D,R{job}
running in [time,time]
x[j] is when running sched(job j) subset W[j]
cumulative(x,D,R,L)
{job j, job k | j<>k} |x[j].end - x[k].end| >= T0
delta[j,t] is whether running.start sched(job j, time t)
phi[j,t] is whether running sched(job j, time t)
```

Solver generates

$$\varepsilon_{jt} + \varepsilon_{kt'} \leq 1, \text{ all } t,t' \text{ with } 0 < t' - t < T_0, \text{ all } j,t \text{ with } j \neq k$$

```
epsilon[j,t] is whether running.end sched(job j, time t)
```

# Interval variables

Variables $\delta_{jt}$ and $\varepsilon_{jt}$ are related by an offset.
Solver associates **`running.`*`end`*** in declaration of $\varepsilon_{jt}$ with **`running.`*`start`*** in declaration of $\delta_{jt}$ and deduces

$$e_{j,t+D_j} = \delta_{jt}, \text{ all } j,t$$

```
delta[j,t] is whether running.start sched(job j, time t)
phi[j,t] is whether running sched(job j, time t)
epsilon[j,t] is whether running.end sched(job j, time t)
```

# Interval variables

Variables $\delta_{jt}$ and $\varepsilon_{jt}$ are related by an offset.
Solver associates **running.*end*** in declaration of $\varepsilon_{jt}$ with **running.*start*** in declaration of $\delta_{jt}$ and deduces

$$e_{j,t+D_j} = \delta_{jt},\ \text{all } j,t$$

Solver also associates **running.*end*** in declaration of $\varepsilon_{jt}$ with **running** in declaration of $\phi_{jt}$ and deduces the redundant constraints

$$\phi_{jt} \geq \varepsilon_{jt'},\ \text{all } t,t' \text{ with } 0 \leq t' - t < D_j,\ \text{all } j$$

```
delta[j,t] is whether running.start sched(job j, time t)
phi[j,t] is whether running sched(job j, time t)
epsilon[j,t] is whether running.end sched(job j, time t)
```

# TSP with Side Constraints

Traveling salesman problem with missing arcs and precedence constraints.

$$\min \sum_i D_{is_i}$$

$$\text{alldiff}(x),\ \text{circuit}(s)$$

$$x_i < x_j,\ \text{all } i, j \text{ with } \text{prec}_{ij} = 1$$

$$s_i \in \text{Succ}_i$$

```
city, position in {1..n}
data D{city, city}
data Prec{city, city}
data Succ{city}
```

`data D{city, city}` Distances

`data Prec{city, city}` **`Prec[i,j]=1`** if *i* must precede *j*

`data Succ{city}` **`Succ[j]`** = set of successors of city *j*

# TSP with Side Constraints

Traveling salesman problem with missing arcs and precedence constraints.

$$\min \sum_i D_{is_i}$$

$$\text{alldiff}(x),\ \text{circuit}(s)$$

$$x_i < x_j,\ \text{all } i, j \text{ with prec}_{ij} = 1$$

$$s_i \in \text{Succ}_i$$

```
city, position in {1..n}
data D{city, city}
data Prec{city, city}
data Succ{city}
```

`data D{city, city}`: Distances

`data Prec{city, city}`: **`Prec[i,j]=1`** if *i* must precede *j*

`data Succ{city}`: **`Succ[j]`** = set of successors of city *j*

Two variable systems:

```
x[i] is which position ordering(city i)
s[i] is successor city ordering(city i) subset Succ[i]
```

# TSP with Side Constraints

Traveling salesman problem with missing arcs and precedence constraints.

$$\min \sum_i D_{i s_i}$$

$$\text{alldiff}(x),\ \text{circuit}(s)$$

$$x_i < x_j,\ \text{all } i, j \text{ with } \text{prec}_{ij} = 1$$

$$s_i \in \text{Succ}_i$$

**`city, position in {1..n}`**

**`data D{city, city}`** Distances

**`data Prec{city, city}`** **`Prec[i,j]=1`** if *i* must precede *j*

**`data Succ{city}`** **`Succ[j]`** = set of successors of city *j*

Two variable systems:

```
x[i] is which position ordering(city i)
s[i] is successor city ordering(city i) subset Succ[i]
```

Precedence constraints require **`x`** variables

```
prec{city i, city j | Prec[i,j] = 1}: x[i] < x[j]
```

Missing arc constraints (implicit in data **`Succ`**) require **`s`** variables

# TSP with Side Constraints

$$\min \sum_i D_{is_i}$$

$$\text{alldiff}(x),\ \text{circuit}(s)$$

$$x_i < x_j,\ \text{all } i, j \text{ with prec}_{ij} = 1$$

$$s_i \in \text{Succ}_i$$

Traveling salesman problem with missing arcs and precedence constraints.

`city, position in {1..n}`

`data D{city, city}` Distances

`data Prec{city, city}` **`Prec[i,j]=1`** if *i* must precede *j*

`data Succ{city}` **`Succ[j]`** = set of successors of city *j*

Two variable systems:

```
x[i] is which position ordering(city i)
s[i] is successor city ordering(city i) subset Succ[i]
```

Precedence constraints require **`x`** variables

```
prec{city i, city j | Prec[i,j] = 1}: x[i] < x[j]
```

Missing arc constraints (implicit in data **`Succ`**) require **`s`** variables

`min sum {city i} D[i,s[i]]` Objective function

# TSP with Side Constraints

The solver can give **`alldiff(x)`** a conventional assignment model using $z_{ik}$ = whether city *i* is in position *k.*

```
z[i,k] is whether ordering(city i, position k)
```

# TSP with Side Constraints

The solver can give **`alldiff(x)`** a conventional assignment model using $z_{ik}$ = whether city *i* is in position *k.*

**`z[i,k] is whether ordering(city i, position k)`**

For **`circuit(s)`**, the solver can introduce
$w_{ij}$ = whether city *i* immediately precedes city *j*.

**`w[i,j]`** ***`is whether successor`*** **`ordering(city i, city j)`**

# TSP with Side Constraints

The solver can give **`alldiff(x)`** a conventional assignment model using $z_{ik}$ = whether city *i* is in position *k.*

**`z[i,k] is whether ordering(city i, position k)`**

For **`circuit(s)`**, the solver can introduce
$w_{ij}$ = whether city *i* immediately precedes city *j*.

**`w[i,j]`** ***`is whether successor`*** **`ordering(city i, city j)`**

Declaration of **`z`** tells solver that predicate is
**`ordering(city,position)`**, not **`ordering(city,city)`**.

# TSP with Side Constraints

The solver can give **`alldiff(x)`** a conventional assignment model using $z_{ik}$ = whether city *i* is in position *k.*

**`z[i,k] is whether ordering(city i, position k)`**

For **`circuit(s)`**, the solver can introduce
$w_{ij}$ = whether city *i* immediately precedes city *j*.

**`w[i,j]`** ***`is whether successor`*** **`ordering(city i, city j)`**

Declaration of **`z`** tells solver that predicate is
**`ordering(city,position)`**, not **`ordering(city,city)`**.
Solver generates cutting planes in **`w`**-space and **`s`**-space.

# TSP with Side Constraints

The solver can give **`alldiff(x)`** a conventional assignment model using $z_{ik}$ = whether city *i* is in position *k.*

```
z[i,k] is whether ordering(city i, position k)
```

For **`circuit(s)`**, the solver can introduce
$w_{ij}$ = whether city *i* immediately precedes city *j*.

```
w[i,j] is whether successor ordering(city i, city j)
```

Declaration of **`z`** tells solver that predicate is **`ordering(city,position)`**, not **`ordering(city,city)`**.
Solver generates cutting planes in **`w`**-space and **`s`**-space.

The **`successor`** keyword tells solver how **`z`** and **`w`** relate.

$$\phi_{jt} \geq \varepsilon_{jt'}, \text{ all } t,t' \text{ with } 0 \leq t' - t < D_j, \text{ all } j$$

# TSP with Side Constraints

Suppose we also have constraints on which city is in position *k*. Simply declare

**`y[k]`** **= *which* city ordering(position k)**

The solver generates the channeling constraints between **`y[k]`** and **`x[i]`** = which position is city *i*

# TSP with Side Constraints

Suppose we also have constraints on which city is in position *k*. Simply declare

**y[k] = *which* city ordering(position k)**

The solver generates the channeling constraints between **y[k]** and **x[i]** = which position is city *i*

The solver can also introduce a second (equivalent) objective function

**min sum{position k} D[y[k],y[k+1]]**

which may improve bounding.

# Pros and Cons of Semantic Typing

- **Pros**
  - Conveys problem structure to the solver(s)
    - …by allowing use of metaconstaints
  - Incorporates state of the art in formulation, valid inequalities
  - Allows solver to expand repertory of techniques
    - Domain filtering, propagation, cutting plane algorithms
  - Good modeling practice
    - Self-documenting
    - Bug detection

# Pros and Cons of Semantic Typing

- **Cons**
  - Modeler must be familiar with a large collection of metaconstraints
    - Rather than few primitive constraints

# Pros and Cons of Semantic Typing

- Cons
  - Modeler must be familiar with a large collection of metaconstraints
    - Rather than few primitive constraints
  - *Response*
    - *Modeler must be familiar with the underlying* ***concepts*** *anyway*
    - *Modeling system can offer sophisticated help, improve modeling*

# Pros and Cons of Semantic Typing

- Cons
  - Modeler must be familiar with a large collection of metaconstraints
    - Rather than few primitive constraints
  - *Response*
    - *Modeler must be familiar with the underlying **concepts** anyway*
    - *Modeling system can offer sophisticated help, improve modeling*
  - OR, SAT community is not accustomed to high-level modeling
    - Typed languages like Ascend never really caught on, resistance to CP.

# Pros and Cons of Semantic Typing

- Cons
  - Modeler must be familiar with a large collection of metaconstraints
    - Rather than few primitive constraints
  - *Response*
    - *Modeler must be familiar with the underlying* ***concepts*** *anyway*
    - *Modeling system can offer sophisticated help, improve modeling*
  - OR, SAT community is not accustomed to high-level modeling
    - Typed languages like Ascend never really caught on, resistance to CP.
  - *Response*
    - *Train the next generation!*