Finding Alternative Musical Scales

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Advantages of Classical Scales

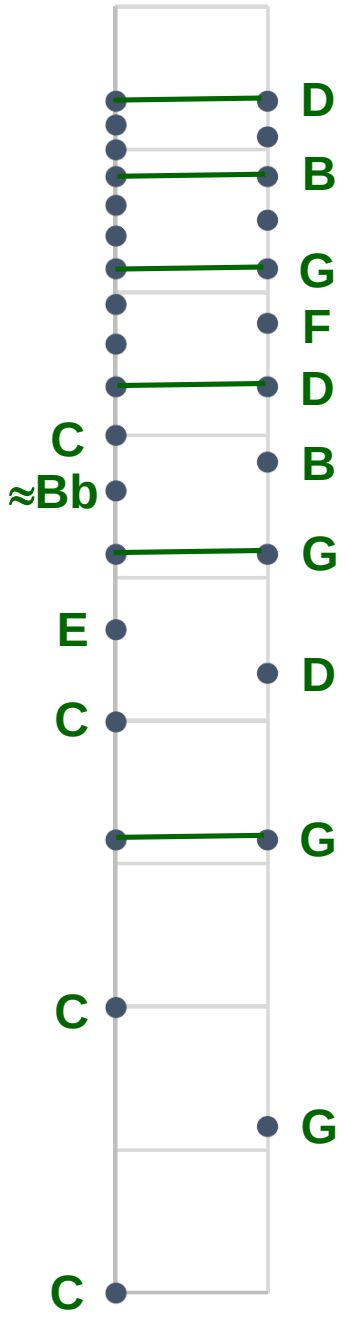
- Pitch frequencies have **simple ratios.**
  - Rich and intelligible harmonies
- **Multiple keys** based on underlying chromatic scale with **tempered tuning**.
  - Can play all keys on instrument with fixed tuning.
  - Complex musical structure.

Advantages of Classical Scales

- Pitch frequencies have **simple ratios.**
  - Rich and intelligible harmonies
- **Multiple keys** based on underlying chromatic scale with **tempered tuning**.
  - Can play all keys on instrument with fixed tuning.
  - Complex musical structure.
- Can we find **new scales** with these same properties?
  - CP is well suited to solve the problem.

Simple Ratios

- Acoustic instruments produce multiple **harmonic partials.**
  - Frequency of partial = integral multiple of frequency of fundamental.
  - Coincidence of partials makes chords with simple ratios easy to recognize.



Perfect fifth
C:G = 2:3

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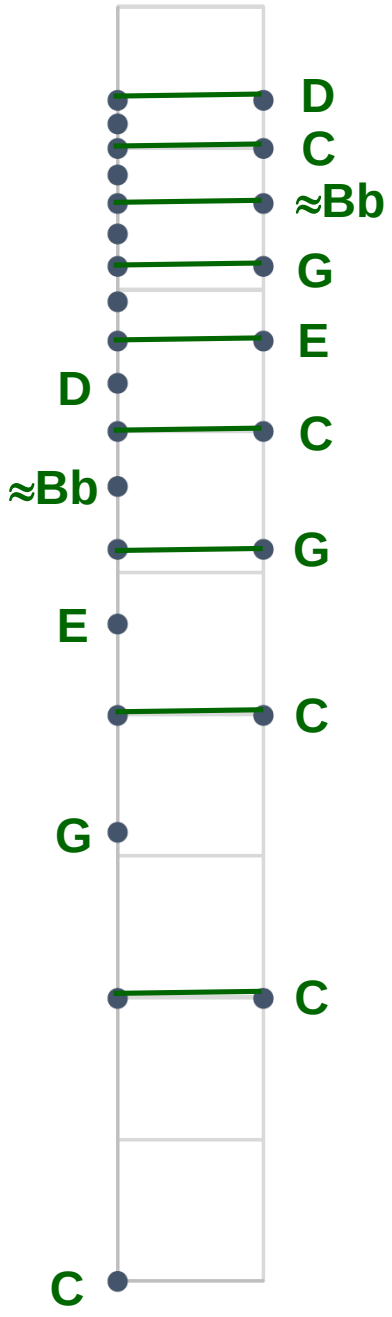

Octave
C:C = 1:2

Simple Ratios

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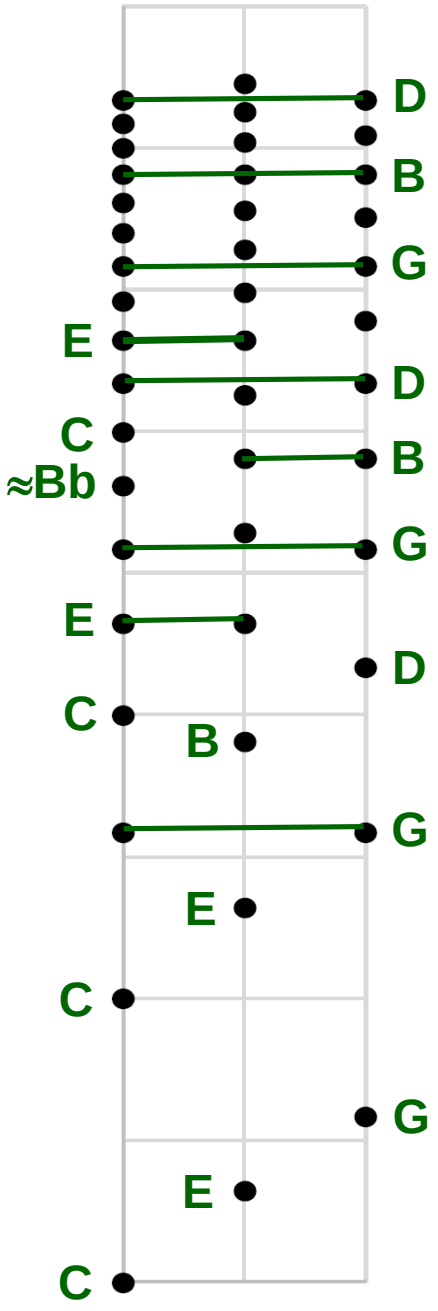
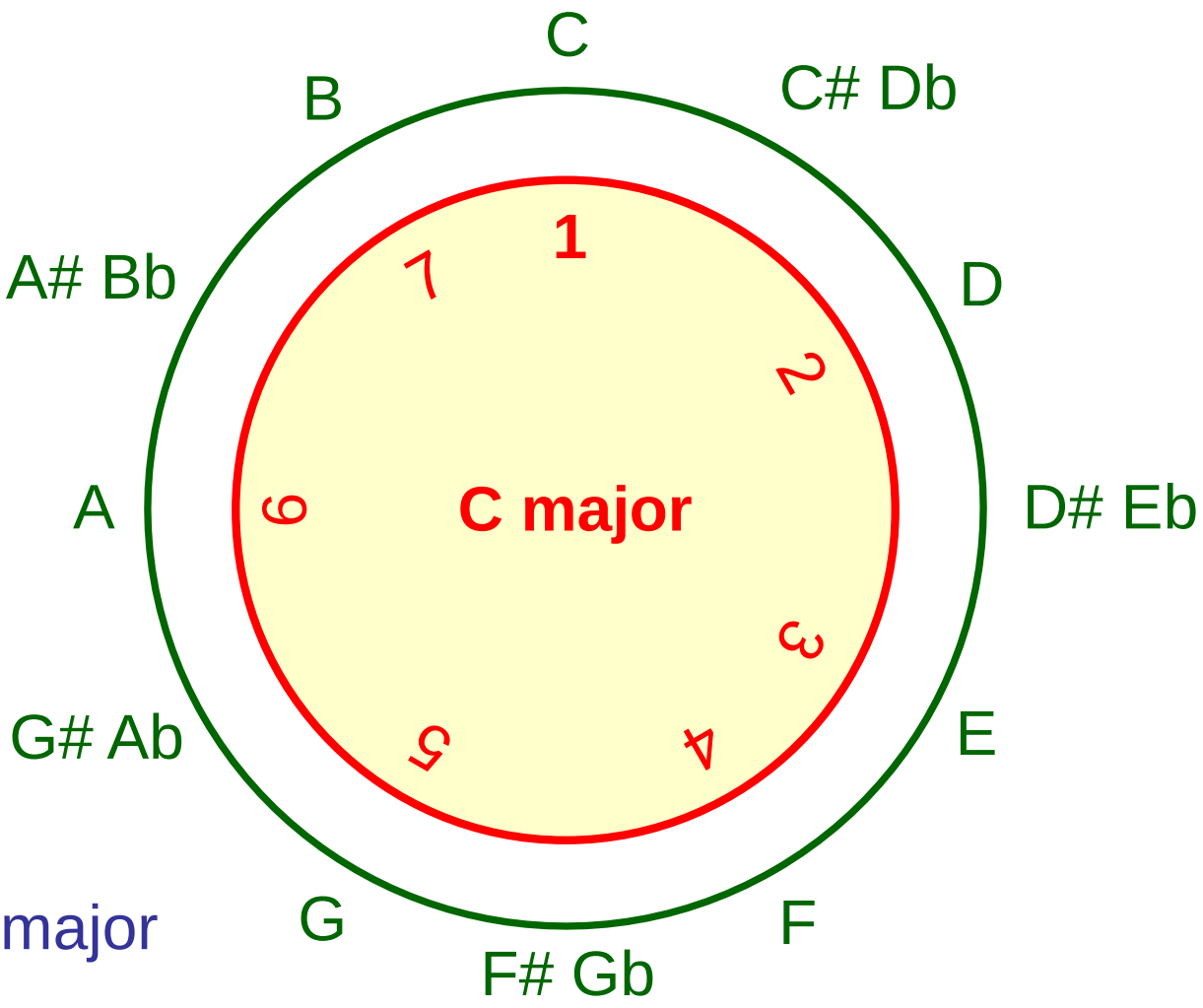


Major triad
C:E:G = 4:5:6

Multiple Keys

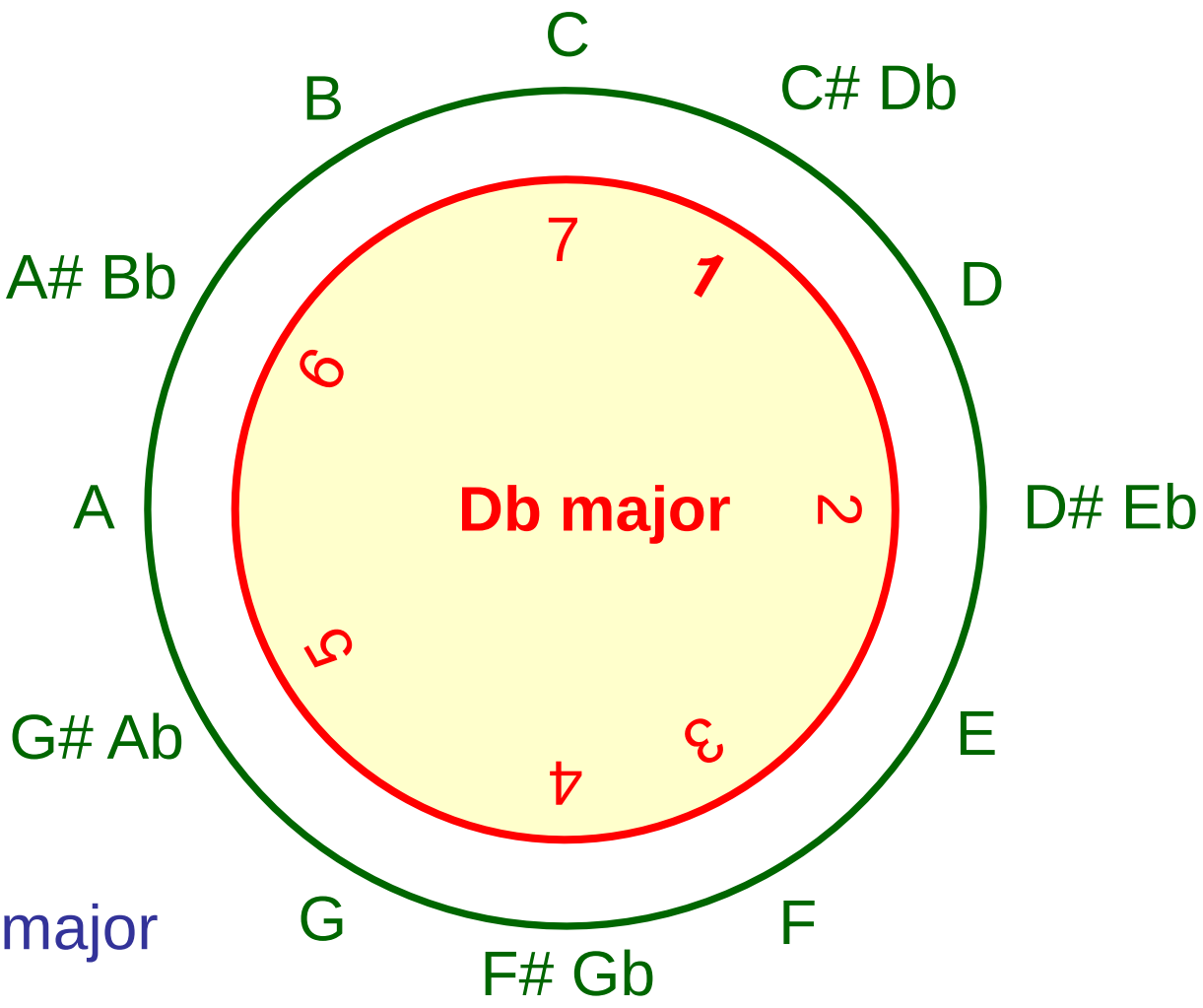
- A classical scale can start from any pitch in a **chromatic** scale with 12 **semitone** intervals.
  - Resulting in 12 **keys**.
  - An instrument with 12 pitches (modulo octaves) can play 12 different keys
  - Can move to a different key by changing only a few notes of the scale.

Multiple Keys



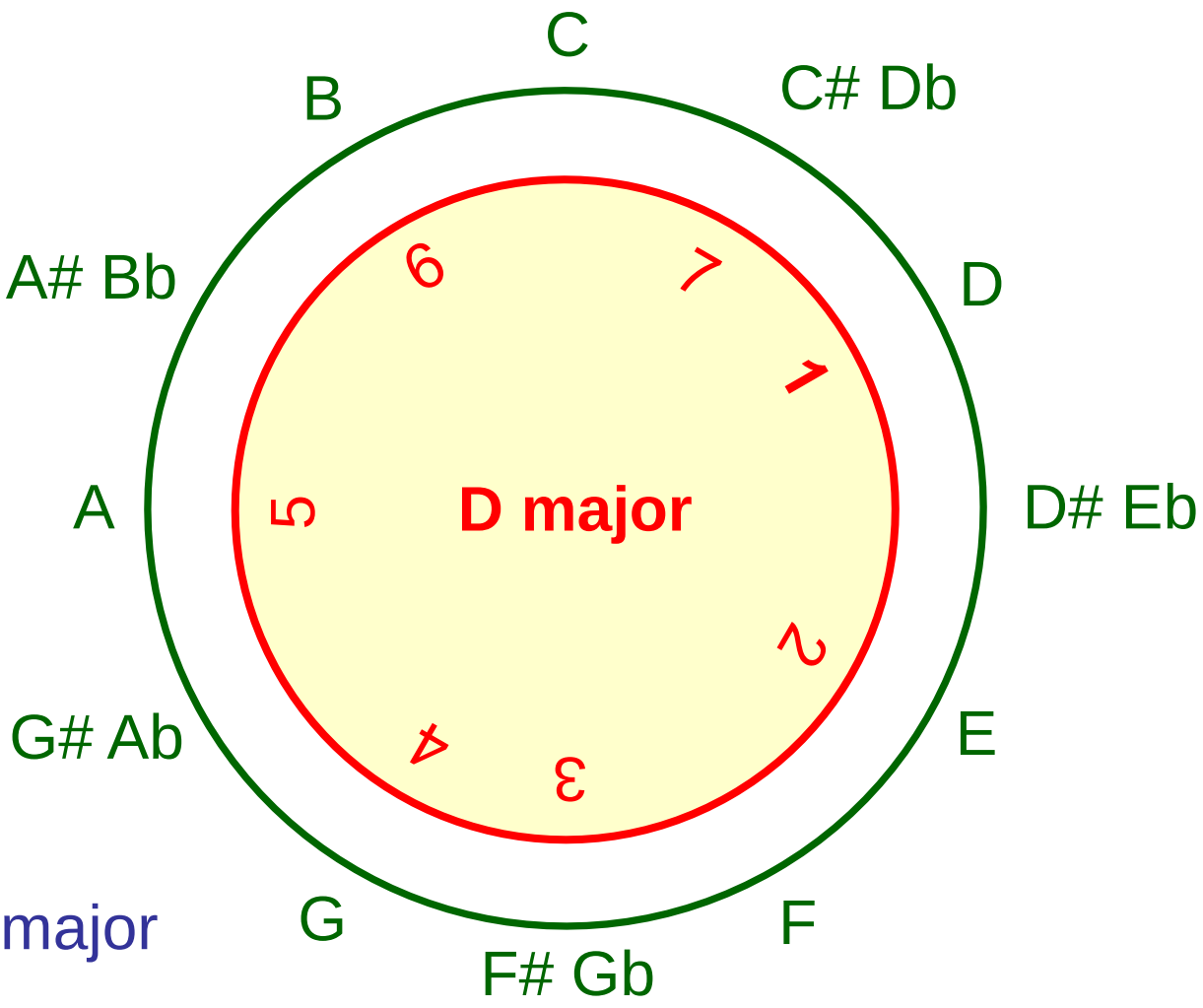
0 notes
not in C major

Multiple Keys



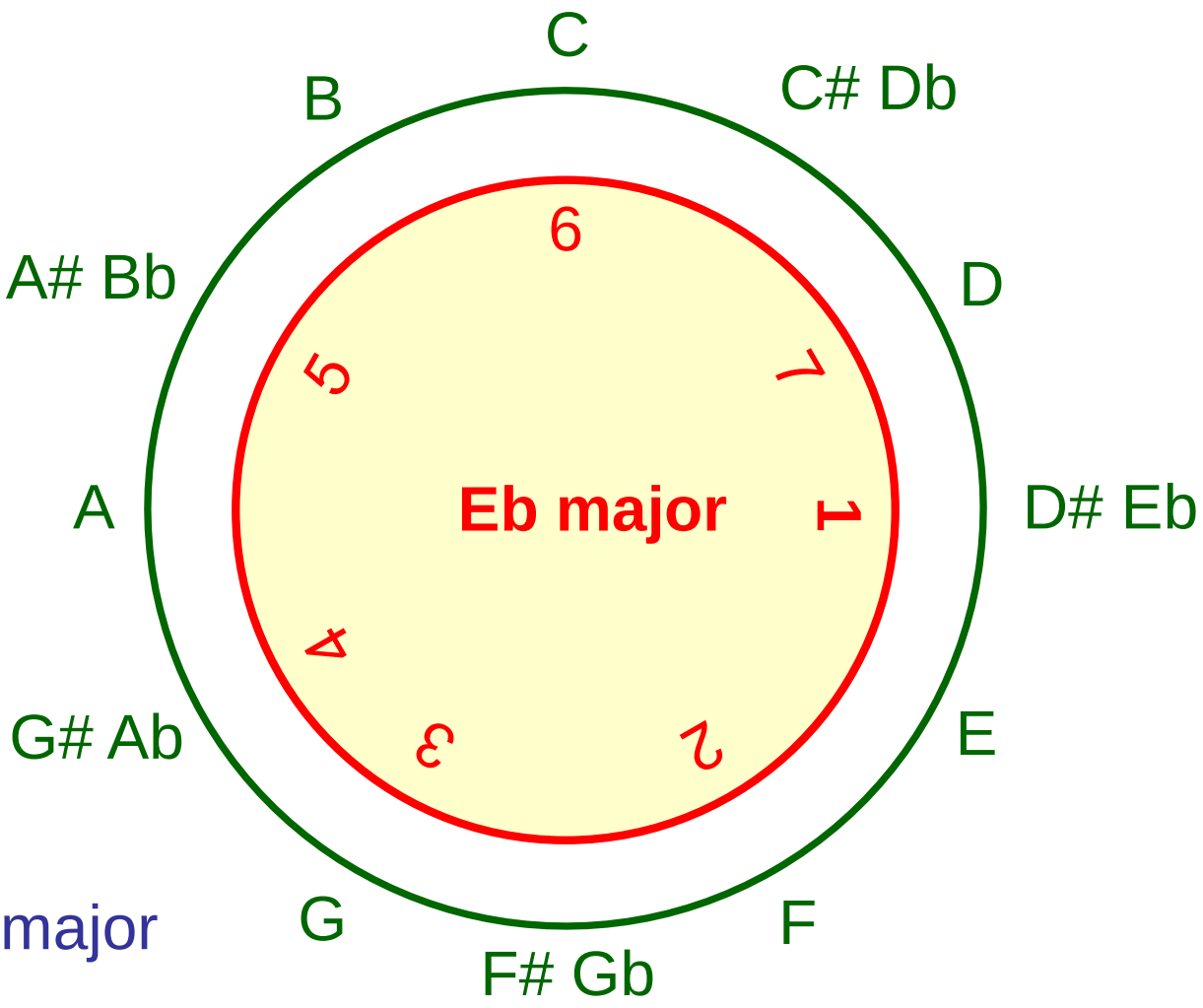
5 notes
not in C major

Multiple Keys



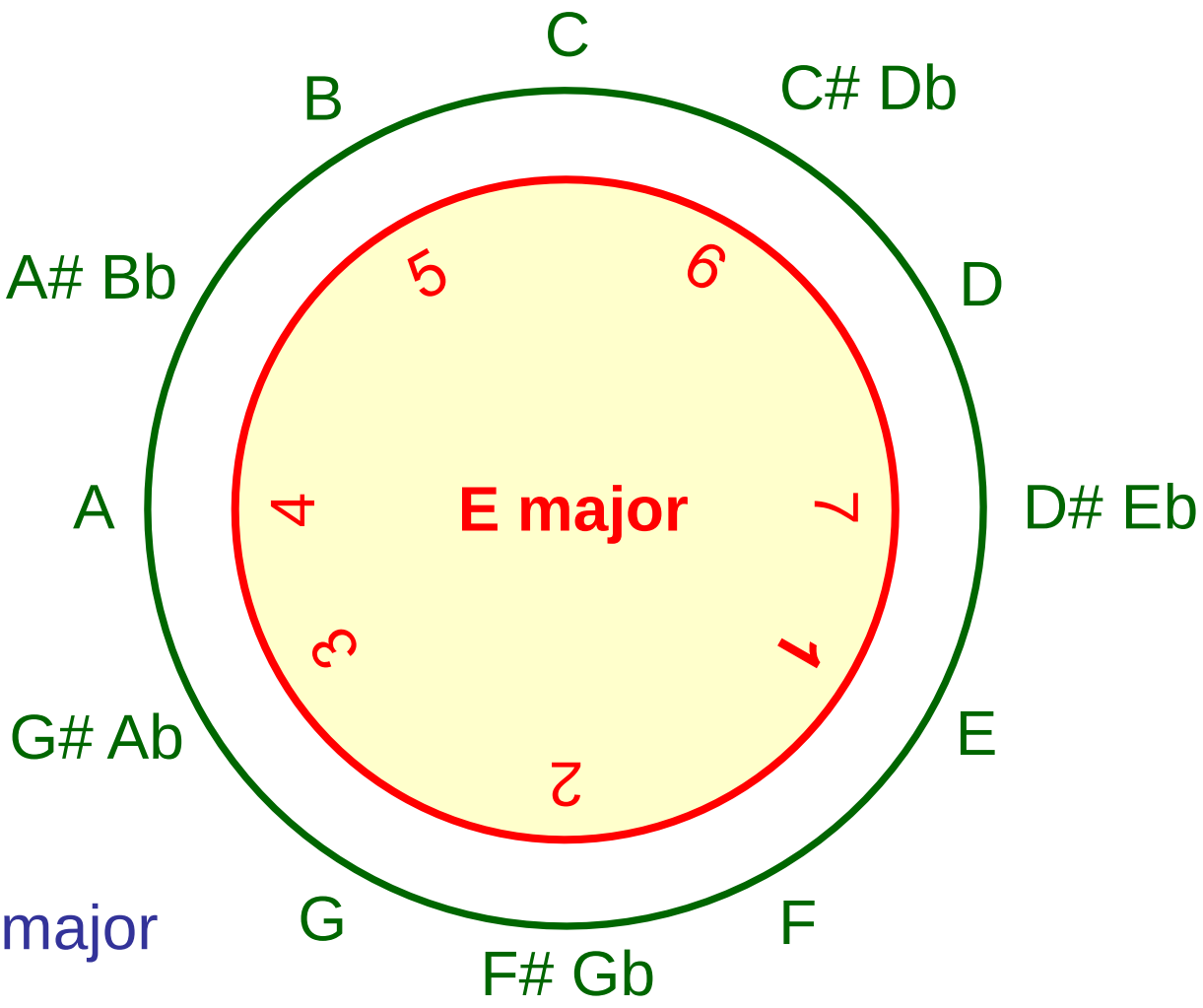
2 notes
not in C major

Multiple Keys



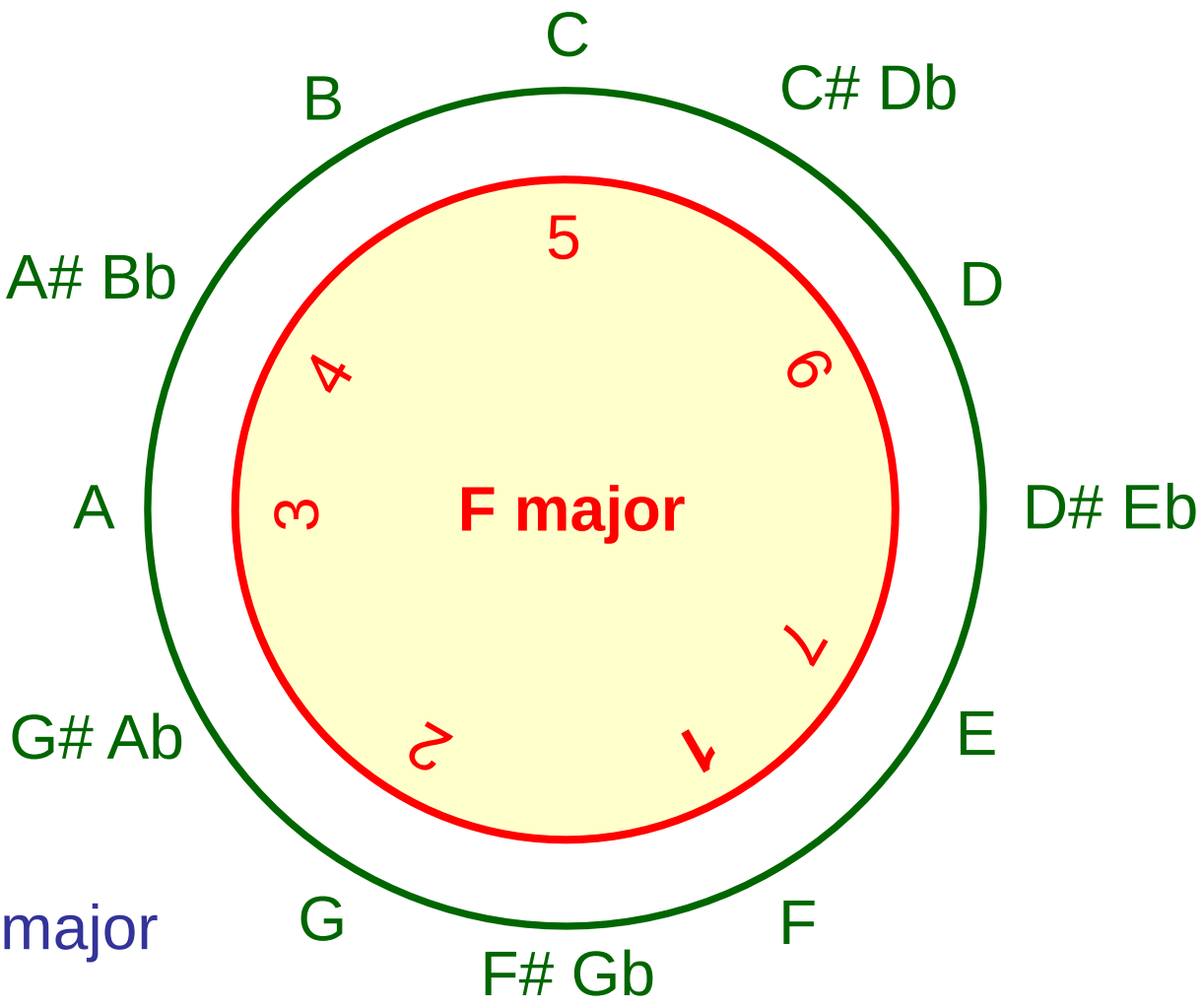
3 notes
not in C major

Multiple Keys



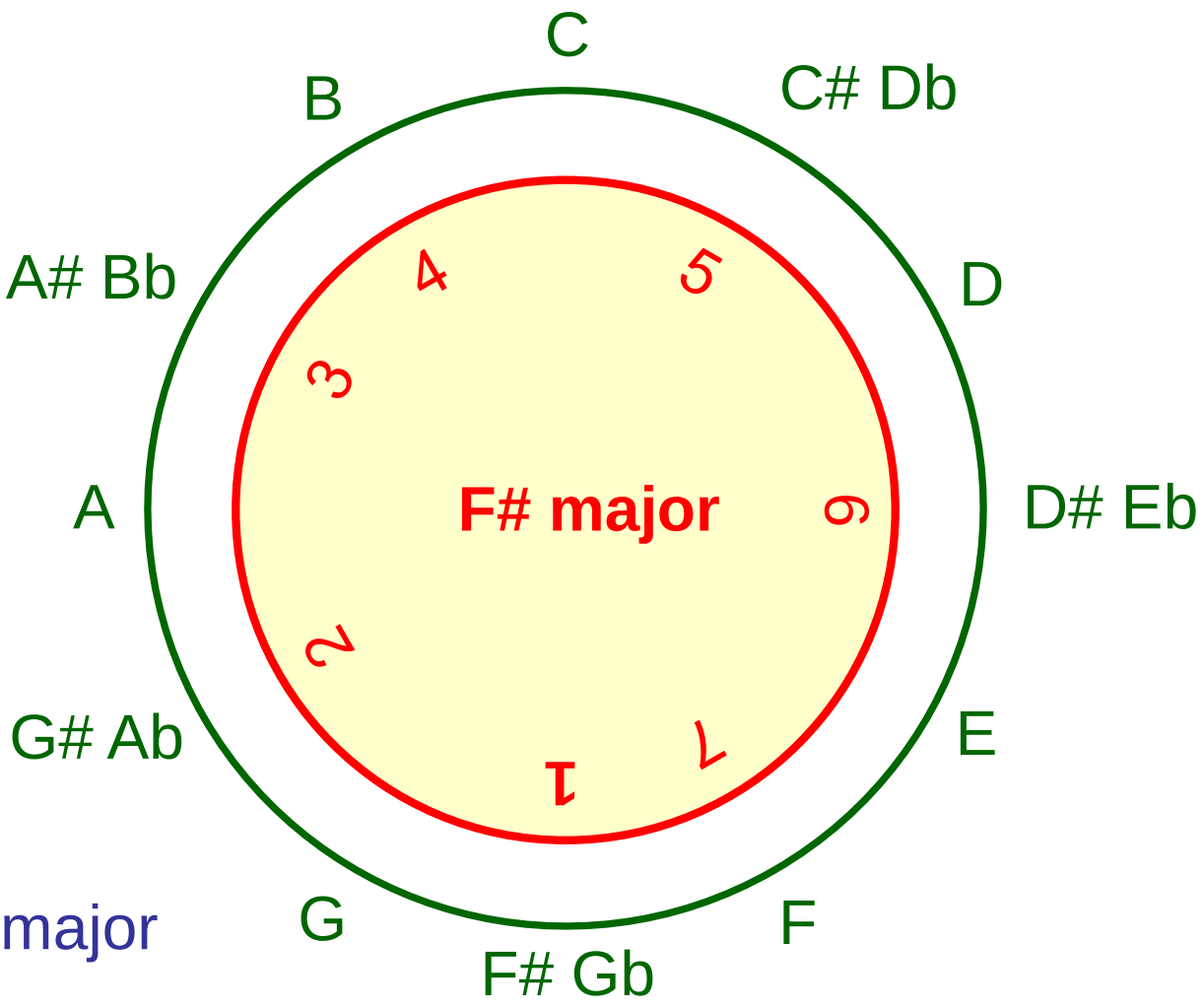
4 notes
not in C major

Multiple Keys



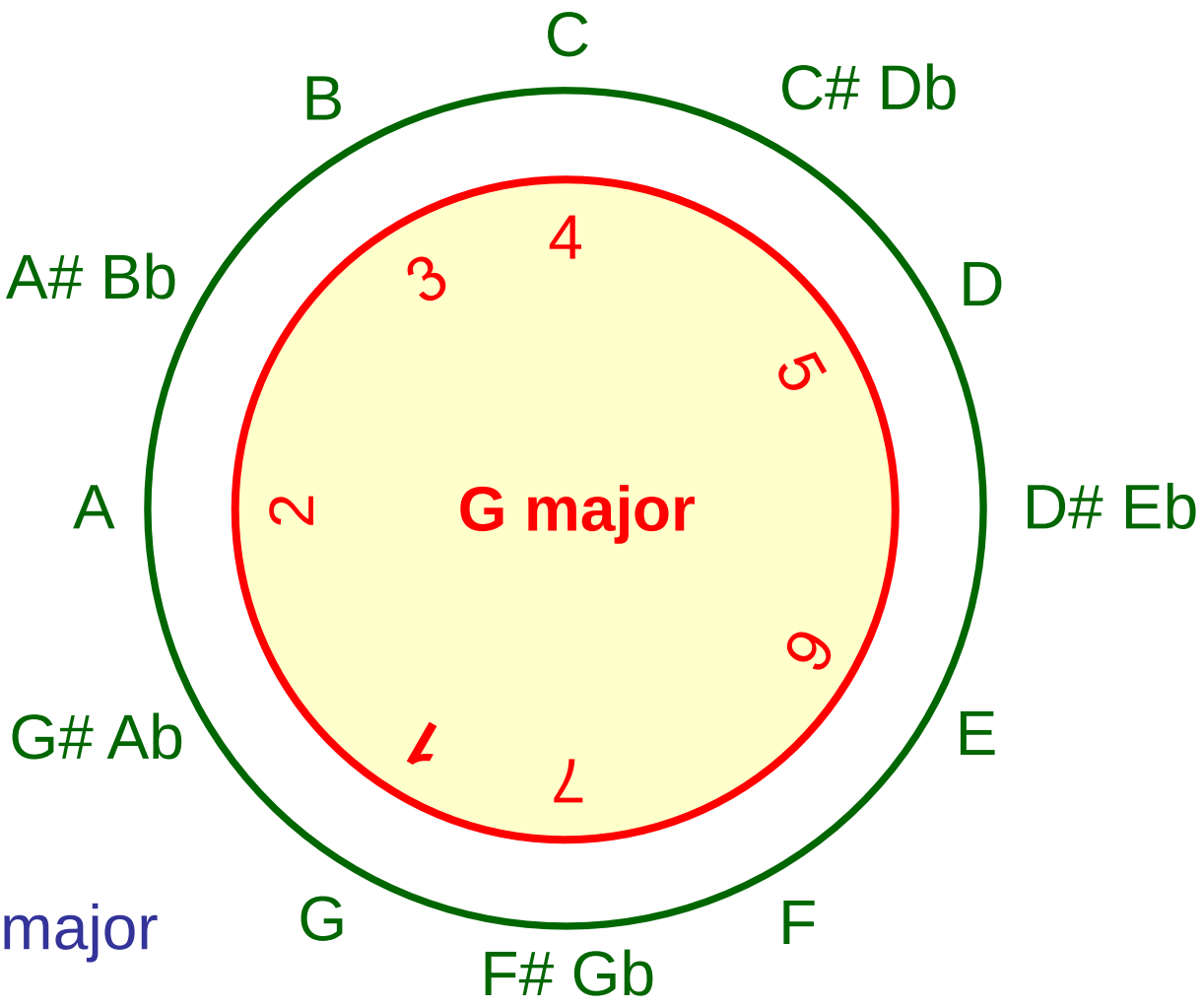
1 note
not in C major

Multiple Keys



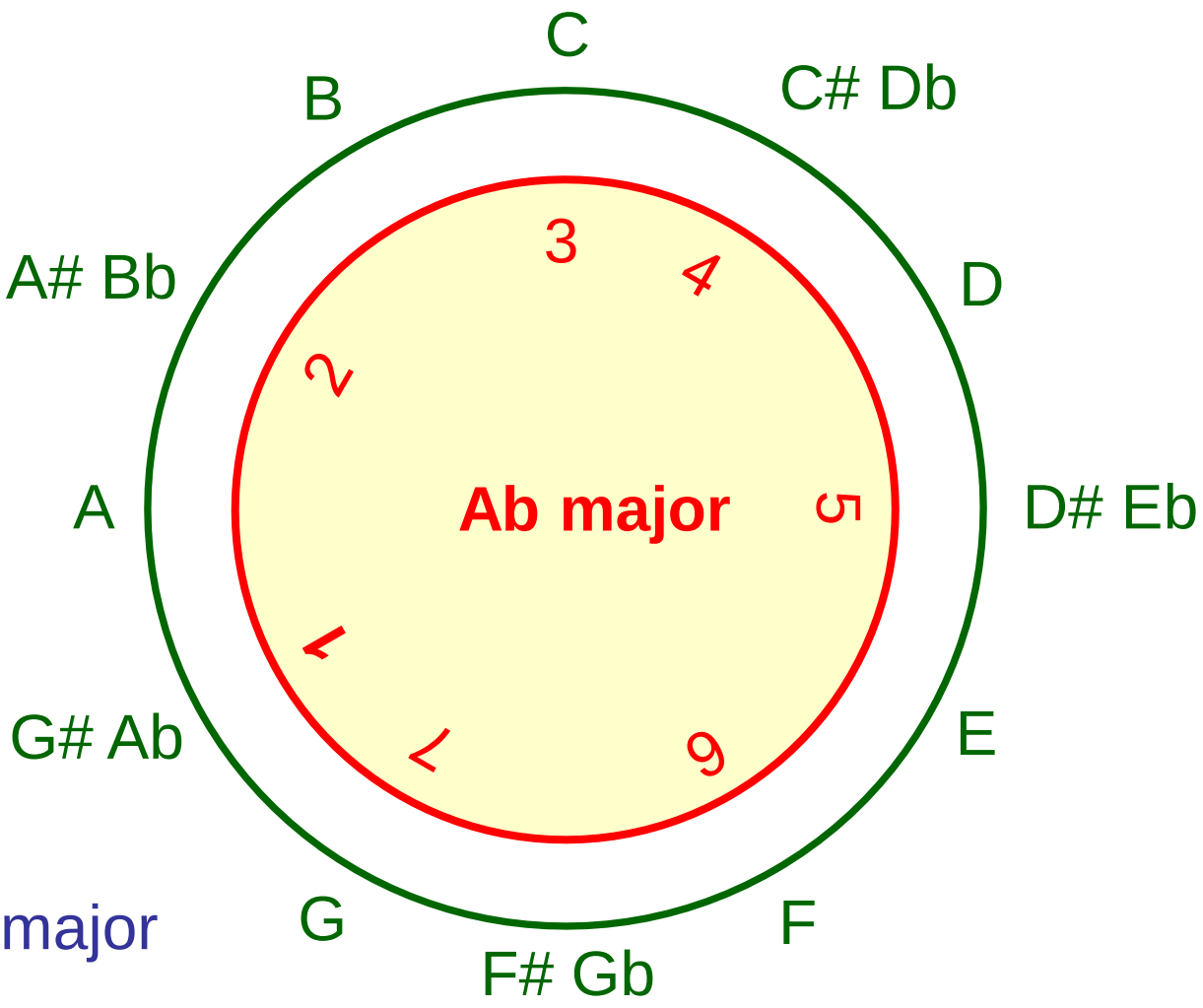
6 notes
not in C major

Multiple Keys



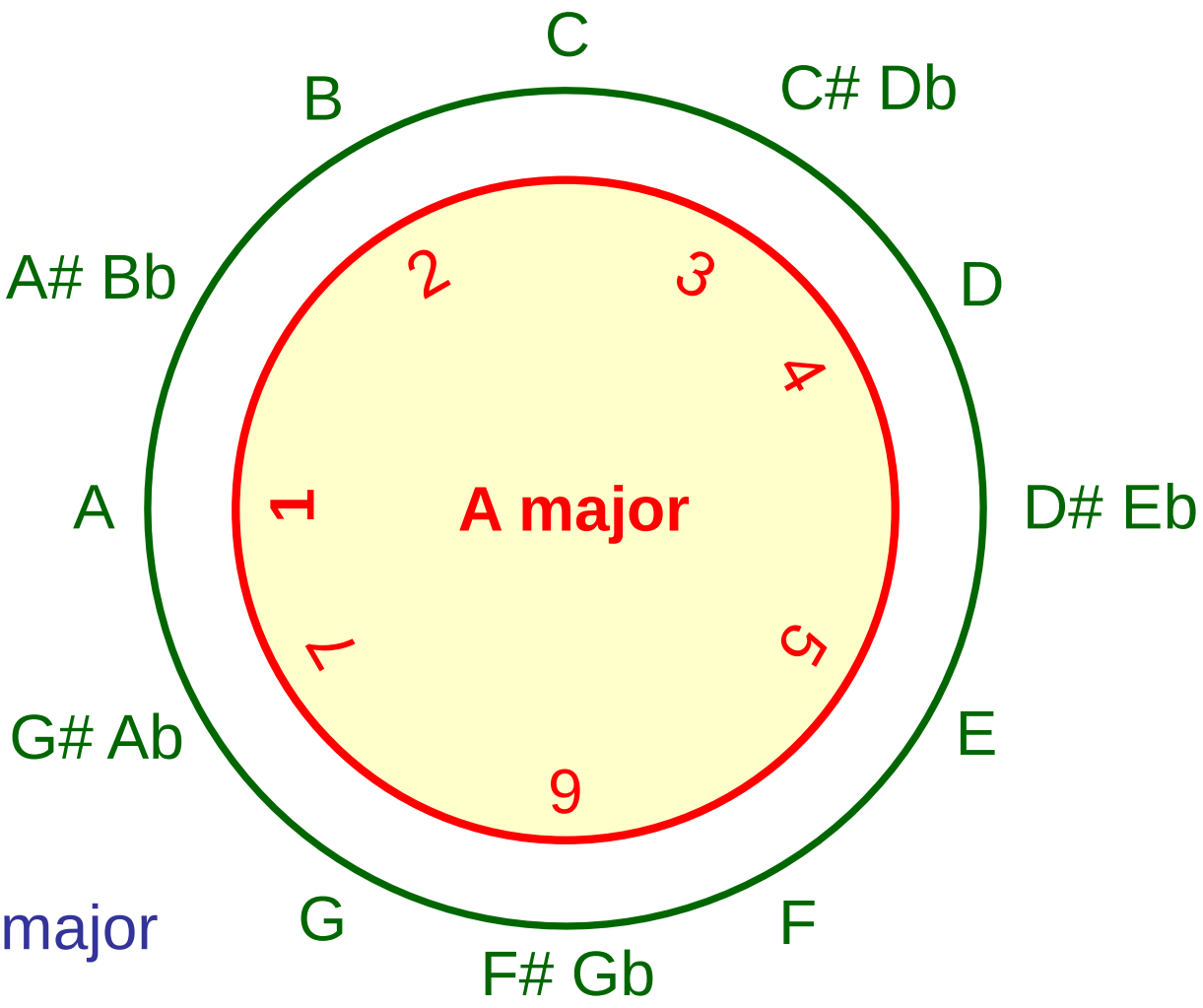
1 note
not in C major

Multiple Keys



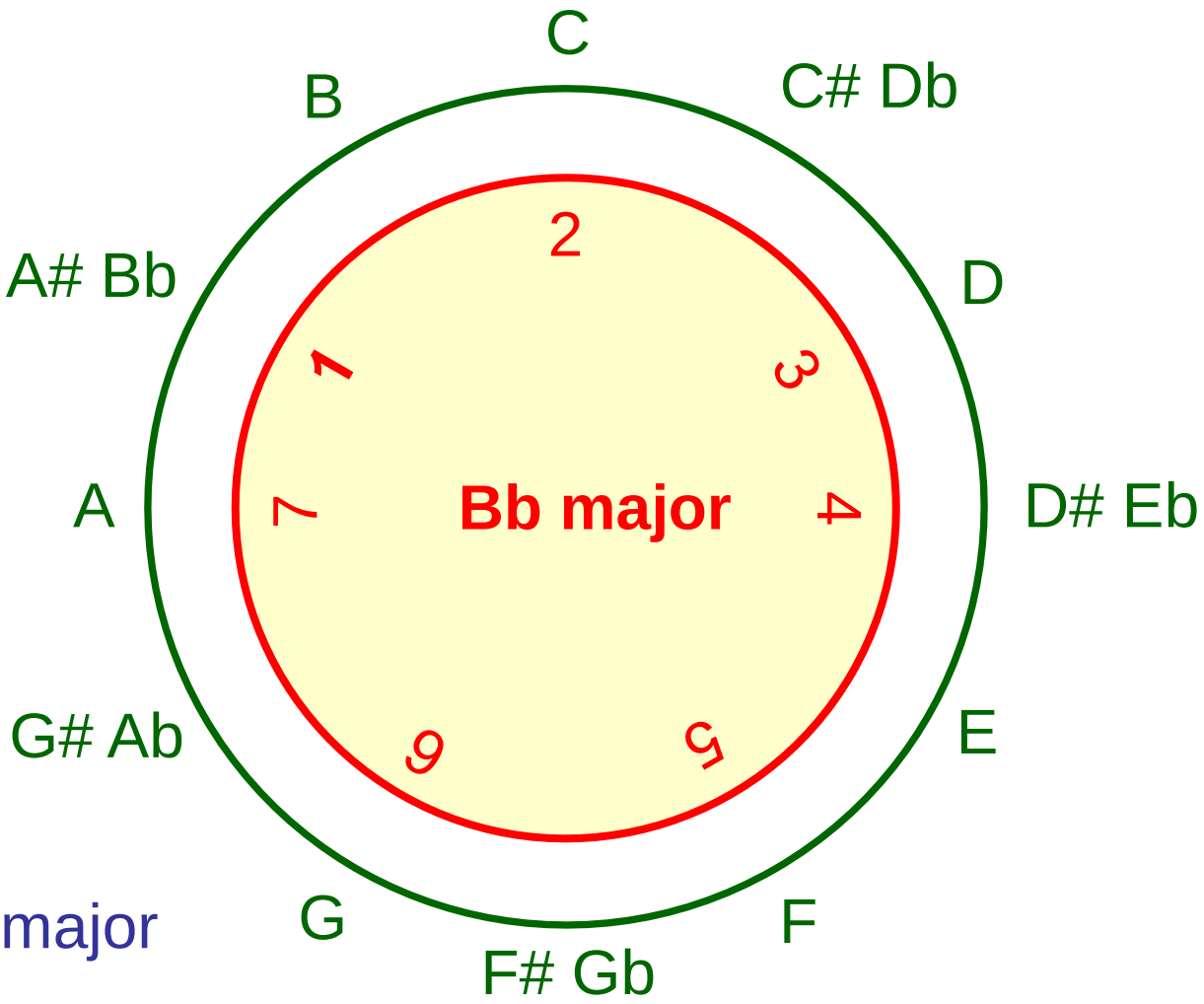
4 notes
not in C major

Multiple Keys



3 notes
not in C major

Multiple Keys



2 notes
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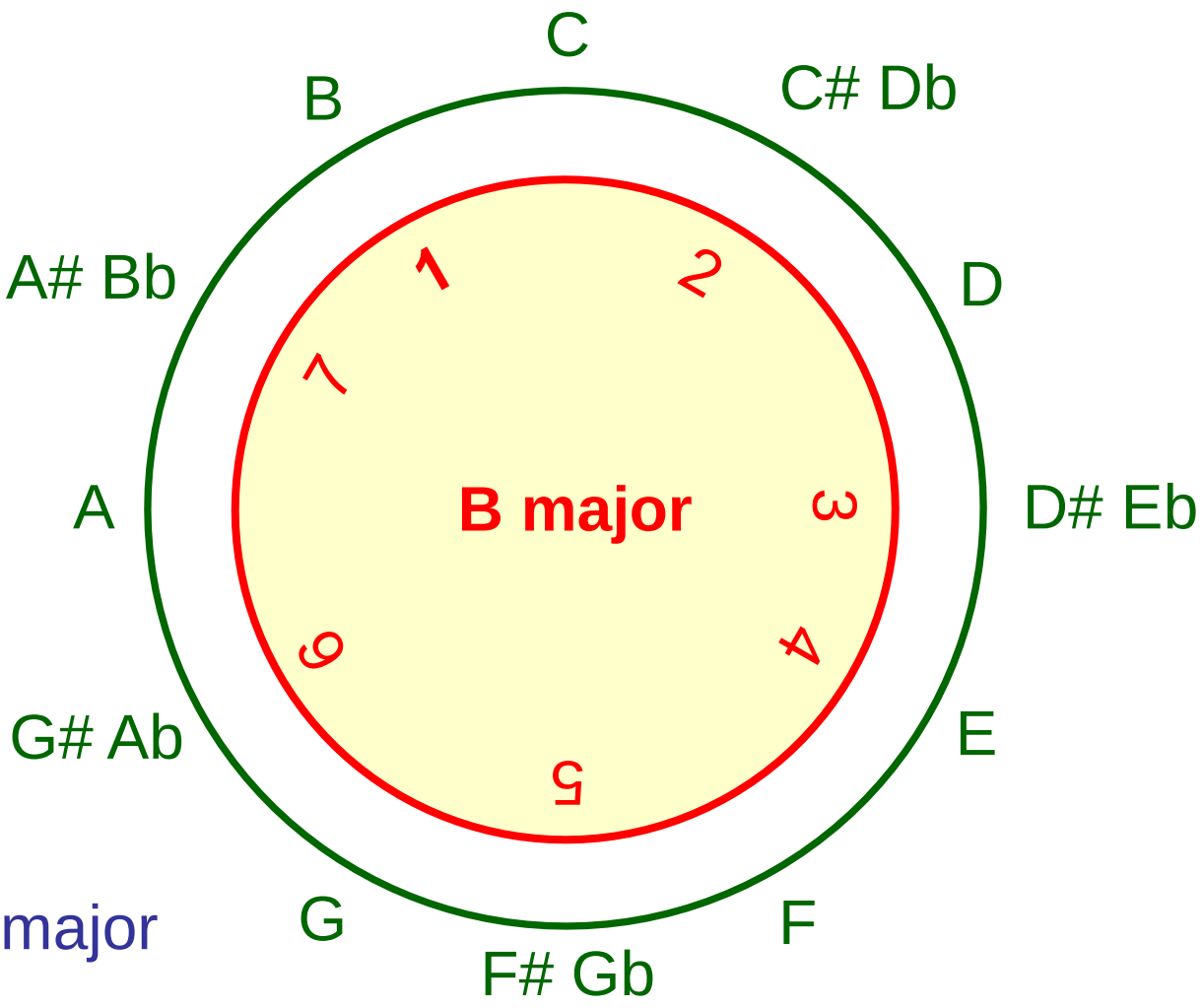
Multiple Keys



5 notes
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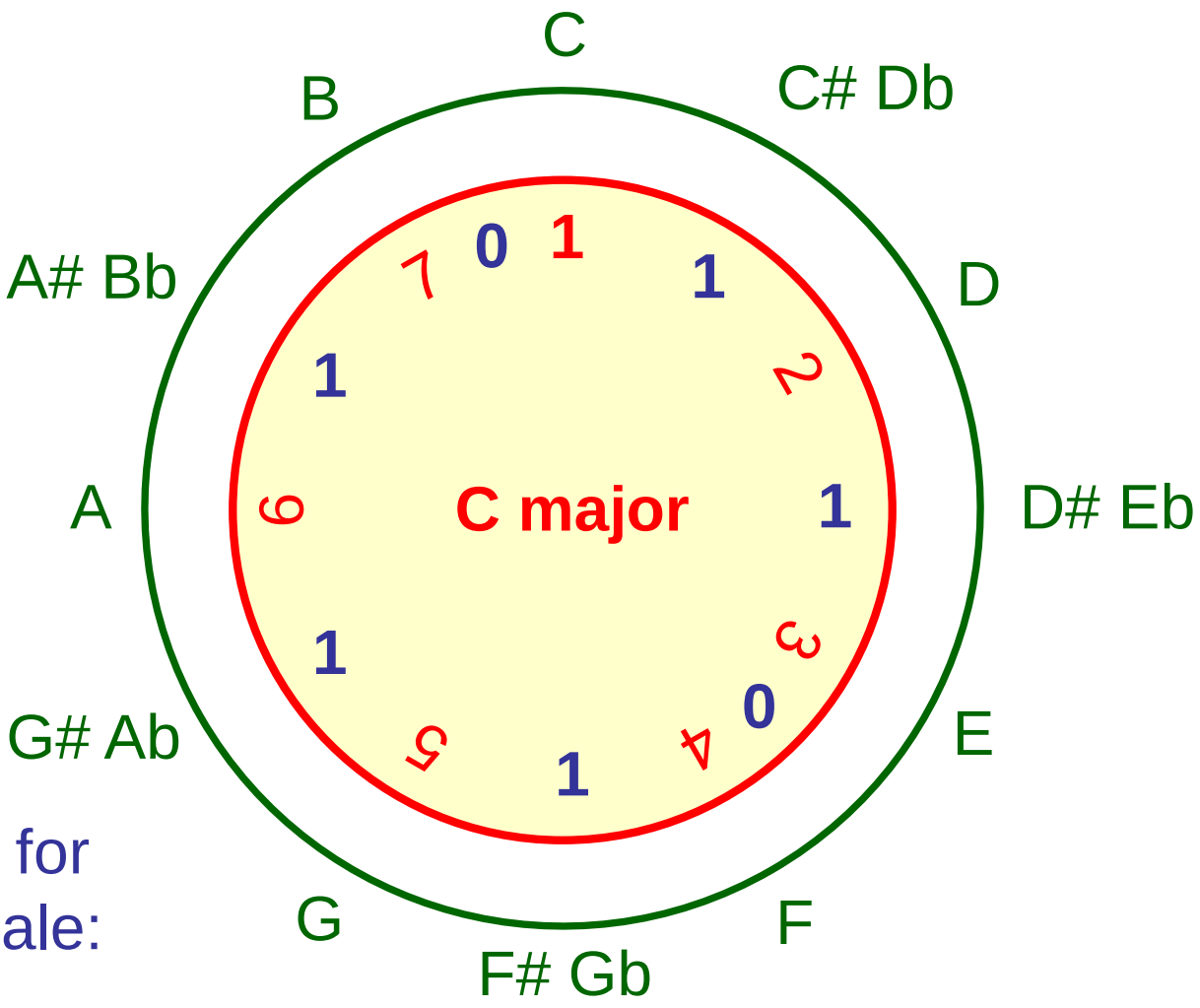
Multiple Keys



Notation for major scale:
1101110

Multiple Keys

- Chromatic pitches are **tempered** so that intervals will have approximately correct ratios in all keys.
  - Modern practice is **equal temperament**.

$$\frac{\text{freq of note } k}{\text{freq of note } 1} = 2^{(k-1)/12}$$

Multiple Keys

- Resulting error is $\leq \pm 0.9\%$

Note	Perfect ratio	Tempered ratio	Error %
C	$1/1$	1.00000	0.000
D	$9/8$	1.12246	-0.226
E	$5/4$	1.25992	$+0.787$
F	$4/3$	1.33484	$+0.113$
G	$3/2$	1.49831	-0.113
A	$5/3$	1.68179	$+0.899$
B	$15/8$	1.88775	$+0.675$

Combinatorial Requirements

- Scales must be **diatonic**
  - Adjacent notes are 1 or 2 semitones apart.

- We consider *m*-note scales on an *n*-tone chromatic
  - In binary representation, let m_0 = number of 0s, m_1 = number of 1s
  - Then $m_0 = 2m - n$, $m_1 = n - m$
    - In a major scale 1101110, there are $m = 7$ notes on an $n = 12$-tone chromatic
    - There are $m_0 = 2 \cdot 7 - 12 = 2$ zeros
    - There are $m_1 = 12 - 7 = 5$ ones

Combinatorial Requirements

- Semitones should not be bunched together.
  - One criterion: **Myhill's property**
  - All intervals of a given size should contain *k* or *k* + 1 semitones.
    - For example, in a major scale:
    - All fifths are 6 or 7 semitones
    - All thirds are 3 or 4 semitones
    - All seconds are 1 or 2 semitones, etc.
  - Few scales satisfy Myhill's property.

Combinatorial Requirements

- Semitones should not be bunched together.
  - We minimize the number of pairs of adjacent 0s and pairs of adjacent 1s.
  - If $m_0 \geq m_1$,

$$\text{number of adjacent 0s} = m_0 - \min\{m_0, m_1\}$$
$$\text{number of adjacent 1s} = 0$$

  - If $m_1 \geq m_0$,

$$\text{number of adjacent 1s} = m_1 - \min\{m_0, m_1\}$$
$$\text{number of adjacent 0s} = 0$$

    - In a major scale 1101110,
      number of adjacent 1s = 5 – min{2,5} = 3
      number of adjacent 0s = 0.

Combinatorial Requirements

- Semitones should not be bunched together.
  - The number of scales satisfying this property is

$$\binom{\max\{m_0, m_1\}}{\min\{m_0, m_1\}} + \binom{\max\{m_0, m_1\} - 1}{\min\{m_0, m_1\} - 1}$$

    - The number of 7-note scales on a 12-tone chromatic satisfying this property is

$$\binom{5}{2} + \binom{4}{1} = 14$$

Combinatorial Requirements

- Can have fewer than *n* keys.
  - A “mode of limited transposition” (Messiaen)
  - Whole tone scale 111111 (Debussy) has 2 keys.
  - Scale 110110110 has 5 keys.
    - Count number of semitones in repeating sequence

Temperament Requirements

- Tolerance for inaccurate tuning
  - At most $\pm$0.9%
  - Don’t exceed tolerance of classical equal temperament.

Simple Ratios

- Frequency of each note should have a simple ratio (between 1 and 2) with some other note, equating notes an octave apart.
  - Let f_i = freq ratio of note i to tonic (note 1), so $f_1 = 1$.
  - For major scale CDEFGAB,

$$(f_1, \ldots, f_7) = \left(1, \tfrac{9}{8}, \tfrac{5}{4}, \tfrac{4}{3}, \tfrac{3}{2}, \tfrac{5}{3}, \tfrac{15}{8}\right)$$

  - For example, B (15/8) has a simple ratio 3/2 with E (5/4)

$$\frac{f_7}{f_3} = \frac{3}{2}$$

  - D octave higher (9/4) has ratio with 3/2 with G (3/2)

$$\frac{2f_2}{f_5} = \frac{3}{2}$$

Simple Ratios

- However, this allows two or more subsets of unrelated pitches.
  - Simple ratios with respect pitches in same subset, but not in other subsets.
  - So we use a **recursive** condition.
  - For some permutation of notes, each note should have simple ratio with previous note.
  - First note in permutation is the tonic.

Simple Ratios

- Let the simple ratios be **generators** $r_1, \ldots, r_p$.
  - Let $(\pi_1, \ldots \pi_m)$ be a permutation of 1, …, m with $\pi_1 = 1$.
  - For each $i \in \{2, \ldots, m\}$, we require

$$1 < f_{\pi_i} < 2$$

and

$$\frac{f_{\pi_i}}{f_{\pi_j}} = r_q \text{ or } \frac{2f_{\pi_j}}{f_{\pi_i}} = r_q \text{ or } \frac{f_{\pi_j}}{f_{\pi_i}} = r_q \text{ or } \frac{2f_{\pi_i}}{f_{\pi_j}} = r_q$$

for some $j \in \{1, \ldots, i-1\}$ and some $q \in \{1, \ldots, p\}$

Simple Ratios

- Ratio with previous note in the permutation π will be a generator.
  - Ratios with previous 2 or 3 notes in the permutation will be simple (product of generators).
  - Ratio with tonic need not be simple.

Simple Ratios

- Observation: No need to consider both r_q and $2/r_q$ as generators.
  - So we consider only reduced fractions with odd numerators (in order of simplicity):

$$\frac{3}{2}, \frac{5}{3}, \frac{5}{4}, \frac{7}{4}, \frac{7}{5}, \frac{9}{5}, \frac{7}{6}, \frac{11}{6}, \frac{9}{7}, \frac{11}{7},$$
$$\frac{13}{7}, \frac{9}{8}, \frac{11}{8}, \frac{13}{8}, \frac{15}{8}, \frac{11}{9}, \frac{13}{9}, \frac{17}{9}, \ldots$$

CP Model

- CP model readily accommodates variable indices

$$f_{\pi_i}$$

- Replace f_i with fraction a_i / b_i in lowest terms

CP Model

$$\text{alldiff}(\pi_1, \ldots, \pi_m)$$

$$\pi_1 = a_1 = b_1 = 1$$

$$1 < \frac{a_i}{b_i} < 2, \;\; \text{coprime}(a_i, b_i), \;\; i = 1, \ldots, m$$

$$\frac{a_{i-1}}{b_{i-1}} < \frac{a_i}{b_i}, \;\; i = 2, \ldots, m$$

$$\bigvee_{j<i} \left[(\pi_i > \pi_j) \Rightarrow \left(\frac{a_{\pi_i}/b_{\pi_i}}{a_{\pi_j}/b_{\pi_j}} \in G \;\vee\; \frac{2a_{\pi_j}/b_{\pi_j}}{a_{\pi_i}/b_{\pi_i}} \in G \right) \right], \;\; i = 2, \ldots, m$$

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$$\frac{|a_i/b_i - 2^{(t_i-1)/n}|}{2^{(t_i-1)/n}} \leq 0.009, \;\; i = 1, \ldots, m$$

$$\pi_i \in \{1, \ldots, m\}, \;\; a_i \in \{1, \ldots, 2M\}, \;\; b_i \in \{1, \ldots, M\}, \;\; i = 1, \ldots, m$$

CP Model

$\text{alldiff}(\pi_1, \ldots, \pi_m)$ ← permutation

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CP Model

$$\text{alldiff}(\pi_1, \ldots, \pi_m)$$

$$\pi_1 = a_1 = b_1 = 1 \quad \longleftarrow \text{tonic note}$$

$$1 < \frac{a_i}{b_i} < 2, \;\; \text{coprime}(a_i, b_i), \;\; i = 1, \ldots, m$$

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$$\pi_i \in \{1, \ldots, m\}, \;\; a_i \in \{1, \ldots, 2M\}, \;\; b_i \in \{1, \ldots, M\}, \;\; i = 1, \ldots, m$$

CP Model

$$\text{alldiff}(\pi_1, \ldots, \pi_m)$$

predefined array

$$\pi_1 = a_1 = b_1 = 1$$

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CP Model

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← symmetry breaking

$$\bigvee_{j<i} \left[(\pi_i > \pi_j) \Rightarrow \left(\frac{a_{\pi_i}/b_{\pi_i}}{a_{\pi_j}/b_{\pi_j}} \in G \;\vee\; \frac{2a_{\pi_j}/b_{\pi_j}}{a_{\pi_i}/b_{\pi_i}} \in G \right) \right], \;\; i = 2, \ldots, m$$

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simple ratios

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set of generators

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$$\frac{|a_i/b_i - 2^{(t_i-1)/n}|}{2^{(t_i-1)/n}} \leq 0.009, \;\; i = 1, \ldots, m$$

tuning tolerance

$$\pi_i \in \{1, \ldots, m\}, \;\; a_i \in \{1, \ldots, 2M\}, \;\; b_i \in \{1, \ldots, M\}, \;\; i = 1, \ldots, m$$

CP Model

$$\text{alldiff}(\pi_1, \ldots, \pi_m)$$

$$\pi_1 = a_1 = b_1 = 1$$

$$1 < \frac{a_i}{b_i} < 2, \;\; \text{coprime}(a_i, b_i), \;\; i = 1, \ldots, m$$

$$\frac{a_{i-1}}{b_{i-1}} < \frac{a_i}{b_i}, \;\; i = 2, \ldots, m$$

$$\bigvee_{j<i} \left[(\pi_i > \pi_j) \Rightarrow \left(\frac{a_{\pi_i}/b_{\pi_i}}{a_{\pi_j}/b_{\pi_j}} \in G \;\vee\; \frac{2a_{\pi_j}/b_{\pi_j}}{a_{\pi_i}/b_{\pi_i}} \in G \right) \right], \;\; i = 2, \ldots, m$$

$$\bigvee_{j<i} \left[(\pi_i < \pi_j) \Rightarrow \left(\frac{a_{\pi_j}/b_{\pi_j}}{a_{\pi_i}/b_{\pi_i}} \in G \;\vee\; \frac{2a_{\pi_i}/b_{\pi_i}}{a_{\pi_j}/b_{\pi_j}} \in G \right) \right], \;\; i = 2, \ldots, m$$

$$\frac{|a_i/b_i - 2^{(t_i-1)/n}|}{2^{(t_i-1)/n}} \leq 0.009, \;\; i = 1, \ldots, m$$

$$\pi_i \in \{1, \ldots, m\}, \;\; a_i \in \{1, \ldots, 2M\}, \;\; b_i \in \{1, \ldots, M\}, \;\; i = 1, \ldots, m$$

chromatic tone corresponding to note *i*

Scales on a 12-note chromatic

- Use the generators mentioned earlier.
  - There are **multiple solutions** for each scale.
  - Select the solution with the **simplest ratios** with tonic and/or **simplest minimal generators**.
  - The 7-note scales with a **single generator 3/2** are **precisely the classical modes!**

- For each note, compute **minimal generator**
  - = simplest ratio with another note.

7-note scales on 12-note chromatic

Scale	Solns	Ratios with tonic	Minimal generators	
1. 0101111	27	$\frac{1}{1}\ \frac{16}{15}\ \frac{6}{5}\ \frac{5}{4}\ \frac{45}{32}\ \frac{8}{5}\ \frac{16}{9}$	$\frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}\ \frac{9}{8}\ \frac{3}{2}\ \frac{5}{3}$	
2. 0110111	10	$\frac{1}{1}\ \frac{18}{17}\ \frac{6}{5}\ \frac{4}{3}\ \frac{24}{17}\ \frac{8}{5}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$	Locrian mode
3. 0111011	18	$\frac{1}{1}\ \frac{16}{15}\ \frac{6}{5}\ \frac{4}{3}\ \frac{3}{2}\ \frac{8}{5}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$	Phrygian mode
4. 0111101	26	$\frac{1}{1}\ \frac{16}{15}\ \frac{6}{5}\ \frac{4}{3}\ \frac{3}{2}\ \frac{5}{3}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}$	
5. 1010111	6	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{45}{32}\ \frac{8}{5}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}$	
6. 1011011	6	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{3}{2}\ \frac{8}{5}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$	Aeolian mode (natural minor)
7. 1011101	10	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{3}{2}\ \frac{5}{3}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}$	Dorian mode
8. 1011110	27	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{3}{2}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{5}{3}$	melodic minor
9. 1101011	14	$\frac{1}{1}\ \frac{9}{8}\ \frac{5}{4}\ \frac{4}{3}\ \frac{3}{2}\ \frac{8}{5}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{9}{8}$	
10. 1101101	9	$\frac{1}{1}\ \frac{9}{8}\ \frac{5}{4}\ \frac{4}{3}\ \frac{3}{2}\ \frac{5}{3}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$	Mixolydian mode
11. 1101110	17	$\frac{1}{1}\ \frac{9}{8}\ \frac{5}{4}\ \frac{4}{3}\ \frac{3}{2}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$	Ionian mode (major)
12. 1110101	10	$\frac{1}{1}\ \frac{9}{8}\ \frac{5}{4}\ \frac{45}{32}\ \frac{3}{2}\ \frac{5}{3}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$	
13. 1110110	16	$\frac{1}{1}\ \frac{9}{8}\ \frac{5}{4}\ \frac{45}{32}\ \frac{3}{2}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$	Lydian mode
14. 1111010	34	$\frac{1}{1}\ \frac{9}{8}\ \frac{5}{4}\ \frac{45}{32}\ \frac{8}{5}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{5}{3}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}$	

7-note scales on 12-note chromatic

Scale	Solns	Ratios with tonic	Minimal generators	
1. 0101111	27	$\frac{1}{1}\ \frac{16}{15}\ \frac{6}{5}\ \frac{5}{4}\ \frac{45}{32}\ \frac{8}{5}\ \frac{16}{9}$	$\frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}\ \frac{9}{8}\ \frac{3}{2}\ \frac{5}{3}$	
2. 0110111	10	$\frac{1}{1}\ \frac{18}{17}\ \frac{6}{5}\ \frac{4}{3}\ \frac{24}{17}\ \frac{8}{5}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$	Locrian mode
3. 0111011	18	$\frac{1}{1}\ \frac{16}{15}\ \frac{6}{5}\ \frac{4}{3}\ \frac{3}{2}\ \frac{8}{5}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$	Phrygian mode
4. 0111101	26	$\frac{1}{1}\ \frac{16}{15}\ \frac{6}{5}\ \frac{4}{3}\ \frac{3}{2}\ \frac{5}{3}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}$	
5. 1010111	6	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{45}{32}\ \frac{8}{5}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}$	
6. 1011011	6	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{3}{2}\ \frac{8}{5}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$	Aeolian mode (natural minor)
7. 1011101	10	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{3}{2}\ \frac{5}{3}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}$	Dorian mode
8. 1011110	27	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{3}{2}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{5}{3}$	melodic minor
9. 1101011	14	$\frac{1}{1}\ \frac{9}{8}\ \frac{5}{4}\ \frac{4}{3}\ \frac{3}{2}\ \frac{8}{5}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{9}{8}$	
10. 1101101	9	$\frac{1}{1}\ \frac{9}{8}\ \frac{5}{4}\ \frac{4}{3}\ \frac{3}{2}\ \frac{5}{3}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$	Mixolydian mode
11. 1101110	17	$\frac{1}{1}\ \frac{9}{8}\ \frac{5}{4}\ \frac{4}{3}\ \frac{3}{2}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$	Ionian mode (major)
12. 1110101	10	$\frac{1}{1}\ \frac{9}{8}\ \frac{5}{4}\ \frac{45}{32}\ \frac{3}{2}\ \frac{5}{3}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$	
13. 1110110	16	$\frac{1}{1}\ \frac{9}{8}\ \frac{5}{4}\ \frac{45}{32}\ \frac{3}{2}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$	Lydian mode
14. 1111010	34	$\frac{1}{1}\ \frac{9}{8}\ \frac{5}{4}\ \frac{45}{32}\ \frac{8}{5}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{5}{3}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}$	

Single generator

Other scales on 12-note chromatic

Scale	Solns	Keys	Ratios with tonic	Minimal generators
1. 111111	6	2	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{9}{5}$
1. 01010101	>50	3	$\frac{1}{1}$ $\frac{16}{15}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{5}{3}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{9}{8}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$
2. 10101010	>50	3	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$
21. 100001010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
22. 100010010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
23. 100010100	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
24. 100100010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
25. 100100100	>50	4	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{9}{5}$ $\frac{3}{2}$
26. 100101000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{9}{8}$ $\frac{3}{2}$
27. 101000010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$
28. 101000100	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
29. 101001000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$
30. 101010000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$

Other scales on 12-note chromatic

Scale	Solns	Keys	Ratios with tonic	Minimal generators
1. 111111	6	2	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{9}{5}$

Whole tone scale. Minimal interest musically

21. 100001010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
22. 100010010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
23. 100010100	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
24. 100100010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
25. 100100100	>50	4	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{9}{5}$ $\frac{3}{2}$
26. 100101000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{9}{8}$ $\frac{3}{2}$
27. 101000010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$
28. 101000100	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
29. 101001000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$
30. 101010000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$

Other scales on 12-note chromatic

Scale	Solns	Keys	Ratios with tonic	Minimal generators
1. 111111	6	2	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{9}{5}$
1. 01010101	>50	3	$\frac{1}{1}$ $\frac{16}{15}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{5}{3}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{9}{8}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$
2. 10101010	>50	3	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$

8-note scales. Only 3 keys.

23. 100010100	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
24. 100100010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
25. 100100100	>50	4	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{9}{5}$ $\frac{3}{2}$
26. 100101000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{9}{8}$ $\frac{3}{2}$
27. 101000010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$
28. 101000100	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
29. 101001000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$
30. 101010000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$

Other scales on 12-note chromatic

Scale	Solns	Keys	Ratios with tonic	Minimal generators
1. 111111	6	2	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{9}{5}$

9-note scales beginning with whole tone interval

21. 100001010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
22. 100010010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
23. 100010100	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
24. 100100010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
25. 100100100	>50	4	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{9}{5}$ $\frac{3}{2}$
26. 100101000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{9}{8}$ $\frac{3}{2}$
27. 101000010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$
28. 101000100	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
29. 101001000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$
30. 101010000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$

Other scales on 12-note chromatic

Scale	Solns	Keys	Ratios with tonic	Minimal generators
1. 111111	6	2	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{9}{5}$

Most appealing scales. Simple ratios, good distribution of semitones.

22. 100010010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
23. 100010100	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
24. 100100010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
25. 100100100	>50	4	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{9}{5}$ $\frac{3}{2}$
26. 100101000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{9}{8}$ $\frac{3}{2}$
27. 101000010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$
28. 101000100	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
29. 101001000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$
30. 101010000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$

Other scales on 12-note chromatic

Scale	Solns	Keys	Ratios with tonic	Minimal generators
1. 111111	6	2	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{9}{5}$

Will illustrate this scale with a Chorale and Fugue for organ

22. 100010010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
23. 100010100	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
24. 100100010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
25. 100100100	>50	4	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{9}{5}$ $\frac{3}{2}$
26. 100101000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{9}{8}$ $\frac{3}{2}$
27. 101000010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$
28. 101000100	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
29. 101001000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$
30. 101010000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$

Other Chromatic Scales

- Which chromatics have the most simple ratios with the tonic, within tuning tolerance?

Ratio	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
3/2	·	·	·	·	·	·	●	·	·	·	·	●	·	●	·	·	●	·	●
4/3	·	·	·	·	·	·	●	·	·	·	·	●	·	●	·	·	●	·	●
5/3	·	·	●	·	·	●	●	·	·	●	●	·	·	●	●	·	●	●	●
5/4	●	·	·	●	·	·	●	·	·	●	●	·	●	●	·	●	·	·	●
7/4	·	·	·	·	●	●	·	·	·	●	●	·	·	·	●	●	·	·	·
6/5	·	·	·	·	·	●	·	·	·	●	·	·	·	●	·	·	●	●	·
7/5	·	·	·	·	·	·	·	·	·	·	·	·	·	●	·	●	·	●	·
8/5	●	·	·	●	·	·	●	·	·	●	●	·	●	●	·	●	●	·	●
9/5	·	●	·	·	·	·	·	●	●	·	·	·	·	●	●	●	·	·	·

Other Chromatic Scales

- Which chromatics have the most simple ratios with the tonic, within tuning tolerance?

Ratio	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
3/2	·	·	·	·	·	·	●	·	·	·	·	●	·	●	·	·	●	·	●
4/3	·	·	·	·	·	·	●	·	·	·	·	●	·	●	·	·	●	·	●
5/3	·	·	●	·	·	●	●	·	·	●	●	·	·	●	●	·	●	●	●
5/4	●	·	·	●	·	·	●	·	·	●	●	·	●	●	·	●	·	·	●
7/4	·	·	·	·	●	●	·	·	·	●	●	·	·	·	●	●	·	·	·
6/5	·	·	·	·	·	●	·	·	·	●	·	·	·	●	·	·	●	●	·
7/5	·	·	·	·	·	·	·	·	·	·	·	·	·	●	·	●	·	●	·
8/5	●	·	·	●	·	·	●	·	·	●	●	·	●	●	·	●	●	·	●
9/5	·	●	·	·	·	·	·	●	●	·	·	·	·	●	●	●	·	·	·

Classical 12-tone chromatic is 2nd best.

Other Chromatic Scales

- Which chromatics have the most simple ratios with the tonic, within tuning tolerance?

Ratio	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
3/2	·	·	·	·	·	·	●	·	·	·	·	●	·	●	·	·	●	·	●
4/3	·	·	·	·	·	·	●	·	·	·	·	●	·	●	·	·	●	·	●
5/3	·	·	●	·	·	●	●	·	·	●	●	·	·	●	●	·	●	●	●
5/4	●	·	·	●	·	·	●	·	·	●	●	·	●	●	·	●	·	·	●
7/4	·	·	·	·	●	●	·	·	·	●	●	·	·	·	●	●	·	·	·
6/5	·	·	·	·	·	●	·	·	·	●	·	·	·	●	·	·	●	●	·
7/5	·	·	·	·	·	·	·	·	·	·	·	·	·	●	·	●	·	●	·
8/5	●	·	·	●	·	·	●	·	·	●	●	·	●	●	·	●	●	·	●
9/5	·	●	·	·	·	·	·	●	●	·	·	·	·	●	●	●	·	·	·

Quarter-tone scale adds nothing.

Other Chromatic Scales

- Which chromatics have the most simple ratios with the tonic?

Ratio	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
3/2	·	·	·	·	·	·	●	·	·	·	·	●	·	●	·	·	●	·	●
4/3	·	·	·	·	·	·	●	·	·	·	·	●	·	●	·	·	●	·	●
5/3	·	·	●	·	·	●	●	·	·	●	●	·	·	●	●	·	●	●	●
5/4	●	·	·	●	·	·	●	·	·	●	●	·	●	●	·	●	·	·	●
7/4	·	·	·	·	●	●	·	·	·	●	●	·	·	·	●	●	·	·	·
6/5	·	·	·	·	·	●	·	·	·	●	·	·	·	●	·	·	●	●	·
7/5	·	·	·	·	·	·	·	·	·	·	·	·	·	●	·	●	·	●	·
8/5	●	·	·	●	·	·	●	·	·	●	●	·	●	●	·	●	●	·	●
9/5	·	●	·	·	·	·	·	●	●	·	·	·	·	●	●	●	·	·	·

19-tone chromatic dominates all others

Historical Sidelight

- Advantage of 19-tone chromatic was discovered during Renaissance.
  - Spanish organist and music theorist **Francisco de Salinas** (1530-1590) recommended 19-tone chromatic due to its tuning properties.
  - He used **meantone temperament** rather than equal temperament.

Scales on 19-tone chromatic

- But what are the best scales on this chromatic?
  - **10-note** scales have only **1 semitone**, not enough for musical interest.
  - **12-note** scales have **5 semitones**, but this makes scale notes very closely spaced.
  - **11-note** scales have 3 semitones, which seems a **good compromise** (1 more semitone than classical scales).

11-note scales on 19-tone chromatic

- There are 77 scales satisfying our requirements.

$$\binom{8}{3} + \binom{7}{2} = 77$$

  - Solve CP problem for all 77.
  - For each scale, determine largest set of simple ratios that occur in at least one solution.
  - 37 different sets of ratios appear in the 77 scales.

Simple ratios in 11-note scales

Ratio	**A**	B	C	D	**E**	F	G	H	I	J	K	L	M	N	O	**P**	Q	R	S	T	U	V	**W**	X	Y	Z	a	b	c	d	e	f	g	h	i	j	k
3/2	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
4/3	●	●	●	●	●	●	●	●	●	●	●	●	·	·	·	·	·	·	·	·	·	·	●	●	●	●	●	●	●	●	●	·	·	·	·	·	·
5/3	●	●	●	●	·	·	·	·	·	·	·	·	●	●	●	●	●	●	·	·	·	·	●	●	●	●	●	·	·	·	·	●	●	●	●	·	·
5/4	●	●	●	●	●	●	●	●	●	●	·	·	●	●	●	·	·	·	●	●	·	·	●	●	●	·	·	●	●	●	·	●	●	·	·	●	·
7/4	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
6/5	·	·	·	·	●	●	·	·	·	·	●	●	·	·	·	●	●	●	·	·	●	●	●	●	·	●	●	●	·	·	·	·	·	●	●	·	●
7/5	●	●	●	·	●	·	●	●	·	·	·	·	●	●	●	●	●	●	●	●	●	●	·	·	·	·	·	·	·	·	·	●	●	●	●	●	●
8/5	●	·	·	●	●	●	●	●	●	●	●	·	●	·	·	●	·	·	●	●	●	·	·	·	·	·	·	●	●	·	●	·	·	·	·	●	●
9/5	●	●	·	●	·	·	●	·	●	·	·	●	●	●	·	●	●	·	●	·	●	●	●	·	·	●	·	·	●	·	·	●	·	●	·	·	·

A - 72
B - 69,70,71
C - 68
D - 74,75
E - 7,8
F - 22,23
G - 73
H - 2
K - 12,43
L - 28
M - 65,66
N - 63,64
O - 62
P - 40,41,55,56
Q - 20,21,38,39,53
R - 19,37,52
U - 57
V - 42
W - 26,27
X - 10,11,25
Y - 5,6
Z - 15,31,32,46,47
a - 14,30,45
b - 9,24
e - 13,29,44
f - 60,61
g - 59
h - 18,35,36,50,51,54
i - 17,34,49
j - 58
k - 16,33,48

Simple ratios in 11-note scales

Ratio	**A**	B	C	D	**E**	F	G	H	I	J	K	L	M	N	O	**P**	Q	R	S	T	U	V	**W**	X	Y	Z	a	b	c	d	e	f	g	h	i	j	k
3/2	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
4/3	●	●	●	●	●	●	●	●	●	●	●	·	·	·	·	·	·	·	·	·	·	·	●	●	●	●	●	●	●	●	·	·	·	·	·	·	·
5/3	●	●	●	●	·	·	·	·	·	·	·	·	●	●	●	●	●	●	·	·	·	·	●	●	●	●	●	·	·	·	●	●	●	●	·	·	·
5/4	●	●	●	●	●	●	●	●	●	●	·	·	●	●	●	·	·	·	●	●	·	·	●	●	●	·	·	●	●	●	·	●	●	·	·	●	·
7/4	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
6/5	·	·	·	·	●	●	·	·	·	·	●	●	·	·	·	●	●	●	·	·	●	●	●	●	·	●	·	●	●	●	·	·	·	·	●	●	·	●
7/5	●	●	●	·	●	·	●	●	·	·	·	·	●	●	●	●	●	●	●	●	●	●	·	·	·	·	·	·	·	·	·	·	·	●	●	●	●	●	●
8/5	●	·	·	●	●	●	●	●	●	●	●	·	●	·	·	●	·	·	●	●	●	·	·	·	·	·	·	●	●	·	●	·	·	·	·	●	●
9/5	●	●	·	●	·	·	●	·	●	·	·	●	●	●	·	●	●	·	●	·	●	●	●	·	·	●	·	·	●	·	·	●	·	●	·	·	·

A - 72
B - 69,70,71
C - 68
D - 74,75
E - 7,8
F - 22,23
G - 73
H - 2

K - 12,43
L - 28
M - 65,66
N - 63,64
O - 62
P - 40,41,55,56
Q - 20,21,38,39,53
R - 19,37,52

U - 57
V - 42
W - 26,27
X - 10,11,25
Y - 5,6
Z - 15,31,32,46,47
a - 14,30,45
b - 9,24

e - 13,29,44
f - 60,61
g - 59
h - 18,35,36,50,51,54
i - 17,34,49
j - 58
k - 16,33,48

These 9 scales dominate all the others.

Simple ratios in 11-note scales

Ratio	**A**	B	C	D	**E**	F	G	H	I	J	K	L	M	N	O	**P**	Q	R	S	T	U	V	**W**	X	Y	Z	a	b	c	d	e	f	g	h	i	j	k
3/2	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
4/3	●	●	●	●	●	●	●	●	●	●	●	●	·	·	·	·	·	·	·	·	·	·	●	●	●	●	●	●	●	●	·	·	·	·	·	·	·
5/3	●	●	●	●	·	·	·	·	·	·	·	·	●	●	●	●	●	●	·	·	·	·	●	●	●	●	●	·	·	·	·	●	●	●	●	·	·
5/4	●	●	●	●	●	●	●	●	●	●	·	·	●	●	●	·	·	·	●	●	·	·	●	●	●	·	·	●	●	●	·	●	●	·	·	●	·
7/4	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
6/5	·	·	·	·	●	●	·	·	·	·	●	●	·	·	·	●	●	●	·	·	●	●	●	●	·	●	·	●	●	●	·	·	·	·	●	●	·	●
7/5	●	●	●	·	●	·	●	●	·	·	·	·	●	●	●	●	●	●	●	●	●	●	·	·	·	·	·	·	·	·	·	●	●	●	●	●	●
8/5	●	·	·	●	●	●	●	●	●	●	·	●	·	·	·	●	·	·	●	●	●	·	·	·	·	·	·	●	●	·	●	·	·	·	●	●	·
9/5	●	●	·	●	·	·	●	·	●	·	·	●	●	●	·	●	●	·	●	·	●	●	●	·	·	●	·	·	●	·	·	●	·	●	·	·	·

A - 72
B - 69,70,71
C - 68
D - 74,75
E - 7 8
F - 22,23
G - 73
H - 2
K - 12,43
L - 28
M - 65,66
N - 63,64
O - 62
P - 40,41,55 56
Q - 20,21,38,39,53
R - 19,37,52
U - 57
V - 42
W - 26,27
X - 10,11,25
Y - 5,6
Z - 15,31,32,46,47
a - 14,30,45
b - 9,24
e - 13,29,44
f - 60,61
g - 59
h - 18,35,36,50,51,54
i - 17,34,49
j - 58
k - 16,33,48

We will focus on 1 scale from each class.

4 attractive 9-note scales

Scale	Class	Ratios with tonic	Minimal generators
7. 01101011111	E	$\frac{1}{1}\ \frac{25}{24}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{7}{5}\ \frac{3}{2}\ \frac{8}{5}\ \frac{12}{7}\ \frac{25}{18}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{7}{4}\ \frac{3}{2}$
		$\frac{1}{1}\ \frac{36}{35}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{7}{5}\ \frac{3}{2}\ \frac{8}{5}\ \frac{12}{7}\ \frac{13}{17}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{4}{2}\ \frac{7}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{13}{7}$
27. 10101111110	W	$\frac{1}{1}\ \frac{15}{14}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{10}{7}\ \frac{54}{35}\ \frac{5}{3}\ \frac{9}{5}\ \frac{27}{14}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}$
		$\frac{1}{1}\ \frac{16}{15}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{10}{7}\ \frac{14}{9}\ \frac{5}{3}\ \frac{9}{5}\ \frac{35}{18}$	$\frac{3}{2}\ \frac{5}{4}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{7}{4}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}$
56. 11011110110	P	$\frac{1}{1}\ \frac{15}{14}\ \frac{7}{6}\ \frac{6}{5}\ \frac{9}{7}\ \frac{7}{5}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{9}{5}\ \frac{27}{14}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}$
		$\frac{1}{1}\ \frac{13}{12}\ \frac{7}{6}\ \frac{6}{5}\ \frac{9}{7}\ \frac{7}{5}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{9}{5}\ \frac{35}{18}$	$\frac{3}{2}\ \frac{13}{7}\ \frac{5}{3}\ \frac{3}{2}\ \frac{7}{5}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{5}{3}$
72. 11110110110	A	$\frac{1}{1}\ \frac{16}{15}\ \frac{7}{6}\ \frac{5}{4}\ \frac{4}{3}\ \frac{7}{5}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{9}{5}\ \frac{35}{18}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{5}{3}$
		$\frac{1}{1}\ \frac{15}{14}\ \frac{7}{6}\ \frac{5}{4}\ \frac{4}{3}\ \frac{7}{5}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{9}{5}\ \frac{27}{14}$	$\frac{3}{2}\ \frac{7}{5}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{5}{3}\ \frac{9}{5}$

Showing 2 simplest solutions for each scale.
One with simplest ratios, one with simplest generators.

Key structure of scales

Classical major scale																			
Note	1	1♯	2	2♯	3	4	4♯	5	5♯	6	6♯	7							
Interval			2^{nd}		3^{rd}	4^{th}		5^{th}		6^{th}		7^{th}							
Distance	0	5	2	3	4	1	5	1	4	3	2	5							
Scale 23 of 9 notes on 12-note chromatic																			
Note	1	1♯	2	3	4	5	5♯	6	7	7♯	8	9							
Interval			2^{nd}	$m3^{rd}$	3^{rd}	4^{th}		5^{th}	$m6^{th}$		$m7^{th}$	7^{th}							
Distance	0	3	3	2	2	2	3	2	2	2	3	3							
Scale 7 of 11 notes on 19-note chromatic																			
Note	1	2	2♯	3	3♯	4	5	5♯	6	7	7♯	8	8♯	9	9♯	10	10♯	11	11♯
Interval				2^{nd}		$m3^{rd}$	3^{rd}		4^{th}			5^{th}		$m6^{th}$					
Distance	0	8	3	5	5	4	5	5	4	5	5	4	5	5	4	5	5	3	8
Scale 27 of 11 notes on 19-note chromatic																			
Note	1	1♯	2	3	3♯	4	5	5♯	6	6♯	7	7♯	8	8♯	9	9♯	10	10♯	11
Interval				2^{nd}		$m3^{rd}$	3^{rd}		4^{th}						6^{th}				
Distance	0	8	3	5	4	6	3	6	4	5	5	4	6	3	6	4	5	3	8
Scale 56 of 11 notes on 19-note chromatic																			
Note	1	1♯	2	2♯	3	4	4♯	5	5♯	6	6♯	7	7♯	8	9	9♯	10	10♯	11
Interval						$m3^{rd}$						5^{th}		$m6^{th}$	6^{th}				
Distance	0	8	3	5	6	2	7	3	6	4	4	6	3	7	2	6	5	3	8
Scale 72 of 11 notes on 19-note chromatic																			
Note	1	1♯	2	2♯	3	3♯	4	4♯	5	6	6♯	7	7♯	8	9	9♯	10	10♯	11
Interval							3^{rd}		4^{th}			5^{th}		$m6^{th}$	6^{th}				
Distance	0	8	3	5	6	2	7	3	6	4	4	6	3	7	2	6	5	3	8

Key structure of scales

Classical major scale

Note	1	1♯	2	2♯	3	4	4♯	5	5♯	6	6♯	7
Interval			2^{nd}		3^{rd}	4^{th}		5^{th}		6^{th}		7^{th}
Distance	0	5	2	3	4	1	5	1	4	3	2	5

No key with distance 1.

Scale 23 of 9 notes on 12-note chromatic

Note	1	1♯	2	3	4	5	5♯	6	7	7♯	8	9
Interval			2^{nd}	$m3^{rd}$	3^{rd}	4^{th}		5^{th}	$m6^{th}$		$m7^{th}$	7^{th}
Distance	0	3	3	2	2	2	3	2	2	2	3	3

Good or bad?

Scale 7 of 11 notes on 19-note chromatic

Note	1	2	2♯	3	3♯	4	5	5♯	6	7	7♯	8	8♯	9	9♯	10	10♯	11	11♯
Interval				2^{nd}		$m3^{rd}$	3^{rd}		4^{th}			5^{th}		$m6^{th}$					
Distance	0	8	3	5	5	4	5	5	4	5	5	4	5	5	4	5	5	3	8

Scale 27 of 11 notes on 19-note chromatic

Note	1	1♯	2	3	3♯	4	5	5♯	6	6♯	7	7♯	8	8♯	9	9♯	10	10♯	11
Interval				2^{nd}		$m3^{rd}$	3^{rd}		4^{th}						6^{th}				
Distance	0	8	3	5	4	6	3	6	4	5	5	4	6	3	6	4	5	3	8

Scale 56 of 11 notes on 19-note chromatic

Note	1	1♯	2	2♯	3	4	4♯	5	5♯	6	6♯	7	7♯	8	9	9♯	10	10♯	11
Interval						$m3^{rd}$						5^{th}		$m6^{th}$	6^{th}				
Distance	0	8	3	5	6	2	7	3	6	4	4	6	3	7	2	6	5	3	8

Scale 72 of 11 notes on 19-note chromatic

Note	1	1♯	2	2♯	3	3♯	4	4♯	5	6	6♯	7	7♯	8	9	9♯	10	10♯	11
Interval							3^{rd}		4^{th}			5^{th}		$m6^{th}$	6^{th}				
Distance	0	8	3	5	6	2	7	3	6	4	4	6	3	7	2	6	5	3	8

4 attractive 9-note scales

Scale	Class	Ratios with tonic	Minimal generators
7. 01101011111	E	$\frac{1}{1}\ \frac{25}{24}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{7}{5}\ \frac{3}{2}\ \frac{8}{5}\ \frac{12}{7}\ \frac{25}{18}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{7}{4}\ \frac{3}{2}$
		$\frac{1}{1}\ \frac{36}{35}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{7}{5}\ \frac{3}{2}\ \frac{8}{5}\ \frac{12}{7}\ \frac{13}{17}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{4}{2}\ \frac{7}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{13}{7}$
27. 10101111110	W	$\frac{1}{1}\ \frac{15}{14}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{10}{7}\ \frac{54}{35}\ \frac{5}{3}\ \frac{9}{5}\ \frac{27}{14}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}$
		$\frac{1}{1}\ \frac{16}{15}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{10}{7}\ \frac{14}{9}\ \frac{5}{3}\ \frac{9}{5}\ \frac{35}{18}$	$\frac{3}{2}\ \frac{5}{4}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{7}{4}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}$
56. 11011110110	P	$\frac{1}{1}\ \frac{15}{14}\ \frac{7}{6}\ \frac{6}{5}\ \frac{9}{7}\ \frac{7}{5}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{9}{5}\ \frac{27}{14}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}$
		$\frac{1}{1}\ \frac{13}{12}\ \frac{7}{6}\ \frac{6}{5}\ \frac{9}{7}\ \frac{7}{5}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{9}{5}\ \frac{35}{18}$	$\frac{3}{2}\ \frac{13}{7}\ \frac{5}{3}\ \frac{3}{2}\ \frac{7}{5}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{5}{3}$
72. 11110110110	A	$\frac{1}{1}\ \frac{16}{15}\ \frac{7}{6}\ \frac{5}{4}\ \frac{4}{3}\ \frac{7}{5}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{9}{5}\ \frac{35}{18}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{5}{3}$
		$\frac{1}{1}\ \frac{15}{14}\ \frac{7}{6}\ \frac{5}{4}\ \frac{4}{3}\ \frac{7}{5}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{9}{5}\ \frac{27}{14}$	$\frac{3}{2}\ \frac{7}{5}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{5}{3}\ \frac{9}{5}$

Focus further on scale 72, which has largest number of simple ratios.

Harmonic Comparison

- Classic major scale
  - Major triad C:E:G = 4:5:6, major 7 chord C:E:G:B = 8:10:12:15
  - Minor triad A:C:E =10:12:15, minor 7 chord A:C:E:G = 10:12:15:18
  - Dominant 7 chord G:B:D:F = 36:45:54:64
  - Tensions (from jazz) C E G B D F# A

- Scale 72
  - Major triad 1-4-7 = 4:5:6
  - Minor triad 5-8-12 = 10:12:15, minor 7 chord 9-12-15-18 = 10:12:15:18
  - New chord 9-12-14-18 = 5:6:7:9
  - New chord 1-3-5-9 = 6:7:8:10
  - New chord 3-5-9-12 = 7:8:10:12
  - New chord 5-9-12-15 = 4:5:6:7
  - Tensions 1-4-7-10-13-15b-16-19-22

Demonstration: 11-note scale

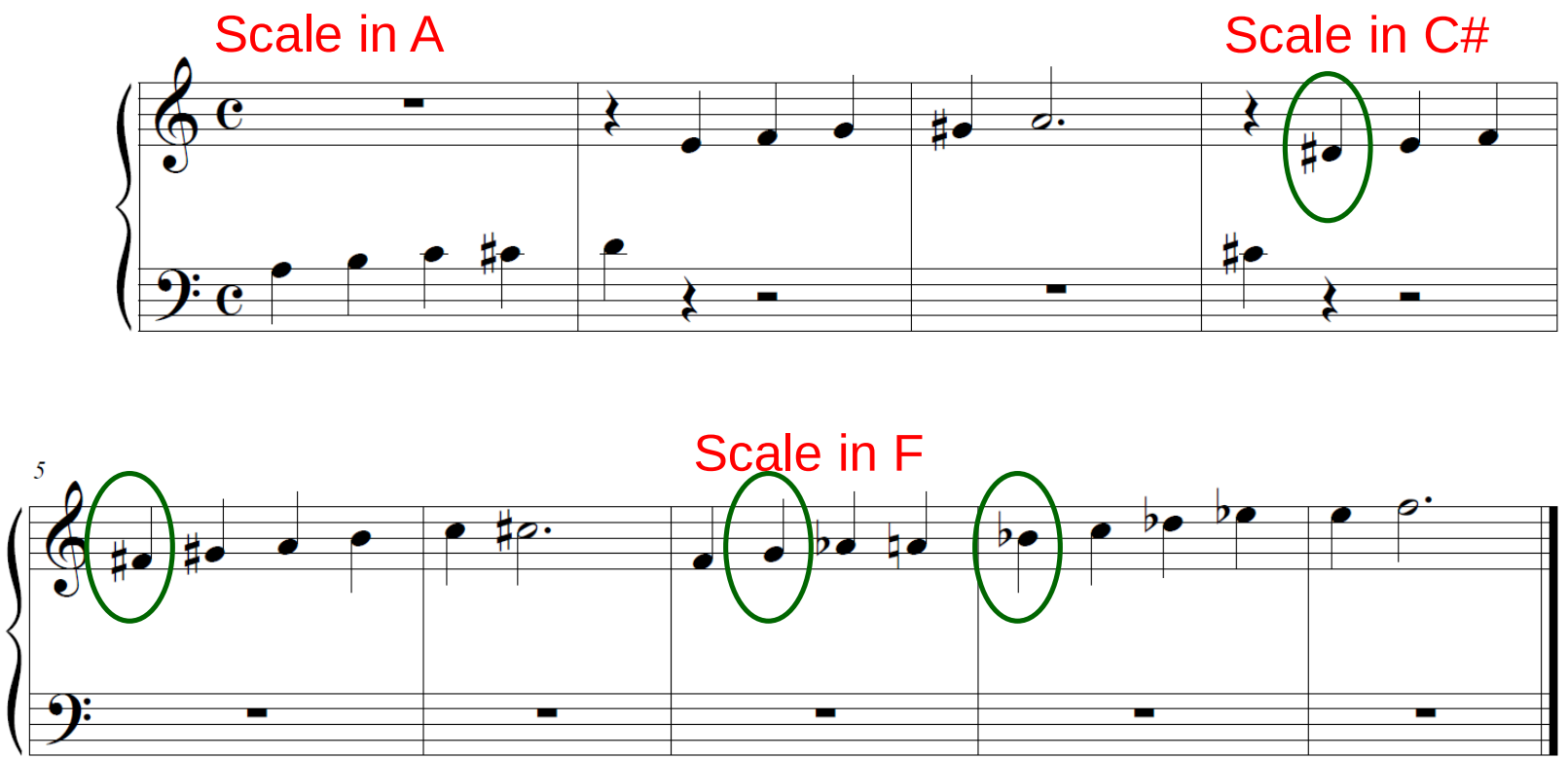
- Software
  - Hex MIDI sequencer for scales satisfying Myhill's property
  - We trick it into generating a 19-tone chromatic
  - Viking synthesizer for use with Hex
  - LoopMIDI virtual MIDI cable

Demonstration: 9-note scale

- Chorale and Fugue for Organ
- Chorale
  - In A, cycles through 2 most closely related keys: A, C#, F, A.
  - Modulate to C# at bar 27
  - Final section starts at bar 72 (5:56)
- Fugue
  - Double fugue
  - First subject enters at pitches A, C#, F
  - Second subject enters at bar 96.
  - Final episode at bar 164 (13:36)
  - Recapitulation at bar 170

Demonstration: 9-note scale

Key of A and 2 most closely related keys.



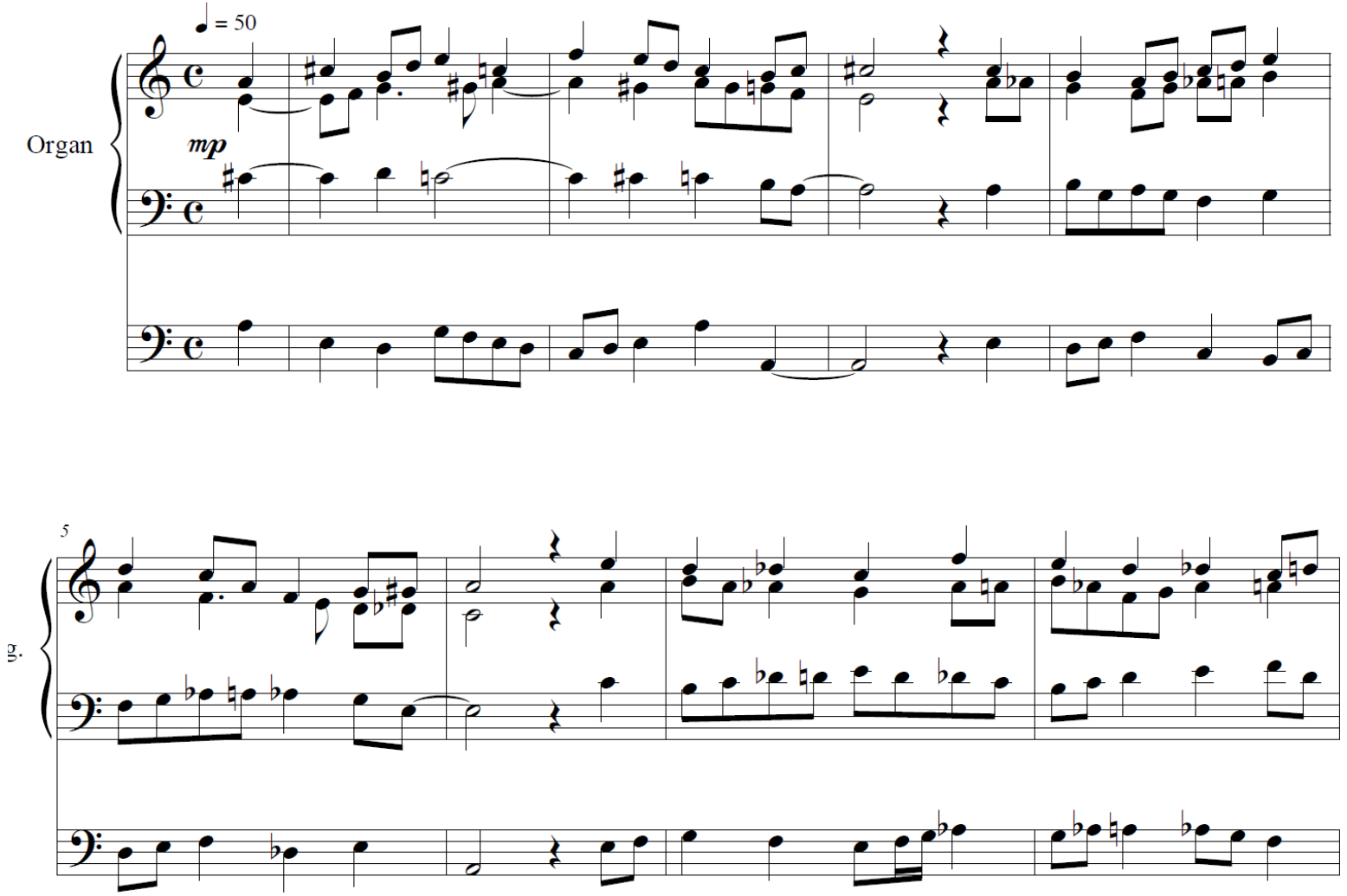
New notes are circled

Chorale and Fugue

On a 9-note Scale

J. N. Hooker
Revised 2013

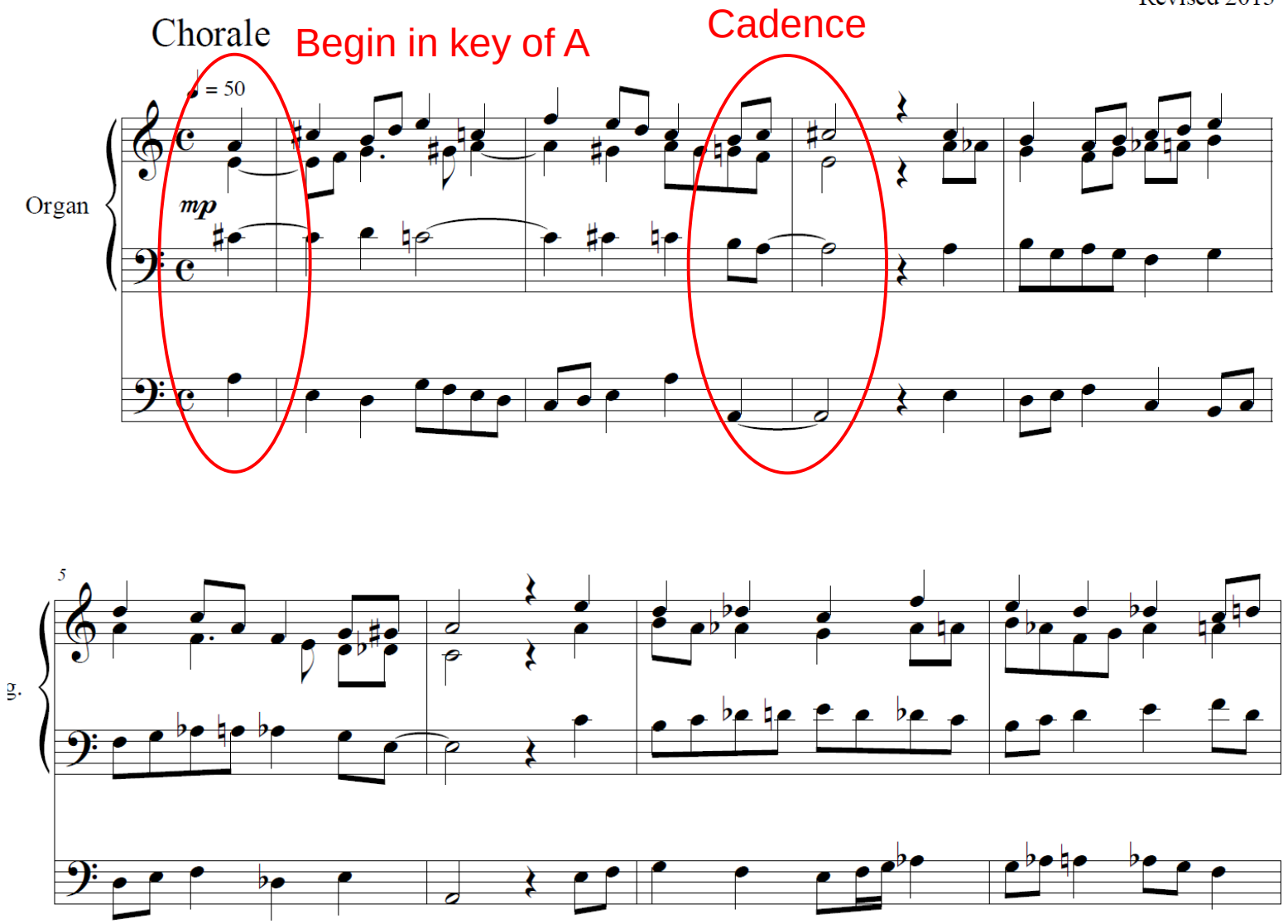
Chorale



Chorale and Fugue

On a 9-note Scale

J. N. Hooker
Revised 2013

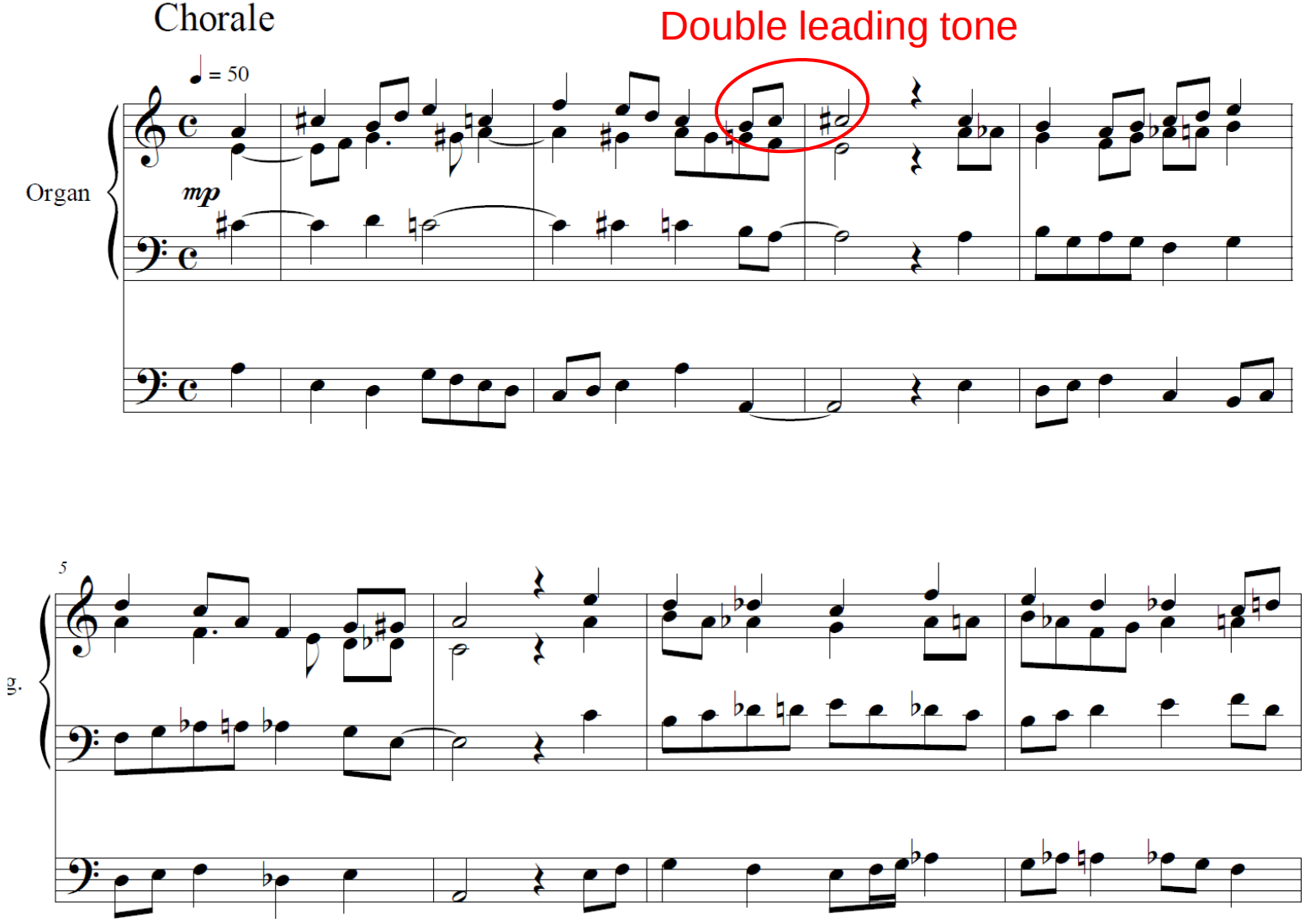


Chorale and Fugue

On a 9-note Scale

J. N. Hooker
Revised 2013

Chorale



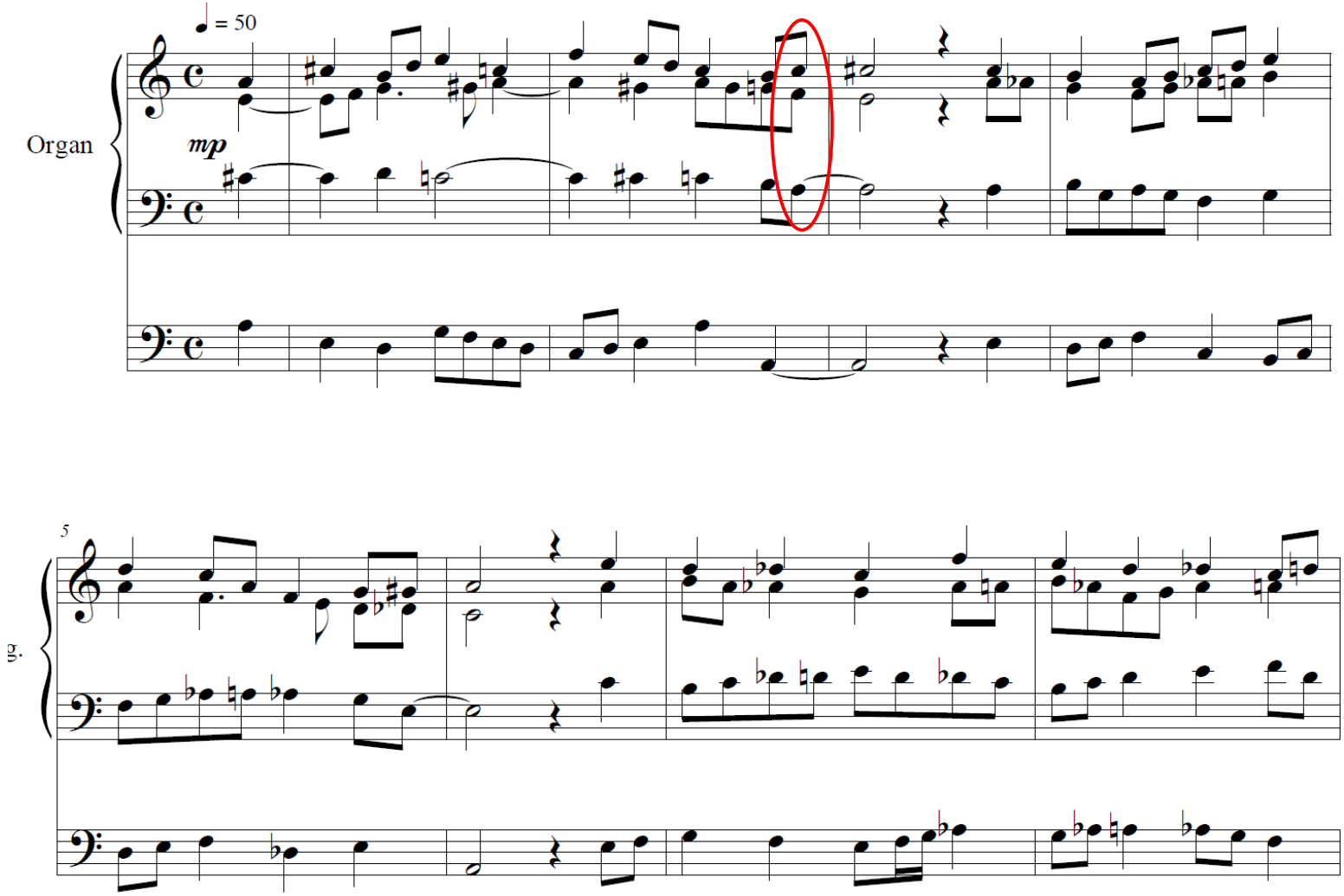
Chorale and Fugue

On a 9-note Scale

J. N. Hooker
Revised 2013

Resolve from lowered submediant (F)

Chorale

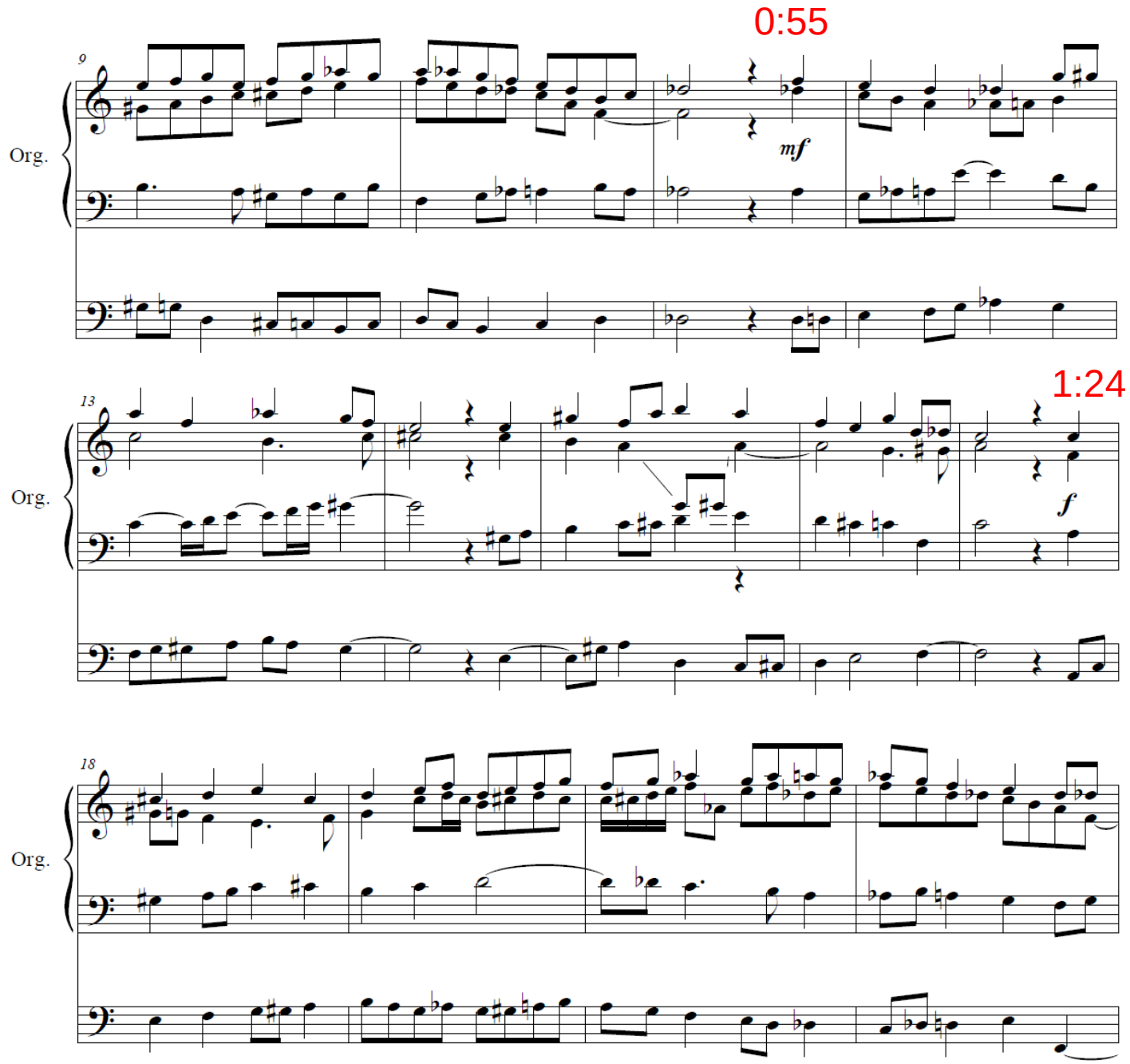


Chorale and Fugue

On a 9-note Scale

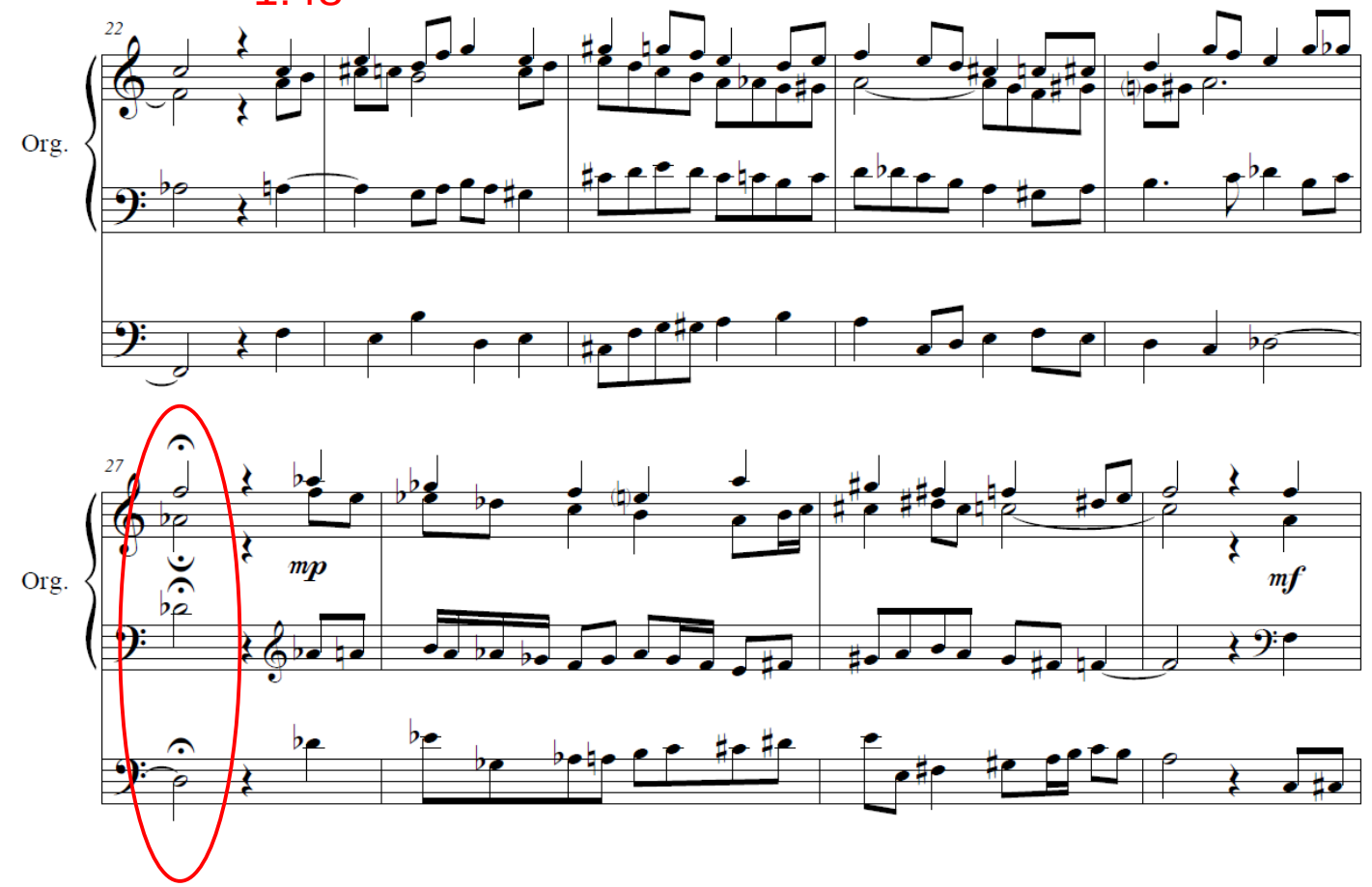
J. N. Hooker
Revised 2013

Chorale


0:55
9
Org.
mf
1:24
13
Org.
f
18
Org.

Where does modulation
to Db actually occur?

1:48

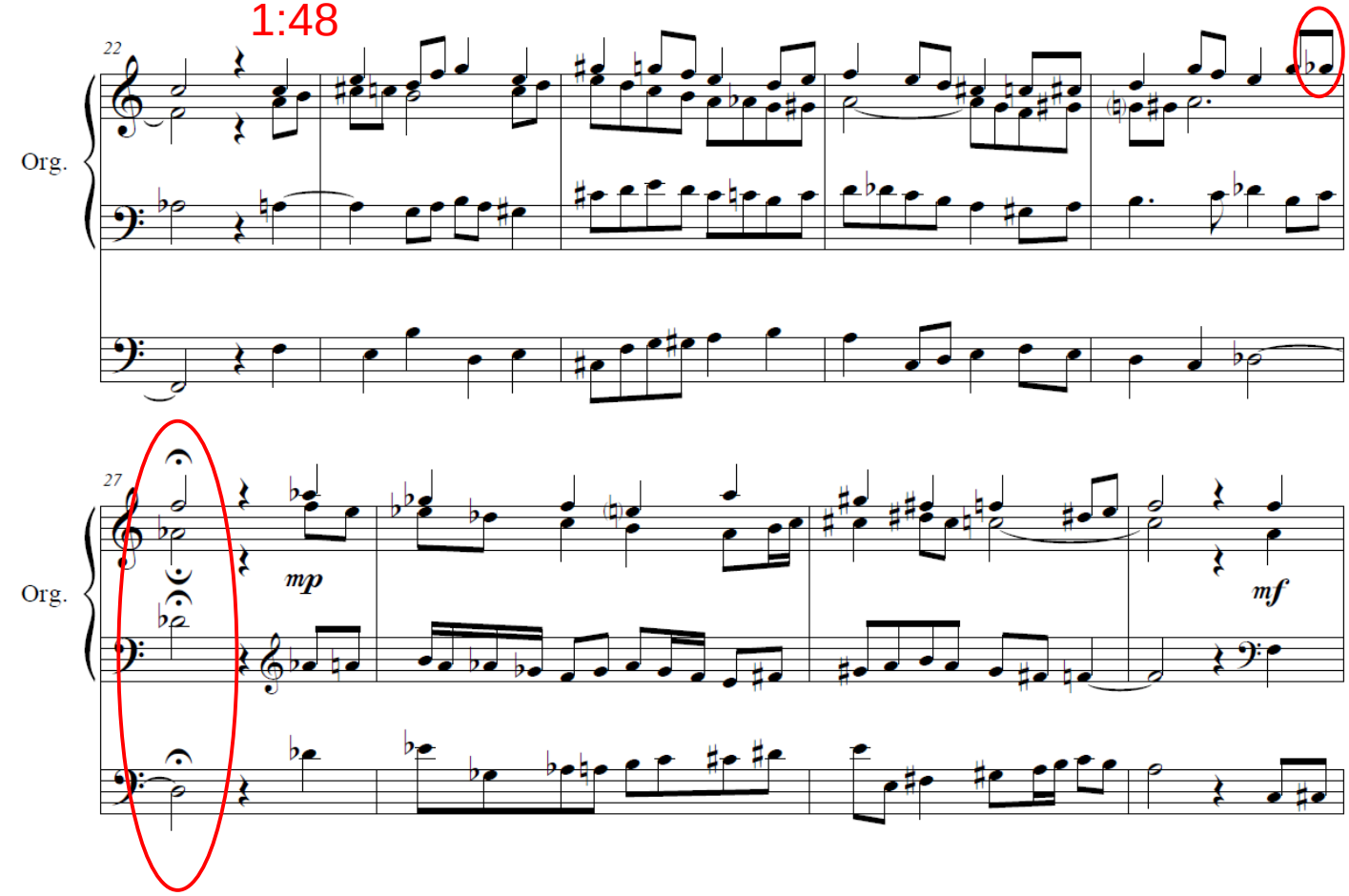


New key (Db = C#)

Where does modulation
to Db actually occur?

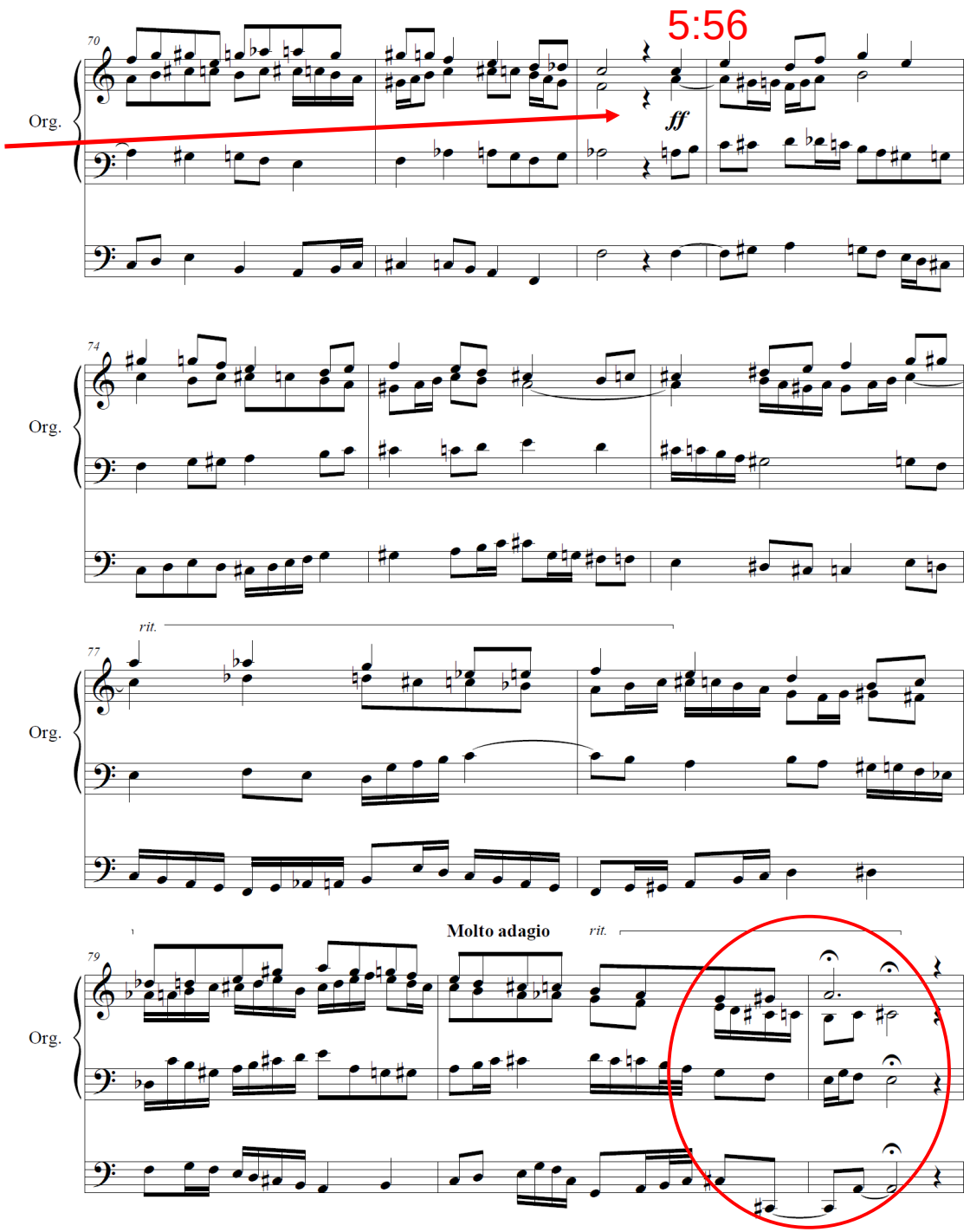
It occurs here

1:48



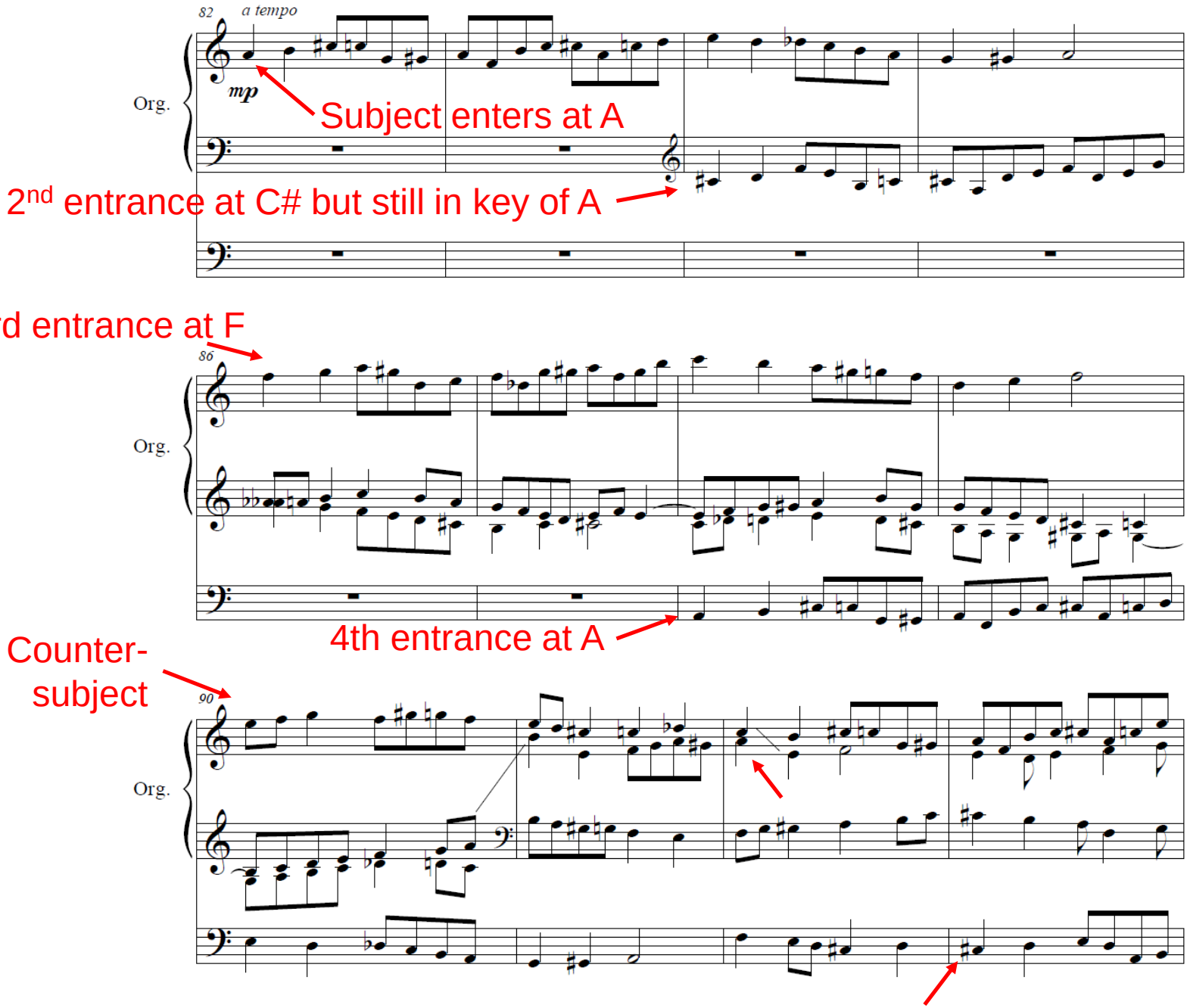
New key (Db = C#)

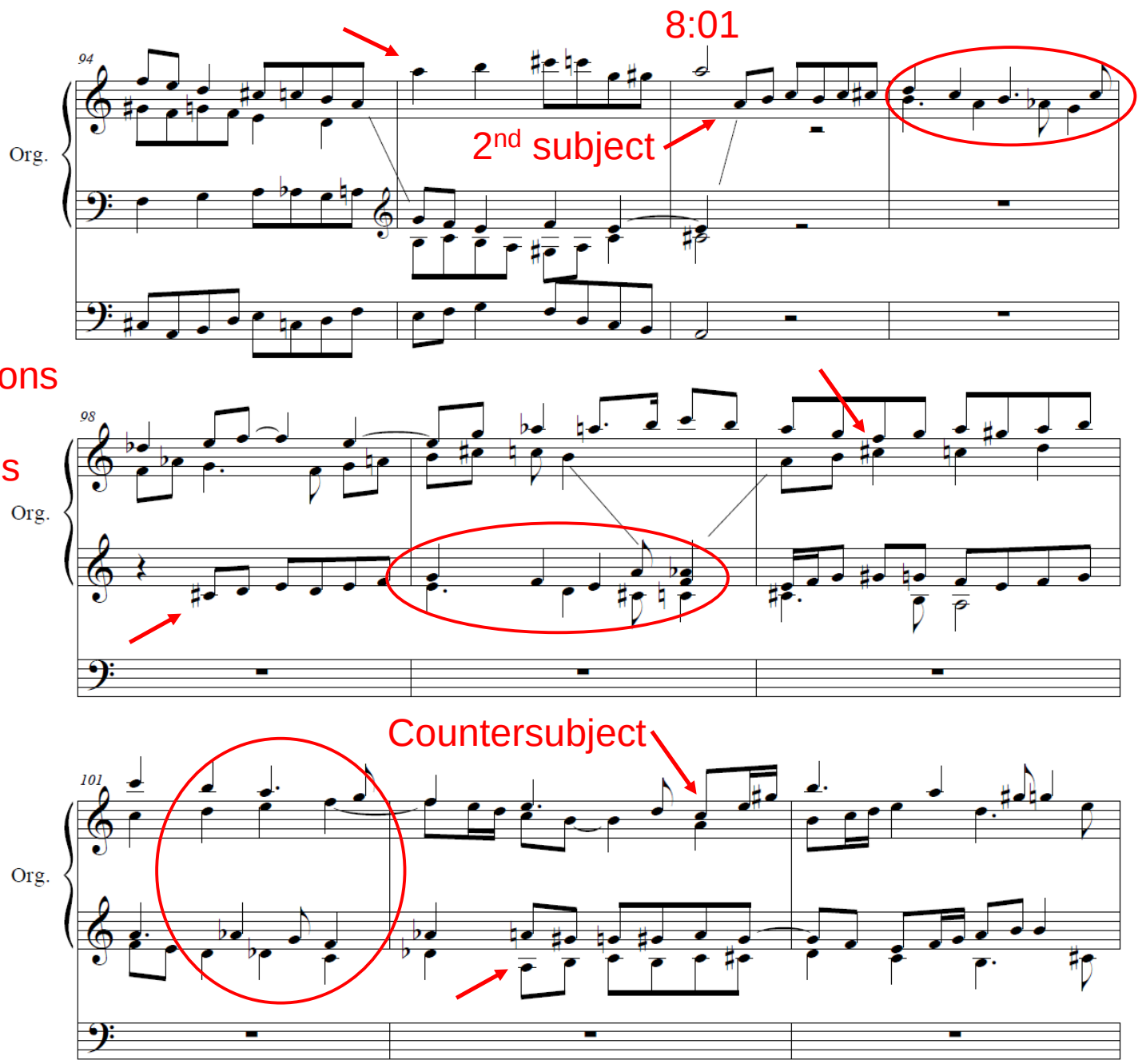
Skip to final section



Final cadence from lowered submediant (F), double leading tone, pivot on tonic

6:53 Fugue


82
a tempo
mp
Org.
Subject enters at A
2nd entrance at C# but still in key of A
3rd entrance at F
86
Org.
4th entrance at A
Counter-subject
90
Org.

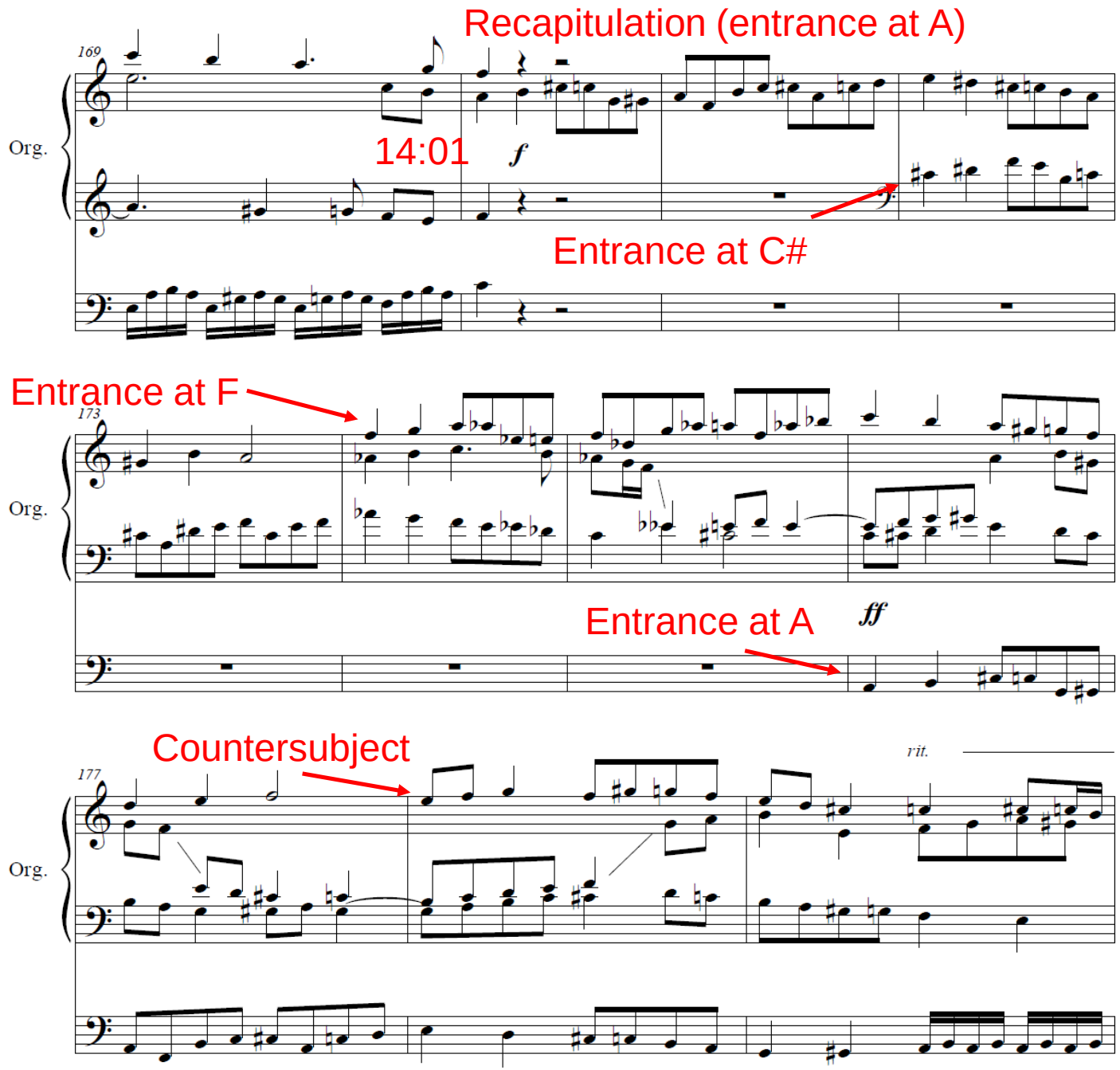


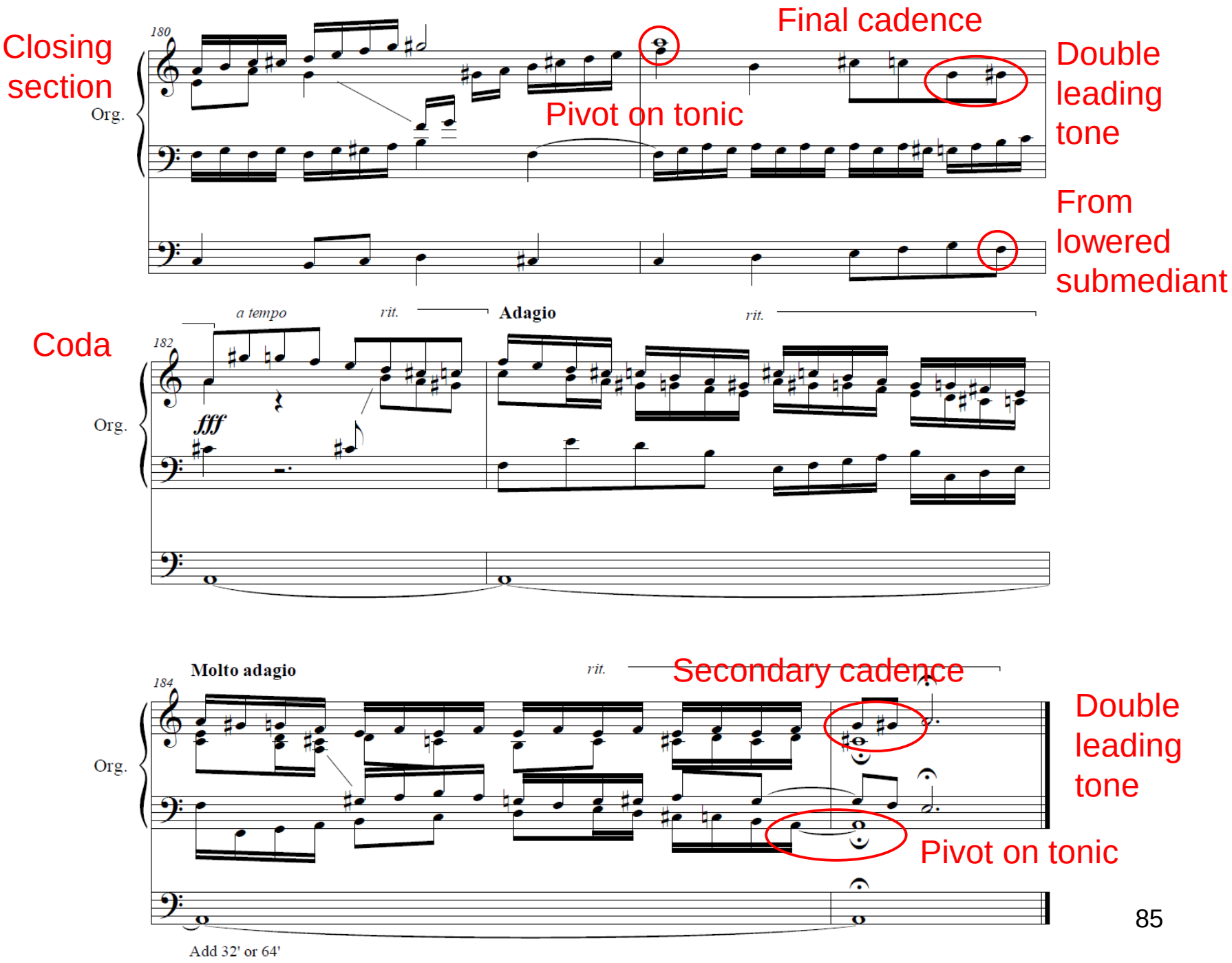
Multiple suspensions on semitones


104
Org.
107
Org.
110
Org.
Countersubject

13:32

Skip to final episode


Recapitulation (entrance at A)
160
Org.
14:01
f
Entrance at C#
Entrance at F
173
Org.
Entrance at A
ff
Countersubject
177
Org.
rit.


Closing section
Final cadence
Pivot on tonic
Double leading tone
From lowered submediant
Coda
Secondary cadence
Double leading tone
Pivot on tonic
180
182
184
Org.
a tempo
rit.
Adagio
fff
Molto adagio
Add 32' or 64'

That’s it.