

# Finding Alternative Musical Scales

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# Advantages of Classical Scales

- Pitch frequencies have **simple ratios**.
  - Rich and intelligible harmonies
- **Multiple keys** based on underlying chromatic scale with **tempered tuning**.
  - Can play all keys on instrument with fixed tuning.
  - Complex musical structure.

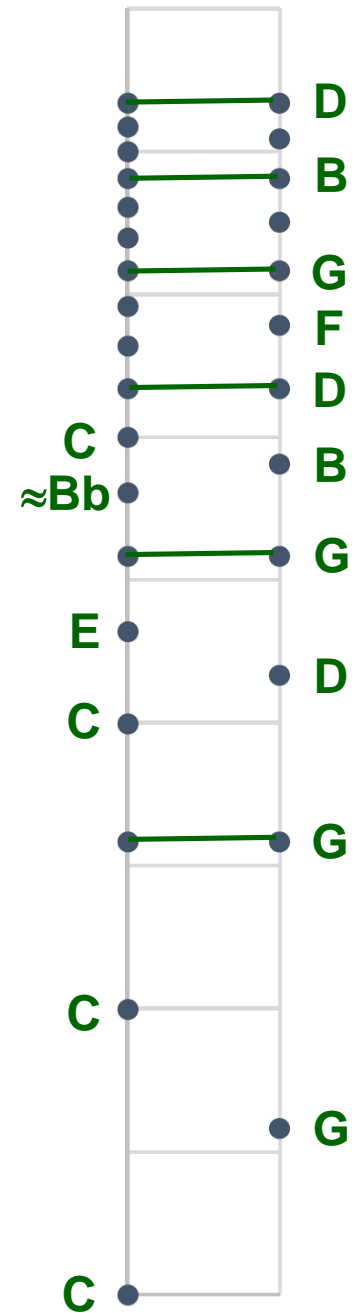
# Advantages of Classical Scales

- Pitch frequencies have **simple ratios**.
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- **Multiple keys** based on underlying chromatic scale with **tempered tuning**.
  - Can play all keys on instrument with fixed tuning.
  - Complex musical structure.
- Can we find **new scales** with these same properties?
  - CP is well suited to solve the problem.

# Simple Ratios

- Acoustic instruments produce multiple **harmonic partials**.
  - Frequency of partial = integral multiple of frequency of fundamental.
  - Coincidence of partials makes chords with simple ratios easy to recognize.

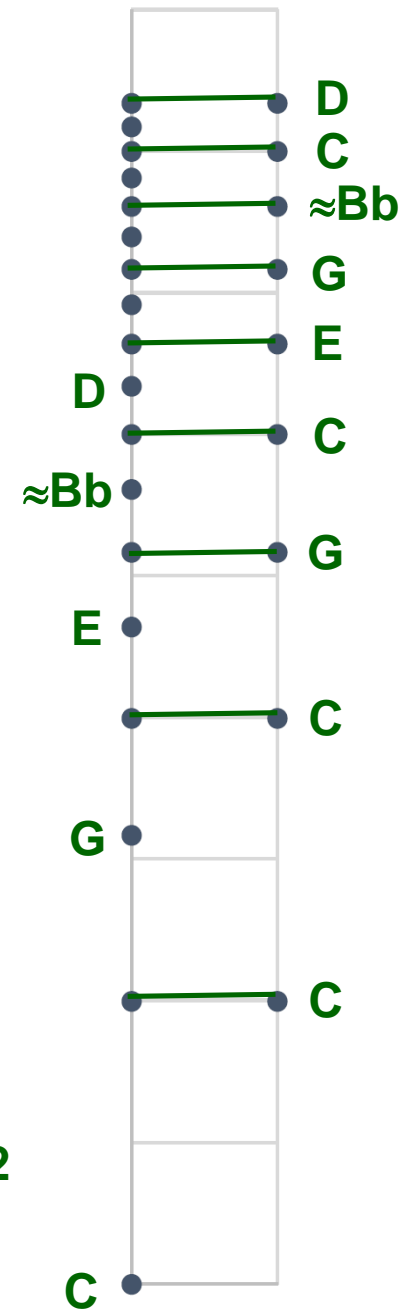
Perfect fifth  
 $C:G = 2:3$



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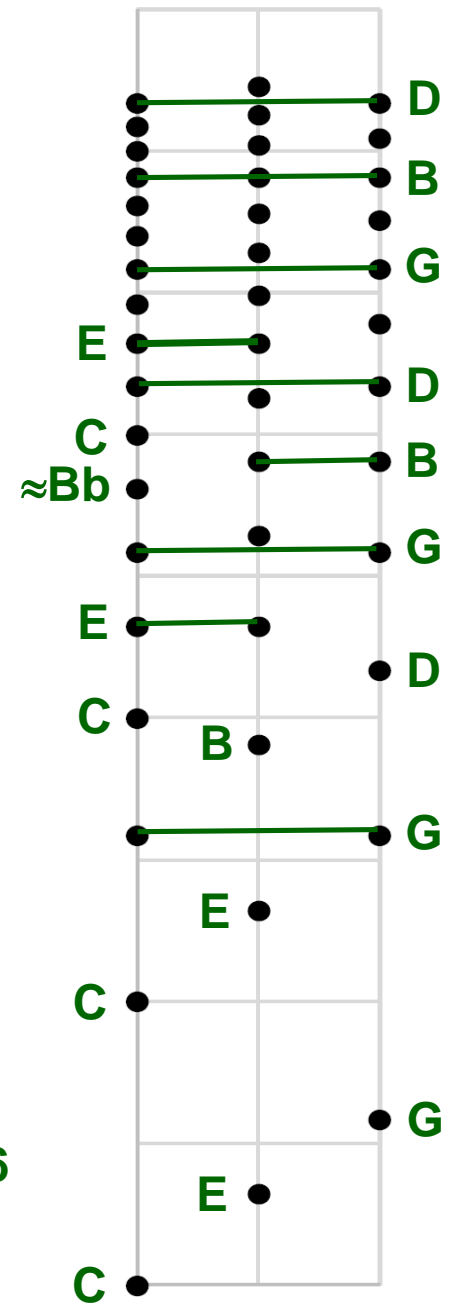
Octave  
 $C:C = 1:2$



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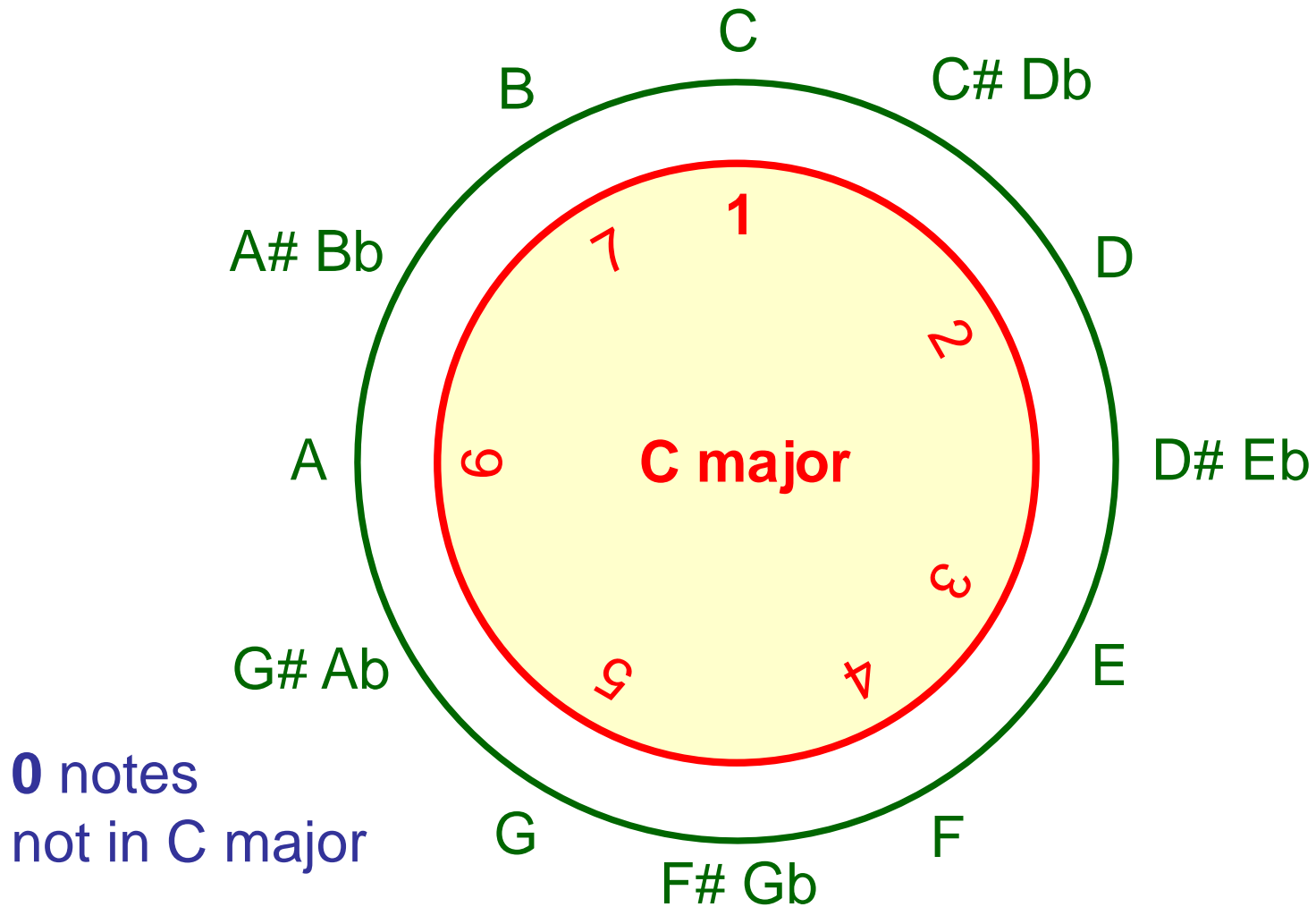
Major triad  
C:E:G = 4:5:6



# Multiple Keys

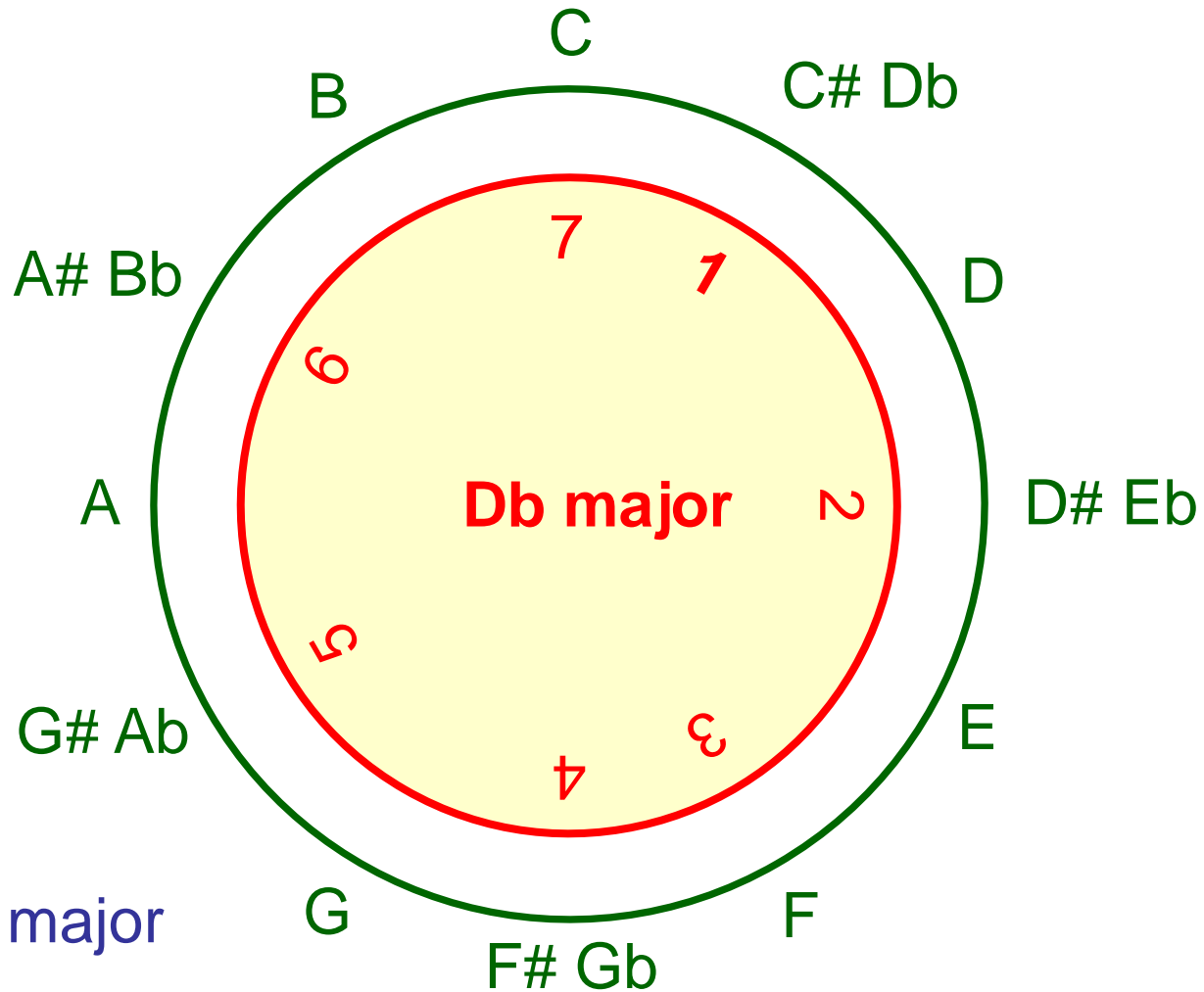
- A classical scale can start from any pitch in a **chromatic** scale with 12 **semitone** intervals.
  - Resulting in 12 **keys**.
  - An instrument with 12 pitches (modulo octaves) can play 12 different keys
  - Can move to a different key by changing only a few notes of the scale.

# Multiple Keys



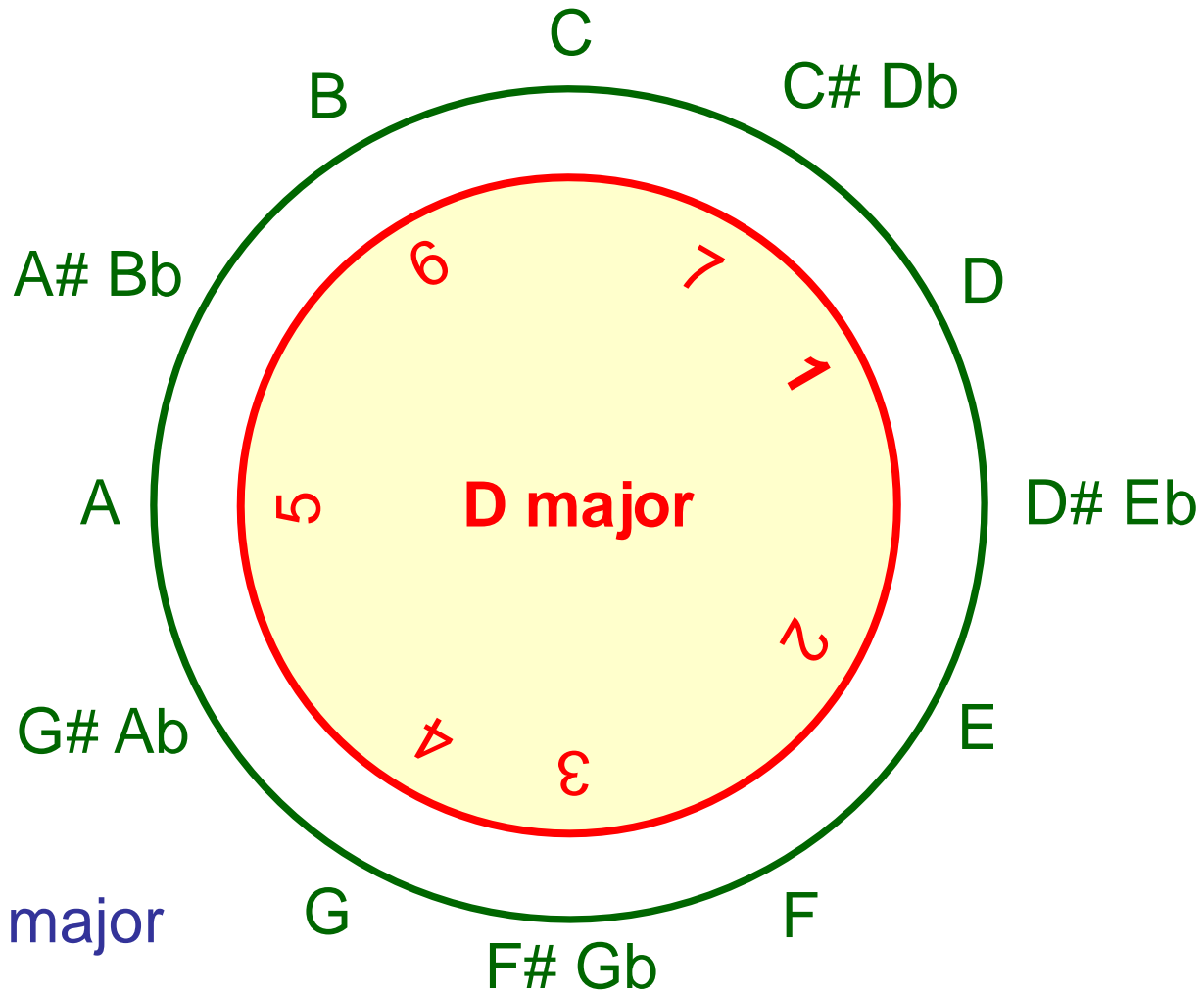


# Multiple Keys



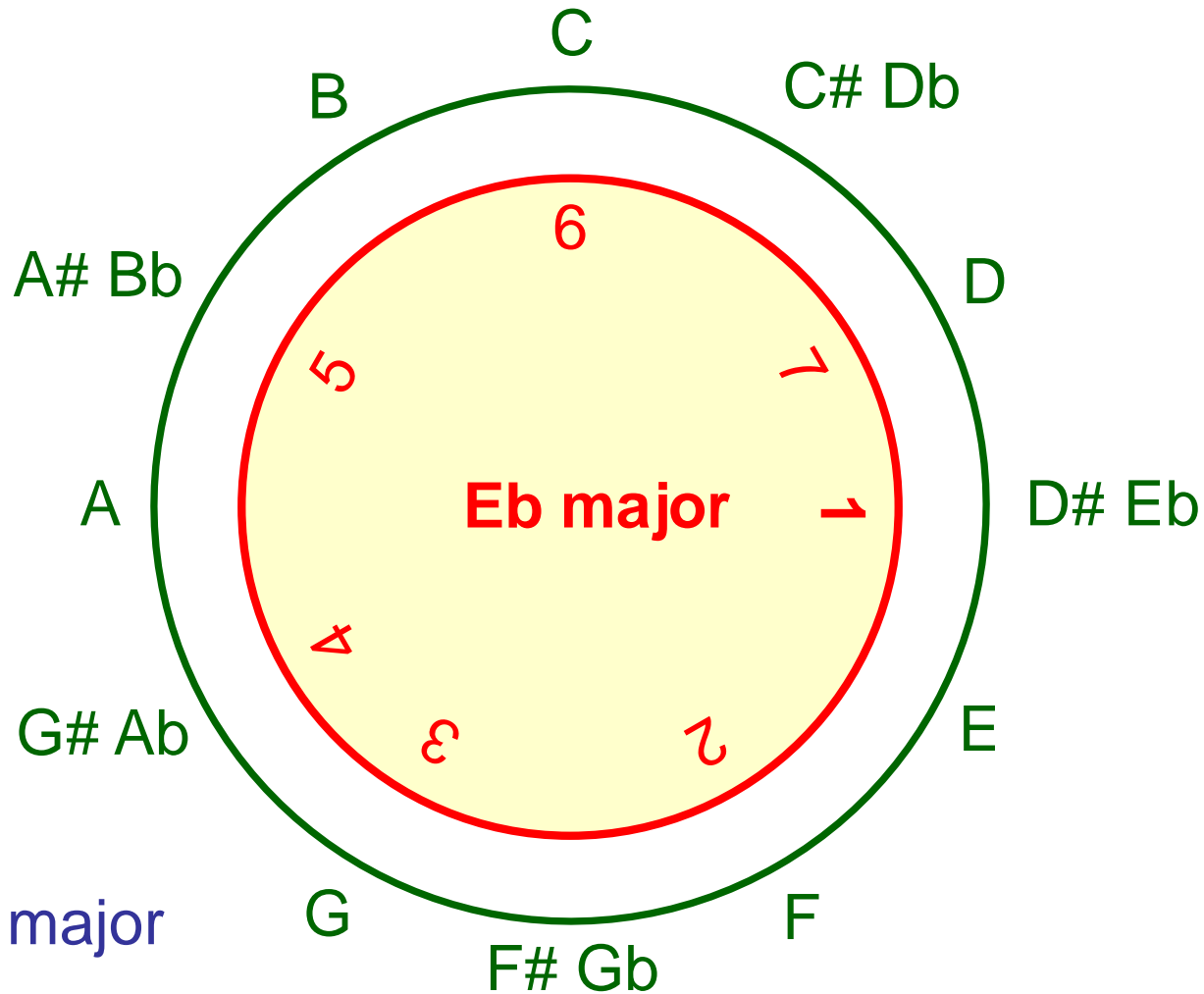
**5** notes  
not in C major

# Multiple Keys



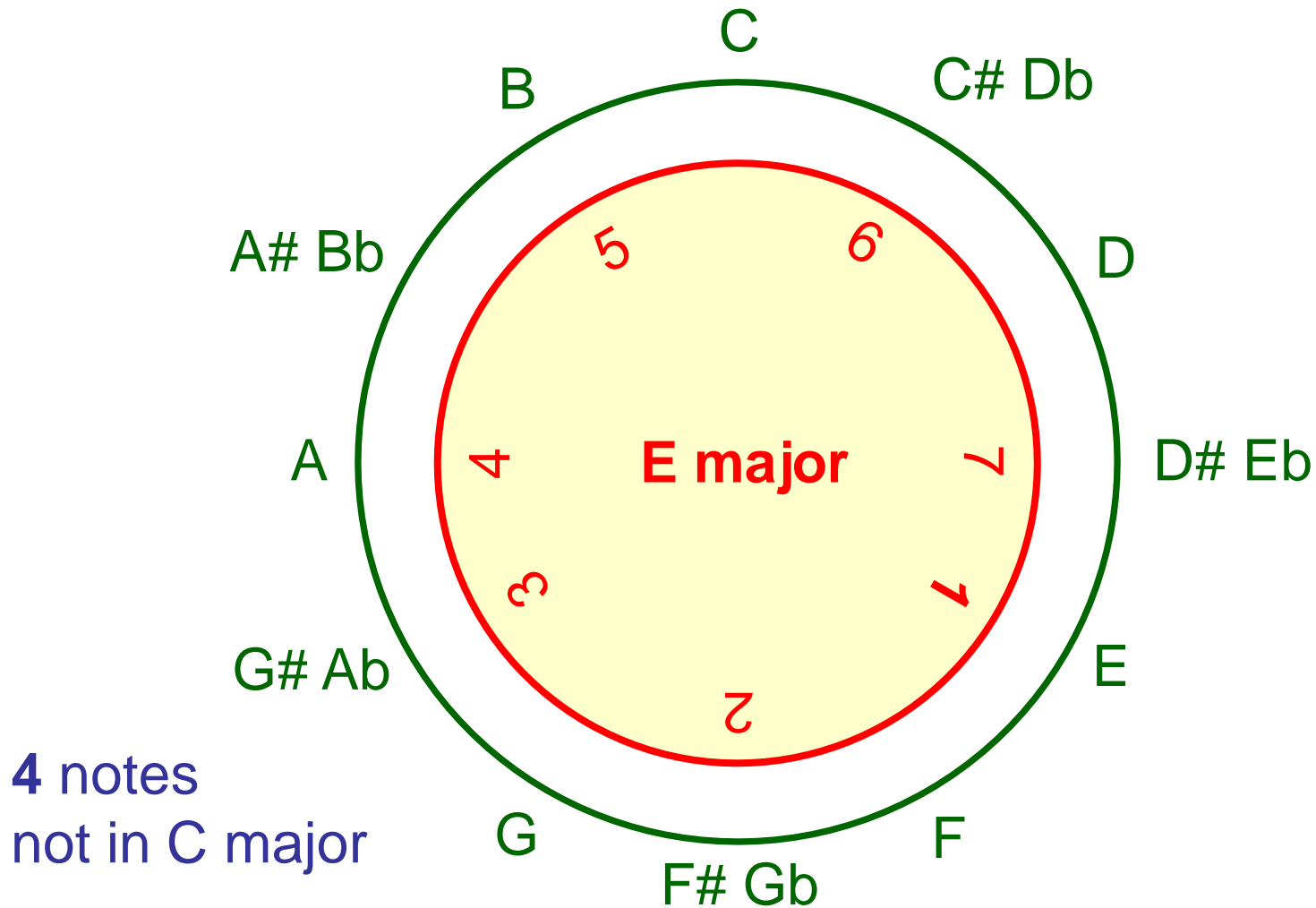
**2 notes**  
not in C major

# Multiple Keys

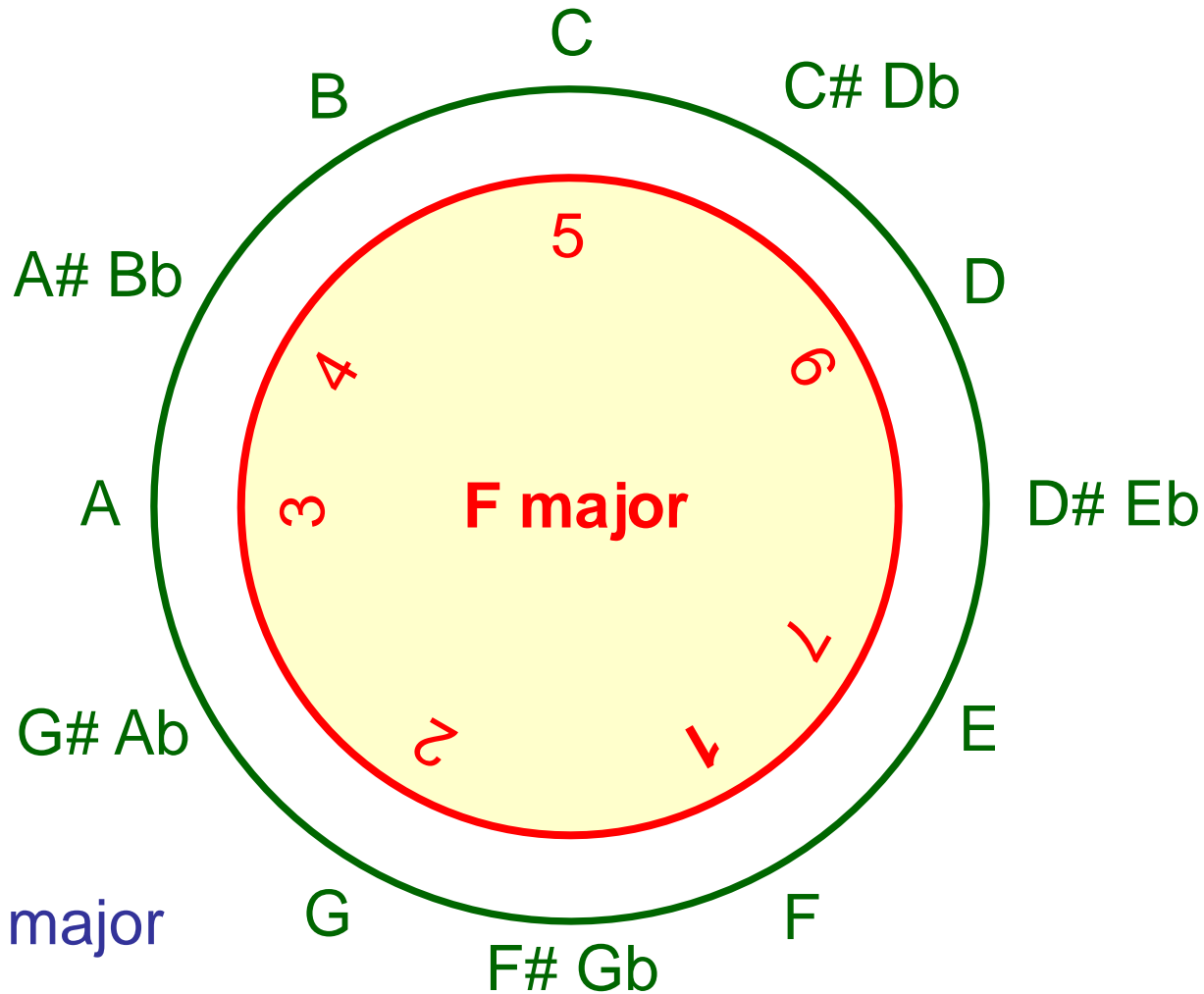


**3** notes  
not in C major

# Multiple Keys

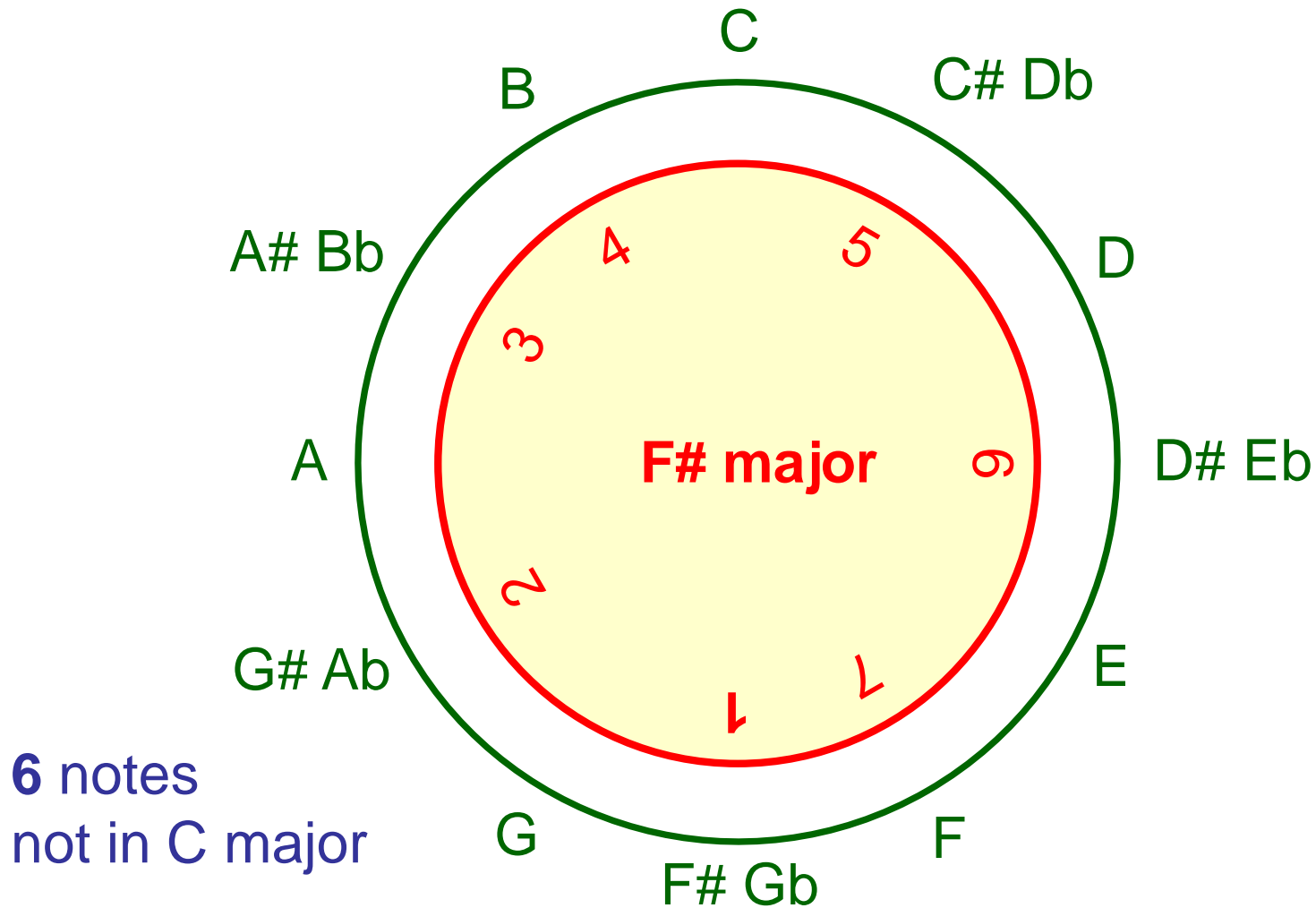


# Multiple Keys

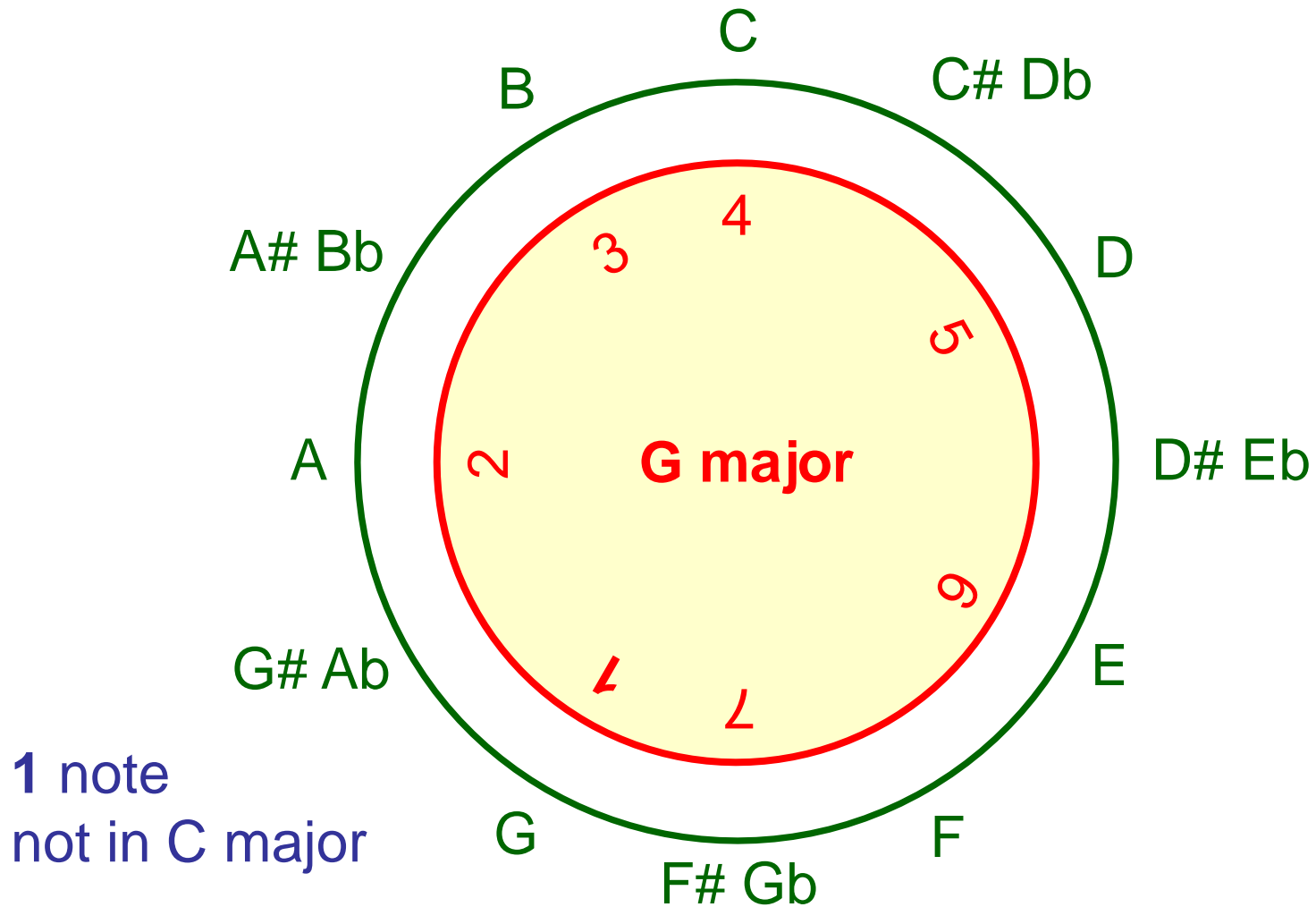


1 note  
not in C major

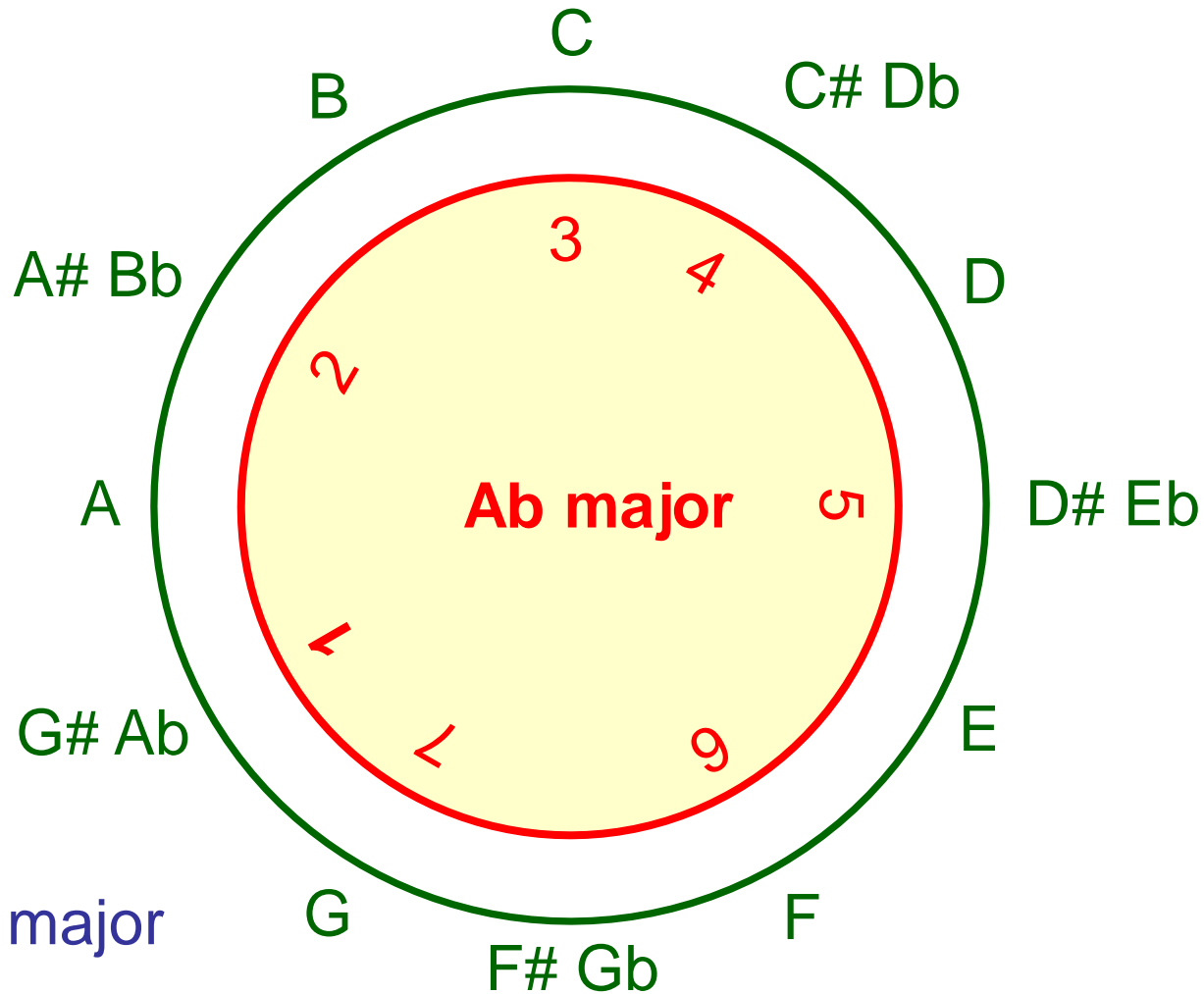
# Multiple Keys



# Multiple Keys



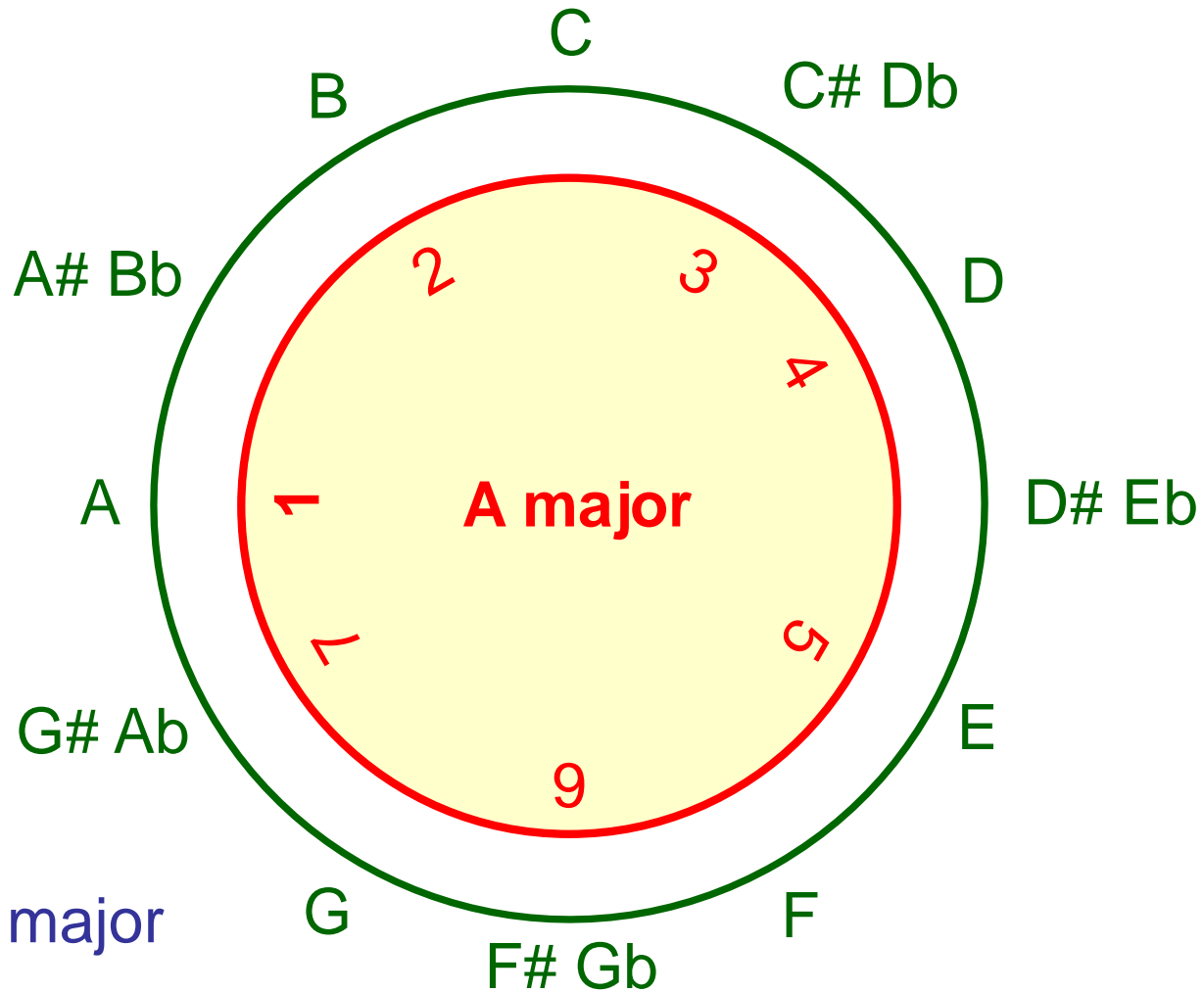
# Multiple Keys



4 notes  
not in C major

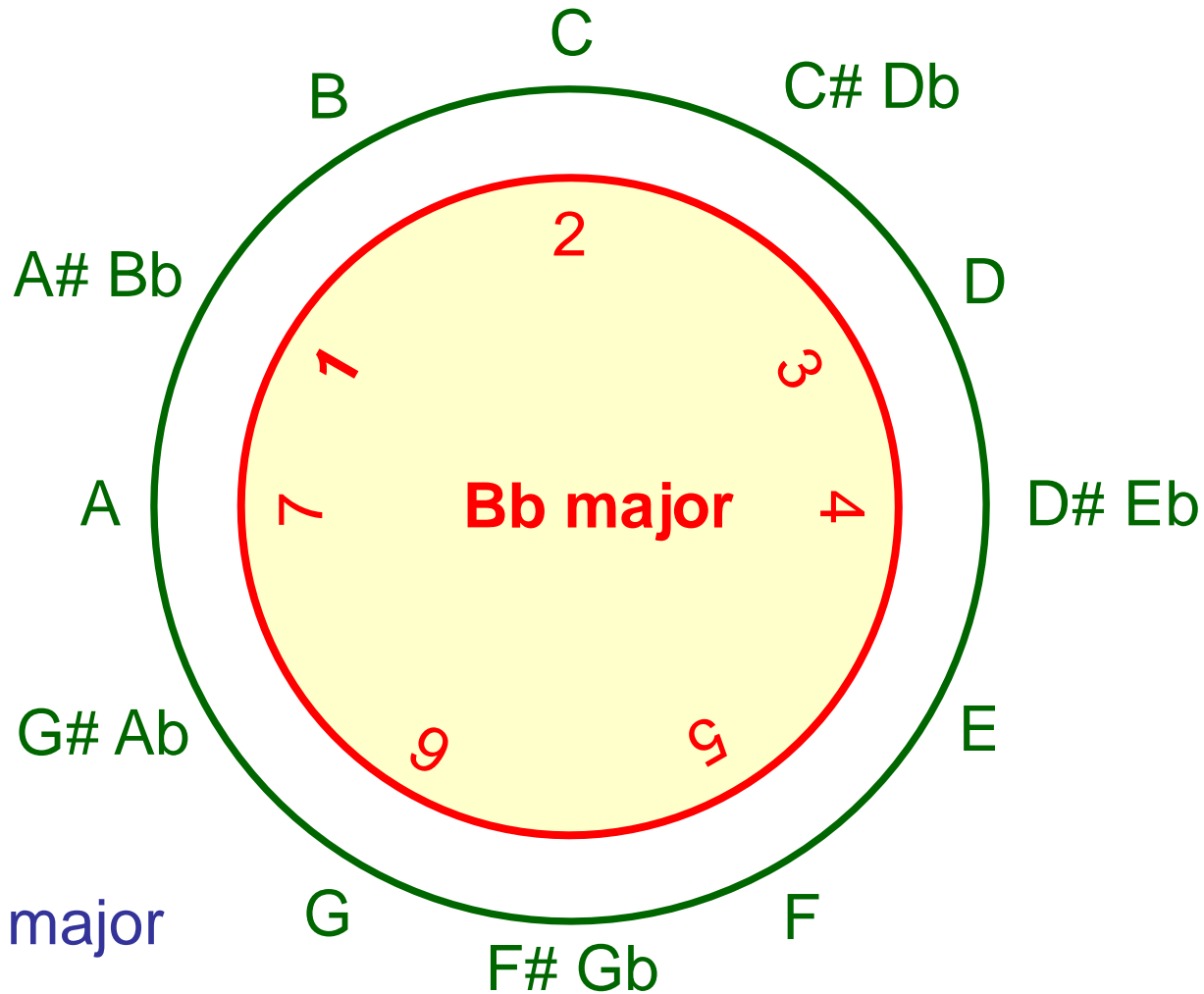


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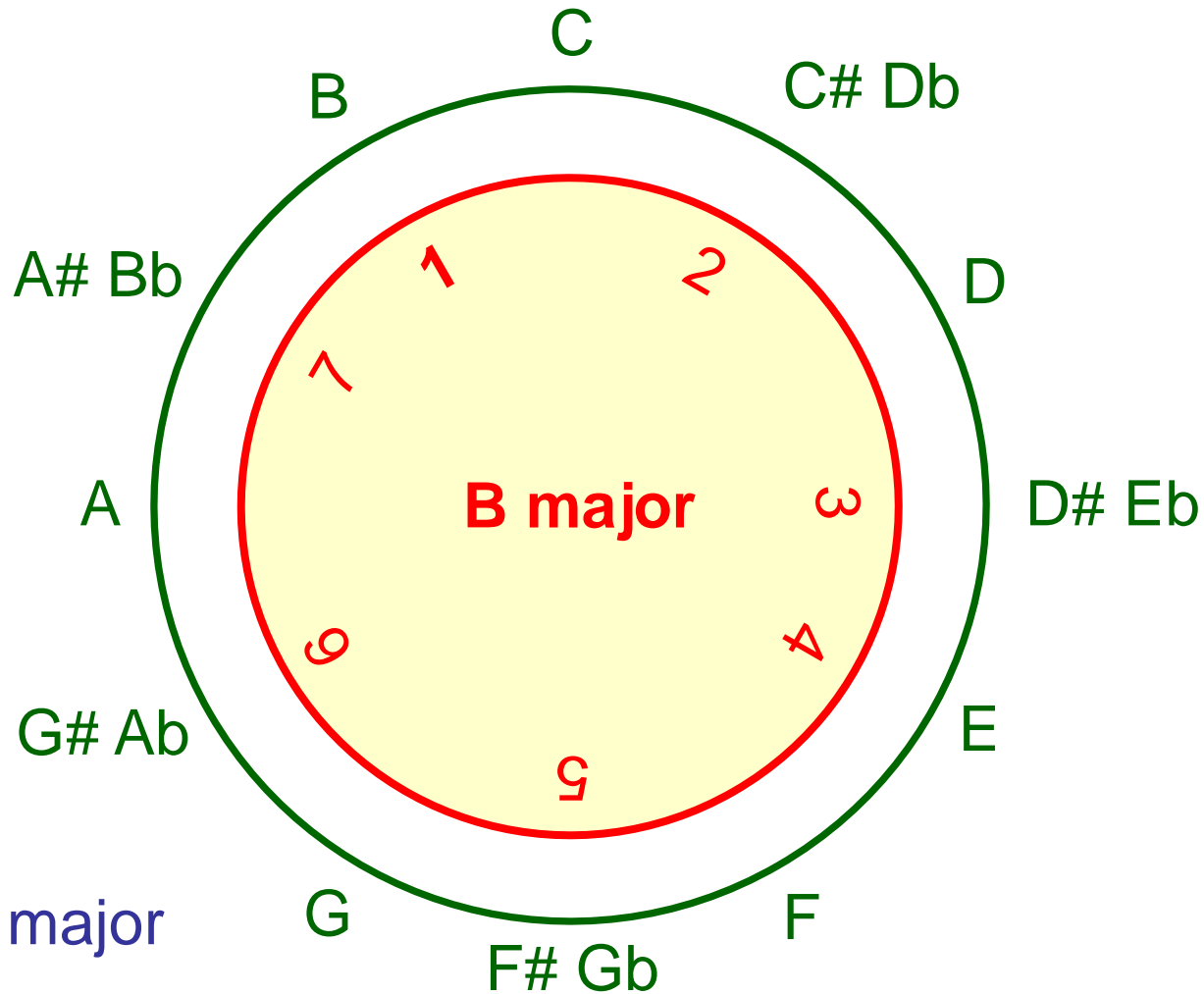


3 notes  
not in C major

# Multiple Keys

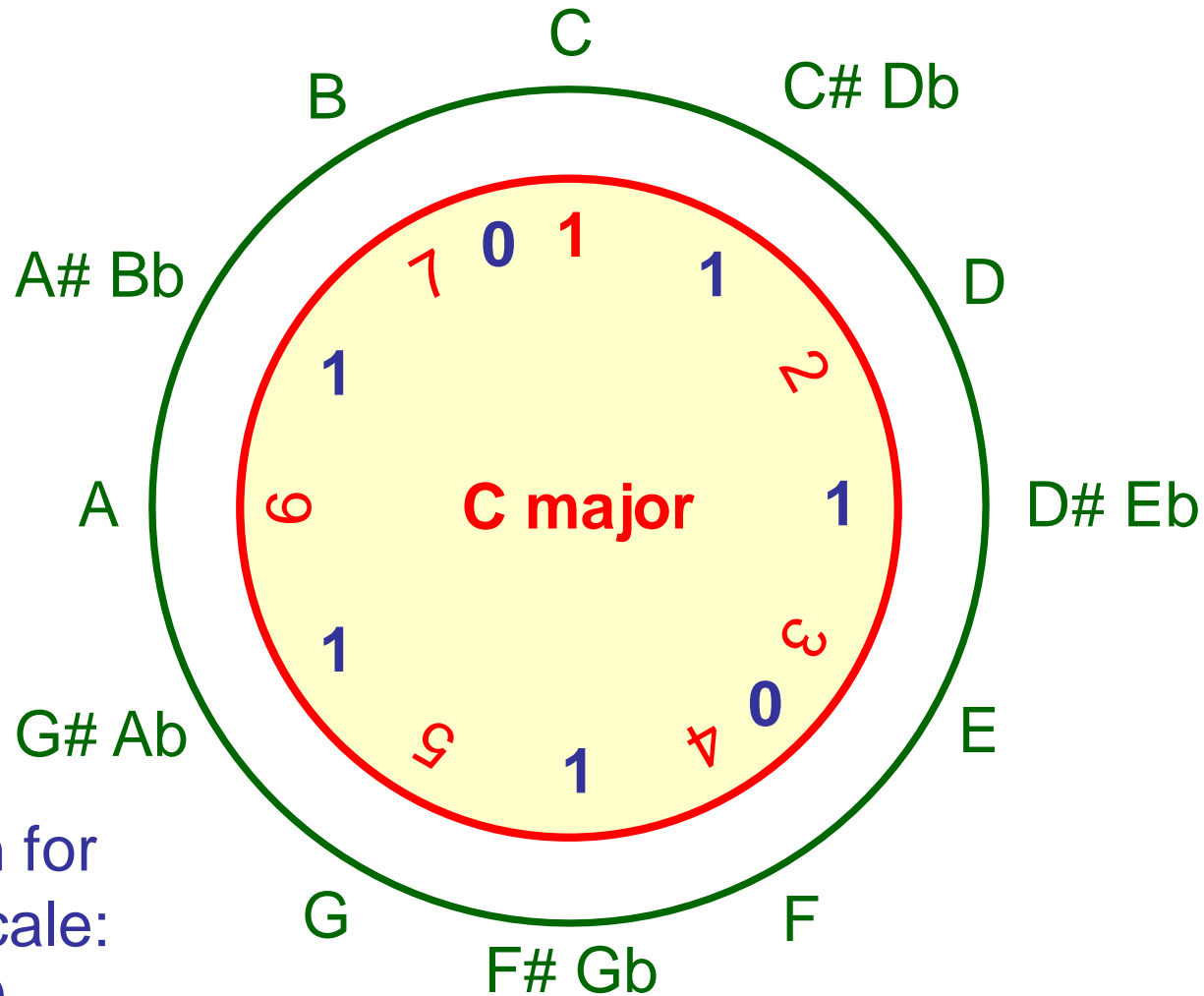


# Multiple Keys



**5** notes  
not in C major

# Multiple Keys



Notation for  
major scale:  
**1101110**

# Multiple Keys

- Chromatic pitches are **tempered** so that intervals will have approximately correct ratios in all keys.
- Modern practice is **equal temperament**.

$$\frac{\text{freq of note } k}{\text{freq of note } 1} = 2^{(k-1)/12}$$

# Multiple Keys

- Resulting error is  $\leq \pm 0.9\%$

Note	Perfect ratio	Tempered ratio	Error %
C	1/1	1.00000	0.000
D	9/8	1.12246	-0.226
E	5/4	1.25992	+0.787
F	4/3	1.33484	+0.113
G	3/2	1.49831	-0.113
A	5/3	1.68179	+0.899
B	15/8	1.88775	+0.675

# Combinatorial Requirements

- Scales must be **diatonic**
  - Adjacent notes are 1 or 2 semitones apart.
- We consider  $m$ -note scales on an  $n$ -tone chromatic
  - In binary representation, let  $m_0$  = number of 0s,  
 $m_1$  = number of 1s
  - Then  $m_0 = 2m - n$ ,  $m_1 = n - m$ 
    - In a major scale 1101110, there are  $m = 7$  notes on an  $n = 12$ -tone chromatic
    - There are  $m_0 = 2 \cdot 7 - 12 = 2$  zeros
    - There are  $m_1 = 12 - 7 = 5$  ones

# Combinatorial Requirements

- Semitones should not be bunched together.
  - One criterion: **Myhill's property**
  - All intervals of a given size should contain  $k$  or  $k + 1$  semitones.
    - For example, in a major scale:
    - All fifths are 6 or 7 semitones
    - All thirds are 3 or 4 semitones
    - All seconds are 1 or 2 semitones, etc.
  - Few scales satisfy Myhill's property.



# Combinatorial Requirements

- Semitones should not be bunched together.
  - We minimize the number of pairs of adjacent 0s and pairs of adjacent 1s.
  - If  $m_0 \geq m_1$ ,  
number of adjacent 0s =  $m_0 - \min\{m_0, m_1\}$   
number of adjacent 1s = 0
  - If  $m_1 \geq m_0$ ,  
number of adjacent 1s =  $m_1 - \min\{m_0, m_1\}$   
number of adjacent 0s = 0
  - In a major scale 1101110,  
number of adjacent 1s =  $5 - \min\{2, 5\} = 3$   
number of adjacent 0s = 0.

# Combinatorial Requirements

- Semitones should not be bunched together.
  - The number of scales satisfying this property is

$$\binom{\max\{m_0, m_1\}}{\min\{m_0, m_1\}} + \binom{\max\{m_0, m_1\} - 1}{\min\{m_0, m_1\} - 1}$$

- The number of 7-note scales on a 12-tone chromatic satisfying this property is

$$\binom{5}{2} + \binom{4}{1} = 14$$

# Combinatorial Requirements

- Can have fewer than  $n$  keys.
  - A “mode of limited transposition” (Messiaen)
  - Whole tone scale 111111 (Debussy) has 2 keys.
  - Scale 110110110 has 5 keys.
    - Count number of semitones in repeating sequence

# Temperament Requirements

- Tolerance for inaccurate tuning
  - At most  $\pm 0.9\%$
  - Don't exceed tolerance of classical equal temperament.

# Simple Ratios

- Frequency of each note should have a simple ratio (between 1 and 2) with some other note, equating notes an octave apart.

- Let  $f_i$  = freq ratio of note  $i$  to tonic (note 1), so  $f_1 = 1$ .

- For major scale CDEFGAB,

$$(f_1, \dots, f_7) = \left(1, \frac{9}{8}, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{15}{8}\right)$$

- For example, B ( $15/8$ ) has a simple ratio  $3/2$  with E ( $5/4$ )

$$\frac{f_7}{f_3} = \frac{3}{2}$$

- D octave higher ( $9/4$ ) has ratio with  $3/2$  with G ( $3/2$ )

$$\frac{2f_2}{f_5} = \frac{3}{2}$$

# Simple Ratios

- However, this allows two or more subsets of unrelated pitches.
  - Simple ratios with respect pitches in same subset, but not in other subsets.
  - So we use a **recursive** condition.
  - For some permutation of notes, each note should have simple ratio with previous note.
  - First note in permutation is the tonic.

# Simple Ratios

- Let the simple ratios be **generators**  $r_1, \dots, r_p$ .
- Let  $(\pi_1, \dots, \pi_m)$  be a permutation of  $1, \dots, m$  with  $\pi_1 = 1$ .
- For each  $i \in \{2, \dots, m\}$ , we require

$$1 < f_{\pi_i} < 2$$

and

$$\frac{f_{\pi_i}}{f_{\pi_j}} = r_q \text{ or } \frac{2f_{\pi_j}}{f_{\pi_i}} = r_q \text{ or } \frac{f_{\pi_j}}{f_{\pi_i}} = r_q \text{ or } \frac{2f_{\pi_i}}{f_{\pi_j}} = r_q$$

for some  $j \in \{1, \dots, i-1\}$  and some  $q \in \{1, \dots, p\}$

# Simple Ratios

- Ratio with previous note in the permutation  $\pi$  will be a generator.
  - Ratios with previous 2 or 3 notes in the permutation will be simple (product of generators).
  - Ratio with tonic need not be simple.



# Simple Ratios

- Observation: No need to consider both  $r_q$  and  $2/r_q$  as generators.
- So we consider only reduced fractions with odd numerators (in order of simplicity):

$$\frac{3}{2}, \frac{5}{3}, \frac{5}{4}, \frac{7}{4}, \frac{7}{5}, \frac{9}{5}, \frac{7}{6}, \frac{11}{6}, \frac{9}{7}, \frac{11}{7},$$
$$\frac{13}{7}, \frac{9}{8}, \frac{11}{8}, \frac{13}{8}, \frac{15}{8}, \frac{11}{9}, \frac{13}{9}, \frac{17}{9}, \dots$$

# CP Model

- CP model readily accommodates variable indices

$$f_{\pi_i}$$

- Replace  $f_i$  with fraction  $a_i/b_i$  in lowest terms

# CP Model

$$\text{alldiff}(\pi_1, \dots, \pi_m)$$

$$\pi_1 = a_1 = b_1 = 1$$

$$1 < \frac{a_i}{b_i} < 2, \quad \text{coprime}(a_i, b_i), \quad i = 1, \dots, m$$

$$\frac{a_{i-1}}{b_{i-1}} < \frac{a_i}{b_i}, \quad i = 2, \dots, m$$

$$\bigvee_{j < i} \left[ (\pi_i > \pi_j) \Rightarrow \left( \frac{a_{\pi_i}/b_{\pi_i}}{a_{\pi_j}/b_{\pi_j}} \in G \vee \frac{2a_{\pi_j}/b_{\pi_j}}{a_{\pi_i}/b_{\pi_i}} \in G \right) \right], \quad i = 2, \dots, m$$

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$$\frac{|a_i/b_i - 2^{(t_i-1)/n}|}{2^{(t_i-1)/n}} \leq 0.009, \quad i = 1, \dots, m$$

$$\pi_i \in \{1, \dots, m\}, \quad a_i \in \{1, \dots, 2M\}, \quad b_i \in \{1, \dots, M\}, \quad i = 1, \dots, m$$

# CP Model

$\text{alldiff}(\pi_1, \dots, \pi_m)$  ← permutation

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# CP Model

$$\text{alldiff}(\pi_1, \dots, \pi_m)$$

$$\pi_1 = a_1 = b_1 = 1 \quad \leftarrow \text{tonic note}$$

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$$\frac{a_{i-1}}{b_{i-1}} < \frac{a_i}{b_i}, \quad i = 2, \dots, m \quad \leftarrow \text{symmetry breaking}$$

$$\bigvee_{j < i} \left[ (\pi_i > \pi_j) \Rightarrow \left( \frac{a_{\pi_i}/b_{\pi_i}}{a_{\pi_j}/b_{\pi_j}} \in G \vee \frac{2a_{\pi_j}/b_{\pi_j}}{a_{\pi_i}/b_{\pi_i}} \in G \right) \right], \quad i = 2, \dots, m$$

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simple ratios



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set of generators



# CP Model

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← tuning tolerance

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chromatic tone corresponding to note  $i$

# Scales on a 12-note chromatic

- Use the generators mentioned earlier.
  - There are **multiple solutions** for each scale.
  - Select the solution with the **simplest ratios** with tonic and/or **simplest minimal generators**.
  - The 7-note scales with a **single generator  $3/2$**  are **precisely the classical modes!**
- For each note, compute **minimal generator**
  - = simplest ratio with another note.

# 7-note scales on 12-note chromatic

Scale	Solns	Ratios with tonic							Minimal generators							
1. 0101111	27	$\frac{1}{1}$	$\frac{16}{15}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{8}{5}$	$\frac{16}{9}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{9}{8}$	$\frac{3}{2}$	$\frac{5}{3}$	
2. 0110111	10	$\frac{1}{1}$	$\frac{18}{17}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{24}{17}$	$\frac{8}{5}$	$\frac{16}{9}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	Locrian mode
3. 0111011	18	$\frac{1}{1}$	$\frac{16}{15}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{16}{9}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	Phrygian mode
4. 0111101	26	$\frac{1}{1}$	$\frac{16}{15}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	
5. 1010111	6	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{8}{5}$	$\frac{16}{9}$	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{3}{2}$	
6. 1011011	6	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{16}{9}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	Aeolian mode (natural minor)
7. 1011101	10	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	Dorian mode
8. 1011110	27	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{5}{3}$	melodic minor
9. 1101011	14	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{16}{9}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{9}{8}$	
10. 1101101	9	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	Mixolydian mode
11. 1101110	17	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	Ionian mode (major)
12. 1110101	10	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	
13. 1110110	16	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	Lydian mode
14. 1111010	34	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{3}{2}$	

# 7-note scales on 12-note chromatic

Scale	Solns	Ratios with tonic	Minimal generators	
1. 0101111	27	$\frac{1}{1}$ $\frac{16}{15}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{4}$ $\frac{9}{8}$ $\frac{3}{2}$ $\frac{5}{3}$	
2. 0110111	10	$\frac{1}{1}$ $\frac{18}{17}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{24}{17}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$	Locrian mode
3. 0111011	18	$\frac{1}{1}$ $\frac{16}{15}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$	Phrygian mode
4. 0111101	26	$\frac{1}{1}$ $\frac{16}{15}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{5}{3}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$	Single generator
5. 1010111	6	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{5}{4}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{4}$ $\frac{3}{2}$ $\frac{3}{2}$	
6. 1011011	6	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$	Aeolian mode (natural minor)
7. 1011101	10	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$	Dorian mode
8. 1011110	27	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{5}{3}$	melodic minor
9. 1101011	14	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{9}{8}$	
10. 1101101	9	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$	Mixolydian mode
11. 1101110	17	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$	Ionian mode (major)
12. 1110101	10	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{4}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$	
13. 1110110	16	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$	Lydian mode
14. 1111010	34	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{5}{3}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{4}$ $\frac{3}{2}$ $\frac{3}{2}$	

# Other scales on 12-note chromatic

Scale	Solns	Keys	Ratios with tonic								Minimal generators								
1. 111111	6	2	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{8}{5}$	$\frac{16}{9}$				$\frac{5}{4}$	$\frac{5}{4}$	$\frac{5}{4}$	$\frac{5}{4}$	$\frac{5}{4}$	$\frac{9}{5}$		
1. 01010101	>50	3	$\frac{1}{1}$	$\frac{16}{15}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{16}{9}$		$\frac{3}{2}$	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{9}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$
2. 10101010	>50	3	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{15}{8}$		$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$
21. 100001010	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
22. 100010010	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
23. 100010100	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
24. 100100010	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
25. 100100100	>50	4	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{9}{5}$	$\frac{3}{2}$
26. 100101000	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{9}{8}$	$\frac{3}{2}$
27. 101000010	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$
28. 101000100	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
29. 101001000	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$
30. 101010000	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$

# Other scales on 12-note chromatic

Scale	Solns	Keys	Ratios with tonic								Minimal generators						
1. 111111	6	2	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{8}{5}$	$\frac{16}{9}$			$\frac{5}{4}$	$\frac{5}{4}$	$\frac{5}{4}$	$\frac{5}{4}$	$\frac{5}{4}$	$\frac{9}{5}$	

Whole tone scale. Minimal interest musically

21. 100001010	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
22. 100010010	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
23. 100010100	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
24. 100100010	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
25. 100100100	>50	4	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{9}{5}$
26. 100101000	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{9}{8}$
27. 101000010	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$
28. 101000100	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
29. 101001000	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$
30. 101010000	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$



# Other scales on 12-note chromatic

Scale	Solns	Keys	Ratios with tonic								Minimal generators							
1. 111111	6	2	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{8}{5}$	$\frac{16}{9}$			$\frac{5}{4}$	$\frac{5}{4}$	$\frac{5}{4}$	$\frac{5}{4}$	$\frac{5}{4}$	$\frac{9}{5}$		
1. 01010101	>50	3	$\frac{1}{1}$	$\frac{16}{15}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{9}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$
2. 10101010	>50	3	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$

8-note scales. Only 3 keys.

23. 100010100	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
24. 100100010	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
25. 100100100	>50	4	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{9}{5}$	$\frac{3}{2}$
26. 100101000	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{9}{8}$	$\frac{3}{2}$
27. 101000010	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$
28. 101000100	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
29. 101001000	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$
30. 101010000	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$

# Other scales on 12-note chromatic

Scale	Solns	Keys	Ratios with tonic								Minimal generators					
1. 111111	6	2	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{8}{5}$	$\frac{16}{9}$			$\frac{5}{4}$	$\frac{5}{4}$	$\frac{5}{4}$	$\frac{5}{4}$	$\frac{5}{4}$	$\frac{9}{5}$

## 9-note scales beginning with whole tone interval

21. 100001010	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
22. 100010010	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
23. 100010100	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
24. 100100010	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
25. 100100100	>50	4	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{9}{5}$
26. 100101000	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{9}{8}$
27. 101000010	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$
28. 101000100	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
29. 101001000	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$
30. 101010000	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$

# Other scales on 12-note chromatic

Scale	Solns	Keys	Ratios with tonic								Minimal generators					
1. 111111	6	2	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{8}{5}$	$\frac{16}{9}$			$\frac{5}{4}$	$\frac{5}{4}$	$\frac{5}{4}$	$\frac{5}{4}$	$\frac{5}{4}$	$\frac{9}{5}$

Most appealing scales. Simple ratios,  
good distribution of semitones.

22. 100010010	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
23. 100010100	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
24. 100100010	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
25. 100100100	>50	4	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{9}{5}$	$\frac{3}{2}$
26. 100101000	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{9}{8}$	$\frac{3}{2}$
27. 101000010	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$
28. 101000100	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
29. 101001000	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$
30. 101010000	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$

## Other scales on 12-note chromatic

Scale	Solns	Keys	Ratios with tonic								Minimal generators						
1. 111111	6	2	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{8}{5}$	$\frac{16}{9}$			$\frac{5}{4}$	$\frac{5}{4}$	$\frac{5}{4}$	$\frac{5}{4}$	$\frac{5}{4}$	$\frac{9}{5}$	

Will illustrate this scale with a Chorale and Fugue for organ

22. 100010010	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
23. 100010100	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
24. 100100010	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
25. 100100100	>50	4	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{9}{5}$	$\frac{3}{2}$
26. 100101000	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{9}{8}$	$\frac{3}{2}$
27. 101000010	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$
28. 101000100	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
29. 101001000	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$
30. 101010000	>50	12	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$

# Other Chromatic Scales

- Which chromatics have the most simple ratios with the tonic, within tuning tolerance?

Ratio	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$3/2$	.	.	.	.	.	.	●	.	.	.	.	●	.	●	.	.	●	.	●
$4/3$	.	.	.	.	.	.	●	.	.	.	.	●	.	●	.	.	●	.	●
$5/3$	.	.	●	.	.	●	●	.	.	●	●	.	.	●	●	.	●	●	●
$5/4$	●	.	.	●	.	.	●	.	.	●	●	.	●	●	.	●	.	.	●
$7/4$	.	.	.	.	●	●	.	.	.	●	●	.	.	.	●	●	.	.	.
$6/5$	.	.	.	.	.	●	.	.	.	●	.	.	.	●	.	.	●	●	.
$7/5$	.	.	.	.	.	.	.	.	.	.	.	.	.	●	.	●	.	●	.
$8/5$	●	.	.	●	.	.	●	.	.	●	●	.	●	●	.	●	●	.	●
$9/5$	.	●	.	.	.	.	.	●	●	.	.	.	.	●	●	●	.	.	.

# Other Chromatic Scales

- Which chromatics have the most simple ratios with the tonic, within tuning tolerance?

Ratio	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$3/2$	.	.	.	.	.	.	●	.	.	.	.	●	.	●	.	.	●	.	●
$4/3$	.	.	.	.	.	.	●	.	.	.	.	●	.	●	.	.	●	.	●
$5/3$	.	.	●	.	.	●	●	.	.	●	●	.	.	●	●	.	●	●	●
$5/4$	●	.	.	●	.	.	●	.	.	●	●	.	●	●	.	●	.	.	●
$7/4$	.	.	.	.	●	●	.	.	.	●	●	.	.	.	●	●	.	.	.
$6/5$	.	.	.	.	.	●	.	.	.	●	.	.	.	●	.	.	●	●	.
$7/5$	.	.	.	.	.	.	.	.	.	.	.	.	.	●	.	●	.	●	.
$8/5$	●	.	.	●	.	.	●	.	.	●	●	.	●	●	.	●	●	.	●
$9/5$	.	●	.	.	.	.	.	●	●	.	.	.	.	●	●	●	.	.	.

Classical 12-tone chromatic is 2nd best.

# Other Chromatic Scales

- Which chromatics have the most simple ratios with the tonic, within tuning tolerance?

Ratio	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$3/2$	.	.	.	.	.	.	●	.	.	.	.	●	.	●	.	.	●	.	●
$4/3$	.	.	.	.	.	.	●	.	.	.	.	●	.	●	.	.	●	.	●
$5/3$	.	.	●	.	.	●	●	.	.	●	●	.	.	●	●	.	●	●	●
$5/4$	●	.	.	●	.	.	●	.	.	●	●	.	●	●	.	●	.	.	●
$7/4$	.	.	.	.	●	●	.	.	.	●	●	.	.	.	●	●	.	.	.
$6/5$	.	.	.	.	.	●	.	.	.	●	.	.	.	●	.	.	●	●	.
$7/5$	.	.	.	.	.	.	.	.	.	.	.	.	.	●	.	●	.	●	.
$8/5$	●	.	.	●	.	.	●	.	.	●	●	.	●	●	.	●	●	.	●
$9/5$	.	●	.	.	.	.	.	●	●	.	.	.	.	●	●	●	.	.	.

Quarter-tone scale adds nothing.

# Other Chromatic Scales

- Which chromatics have the most simple ratios with the tonic?

Ratio	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$3/2$	.	.	.	.	.	.	•	.	.	.	.	•	.	•	.	.	•	.	•
$4/3$	.	.	.	.	.	.	•	.	.	.	.	•	.	•	.	.	•	.	•
$5/3$	.	.	•	.	.	•	•	.	.	•	•	.	.	•	•	.	•	•	•
$5/4$	•	.	.	•	.	.	•	.	.	•	•	.	•	•	.	•	.	.	•
$7/4$	.	.	.	.	•	•	.	.	.	•	•	.	.	.	•	•	.	.	.
$6/5$	.	.	.	.	.	•	.	.	.	•	.	.	.	•	.	.	•	•	.
$7/5$	.	.	.	.	.	.	.	.	.	.	.	.	.	•	.	•	.	•	.
$8/5$	•	.	.	•	.	.	•	.	.	•	•	.	•	•	.	•	•	.	•
$9/5$	.	•	.	.	.	.	.	•	•	.	.	.	.	•	•	•	.	.	.

19-tone chromatic dominates all others



# Historical Sidelight

- Advantage of 19-tone chromatic was discovered during Renaissance.
  - Spanish organist and music theorist **Francisco de Salinas** (1530-1590) recommended 19-tone chromatic due to its tuning properties.
  - He used **meantone temperament** rather than equal temperament.



# Scales on 19-tone chromatic

- But what are the best scales on this chromatic?
  - **10-note** scales have only **1 semitone**, not enough for musical interest.
  - **12-note** scales have **5 semitones**, but this makes scale notes very closely spaced.
  - **11-note** scales have 3 semitones, which seems a **good compromise** (1 more semitone than classical scales).

# 11-note scales on 19-tone chromatic

- There are 77 scales satisfying our requirements.

$$\binom{8}{3} + \binom{7}{2} = 77$$

- Solve CP problem for all 77.
- For each scale, determine largest set of simple ratios that occur in at least one solution.
- 37 different sets of ratios appear in the 77 scales.

# Simple ratios in 11-note scales

Ratio	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	a	b	c	d	e	f	g	h	i	j	k		
3/2	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●
4/3	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●
5/3	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●
5/4	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●
7/4	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●
6/5	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●
7/5	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●
8/5	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●
9/5	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●

<b>A</b> - 72	<b>K</b> - 12,43	<b>U</b> - 57	<b>e</b> - 13,29,44
<b>B</b> - 69,70,71	<b>L</b> - 28	<b>V</b> - 42	<b>f</b> - 60,61
<b>C</b> - 68	<b>M</b> - 65,66	<b>W</b> - 26,27	<b>g</b> - 59
<b>D</b> - 74,75	<b>N</b> - 63,64	<b>X</b> - 10,11,25	<b>h</b> - 18,35,36,50,51,54
<b>E</b> - 7,8	<b>O</b> - 62	<b>Y</b> - 5,6	<b>i</b> - 17,34,49
<b>F</b> - 22,23	<b>P</b> - 40,41,55,56	<b>Z</b> - 15,31,32,46,47	<b>j</b> - 58
<b>G</b> - 73	<b>Q</b> - 20,21,38,39,53	<b>a</b> - 14,30,45	<b>k</b> - 16,33,48
<b>H</b> - 2	<b>R</b> - 19,37,52	<b>b</b> - 9,24	

## Simple ratios in 11-note scales

Ratio	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	a	b	c	d	e	f	g	h	i	j	k		
3/2	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•
4/3	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•
5/3	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•
5/4	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•
7/4	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•
6/5	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•
7/5	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•
8/5	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•
9/5	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•

A - 72

B - 69,70,71

C - 68

D - 74,75

E - 7,8

F - 22,23

G - 73

H - 2

K - 12,43

L - 28

M - 65,66

N - 63,64

O - 62

P - 40,41,55,56

Q - 20,21,38,39,53

R - 19,37,52

U - 57

V - 42

W - 26,27

X - 10,11,25

Y - 5,6

Z - 15,31,32,46,47

a - 14,30,45

b - 9,24

e - 13,29,44

f - 60,61

g - 59

h - 18,35,36,50,51,54

i - 17,34,49

j - 58

k - 16,33,48

These 9 scales dominate all the others.

## Simple ratios in 11-note scales

Ratio	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	a	b	c	d	e	f	g	h	i	j	k			
3/2	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	
4/3	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	
5/3	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	
5/4	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	
7/4	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	
6/5	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	
7/5	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	
8/5	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	
9/5	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	
A - 72				K - 12,43												U - 57												e - 13,29,44												
B - 69,70,71				L - 28												V - 42												f - 60,61												
C - 68				M - 65,66												W - 26,27												g - 59												
D - 74,75				N - 63,64												X - 10,11,25												h - 18,35,36,50,51,54												
E - 78				O - 62												Y - 5,6												i - 17,34,49												
F - 22,23				P - 40,41,55,56												Z - 15,31,32,46,47												j - 58												
G - 73				Q - 20,21,38,39,53												a - 14,30,45												k - 16,33,48												
H - 2				R - 19,37,52												b - 9,24																								

We will focus on 1 scale from each class.

## 4 attractive 9-note scales

Scale	Class	Ratios with tonic											Minimal generators										
7. 01101011111	E	$\frac{1}{1}$	$\frac{25}{24}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{7}{5}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{12}{7}$	$\frac{25}{18}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{7}{4}$	$\frac{3}{2}$
		$\frac{1}{1}$	$\frac{36}{35}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{7}{5}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{12}{7}$	$\frac{13}{17}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{4}{2}$	$\frac{7}{4}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{13}{7}$
27. 101011111110	W	$\frac{1}{1}$	$\frac{15}{14}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{10}{7}$	$\frac{54}{35}$	$\frac{5}{3}$	$\frac{9}{5}$	$\frac{27}{14}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{4}$
		$\frac{1}{1}$	$\frac{16}{15}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{10}{7}$	$\frac{14}{9}$	$\frac{5}{3}$	$\frac{9}{5}$	$\frac{35}{18}$	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{7}{4}$	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{4}$
56. 11011110110	P	$\frac{1}{1}$	$\frac{15}{14}$	$\frac{7}{6}$	$\frac{6}{5}$	$\frac{9}{7}$	$\frac{7}{5}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{9}{5}$	$\frac{27}{14}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$
		$\frac{1}{1}$	$\frac{13}{12}$	$\frac{7}{6}$	$\frac{6}{5}$	$\frac{9}{7}$	$\frac{7}{5}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{9}{5}$	$\frac{35}{18}$	$\frac{3}{2}$	$\frac{13}{7}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{7}{5}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{5}{3}$
72. 11110110110	A	$\frac{1}{1}$	$\frac{16}{15}$	$\frac{7}{6}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{7}{5}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{9}{5}$	$\frac{35}{18}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{5}{3}$
		$\frac{1}{1}$	$\frac{15}{14}$	$\frac{7}{6}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{7}{5}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{9}{5}$	$\frac{27}{14}$	$\frac{3}{2}$	$\frac{7}{5}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{9}{5}$

Showing 2 simplest solutions for each scale.  
One with simplest ratios, one with simplest generators.

# Key structure of scales

## *Classical major scale*

Note	1	1 $\sharp$	2	2 $\sharp$	3	4	4 $\sharp$	5	5 $\sharp$	6	6 $\sharp$	7
Interval			2 <sup>nd</sup>		3 <sup>rd</sup>	4 <sup>th</sup>		5 <sup>th</sup>		6 <sup>th</sup>		7 <sup>th</sup>
Distance	0	5	2	3	4	1	5	1	4	3	2	5

## *Scale 23 of 9 notes on 12-note chromatic*

Note	1	1 $\sharp$	2	3	4	5	5 $\sharp$	6	7	7 $\sharp$	8	9
Interval			2 <sup>nd</sup>	m3 <sup>rd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>		5 <sup>th</sup>	m6 <sup>th</sup>		m7 <sup>th</sup>	7 <sup>th</sup>
Distance	0	3	3	2	2	2	3	2	2	2	3	3

## *Scale 7 of 11 notes on 19-note chromatic*

Note	1	2	2 $\sharp$	3	3 $\sharp$	4	5	5 $\sharp$	6	7	7 $\sharp$	8	8 $\sharp$	9	9 $\sharp$	10	10 $\sharp$	11	11 $\sharp$
Interval				2 <sup>nd</sup>		m3 <sup>rd</sup>	3 <sup>rd</sup>		4 <sup>th</sup>			5 <sup>th</sup>		m6 <sup>th</sup>					
Distance	0	8	3	5	5	4	5	5	4	5	5	4	5	5	4	5	5	3	8

## *Scale 27 of 11 notes on 19-note chromatic*

Note	1	1 $\sharp$	2	3	3 $\sharp$	4	5	5 $\sharp$	6	6 $\sharp$	7	7 $\sharp$	8	8 $\sharp$	9	9 $\sharp$	10	10 $\sharp$	11
Interval				2 <sup>nd</sup>		m3 <sup>rd</sup>	3 <sup>rd</sup>		4 <sup>th</sup>						6 <sup>th</sup>				
Distance	0	8	3	5	4	6	3	6	4	5	5	4	6	3	6	4	5	3	8

## *Scale 56 of 11 notes on 19-note chromatic*

Note	1	1 $\sharp$	2	2 $\sharp$	3	4	4 $\sharp$	5	5 $\sharp$	6	6 $\sharp$	7	7 $\sharp$	8	9	9 $\sharp$	10	10 $\sharp$	11
Interval						m3 <sup>rd</sup>						5 <sup>th</sup>		m6 <sup>th</sup>	6 <sup>th</sup>				
Distance	0	8	3	5	6	2	7	3	6	4	4	6	3	7	2	6	5	3	8

## *Scale 72 of 11 notes on 19-note chromatic*

Note	1	1 $\sharp$	2	2 $\sharp$	3	3 $\sharp$	4	4 $\sharp$	5	6	6 $\sharp$	7	7 $\sharp$	8	9	9 $\sharp$	10	10 $\sharp$	11
Interval							3 <sup>rd</sup>		4 <sup>th</sup>			5 <sup>th</sup>		m6 <sup>th</sup>	6 <sup>th</sup>				
Distance	0	8	3	5	6	2	7	3	6	4	4	6	3	7	2	6	5	3	8



# Key structure of scales

## *Classical major scale*

Note	1	1 $\sharp$	2	2 $\sharp$	3	4	4 $\sharp$	5	5 $\sharp$	6	6 $\sharp$	7
Interval			2 <sup>nd</sup>		3 <sup>rd</sup>	4 <sup>th</sup>		5 <sup>th</sup>		6 <sup>th</sup>		7 <sup>th</sup>
Distance	0	5	2	3	4	1	5	1	4	3	2	5

No key with distance 1.

## *Scale 23 of 9 notes on 12-note chromatic*

Note	1	1 $\sharp$	2	3	4	5	5 $\sharp$	6	7	7 $\sharp$	8	9
Interval			2 <sup>nd</sup>	m3 <sup>rd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>		5 <sup>th</sup>	m6 <sup>th</sup>		m7 <sup>th</sup>	7 <sup>th</sup>
Distance	0	3	3	2	2	2	3	2	2	2	3	3

Good or bad?

## *Scale 7 of 11 notes on 19-note chromatic*

Note	1	2	2 $\sharp$	3	3 $\sharp$	4	5	5 $\sharp$	6	7	7 $\sharp$	8	8 $\sharp$	9	9 $\sharp$	10	10 $\sharp$	11	11 $\sharp$
Interval				2 <sup>nd</sup>		m3 <sup>rd</sup>	3 <sup>rd</sup>		4 <sup>th</sup>			5 <sup>th</sup>		m6 <sup>th</sup>					
Distance	0	8	3	5	5	4	5	5	4	5	5	4	5	5	4	5	5	3	8

## *Scale 27 of 11 notes on 19-note chromatic*

Note	1	1 $\sharp$	2	3	3 $\sharp$	4	5	5 $\sharp$	6	6 $\sharp$	7	7 $\sharp$	8	8 $\sharp$	9	9 $\sharp$	10	10 $\sharp$	11
Interval				2 <sup>nd</sup>		m3 <sup>rd</sup>	3 <sup>rd</sup>		4 <sup>th</sup>						6 <sup>th</sup>				
Distance	0	8	3	5	4	6	3	6	4	5	5	4	6	3	6	4	5	3	8

## *Scale 56 of 11 notes on 19-note chromatic*

Note	1	1 $\sharp$	2	2 $\sharp$	3	4	4 $\sharp$	5	5 $\sharp$	6	6 $\sharp$	7	7 $\sharp$	8	9	9 $\sharp$	10	10 $\sharp$	11
Interval						m3 <sup>rd</sup>						5 <sup>th</sup>		m6 <sup>th</sup>	6 <sup>th</sup>				
Distance	0	8	3	5	6	2	7	3	6	4	4	6	3	7	2	6	5	3	8

## *Scale 72 of 11 notes on 19-note chromatic*

Note	1	1 $\sharp$	2	2 $\sharp$	3	3 $\sharp$	4	4 $\sharp$	5	6	6 $\sharp$	7	7 $\sharp$	8	9	9 $\sharp$	10	10 $\sharp$	11
Interval							3 <sup>rd</sup>		4 <sup>th</sup>			5 <sup>th</sup>		m6 <sup>th</sup>	6 <sup>th</sup>				
Distance	0	8	3	5	6	2	7	3	6	4	4	6	3	7	2	6	5	3	8

## 4 attractive 9-note scales

Scale	Class	Ratios with tonic												Minimal generators											
7. 01101011111	E	$\frac{1}{1}$	$\frac{25}{24}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{7}{5}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{12}{7}$	$\frac{25}{18}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{7}{4}$	$\frac{3}{2}$		
		$\frac{1}{1}$	$\frac{36}{35}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{7}{5}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{12}{7}$	$\frac{13}{17}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{4}{2}$	$\frac{7}{4}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{13}{7}$		
27. 10101111110	W	$\frac{1}{1}$	$\frac{15}{14}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{10}{7}$	$\frac{54}{35}$	$\frac{5}{3}$	$\frac{9}{5}$	$\frac{27}{14}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{4}$		
		$\frac{1}{1}$	$\frac{16}{15}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{10}{7}$	$\frac{14}{9}$	$\frac{5}{3}$	$\frac{9}{5}$	$\frac{35}{18}$	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{7}{4}$	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{4}$		
56. 11011110110	P	$\frac{1}{1}$	$\frac{15}{14}$	$\frac{7}{6}$	$\frac{6}{5}$	$\frac{9}{7}$	$\frac{7}{5}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{9}{5}$	$\frac{27}{14}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$		
		$\frac{1}{1}$	$\frac{13}{12}$	$\frac{7}{6}$	$\frac{6}{5}$	$\frac{9}{7}$	$\frac{7}{5}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{9}{5}$	$\frac{35}{18}$	$\frac{3}{2}$	$\frac{13}{7}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{7}{5}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{5}{3}$		
72. 11110110110	A	$\frac{1}{1}$	$\frac{16}{15}$	$\frac{7}{6}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{7}{5}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{9}{5}$	$\frac{35}{18}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{5}{3}$		
		$\frac{1}{1}$	$\frac{15}{14}$	$\frac{7}{6}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{7}{5}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{9}{5}$	$\frac{27}{14}$	$\frac{3}{2}$	$\frac{7}{5}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{9}{5}$		

Focus further on scale 72, which has largest number of simple ratios.

# Harmonic Comparison

- Classic major scale

- Major triad C:E:G = 4:5:6, major 7 chord C:E:G:B = 8:10:12:15
- Minor triad A:C:E = 10:12:15, minor 7 chord A:C:E:G = 10:12:15:18
- Dominant 7 chord G:B:D:F = 36:45:54:64
- Tensions (from jazz) C E G B D F# A

- Scale 72

- Major triad 1-4-7 = 4:5:6
- Minor triad 5-8-12 = 10:12:15, minor 7 chord 9-12-15-18 = 10:12:15:18
- New chord 9-12-14-18 = 5:6:7:9
- New chord 1-3-5-9 = 6:7:8:10
- New chord 3-5-9-12 = 7:8:10:12
- New chord 5-9-12-15 = 4:5:6:7
- Tensions 1-4-7-10-13-15b-16-19-22

# Demonstration: 11-note scale

- Software
  - Hex MIDI sequencer for scales satisfying Myhill's property
  - We trick it into generating a 19-tone chromatic
  - Viking synthesizer for use with Hex
  - LoopMIDI virtual MIDI cable

# Demonstration: 9-note scale

- Chorale and Fugue for Organ
- Chorale
  - In A, cycles through 2 most closely related keys: A, C#, F, A.
  - Modulate to C# at bar 27
  - Final section starts at bar 72 (5:56)
- Fugue
  - Double fugue
  - First subject enters at pitches A, C#, F
  - Second subject enters at bar 96.
  - Final episode at bar 164 (13:36)
  - Recapitulation at bar 170

# Demonstration: 9-note scale

Key of A and 2 most closely related keys.


Scale in A

Scale in C#



This musical notation shows a 9-note scale in A major and C# minor. The scale is written in treble and bass clefs with a common time signature. The notes are A, B, C, D, E, F#, G, A, B. The notes A, B, C, D, E, F#, and G are circled in green. The notes A and B are circled in green in the final measure.

Scale in F



This musical notation shows a 9-note scale in F major and D minor. The scale is written in treble and bass clefs with a common time signature. The notes are F, G, A, B, C, D, E, F, G. The notes F, G, A, B, C, D, and E are circled in green. The notes F and G are circled in green in the final measure.

New notes are circled

# Chorale and Fugue

## On a 9-note Scale

J. N. Hooker  
Revised 2013

### Chorale

Organ *mp*  $\text{♩} = 50$

The musical score is divided into two systems. The first system is labeled 'Organ' and 'mp' (mezzo-piano). It begins with a tempo marking of a quarter note equal to 50 beats per minute. The music is in common time (C) and features a 9-note scale in the right hand, with the notes G4, A4, B4, C5, B4, A4, G4, F#4, and E4. The left hand provides a harmonic accompaniment. The second system continues the piece, with a measure rest at the beginning of the first staff. The notation includes various musical symbols such as clefs, time signatures, notes, rests, and accidentals.

# Chorale and Fugue

## On a 9-note Scale

J. N. Hooker  
Revised 2013

Chorale **Begin in key of A** Cadence

Organ *mp* = 50

5

The musical score is presented in two systems. The first system is for the Organ, marked 'mp' and '50'. It features a treble and bass staff. A red oval highlights the beginning of the chorale in the key of A, and another red oval highlights a cadence. The second system is for the Fugue, marked '5'. It also features a treble and bass staff. The score is written in common time (C) and uses a 9-note scale.



# Chorale and Fugue

## On a 9-note Scale

J. N. Hooker  
Revised 2013

Chorale

Double leading tone

Organ *mp*  $\text{♩} = 50$

5

The musical score is written for organ and features three systems of music. The first system is labeled 'Organ' and 'mp' (mezzo-piano), with a tempo marking of quarter note = 50. It consists of a grand staff (treble and bass clefs) and a single bass line. The second system is a grand staff with treble and bass clefs. The third system is also a grand staff with treble and bass clefs. A red circle highlights a 'Double leading tone' in the first system, specifically a chromatic pair of notes (F# and G) in the treble staff. The key signature has one sharp (F#). The time signature is common time (C). The tempo is marked as quarter note = 50. The dynamics are marked as mezzo-piano (mp). The score is numbered 5 at the beginning of the second system.

# Chorale and Fugue

On a 9-note Scale

J. N. Hooker  
Revised 2013

Chorale

Organ *mp*  $\text{♩} = 50$

Resolve from lowered submediant (F)

5

# Chorale and Fugue

## On a 9-note Scale

J. N. Hooker  
Revised 2013

Chorale

Organ *mp*

$\text{♩} = 50$

Pivot on tonic 0:16

5

0:55

Org.

*mf*

1:24

Org.

*f*

Org.

Where does modulation  
to Db actually occur?

1:48

Org.

Org.

*mp*

*mf*

New key (Db = C#)

Where does modulation  
to Db actually occur?

It occurs here

1:48

Org.

mp

mf

New key (Db = C#)

Skip to  
final section

5:56

Org.

ff

Org.

Org.

rit.

Org.

Molto adagio

rit.

Final cadence from lowered submediant (F), double leading tone, pivot on tonic

# 6:53 Fugue

## Chorale and Fugue

7

82 *a tempo*  
Org. *mp*

Subject enters at A

2<sup>nd</sup> entrance at C# but still in key of A

3<sup>rd</sup> entrance at F

86

Org.

4<sup>th</sup> entrance at A

Counter-subject

90

Org.



94 8:01

Org.

2<sup>nd</sup> subject

Multiple suspensions on semitones

98

Org.

Countersubject

101

Org.

Countersubject

104

Org.

107

Org.

110

Org.

Countersubject

13:32

Org.

163

Skip to final episode

Org.

165

*p*

Org.

167

Recapitulation (entrance at A)

169

Org.

14:01 *f*

Entrance at C#

This block shows the first system of the organ score, measures 169 to 172. The right hand (treble clef) contains a melodic line with a red arrow pointing to a C# note in measure 172. The left hand (bass clef) contains a rhythmic accompaniment. A time stamp '14:01' and dynamic 'f' are present. The text 'Recapitulation (entrance at A)' is at the top, and 'Entrance at C#' is below the right hand.

Entrance at F

173

Org.

Entrance at A *ff*

This block shows the second system of the organ score, measures 173 to 176. The right hand (treble clef) contains a melodic line with a red arrow pointing to an F note in measure 173. The left hand (bass clef) contains a rhythmic accompaniment. A dynamic 'ff' is present. The text 'Entrance at F' is at the top left, and 'Entrance at A' is below the left hand.

Countersubject

177

Org.

*rit.*

This block shows the third system of the organ score, measures 177 to 180. The right hand (treble clef) contains a melodic line with a red arrow pointing to a note in measure 177. The left hand (bass clef) contains a rhythmic accompaniment. A 'rit.' marking is present. The text 'Countersubject' is at the top left.

Closing  
section

Org.

180

Final cadence

Pivot on tonic

Double leading tone

From lowered submediant

Coda

Org.

182

*a tempo* *rit.* **Adagio** *rit.*

*fff*

Org.

184

**Molto adagio** *rit.*

Secondary cadence

Double leading tone

Pivot on tonic

Add 32' or 64'

**That's it.**