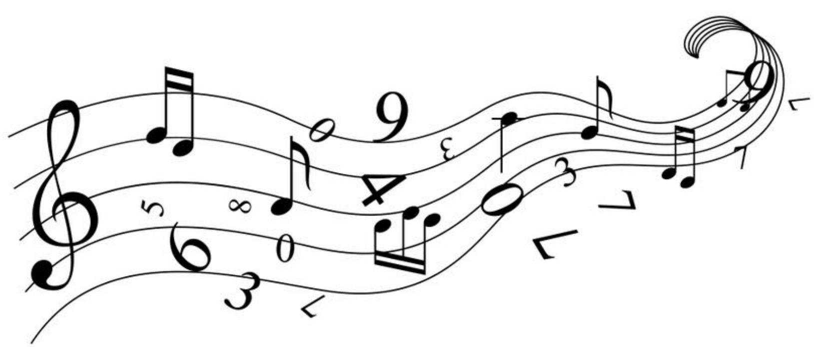
Searching for the Perfect Musical Scale

John Hooker
Carnegie Mellon University

October 2022

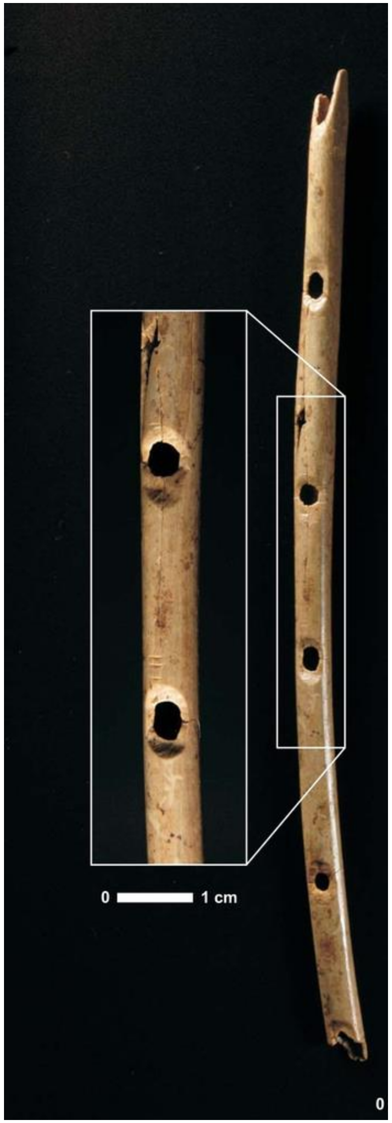
Music and Mathematics

- Music is much more than mathematics and combinatorics.
  - But almost all music relies on **mathematical structure**.
  - … even when we are not aware of it.

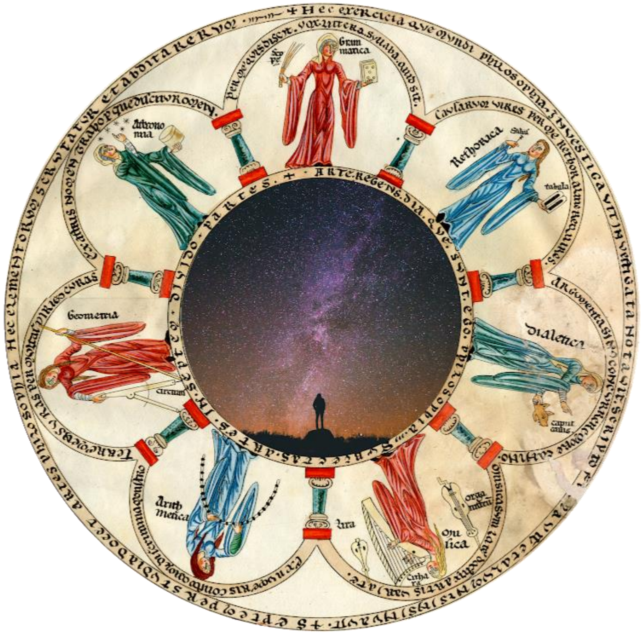


Music and Mathematics

- Oldest known musical instrument uses tones with **mathematical relationships**.
  - Prehistoric flute, from ice-age cave in Germany, 40,000 bce.
  - Based on notes of pentatonic scale: frequency ratios $1, \frac{9}{8}, \frac{5}{4}, \frac{3}{2}, \frac{5}{3}$
  - Same notes are in our **modern scales!**



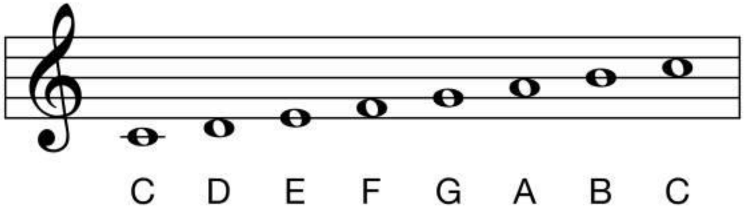
Music and Mathematics



- The 7 liberal arts
  - ***Trivium*** – arts of the mind
    - logic
    - grammar
    - rhetoric
  - ***Quadrivium*** – arts of matter
    - mathematics
    - **music (viewed as applied math!)**
    - geometry
    - astronomy (applied geometry)

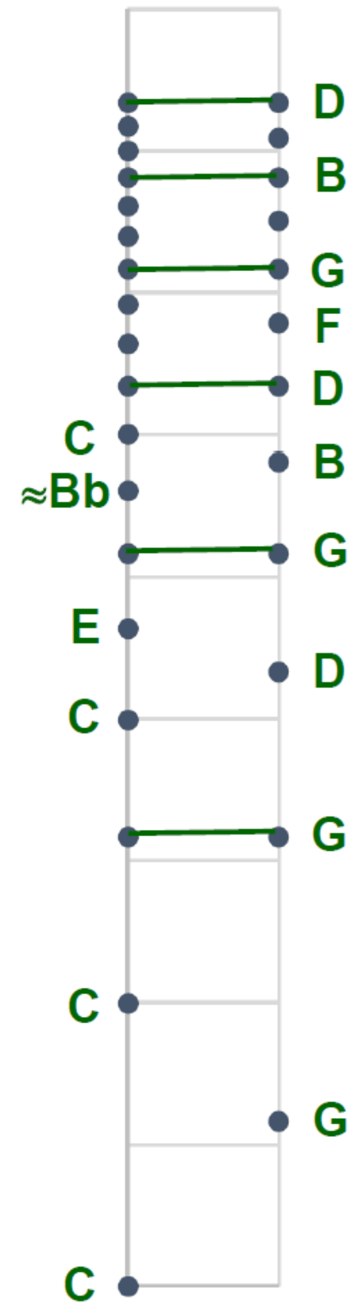
Music and Mathematics

- All elements of music are based on mathematical structure:
  - **Harmony** – mathematics of overtone series.
  - **Rhythm** – e.g., Indian ragas
  - **Melody** – combinatorial structure of Western polyphonic music.
  - **Scales** – foundation for harmony, melody, counterpoint, key relationships, etc.



Harmony

- Acoustic instruments produce multiple **harmonic partials**.
  - Frequency of partial = integral multiple of frequency of fundamental.
  - Coincidence of partials makes chords with simple ratios easy to recognize.



Perfect fifth
C:G = 2:3

Harmony

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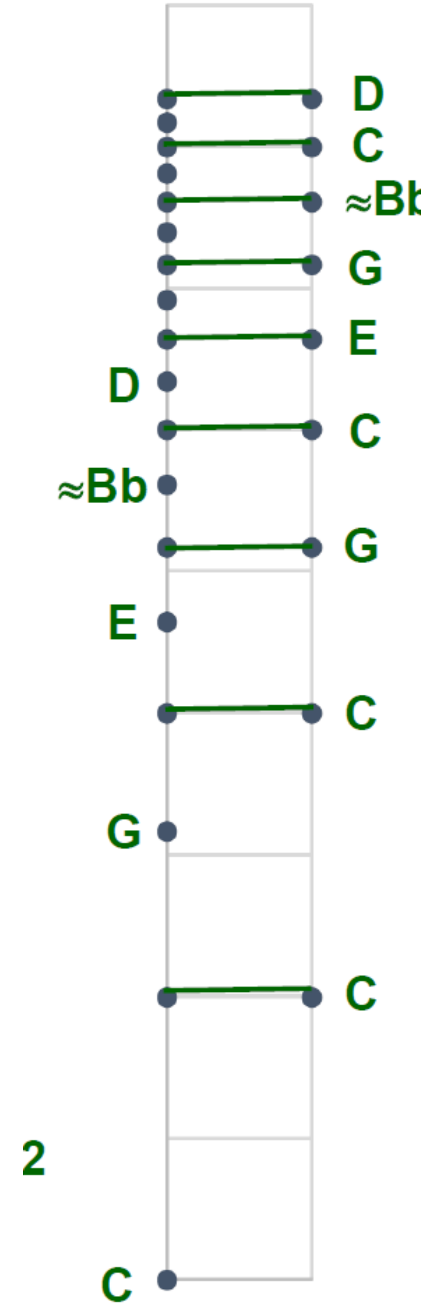
Octave
C:C = 1:2



Harmony

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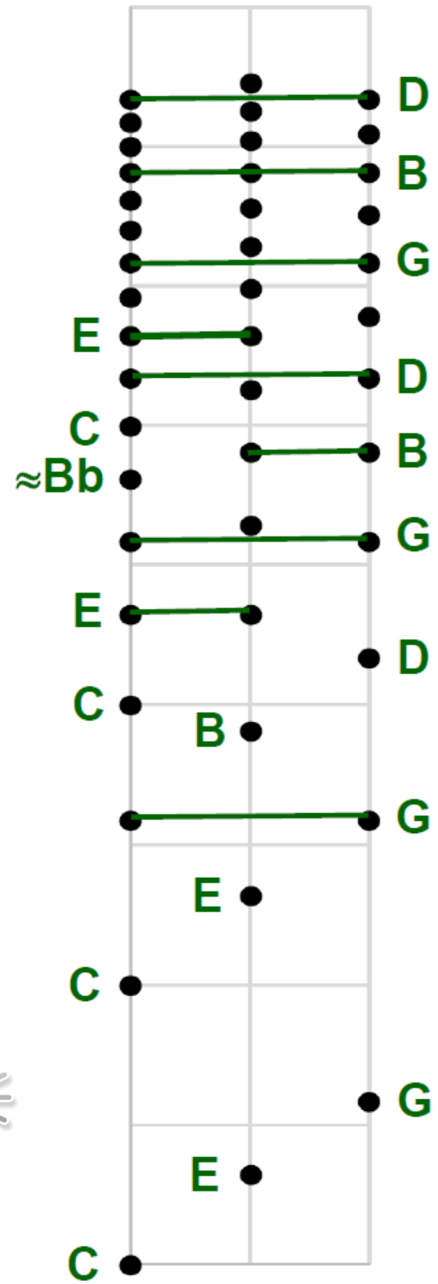


Major triad
C:E:G = 4:5:6

Polyphony

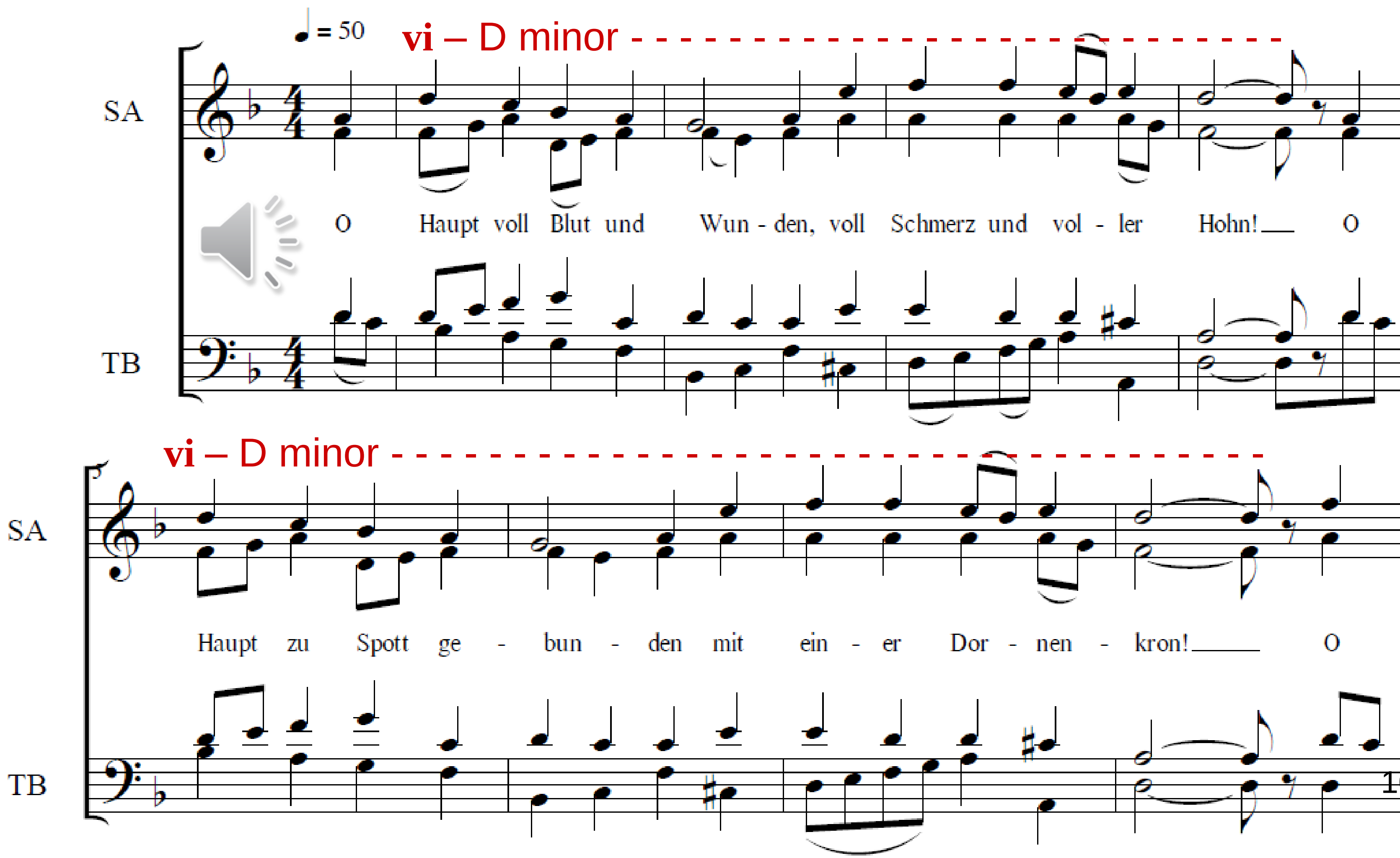
- A challenging combinatorial problem.
  - Relationships among multiple voices must be intelligible to the ear.
- Classic example: Bach's chorale harmonizations.
  - AI-based harmonization: follows some 350 rules, result tends to be mediocre.
  - Human harmonization: requires a highly skilled composer, result can be beautiful and inspiring.

Harmonization: Bach

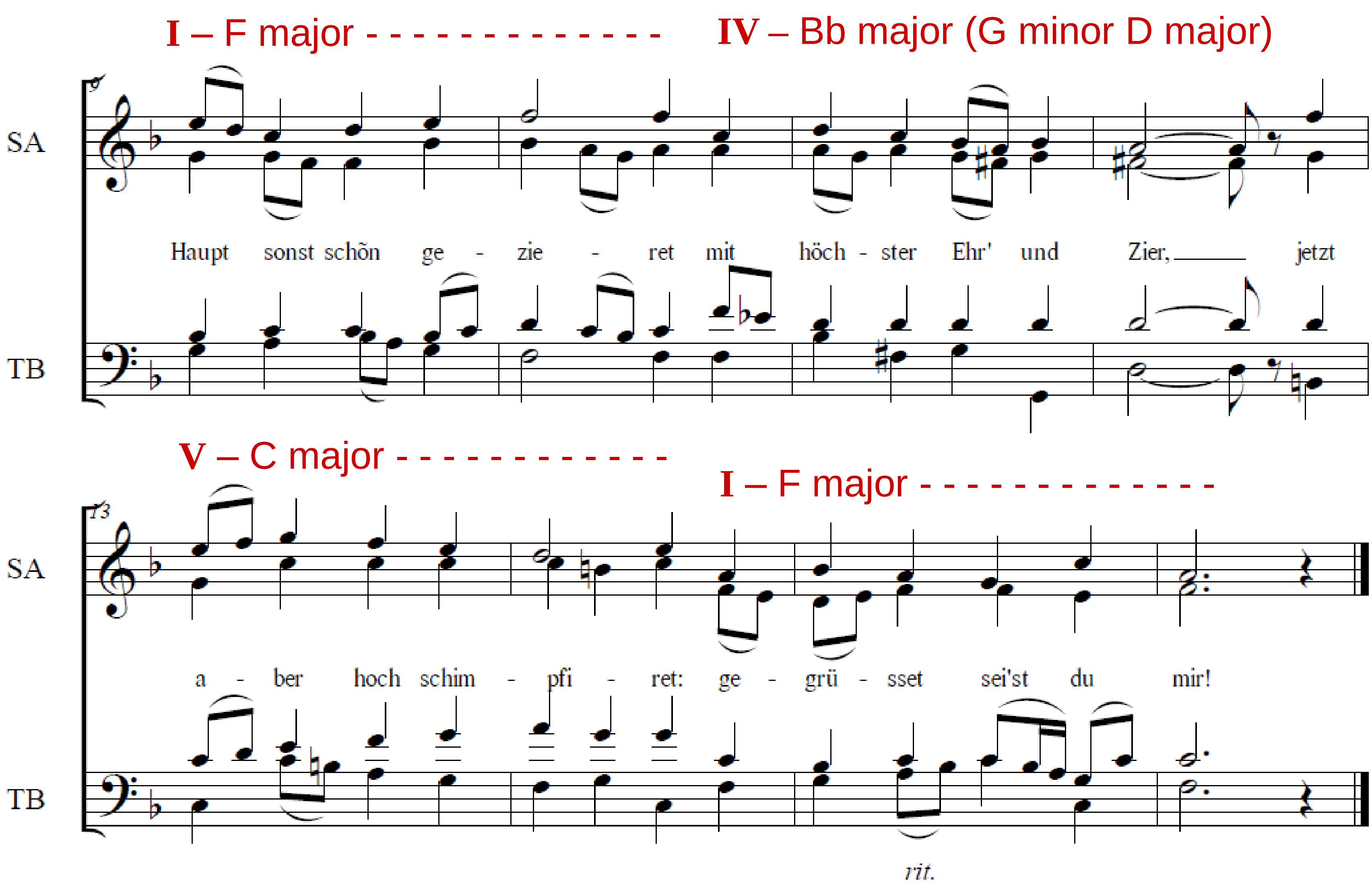
Passion Chorale

From St Matthew Passion (1727)

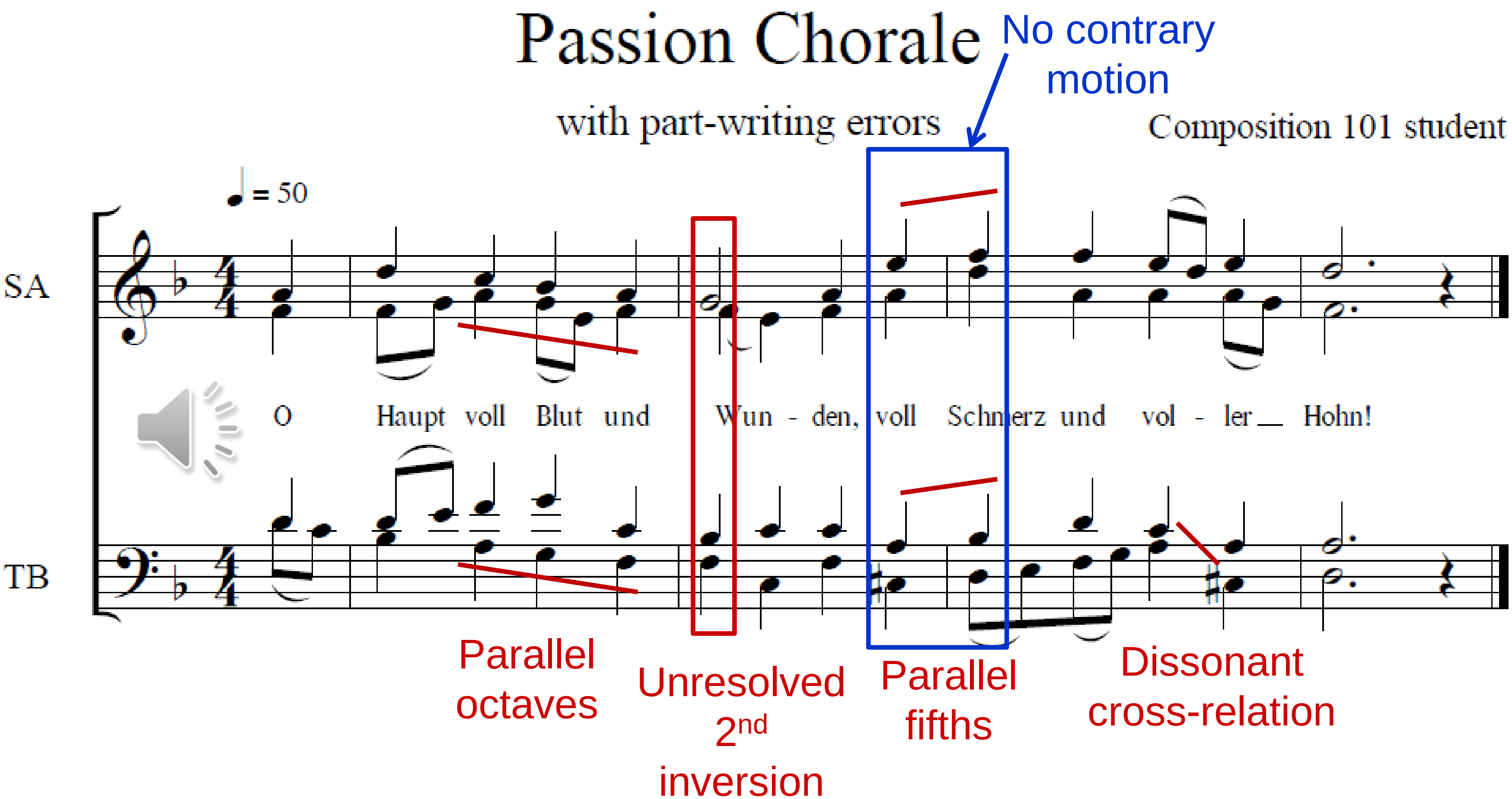
J. S. Bach



Harmonization: Bach



Harmonization: Amateur



Advantages of Classical Scales

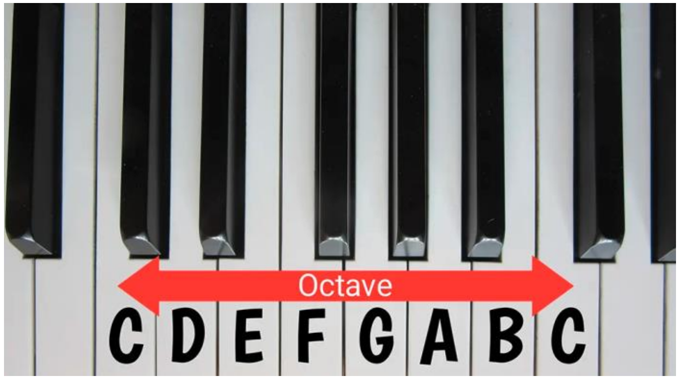
- Pitch frequencies have **simple ratios**.
  - Rich and intelligible harmonies

- **Multiple keys** based on underlying chromatic scale with **tempered tuning**.
  - Can play all keys on instrument with fixed tuning.
  - Complex musical structure.

- Can we find **new scales** with these same properties?
  - Constraint programming is well suited to solve the problem.

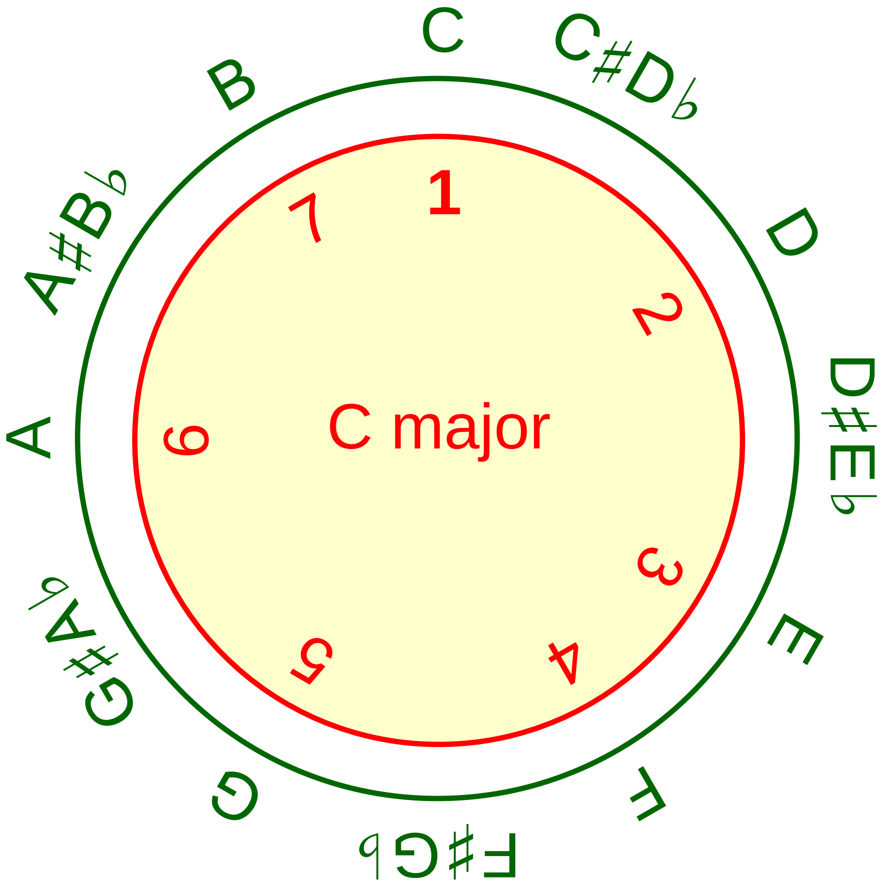
Multiple Keys

- A classical scale can start from any pitch in a **chromatic** scale with 12 **semitone** intervals.
  - Resulting in 12 **keys**.
  - An instrument with 12 pitches (modulo octaves) can play 12 different keys.
  - Can move to a different key by changing only a few notes of the scale.



Multiple Keys

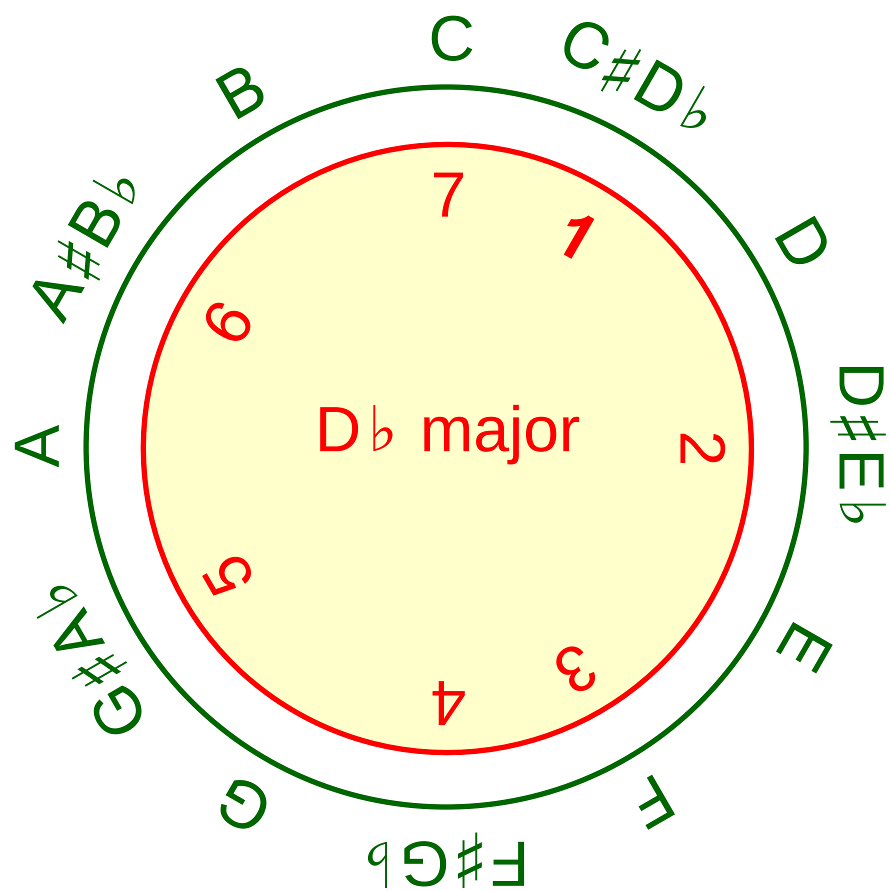
Let C major be the tonic key



distance **0** from C major

Multiple Keys

Let C major be
the tonic key

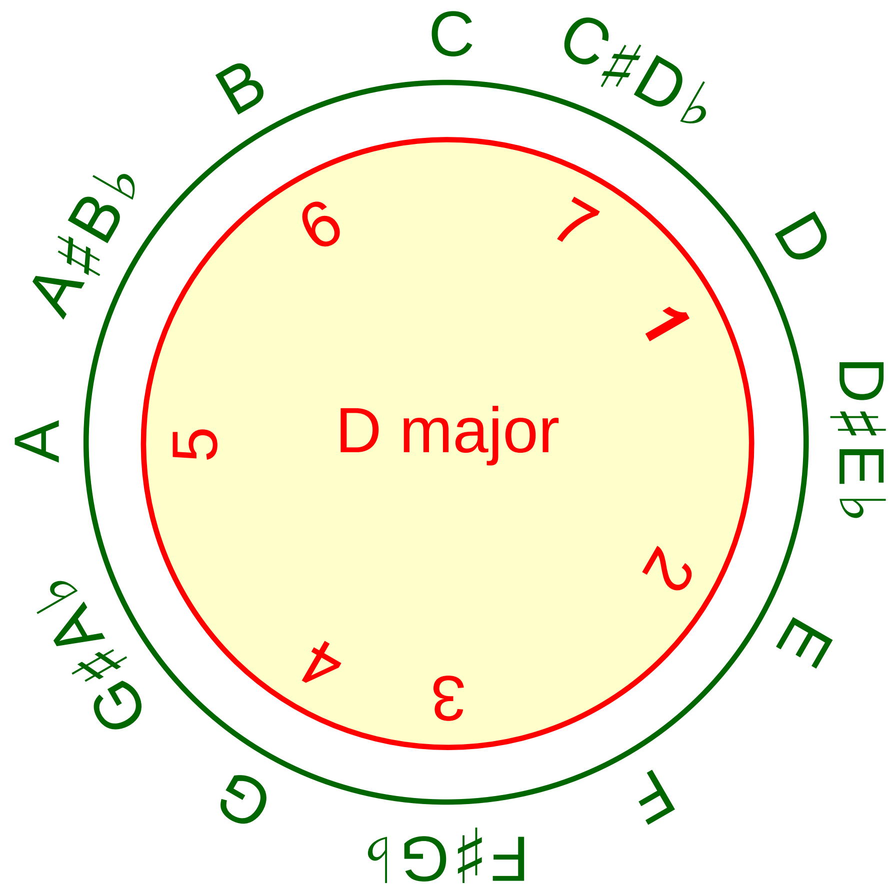


distance **5**
from C major

i.e., 5 notes do not
occur in C major

Multiple Keys

Let C major be
the tonic key



distance **2**
from C major

Multiple Keys

Let C major be
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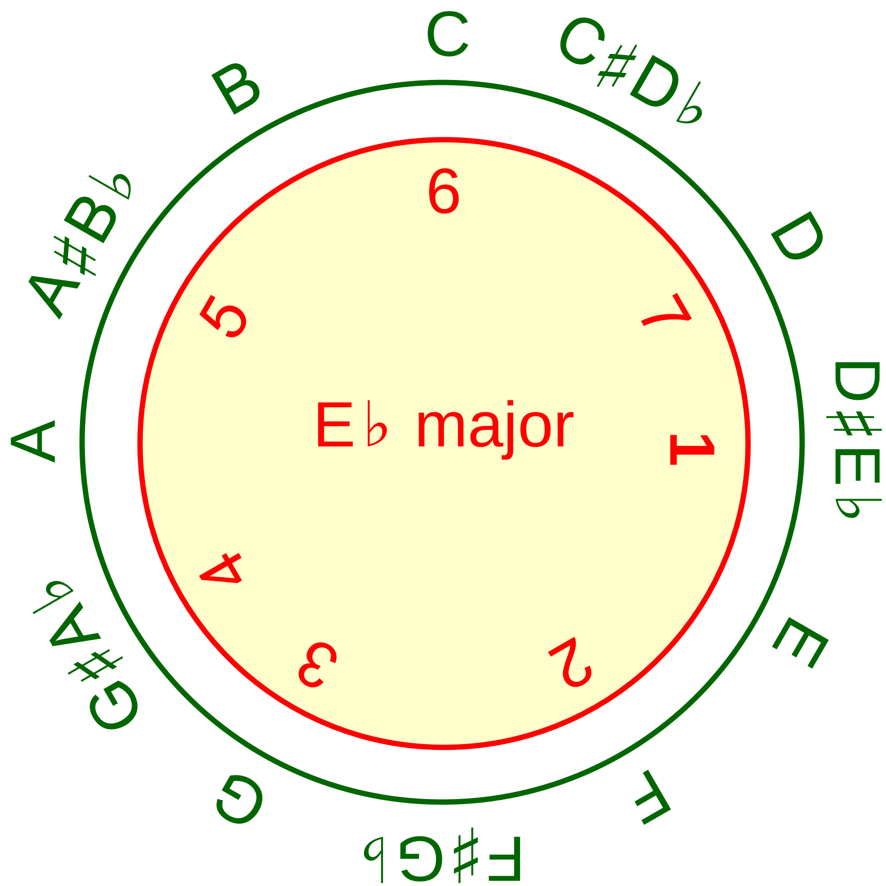

distance **3**
from C major

Multiple Keys

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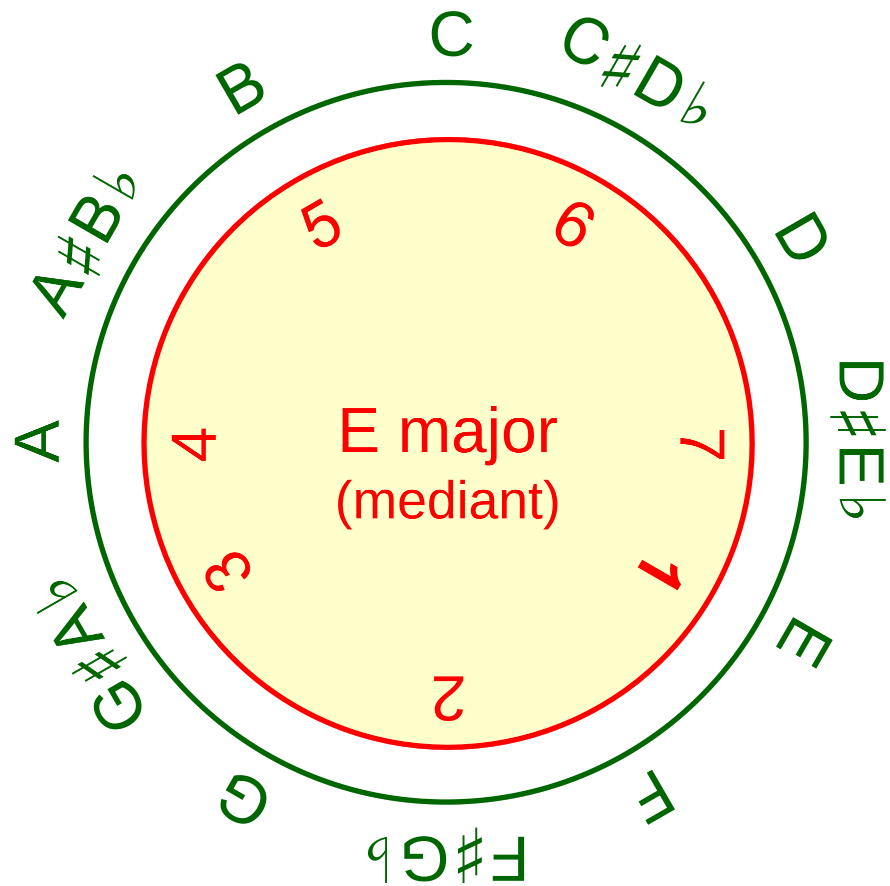
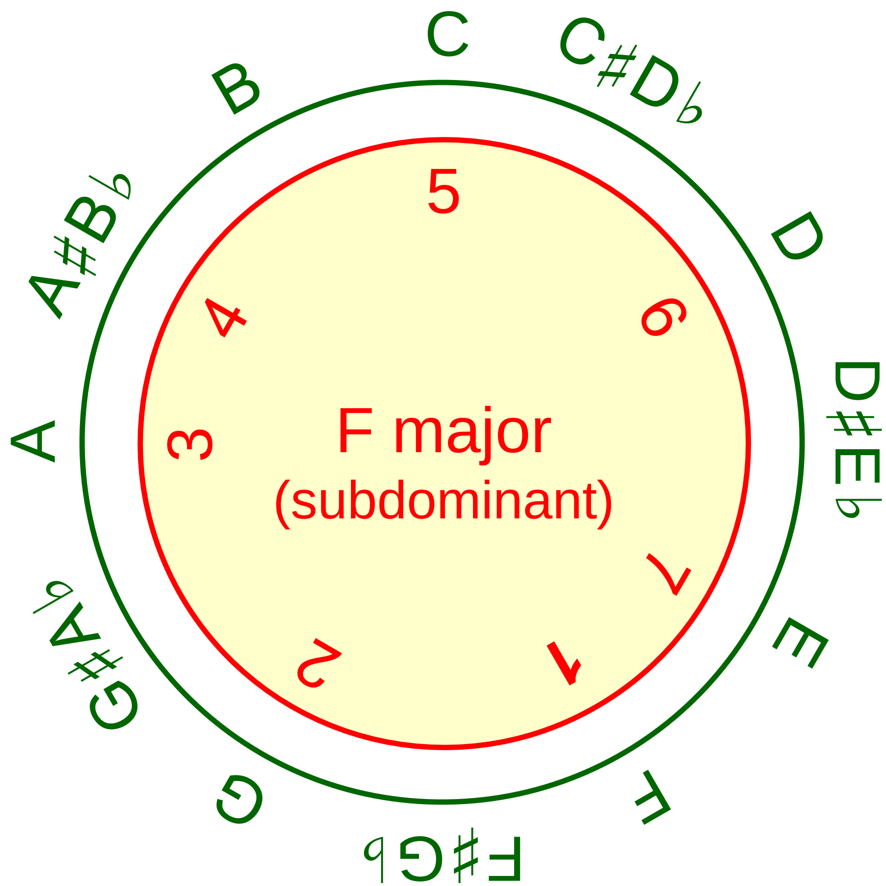


distance **4**
from C major

Multiple Keys

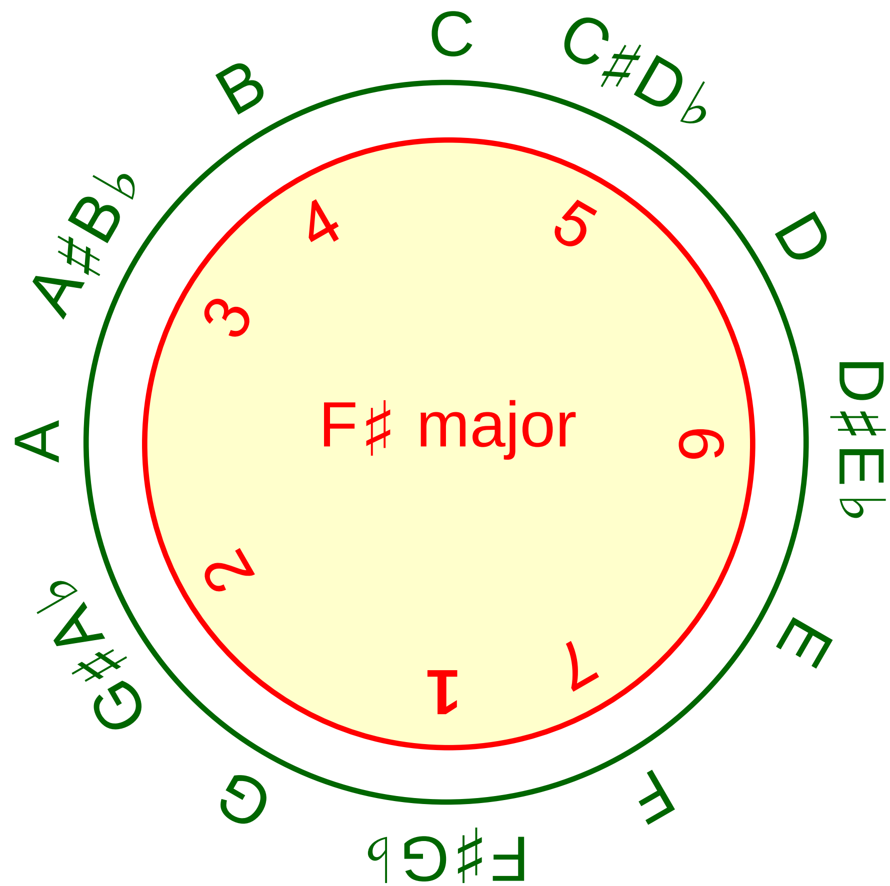
Let C major be
the tonic key



distance **1**
from C major

Multiple Keys

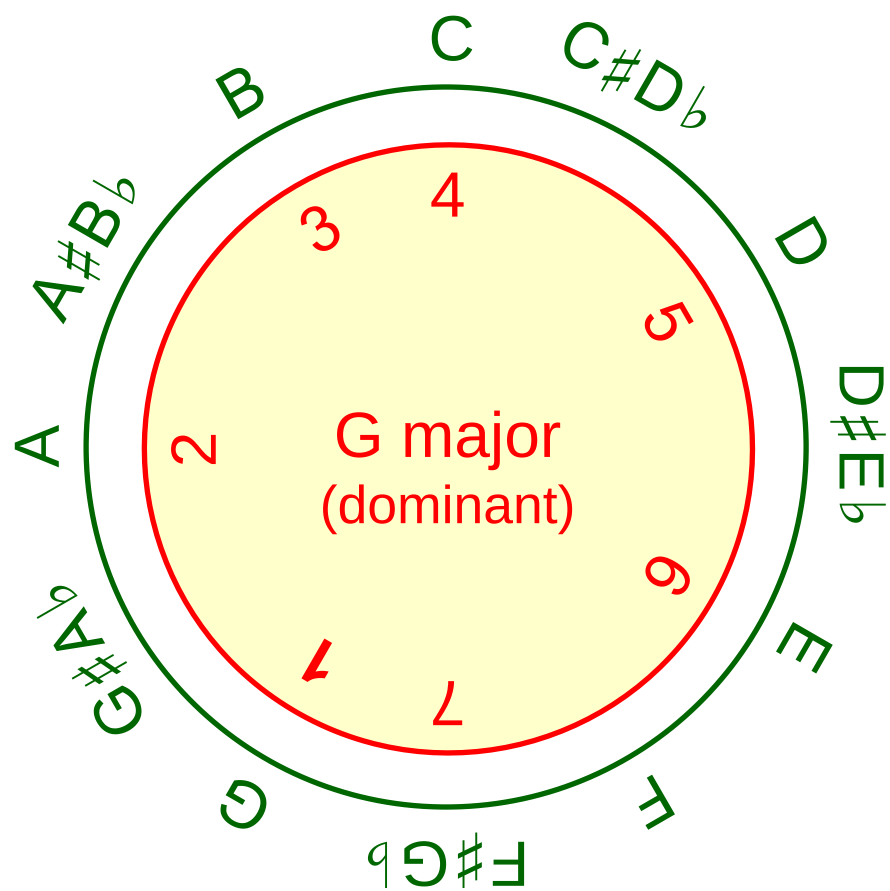
Let C major be the tonic key



distance **6** from C major

Multiple Keys

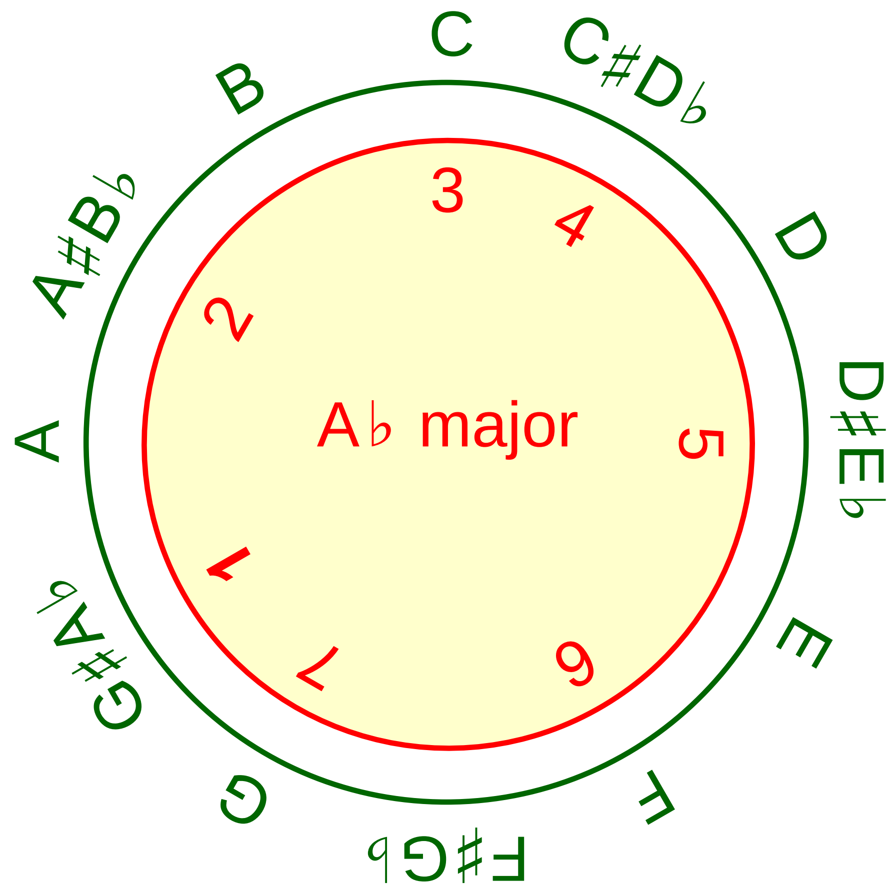
Let C major be
the tonic key



distance **1**
from C major

Multiple Keys

Let C major be
the tonic key



distance **4**
from C major

Multiple Keys

Let C major be
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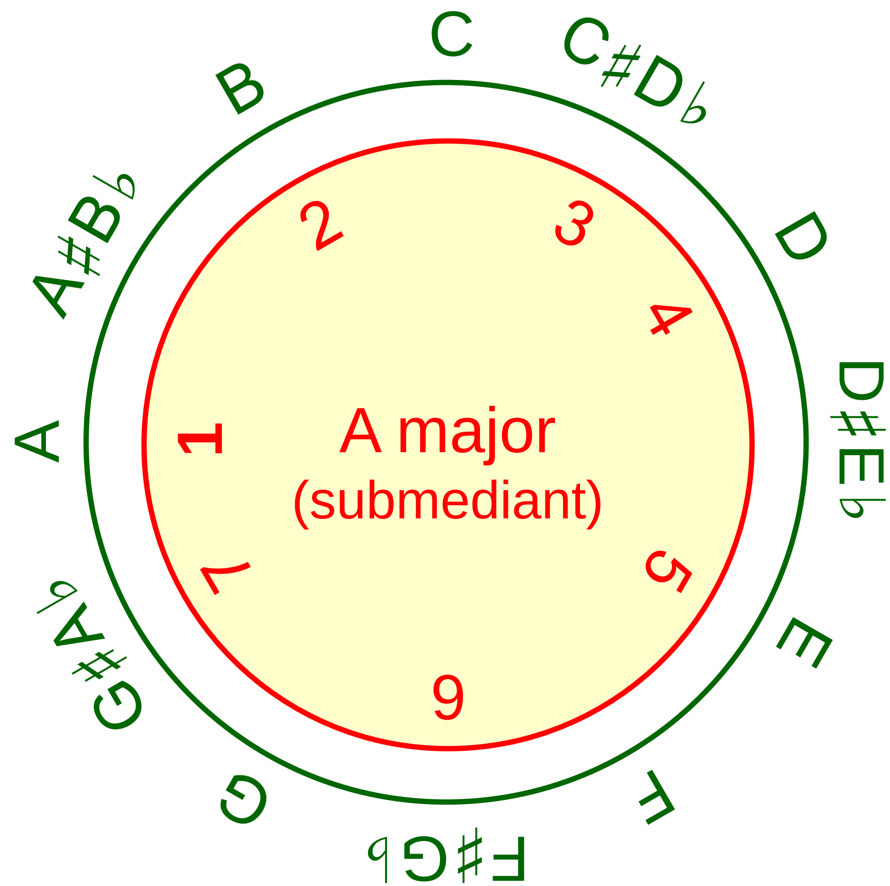
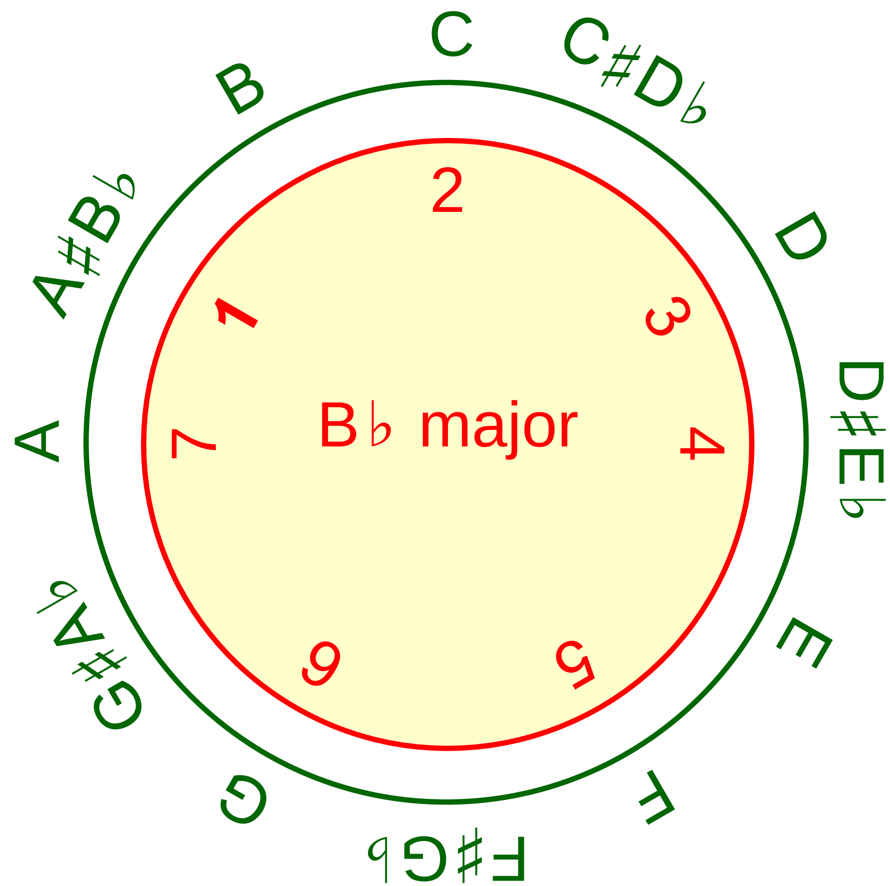


distance **3**
from C major

Multiple Keys

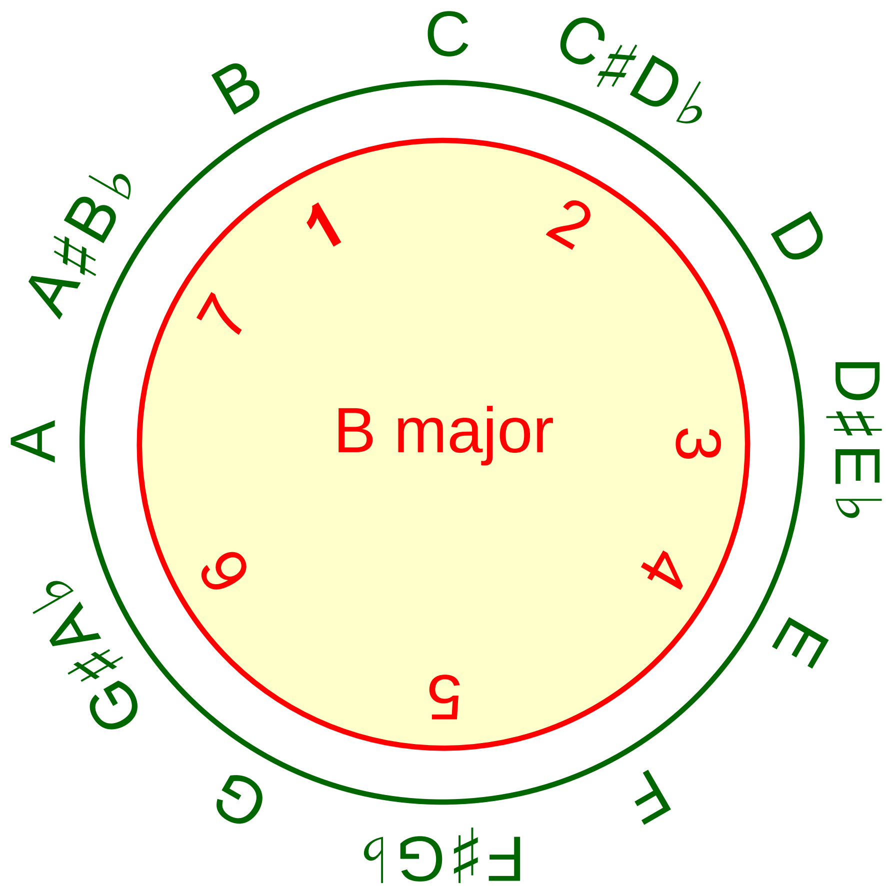
Let C major be
the tonic key



distance **2**
from C major

Multiple Keys

Let C major be
the tonic key



distance **5**
from C major

Multiple Keys

- Chromatic pitches ae **tempered** so that intervals will have approximately correct ratios in all keys.
  - Modern practice is **equal temperament**.

$$\frac{\text{freq of note } k}{\text{freq of note 1}} = 2^{(k-1)/12}$$

Multiple Keys

– Resulting error is ≤ ±0.9%

Note	Perfect ratio	Tempered ratio	Error %
C	1/1	1.00000	0.000
D	9/8	1.12246	−0.226
E	5/4	1.25992	+0.787
F	4/3	1.33484	+0.113
G	3/2	1.49831	−0.113
A	5/3	1.68179	+0.899
B	15/8	1.88775	+0.675

Combinatorial Requirements

- Scales must be **diatonic**
  - Adjacent notes are 1 or 2 semitones apart.

- We consider *m*-note scales on an *n*-tone chromatic
  - In binary representation, let m_0 = number of 0s
    m_1 = number of 1s
  - Then $m_0 = 2m - n$, $m_1 = n - \mathrm{m}$
    - In a major scale 1101110, there are $m = 7$ notes on an $n = 12$-tone chromatic
    - There are $m_0 = 2 \cdot 7 - 12 = 2$ zeros
    - There are $m_1 = 12 - 7 = 5$ ones

0 = semitone interval
1 = whole tone interval (2 semitones)

Combinatorial Requirements

- Semitones should not be bunched together.
  - One criterion: **Myhill's property**
  - All intervals of a given size should contain k or $k + 1$ semitones for some k.
    - For example, in a major scale:
    - All fifths are 6 or 7 semitones
    - All thirds are 3 or 4 semitones
    - All seconds are 1 or 2 semitones, etc.
  - Few scales satisfy Myhill's property

Combinatorial Requirements

- Semitones should not be bunched together.
  - We minimize the number of pairs of adjacent 0s and pairs of adjacent 1s.
  - If $m_0 \geq m_1$,

$$\text{number of adjacent 0s} = m_0 - \min\{m_0, m_1\}$$
$$\text{number of adjacent 1s} = 0$$

  - If $m_1 \geq m_0$,

$$\text{number of adjacent 1s} = m_1 - \min\{m_0, m_1\}$$
$$\text{number of adjacent 0s} = 0$$

    - In a major scale 1101110,
      number of pairs of adjacent 0s = 0
      number of pairs of adjacent 1s = 5 – min{2,5} = 3

Combinatorial Requirements

- Semitones should not be bunched together.
  - The number of scales satisfying this property is

$$\binom{\max\{m_0, m_1\}}{\min\{m_0, m_1\}} + \binom{\max\{m_0, m_1\} - 1}{\min\{m_0, m_1\} - 1}$$

    - The number of 7-note scales on a 12-tone chromatic satisfying this property is

$$\binom{5}{2} + \binom{4}{1} = 14$$

Combinatorial Requirements

- Can have fewer than *n* keys.
  - A “mode of limited transposition”
  - Whole tone scale 111111 (Debussy) has 2 keys
  - Scale 110110110 has 5 keys
    - Count number of semitones in repeating sequence

Temperament Requirements

- Tolerance for inaccurate tuning
  - At most ±0.9%
  - Don't exceed tolerance of classical equal temperament

Previous Work

- Scales on a tempered chromatic
  - Bohlen-Pierce scale (1978, Mathews et al. 1988)
    - 9 notes on 13-note chromatic spanning a 12^{th}
  - Music for Bohlen-Pierce scale
    - R.Boulanger, A. Radunskaya, J. Appleton
  - Scales of limited transposition
    - O. Messiaen
- Microtonal scales
  - Quarter-tone scale (24-tone equally tempered chromatic)
    - Bartok, Berg, Bloch, Boulez, Copeland, Enescu, Ives, Mancini.
  - 15- or 19-tone equally tempered chromatic
    - E. Blackwood

Previous Work

- “Super just” scales (perfect intervals, 1 key)
  - H. Partch (43 tones)
  - W. Carlos (12 tones)
  - L. Harrison (16 tones)
  - W. Perret (19 tones)
  - J. Chalmers (19 tones)
  - M. Harison (24 tones)
- Combinatorial properties
  - G. J. Balzano (1980)
  - T. Noll (2005, 2007, 2014)
  - E. Chew (2014), M. Pearce (2002), Zweifel (1996)

Simple Ratios

- Frequency of each note should have a simple ratio (between 1 and 2) with some other note
  - Equating notes an octave apart.
  - Let f_i = freq ratio of note i to tonic (note 1), $f_1 = 1$.
  - For major scale CDEFGAB,

$$(f_1, \ldots, f_7) = \left(1, \frac{9}{8}, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{15}{8}\right)$$

  - For example, B (15/8) has a simple ratio 3/2 with E (5/4)

$$\frac{f_7}{f_3} = \frac{3}{2}$$

  - D octave higher (9/4) has ratio 3/2 with G (3/2)

$$\frac{2f_2}{f_5} = \frac{3}{2}$$

Simple Ratios

- However, this allows two or more subsets of unrelated pitches.
  - Simple ratios with respect to pitches in same subset, but not in other subsets.
  - So we use a **recursive** condition.
  - For some permutation of notes, each note should have simple ratio with previous note.
  - First note in the permutation is the tonic.

Simple Ratios

- Let the simple ratios be **generators** $r_1, \ldots, r_p$.
  - Let $(\pi_1, \ldots, \pi_m)$ be a permutation of $1, \ldots, m$ with $\pi_1 = 1$.
  - For each $i \in \{2, \ldots, m\}$, we require

$$1 < f_{\pi_i} < 2$$

and

$$\frac{f_{\pi_i}}{f_{\pi_j}} = r_q \text{ or } \frac{2f_{\pi_j}}{f_{\pi_i}} = r_q \text{ or } \frac{f_{\pi_j}}{f_{\pi_i}} = r_q \text{ or } \frac{2f_{\pi_i}}{f_{\pi_j}} = r_q$$

for some $j \in \{1, \ldots, i-1\}$ and some $q \in \{1, \ldots, p\}$.

Simple Ratios

- Ratio with previous note in the permutation π must be a generator.
  - Ratios with previous 2 or 3 notes in the permutation will be simple (product of generators)
  - Ratio with tonic need not be simple.

Simple Ratios

- Observation: No need to consider both r_q and $2/r_q$ as generators.
  - So we consider only reduced fractions with odd numerators (in order of simplicity):

$$\frac{3}{2}, \frac{5}{3}, \frac{5}{4}, \frac{7}{4}, \frac{7}{5}, \frac{9}{5}, \frac{7}{6}, \frac{11}{6}, \frac{9}{7}, \frac{11}{7}, \frac{13}{7}, \frac{9}{8}, \frac{11}{8}, \frac{13}{8}, \frac{15}{8}, \frac{11}{9}, \frac{13}{9}, \frac{17}{9}, \ldots$$

Simple Ratios

- CP model readily accommodates variable indices

$$f_{\pi_i}$$

- Replace f_i with fraction a_i/b_i in lowest terms.

$$\frac{3}{2}, \frac{5}{3}, \frac{5}{4}, \frac{7}{4}, \frac{7}{5}, \frac{9}{5}, \frac{7}{6}, \frac{11}{6}, \frac{9}{7}, \frac{11}{7}, \frac{13}{7}, \frac{9}{8}, \frac{11}{8}, \frac{13}{8}, \frac{15}{8}, \frac{11}{9}, \frac{13}{9}, \frac{17}{9}, \ldots$$

CP Model

$$\text{alldiff}(\pi_1, \ldots, \pi_m)$$

$$\pi_1 = a_1 = b_1 = 1$$

$$1 < \frac{a_i}{b_i} < 2, \quad \text{coprime}(a_i, b_i), \ i = 1, \ldots, m$$

$$\frac{a_{i-1}}{b_{i-1}} < \frac{a_i}{b_i}, \ i = 2, \ldots, m$$

$$\bigvee_{j<i} \left[(\pi_i > \pi_j) \Rightarrow \left(\frac{a_{\pi_i}/b_{\pi_i}}{a_{\pi_j}/b_{\pi_j}} \in G \ \vee \ \frac{2a_{\pi_j}/b_{\pi_j}}{a_{\pi_i}/b_{\pi_i}} \in G \right) \right], \ i = 2, \ldots, m$$

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$$\frac{|a_i/b_i - 2^{(t_i-1)/n}|}{2^{(t_i-1)/n}} \le 0.009, \ i = 1, \ldots, m$$

$$\pi_i \in \{1, \ldots, m\}, \ a_i \in \{1, \ldots, 2M\}, \ b_i \in \{1, \ldots, M\}, \ i = 1, \ldots, m$$

CP Model

$$\text{alldiff}(\pi_1, \ldots, \pi_m)$$ ← permutation

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CP Model

$$\text{alldiff}(\pi_1, \ldots, \pi_m)$$

$$\pi_1 = a_1 = b_1 = 1$$ ← tonic note

$$1 < \frac{a_i}{b_i} < 2, \quad \text{coprime}(a_i, b_i), \; i = 1, \ldots, m$$

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CP Model

$$\text{alldiff}(\pi_1, \ldots, \pi_m)$$

$$\pi_1 = a_1 = b_1 = 1$$

predefined array

$$1 < \frac{a_i}{b_i} < 2, \ \boxed{\text{coprime}(a_i, b_i)}, \ i = 1, \ldots, m$$

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CP Model

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$$\frac{a_{i-1}}{b_{i-1}} < \frac{a_i}{b_i}, \; i = 2, \ldots, m$$

← symmetry breaking

$$\bigvee_{j<i} \left[(\pi_i > \pi_j) \Rightarrow \left(\frac{a_{\pi_i}/b_{\pi_i}}{a_{\pi_j}/b_{\pi_j}} \in G \;\vee\; \frac{2a_{\pi_j}/b_{\pi_j}}{a_{\pi_i}/b_{\pi_i}} \in G \right) \right], \; i = 2, \ldots, m$$

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CP Model

$$\text{alldiff}(\pi_1, \ldots, \pi_m)$$

$$\pi_1 = a_1 = b_1 = 1$$

$$1 < \frac{a_i}{b_i} < 2, \quad \text{coprime}(a_i, b_i), \; i = 1, \ldots, m$$

$$\frac{a_{i-1}}{b_{i-1}} < \frac{a_i}{b_i}, \; i = 2, \ldots, m$$

simple ratios

$$\bigvee_{j<i} \left[(\pi_i > \pi_j) \Rightarrow \left(\frac{a_{\pi_i}/b_{\pi_i}}{a_{\pi_j}/b_{\pi_j}} \in G \;\vee\; \frac{2a_{\pi_j}/b_{\pi_j}}{a_{\pi_i}/b_{\pi_i}} \in G \right) \right], \; i = 2, \ldots, m$$

$$\bigvee_{j<i} \left[(\pi_i < \pi_j) \Rightarrow \left(\frac{a_{\pi_j}/b_{\pi_j}}{a_{\pi_i}/b_{\pi_i}} \in G \;\vee\; \frac{2a_{\pi_i}/b_{\pi_i}}{a_{\pi_j}/b_{\pi_j}} \in G \right) \right], \; i = 2, \ldots, m$$

$$\frac{|a_i/b_i - 2^{(t_i-1)/n}|}{2^{(t_i-1)/n}} \leq 0.009, \; i = 1, \ldots, m$$

$$\pi_i \in \{1, \ldots, m\}, \; a_i \in \{1, \ldots, 2M\}, \; b_i \in \{1, \ldots, M\}, \; i = 1, \ldots, m$$

CP Model

$$\text{alldiff}(\pi_1, \ldots, \pi_m)$$

$$\pi_1 = a_1 = b_1 = 1$$

$$1 < \frac{a_i}{b_i} < 2, \quad \text{coprime}(a_i, b_i), \; i = 1, \ldots, m$$

$$\frac{a_{i-1}}{b_{i-1}} < \frac{a_i}{b_i}, \; i = 2, \ldots, m$$

set of generators

$$\bigvee_{j<i} \left[(\pi_i > \pi_j) \Rightarrow \left(\frac{a_{\pi_i}/b_{\pi_i}}{a_{\pi_j}/b_{\pi_j}} \in \boxed{G} \vee \frac{2a_{\pi_j}/b_{\pi_j}}{a_{\pi_i}/b_{\pi_i}} \in G \right) \right], \; i = 2, \ldots, m$$

$$\bigvee_{j<i} \left[(\pi_i < \pi_j) \Rightarrow \left(\frac{a_{\pi_j}/b_{\pi_j}}{a_{\pi_i}/b_{\pi_i}} \in G \vee \frac{2a_{\pi_i}/b_{\pi_i}}{a_{\pi_j}/b_{\pi_j}} \in G \right) \right], \; i = 2, \ldots, m$$

$$\frac{|a_i/b_i - 2^{(t_i-1)/n}|}{2^{(t_i-1)/n}} \leq 0.009, \; i = 1, \ldots, m$$

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$$\bigvee_{j<i} \left[(\pi_i > \pi_j) \Rightarrow \left(\frac{a_{\pi_i}/b_{\pi_i}}{a_{\pi_j}/b_{\pi_j}} \in G \; \vee \; \frac{2a_{\pi_j}/b_{\pi_j}}{a_{\pi_i}/b_{\pi_i}} \in G \right) \right], \; i = 2, \ldots, m$$

$$\bigvee_{j<i} \left[(\pi_i < \pi_j) \Rightarrow \left(\frac{a_{\pi_j}/b_{\pi_j}}{a_{\pi_i}/b_{\pi_i}} \in G \; \vee \; \frac{2a_{\pi_i}/b_{\pi_i}}{a_{\pi_j}/b_{\pi_j}} \in G \right) \right], \; i = 2, \ldots, m$$

$$\frac{|a_i/b_i - 2^{(t_i-1)/n}|}{2^{(t_i-1)/n}} \leq 0.009, \; i = 1, \ldots, m$$

← tuning tolerance

$$\pi_i \in \{1, \ldots, m\}, \; a_i \in \{1, \ldots, 2M\}, \; b_i \in \{1, \ldots, M\}, \; i = 1, \ldots, m$$

CP Model

$$\text{alldiff}(\pi_1, \ldots, \pi_m)$$

$$\pi_1 = a_1 = b_1 = 1$$

$$1 < \frac{a_i}{b_i} < 2, \quad \text{coprime}(a_i, b_i), \; i = 1, \ldots, m$$

$$\frac{a_{i-1}}{b_{i-1}} < \frac{a_i}{b_i}, \; i = 2, \ldots, m$$

$$\bigvee_{j<i} \left[(\pi_i > \pi_j) \Rightarrow \left(\frac{a_{\pi_i}/b_{\pi_i}}{a_{\pi_j}/b_{\pi_j}} \in G \;\vee\; \frac{2a_{\pi_j}/b_{\pi_j}}{a_{\pi_i}/b_{\pi_i}} \in G \right) \right], \; i = 2, \ldots, m$$

$$\bigvee_{j<i} \left[(\pi_i < \pi_j) \Rightarrow \left(\frac{a_{\pi_j}/b_{\pi_j}}{a_{\pi_i}/b_{\pi_i}} \in G \;\vee\; \frac{2a_{\pi_i}/b_{\pi_i}}{a_{\pi_j}/b_{\pi_j}} \in G \right) \right], \; i = 2, \ldots, m$$

$$\frac{|a_i/b_i - 2^{(t_i-1)/n}|}{2^{(t_i-1)/n}} \leq 0.009, \; i = 1, \ldots, m$$

$$\pi_i \in \{1, \ldots, m\}, \; a_i \in \{1, \ldots, 2M\}, \; b_i \in \{1, \ldots, M\}, \; i = 1, \ldots, m$$

chromatic tone corresponding to note *i*

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Scales on a 12-note chromatic

- Use the generators mentioned earlier.
  - There are **multiple solutions** for each scale.
  - For each note, compute the **minimal generator**, or the simplest ratio with another note.
  - Select the solution with the **simplest ratios** with the tonic and/or **simplest minimal generators**.
  - The 7-note scales with a **single generator 3/2** are **precisely the classical modes!**

7-note scales on a 12-note chromatic

Scale	Solns	Ratios with tonic	Minimal generators	
1. 0101111	27	$\frac{1}{1}$ $\frac{16}{15}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{4}$ $\frac{9}{8}$ $\frac{3}{2}$ $\frac{5}{3}$	
2. 0110111	10	$\frac{1}{1}$ $\frac{18}{17}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{24}{17}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$	Locrian mode
3. 0111011	18	$\frac{1}{1}$ $\frac{16}{15}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$	Phrygian mode
4. 0111101	26	$\frac{1}{1}$ $\frac{16}{15}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{5}{3}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$	
5. 1010111	6	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{5}{4}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{4}$ $\frac{3}{2}$ $\frac{3}{2}$	
6. 1011011	6	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$	Aeolian mode (natural minor)
7. 1011101	10	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$	Dorian mode
8. 1011110	27	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{5}{3}$	melodic minor
9. 1101011	14	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{9}{8}$	
10. 1101101	9	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$	Mixolydian mode
11. 1101110	17	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$	Ionian mode (major)
12. 1110101	10	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{4}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$	
13. 1110110	16	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$	Lydian mode
14. 1111010	34	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{5}{3}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{4}$ $\frac{3}{2}$ $\frac{3}{2}$	

7-note scales on a 12-note chromatic

Scale	Solns	Ratios with tonic	Minimal generators	
1. 0101111	27	$\frac{1}{1}$ $\frac{16}{15}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{4}$ $\frac{9}{8}$ $\frac{3}{2}$ $\frac{5}{3}$	
2. 0110111	10	$\frac{1}{1}$ $\frac{18}{17}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{24}{17}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$	Locrian mode
3. 0111011	18	$\frac{1}{1}$ $\frac{16}{15}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$	Phrygian mode
4. 0111101	26	$\frac{1}{1}$ $\frac{16}{15}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{5}{3}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$	
5. 1010111	6	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{5}{4}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{4}$ $\frac{3}{2}$ $\frac{3}{2}$	
6. 1011011	6	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$	Aeolian mode (natural minor)
7. 1011101	10	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$	Dorian mode
8. 1011110	27	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{5}{3}$	melodic minor
9. 1101011	14	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{9}{8}$	
10. 1101101	9	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$	Mixolydian mode
11. 1101110	17	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$	Ionian mode (major)
12. 1110101	10	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{4}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$	
13. 1110110	16	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$	Lydian mode
14. 1111010	34	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{5}{3}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{4}$ $\frac{3}{2}$ $\frac{3}{2}$	

Single generator

Other scales on a 12-note chromatic

Scale	Solns	Keys	Ratios with tonic	Minimal generators
1. 111111	6	2	$\frac{1}{1}\ \frac{9}{8}\ \frac{5}{4}\ \frac{45}{32}\ \frac{8}{5}\ \frac{16}{9}$	$\frac{5}{4}\ \frac{5}{4}\ \frac{5}{4}\ \frac{5}{4}\ \frac{5}{4}\ \frac{9}{5}$
1. 01010101	>50	3	$\frac{1}{1}\ \frac{16}{15}\ \frac{6}{5}\ \frac{5}{4}\ \frac{45}{32}\ \frac{3}{2}\ \frac{5}{3}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{5}{3}\ \frac{3}{2}\ \frac{9}{8}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}$
2. 10101010	>50	3	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{45}{32}\ \frac{8}{5}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}$
21. 100001010	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{45}{32}\ \frac{8}{5}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$
22. 100010010	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$
23. 100010100	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{3}{2}\ \frac{8}{5}\ \frac{16}{9}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$
24. 100100010	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{45}{32}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$
25. 100100100	>50	4	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{45}{32}\ \frac{3}{2}\ \frac{8}{5}\ \frac{16}{9}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{9}{5}\ \frac{3}{2}$
26. 100101000	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{45}{32}\ \frac{3}{2}\ \frac{5}{3}\ \frac{16}{9}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{9}{8}\ \frac{3}{2}$
27. 101000010	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{45}{32}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}$
28. 101000100	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{45}{32}\ \frac{3}{2}\ \frac{8}{5}\ \frac{16}{9}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$
29. 101001000	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{45}{32}\ \frac{3}{2}\ \frac{5}{3}\ \frac{16}{9}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}$
30. 101010000	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{45}{32}\ \frac{8}{5}\ \frac{5}{3}\ \frac{16}{9}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}$

Other scales on a 12-note chromatic

Scale	Solns	Keys	Ratios with tonic	Minimal generators
1. 111111	6	2	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{9}{5}$

Whole tone scale. Minimal interest musically

Scale	Solns	Keys	Ratios with tonic	Minimal generators
21. 100001010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
22. 100010010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
23. 100010100	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
24. 100100010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
25. 100100100	>50	4	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{9}{5}$ $\frac{3}{2}$
26. 100101000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{9}{8}$ $\frac{3}{2}$
27. 101000010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$
28. 101000100	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
29. 101001000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$
30. 101010000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$

Other scales on a 12-note chromatic

Scale	Solns	Keys	Ratios with tonic	Minimal generators
1. 111111	6	2	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{9}{5}$
1. 01010101	>50	3	$\frac{1}{1}$ $\frac{16}{15}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$	$\frac{3}{2}$ $\frac{5}{3}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{9}{8}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$
2. 10101010	>50	3	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$

8-note scales. Only 3 keys.

⋮

23. 100010100	>50	12	$\frac{1}{1}$ [illegible]/8 [illegible]/5 [illegible]/4 [illegible]/3 [illegible]/2 [illegible]/5 [illegible]/9 [illegible]/8	[illegible]/2 [illegible]/2 [illegible]/2 [illegible]/2 [illegible]/2 [illegible]/2 [illegible]/2 [illegible]/2 [illegible]/2
24. 100100010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
25. 100100100	>50	4	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{9}{5}$ $\frac{3}{2}$
26. 100101000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{9}{8}$ $\frac{3}{2}$
27. 101000010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$
28. 101000100	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
29. 101001000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$
30. 101010000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$

Other scales on a 12-note chromatic

Scale	Solns	Keys	Ratios with tonic	Minimal generators
1. 111111	6	2	$\frac{1}{1}\ \frac{9}{8}\ \frac{5}{4}\ \frac{45}{32}\ \frac{8}{5}\ \frac{16}{9}$	$\frac{5}{4}\ \frac{5}{4}\ \frac{5}{4}\ \frac{5}{4}\ \frac{5}{4}\ \frac{9}{5}$

9-note scales beginning with whole tone interval

21. 100001010	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{45}{32}\ \frac{8}{5}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$
22. 100010010	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$
23. 100010100	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{3}{2}\ \frac{8}{5}\ \frac{16}{9}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$
24. 100100010	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{45}{32}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$
25. 100100100	>50	4	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{45}{32}\ \frac{3}{2}\ \frac{8}{5}\ \frac{16}{9}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{9}{5}\ \frac{3}{2}$
26. 100101000	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{45}{32}\ \frac{3}{2}\ \frac{5}{3}\ \frac{16}{9}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{9}{8}\ \frac{3}{2}$
27. 101000010	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{45}{32}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}$
28. 101000100	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{45}{32}\ \frac{3}{2}\ \frac{8}{5}\ \frac{16}{9}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$
29. 101001000	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{45}{32}\ \frac{3}{2}\ \frac{5}{3}\ \frac{16}{9}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}$
30. 101010000	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{45}{32}\ \frac{8}{5}\ \frac{5}{3}\ \frac{16}{9}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}$

Other scales on a 12-note chromatic

Scale	Solns	Keys	Ratios with tonic	Minimal generators
1. 111111	6	2	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{9}{5}$

Most appealing scales. Simple ratios, good distribution of semitones.

Scale	Solns	Keys	Ratios with tonic	Minimal generators
22. 100010010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
23. 100010100	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
24. 100100010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
25. 100100100	>50	4	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{9}{5}$ $\frac{3}{2}$
26. 100101000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{9}{8}$ $\frac{3}{2}$
27. 101000010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$
28. 101000100	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
29. 101001000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$
30. 101010000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$

Other scales on a 12-note chromatic

Scale	Solns	Keys	Ratios with tonic	Minimal generators
1. 111111	6	2	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{16}{9}$	$\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{5}{4}$ $\frac{9}{5}$

Will illustrate this scale with a Chorale and Fugue for organ

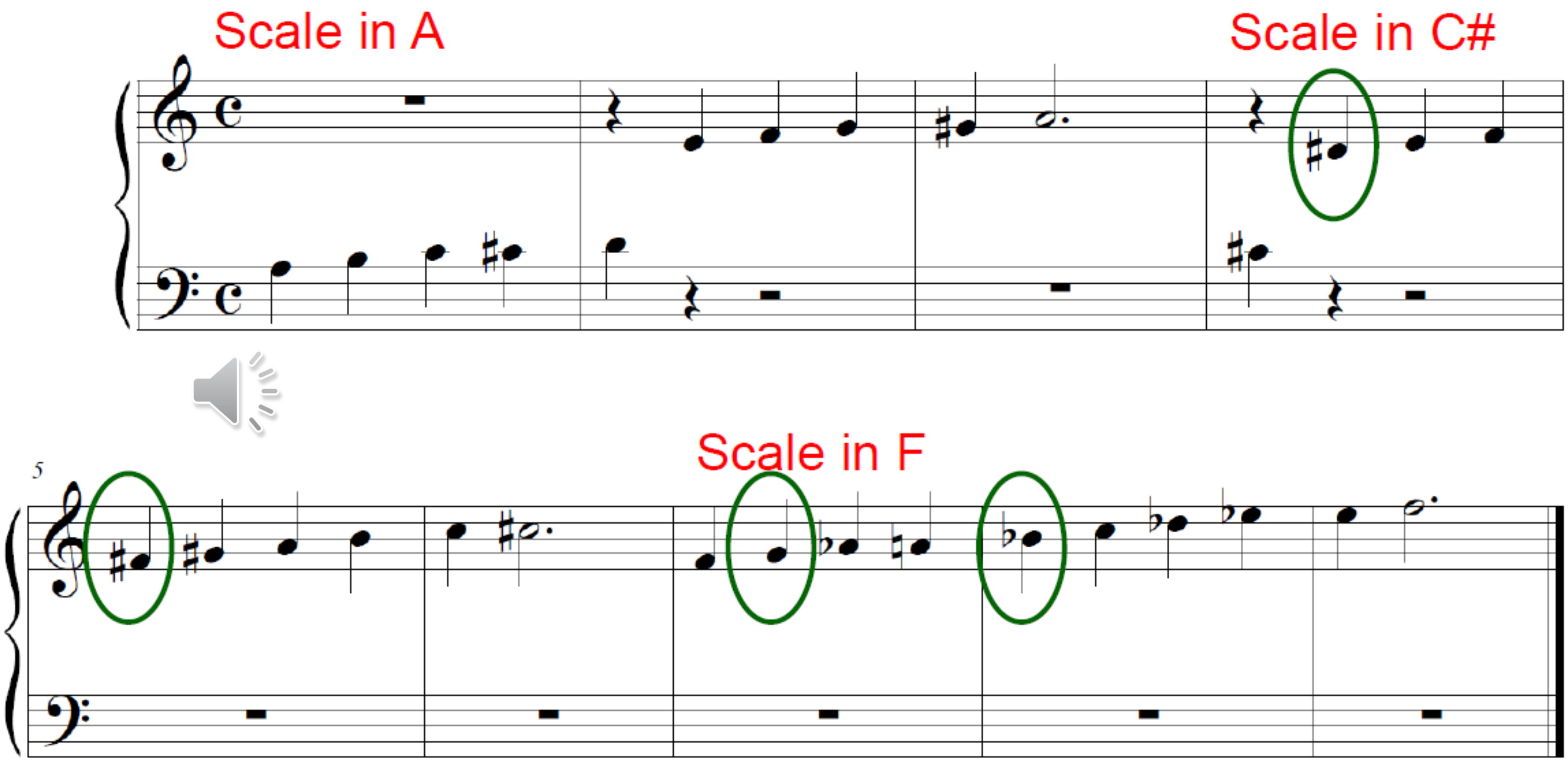
22. 100010010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
23. 100010100	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{4}{3}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
24. 100100010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
25. 100100100	>50	4	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{9}{5}$ $\frac{3}{2}$
26. 100101000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{5}{4}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{9}{8}$ $\frac{3}{2}$
27. 101000010	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$
28. 101000100	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{8}{5}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$
29. 101001000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$
30. 101010000	>50	12	$\frac{1}{1}$ $\frac{9}{8}$ $\frac{6}{5}$ $\frac{4}{3}$ $\frac{45}{32}$ $\frac{8}{5}$ $\frac{5}{3}$ $\frac{16}{9}$ $\frac{15}{8}$	$\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{3}{2}$ $\frac{3}{2}$

Demonstration: 9-note scale

- Chorale and Fugue for organ
- Chorale
  - In A, cycles through 2 most closely related keys: A, C$\sharp$, F, A
  - Modulate to C$\sharp$ at bar 27
  - Final sections starts at bar 72 (5:56)
- Fugue
  - Double fugue
  - First subject enters at pitches A, C$\sharp$, F
  - Second subject enters at bar 96.
  - Final episode at bar 164 (13:36)
  - Recapitulation at bar 170

Demonstration: 9-note scale

Key of A and 2 most closely related keys.



New notes are circled

Chorale and Fugue

On a 9-note Scale

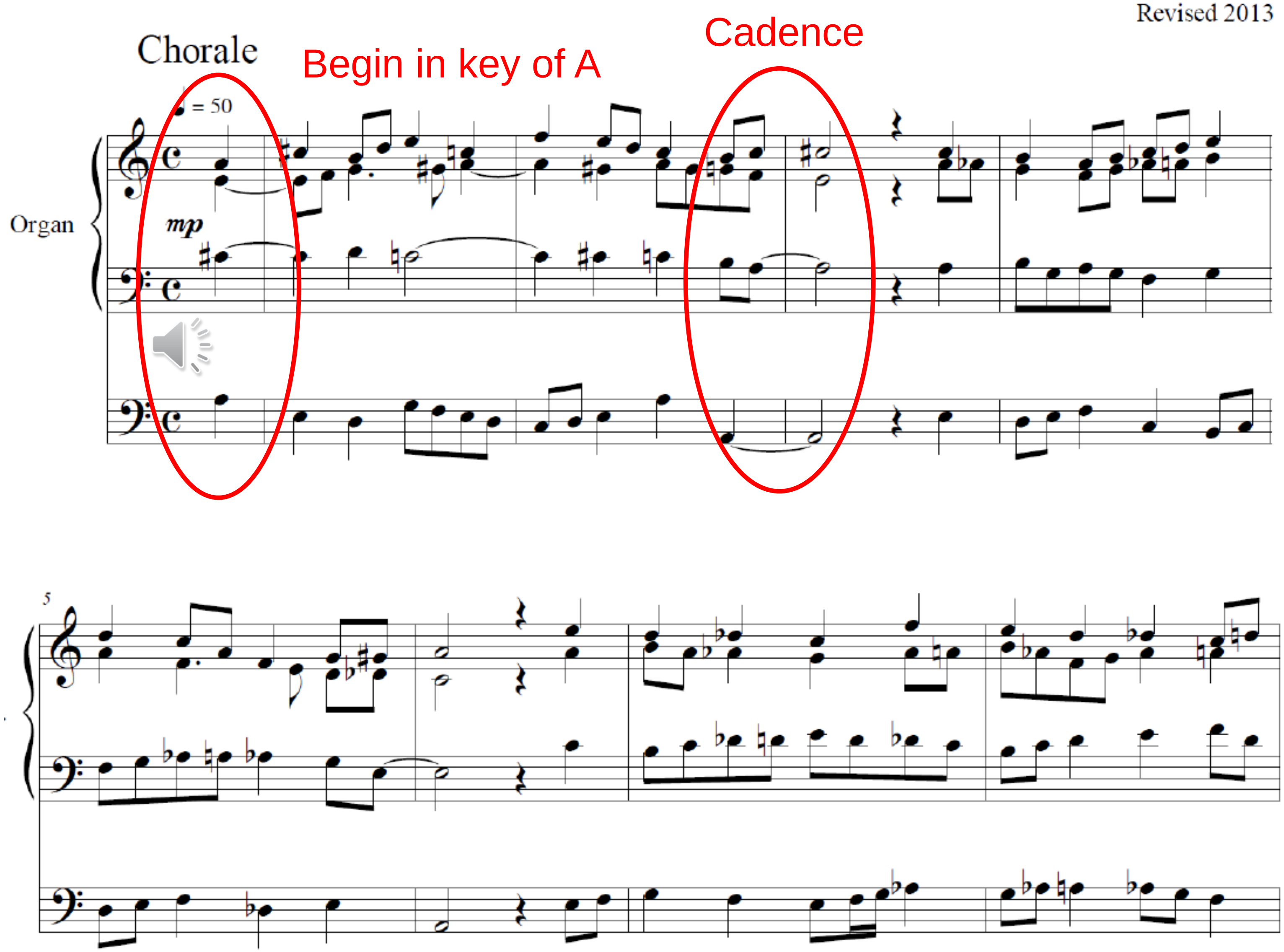
J. N. Hooker
Revised 2013

Chorale

Chorale and Fugue

On a 9-note Scale

J. N. Hooker
Revised 2013



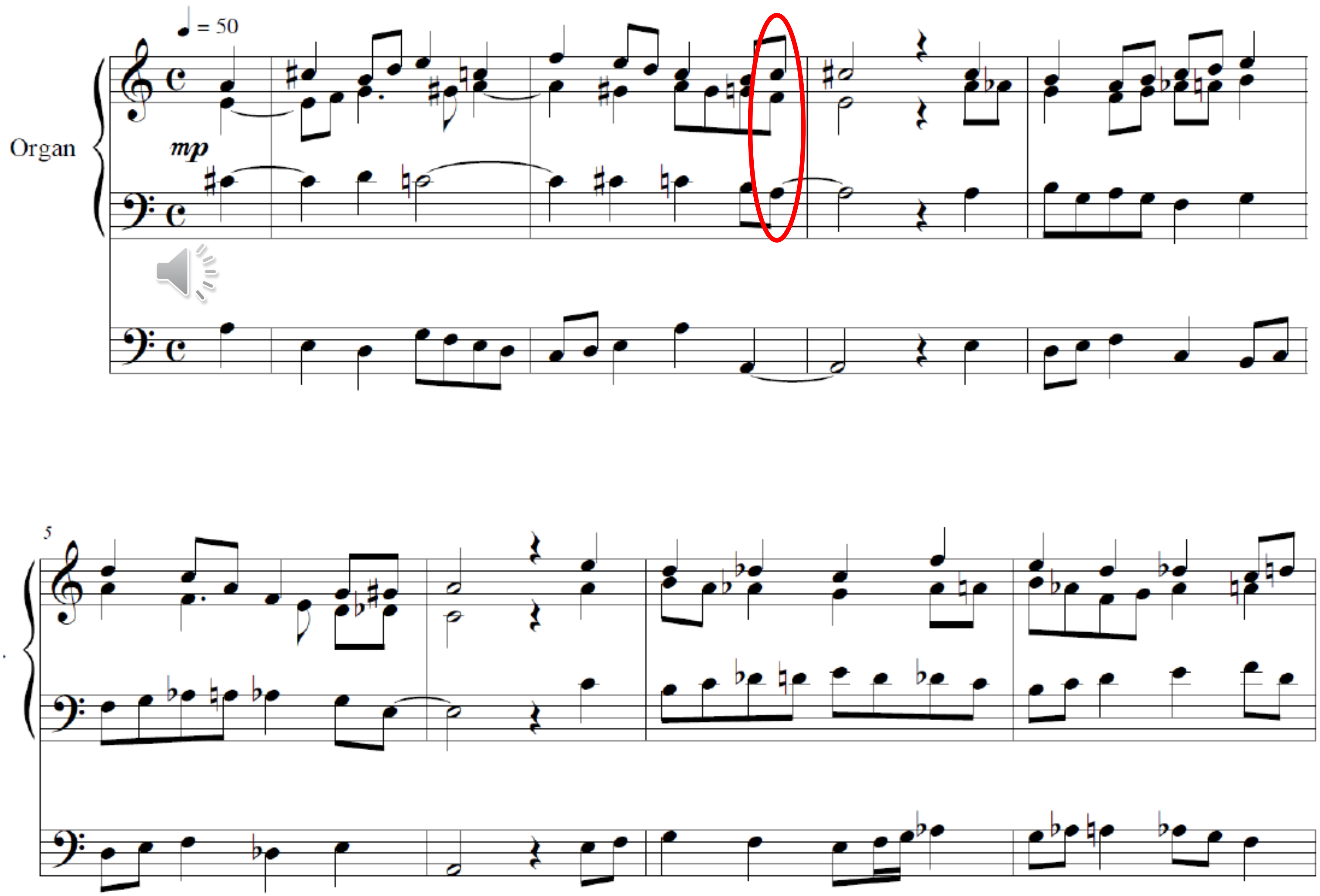
Chorale and Fugue

On a 9-note Scale

J. N. Hooker
Revised 2013

Resolve from lowered submediant (F)

Chorale



Chorale and Fugue

On a 9-note Scale

J. N. Hooker
Revised 2013

Chorale

Pivot on tonic

0:16


0:55
9
Org.
mf
1:24
13
Org.
f
18
Org.

Where does modulation
to Db actually occur?

1:48

New key (Db = C#)

Where does modulation to Db actually occur?

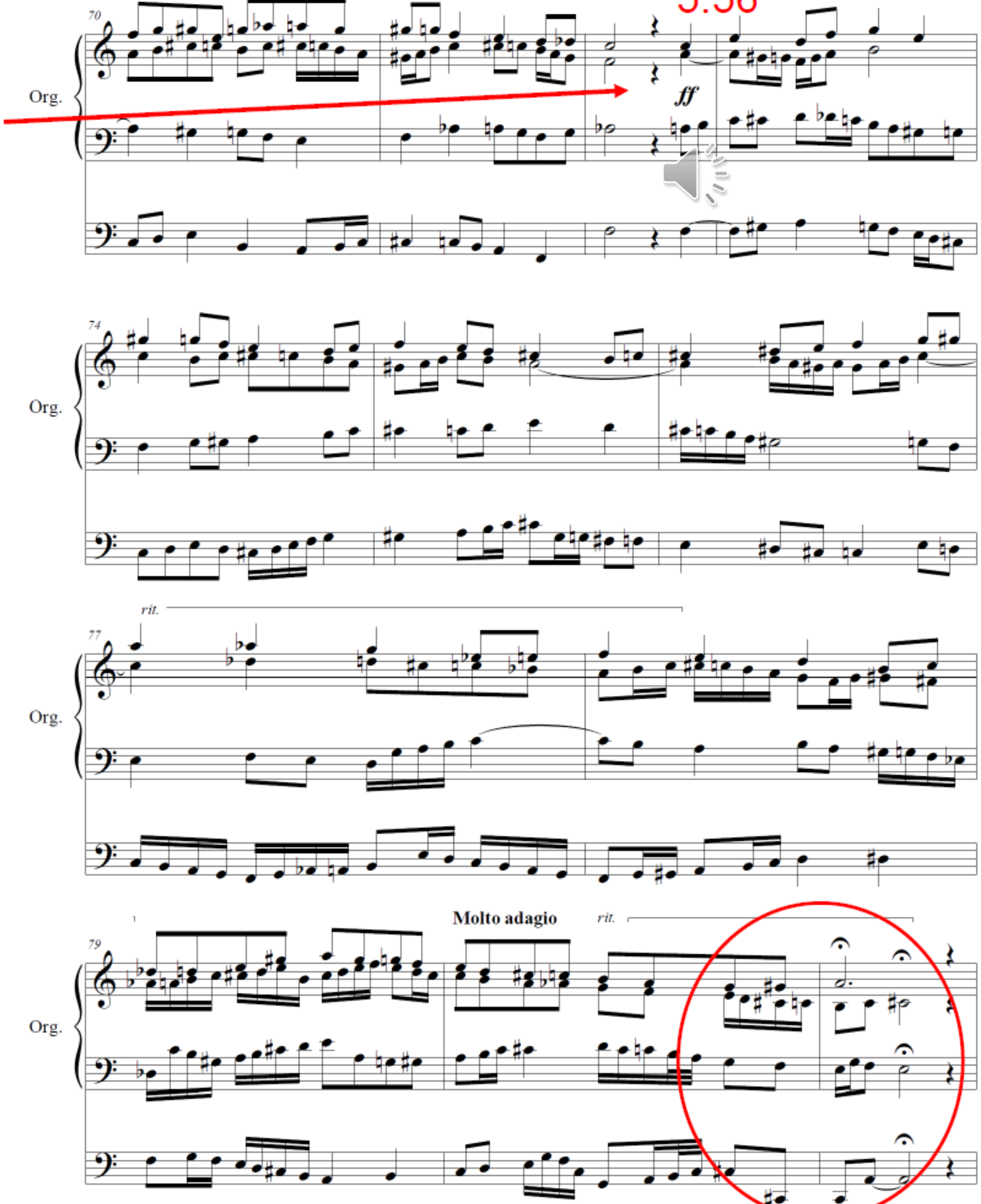
1:48

It occurs here

New key (Db = C#)

Skip to final section

5:56



Final cadence from lowered submediant (F), double leading tone, pivot on tonic

6:53 Fugue

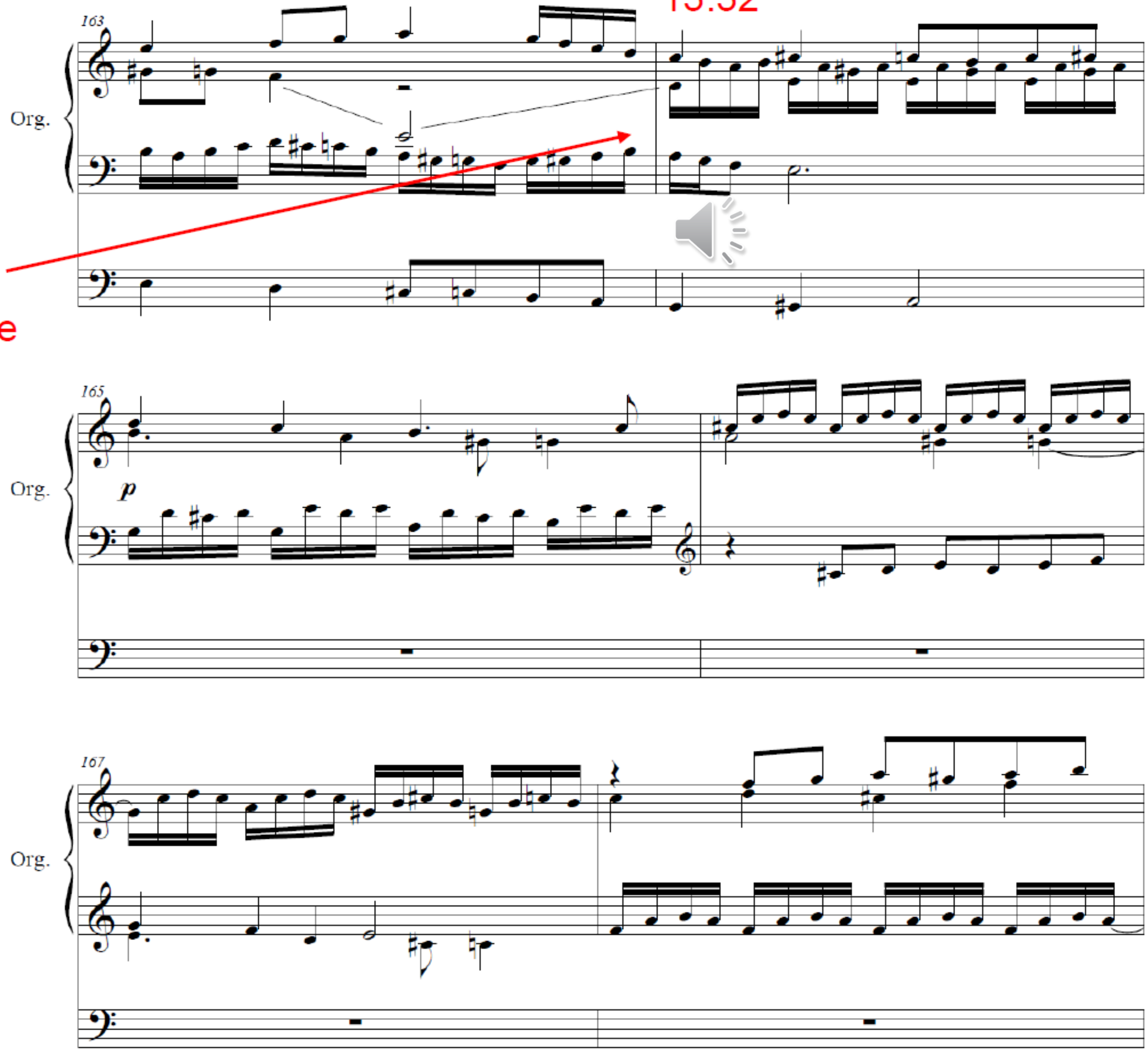

82
a tempo
mp
Org.
Subject enters at A
2nd entrance at C# but still in key of A
3rd entrance at F
86
Org.
4th entrance at A
Counter-
subject
90
Org.


8:01
2nd subject
Multiple suspensions on semitones
Countersubject
94
98
101
Org.
Org.
Org.


104
Org.
107
Org.
110
Org.
Countersubject

13:32

Skip to final episode




Recapitulation (entrance at A)
169
Org.
14:01
f
Entrance at C#
Entrance at F
173
Org.
Entrance at A
ff
Countersubject
177
Org.
rit.


Closing section
180
Org.
Final cadence
Double leading tone
Pivot on tonic
From lowered submediant
Coda
182
a tempo
rit.
Adagio
rit.
fff
Org.
Molto adagio
rit.
Secondary cadence
184
Org.
Double leading tone
Pivot on tonic
Add 32' or 64'

Other Chromatic Scales

- Which chromatics have the most simple ratios with the tonic, within tuning tolerance?

Ratio	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
3/2	·	·	·	·	·	·	●	·	·	·	·	●	·	●	·	·	●	·	●
4/3	·	·	·	·	·	·	●	·	·	·	·	●	·	●	·	·	●	·	●
5/3	·	·	●	·	·	●	●	·	·	●	●	·	·	●	●	·	●	●	●
5/4	●	·	·	●	·	·	●	·	·	●	●	·	●	●	·	●	·	·	●
7/4	·	·	·	·	●	●	·	·	·	●	●	·	·	·	●	●	·	·	·
6/5	·	·	·	·	·	●	·	·	·	●	·	·	·	●	·	·	●	●	·
7/5	·	·	·	·	·	·	·	·	·	·	·	·	·	●	·	●	·	●	·
8/5	●	·	·	●	·	·	●	·	·	●	●	·	●	●	·	●	●	·	●
9/5	·	●	·	·	·	·	·	●	●	·	·	·	·	●	●	●	·	·	·

Other Chromatic Scales

- Which chromatics have the most simple ratios with the tonic, within tuning tolerance?

Ratio	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
3/2	·	·	·	·	·	·	●	·	·	·	·	●	·	●	·	·	●	·	●
4/3	·	·	·	·	·	·	●	·	·	·	·	●	·	●	·	·	●	·	●
5/3	·	·	●	·	·	●	●	·	·	●	●	·	·	●	●	·	●	●	●
5/4	●	·	·	●	·	·	●	·	·	●	●	·	●	●	·	●	·	·	●
7/4	·	·	·	·	●	●	·	·	·	●	●	·	·	·	●	●	·	·	·
6/5	·	·	·	·	·	●	·	·	·	●	·	·	·	●	·	·	●	●	·
7/5	·	·	·	·	·	·	·	·	·	·	·	·	·	●	·	●	·	●	·
8/5	●	·	·	●	·	·	●	·	·	●	●	·	●	●	·	●	●	·	●
9/5	·	●	·	·	·	·	·	●	●	·	·	·	·	●	●	●	·	·	·

Classical 12-tone chromatic is 2nd best

Other Chromatic Scales

- Which chromatics have the most simple ratios with the tonic, within tuning tolerance?

Ratio	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
3/2	·	·	·	·	·	·	●	·	·	·	·	●	·	●	·	·	●	·	●
4/3	·	·	·	·	·	·	●	·	·	·	·	●	·	●	·	·	●	·	●
5/3	·	·	●	·	·	●	●	·	·	●	●	·	·	●	●	·	●	●	●
5/4	●	·	·	●	·	·	●	·	·	●	●	·	●	●	·	●	·	·	●
7/4	·	·	·	·	●	●	·	·	·	●	●	·	·	·	●	●	·	·	·
6/5	·	·	·	·	·	●	·	·	·	●	·	·	·	●	·	·	●	●	·
7/5	·	·	·	·	·	·	·	·	·	·	·	·	·	●	·	●	·	●	·
8/5	●	·	·	●	·	·	●	·	·	●	●	·	●	●	·	●	●	·	●
9/5	·	●	·	·	·	·	·	●	●	·	·	·	·	●	●	●	·	·	·

Quarter-tone scale adds nothing

Other Chromatic Scales

- Which chromatics have the most simple ratios with the tonic, within tuning tolerance?

Ratio	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
3/2	·	·	·	·	·	·	●	·	·	·	·	●	·	●	·	·	●	·	●
4/3	·	·	·	·	·	·	●	·	·	·	·	●	·	●	·	·	●	·	●
5/3	·	·	●	·	·	●	●	·	·	●	●	·	·	●	●	·	●	●	●
5/4	●	·	·	●	·	·	●	·	·	●	●	·	●	●	·	●	·	·	●
7/4	·	·	·	·	●	●	·	·	·	●	●	·	·	·	●	●	·	·	·
6/5	·	·	·	·	·	●	·	·	·	●	·	·	·	●	·	·	●	●	·
7/5	·	·	·	·	·	·	·	·	·	·	·	·	·	●	·	●	·	●	·
8/5	●	·	·	●	·	·	●	·	·	●	●	·	●	●	·	●	●	·	●
9/5	·	●	·	·	·	·	·	●	●	·	·	·	·	●	●	●	·	·	·

19-tone chromatic dominates all others

Historical Sidelight

- Advantage of 19-tone chromatic was discovered during Renaissance.
  - Spanish organist and music theorist **Franciso de Salinas** (1530-1590) recommended 19-tone chromatic due to desirable tuning properties for traditional intervals.
  - He used **meantone temperament** rather than equal temperament.

Historical Sidelight

- 19-tone chromatic has received some additional attention over the years
  - W. S. B. Woolhouse (1835)
  - M. J. Mandelbaum (1961)
  - E. Blackwood (1992)
  - W. A. Sethares (2005)

Demonstration: 19-note chromatic

- "Etude" by Easley Blackwood, 1980 (41:59)
  - Uses entire 19-note scale
  - Emphasis on traditional intervals
  - Renaissance/Baroque sound
  - Musical syntax is **basically tonal**
  - We wish to introduce **new intervals** and a **new syntax** by using 11-note or other scales on the 19-note chromatic

Scales on 19-note chromatic

- But what are the **best scales** on this chromatic?
  - **10-note** scales have only 1 semitone, not enough for musical interest.
  - **12-note** scales have 5 semitones, but this makes scale notes very closely spaced.
  - 11-note scales have 3 semitones, which seems a **good compromise** (1 more semitone than classical scales).

11-note scales on 19-note chromatic

- There are 77 scales satisfying our requirements

$$\binom{8}{3} + \binom{7}{2} = 77$$

  - Solve CP problem for all 77.
  - For each scale, determine largest set of simple ratios that occur in at least one solution.
  - 37 different sets of ratios appear in the 77 scales.

Simple ratios in 11-note scales

Ratio	**A**	B	C	D	**E**	F	G	H	I	J	K	L	M	N	O	**P**	Q	R	S	T	U	V	**W**	X	Y	Z	a	b	c	d	e	f	g	h	i	j	k
3/2	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
4/3	●	●	●	●	●	●	●	●	●	●	●	●	·	·	·	·	·	·	·	·	·	·	●	●	●	●	●	●	●	●	●	·	·	·	·	·	·
5/3	●	●	●	●	·	·	·	·	·	·	·	·	●	●	●	●	●	●	·	·	·	·	●	●	●	●	●	·	·	·	·	●	●	●	●	·	·
5/4	●	●	●	●	●	●	●	●	●	●	·	·	●	●	●	·	·	·	●	●	·	·	●	●	●	·	·	●	●	●	·	●	●	·	·	●	·
7/4	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
6/5	·	·	·	·	●	●	·	·	·	·	●	●	·	·	·	●	●	●	·	·	●	●	●	●	·	●	●	●	·	·	·	·	·	●	●	·	●
7/5	●	●	●	·	●	·	●	●	·	·	·	·	●	●	●	●	●	●	●	●	●	●	·	·	·	·	·	·	·	·	·	●	●	●	●	●	●
8/5	●	·	·	●	●	●	●	●	●	●	●	·	●	·	·	●	·	·	●	●	●	·	·	·	·	·	·	●	●	·	●	·	·	·	·	●	●
9/5	●	●	·	●	·	·	●	·	●	·	·	●	●	●	·	●	●	·	●	·	●	●	●	·	·	●	·	·	●	·	·	●	·	●	·	·	·

A - 72
B - 69,70,71
C - 68
D - 74,75
E - 7,8
F - 22,23
G - 73
H - 2

K - 12,43
L - 28
M - 65,66
N - 63,64
O - 62
P - 40,41,55,56
Q - 20,21,38,39,53
R - 19,37,52

U - 57
V - 42
W - 26,27
X - 10,11,25
Y - 5,6
Z - 15,31,32,46,47
a - 14,30,45
b - 9,24

e - 13,29,44
f - 60,61
g - 59
h - 18,35,36,50,51,54
i - 17,34,49
j - 58
k - 16,33,48

Simple ratios in 11-note scales

Ratio	**A**	B	C	D	**E**	F	G	H	I	J	K	L	M	N	O	**P**	Q	R	S	T	U	V	**W**	X	Y	Z	a	b	c	d	e	f	g	h	i	j	k
3/2	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
4/3	●	●	●	●	●	●	●	●	●	●	●	●	·	·	·	·	·	·	·	·	·	·	●	●	●	●	●	●	●	●	●	·	·	·	·	·	·
5/3	●	●	●	●	·	·	·	·	·	·	·	·	●	●	●	●	●	●	·	·	·	·	●	●	●	●	●	·	·	·	·	●	●	●	●	·	·
5/4	●	●	●	●	●	●	●	●	●	●	·	·	●	●	●	·	·	·	●	●	·	·	●	●	●	·	·	●	●	●	·	●	●	·	·	●	·
7/4	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
6/5	·	·	·	·	●	●	·	·	·	·	●	●	·	·	·	●	●	●	·	·	●	●	●	●	·	●	●	●	·	·	·	·	·	●	●	·	●
7/5	●	●	●	·	●	·	●	●	·	·	·	·	●	●	●	●	●	●	●	●	●	●	·	·	·	·	·	·	·	·	·	●	●	●	●	●	●
8/5	●	·	·	●	●	●	●	●	●	●	●	·	●	·	·	●	·	·	●	●	●	·	·	·	·	·	·	●	●	·	●	·	·	·	·	●	●
9/5	●	●	·	●	·	·	●	·	●	·	·	●	●	●	·	●	●	·	●	·	●	●	●	·	·	●	·	·	●	·	·	●	·	●	·	·	·

A - 72
B - 69,70,71
C - 68
D - 74,75
E - 7,8
F - 22,23
G - 73
H - 2

K - 12,43
L - 28
M - 65,66
N - 63,64
O - 62
P - 40,41,55,56
Q - 20,21,38,39,53
R - 19,37,52

U - 57
V - 42
W - 26,27
X - 10,11,25
Y - 5,6
Z - 15,31,32,46,47
a - 14,30,45
b - 9,24

e - 13,29,44
f - 60,61
g - 59
h - 18,35,36,50,51,54
i - 17,34,49
j - 58
k - 16,33,48

These 9 scales dominate all the others.

Simple ratios in 11-note scales

Ratio	**A**	B	C	D	**E**	F	G	H	I	J	K	L	M	N	O	**P**	Q	R
3/2	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●
4/3	●	●	●	●	●	●	●	●	●	●	●	●	·	·	·	·	·	·
5/3	●	●	●	●	·	·	·	·	·	·	·	·	●	●	●	●	●	●
5/4	●	●	●	●	●	●	●	●	●	●	·	·	●	●	●	·	·	·
7/4	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
6/5	·	·	·	·	●	●	·	·	·	·	●	●	·	·	·	●	●	●
7/5	●	●	●	·	●	·	●	●	·	·	·	·	●	●	●	●	●	●
8/5	●	·	·	●	●	●	●	●	●	●	●	·	●	·	·	●	·	·
9/5	●	●	·	●	·	·	●	·	●	·	·	●	●	●	·	●	●	·

Ratio	S	T	U	V	**W**	X	Y	Z	a	b	c	d	e	f	g	h	i	j	k
3/2	●	●	●	●	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
4/3	·	·	·	·	●	●	●	●	●	●	●	●	●	·	·	·	·	·	·
5/3	·	·	·	·	●	●	●	●	●	·	·	·	·	●	●	●	●	·	·
5/4	●	●	·	·	●	●	●	·	·	●	●	●	·	●	●	·	·	●	·
7/4	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
6/5	·	·	●	●	●	●	·	●	●	●	·	·	·	·	·	●	●	·	●
7/5	●	●	●	●	·	·	·	·	·	·	·	·	·	●	●	●	●	●	●
8/5	●	●	●	·	·	·	·	·	·	●	●	·	●	·	·	·	·	●	●
9/5	●	·	●	●	●	·	·	●	·	·	●	·	·	●	·	●	·	·	·

A - 72
B - 69,70,71
C - 68
D - 74,75
E - 7,8
F - 22,23
G - 73
H - 2

K - 12,43
L - 28
M - 65,66
N - 63,64
O - 62
P - 40,41,55,56
Q - 20,21,38,39,53
R - 19,37,52

U - 57
V - 42
W - 26,27
X - 10,11,25
Y - 5,6
Z - 15,31,32,46,47
a - 14,30,45
b - 9,24

e - 13,29,44
f - 60,61
g - 59
h - 18,35,36,50,51,54
i - 17,34,49
j - 58
k - 16,33,48

We will focus on 1 scale from each class.

4 attractive 11-note scales

Scale	Class	Ratios with tonic	Minimal generators
7. 01101011111	E	1/1 25/24 9/8 6/5 5/4 4/3 7/5 3/2 8/5 12/7 25/18	3/2 5/3 3/2 3/2 5/3 3/3 3/2 3/2 3/2 7/4 3/2
		1/1 36/35 9/8 6/5 5/4 4/3 7/5 3/2 8/5 12/7 13/17	3/2 5/3 3/2 3/2 5/3 4/2 7/4 3/2 3/2 5/3 13/7
27. 10101111110	W	1/1 15/14 9/8 6/5 5/4 4/3 10/7 54/35 5/3 9/5 27/14	3/2 3/2 5/4 3/2 3/2 3/2 3/2 5/4 3/2 3/2 5/4
		1/1 16/15 9/8 6/5 5/4 4/3 10/7 14/9 5/3 9/5 35/18	3/2 5/4 5/4 3/2 3/2 3/2 7/4 5/4 3/2 3/2 5/4
56. 11011110110	P	1/1 15/14 7/6 6/5 9/7 7/5 3/2 8/5 5/3 9/5 27/14	3/2 5/3 5/3 3/2 3/2 5/3 3/2 3/2 5/3 3/2 3/2
		1/1 13/12 7/6 6/5 9/7 7/5 3/2 8/5 5/3 9/5 35/18	3/2 13/7 5/3 3/2 7/5 5/3 3/2 3/2 5/3 3/2 5/3
72. 11110110110	A	1/1 16/15 7/6 5/4 4/3 7/5 3/2 8/5 5/3 9/5 35/18	3/2 3/2 5/3 3/2 3/2 5/3 3/2 3/2 3/2 5/3 5/3
		1/1 15/14 7/6 5/4 4/3 7/5 3/2 8/5 5/3 9/5 27/14	3/2 7/5 5/3 3/2 3/2 5/3 3/2 5/3 3/2 5/3 9/5

Showing 2 simplest solutions for each scale.
One with simplest generators, one with simplest ratios to tonic.

Key structure of scales

Classical major scale																			
Note	1	1♯	2	2♯	3	4	4♯	5	5♯	6	6♯	7							
Interval			2^{nd}		3^{rd}	4^{th}		5^{th}		6^{th}		7^{th}							
Distance	0	5	2	3	4	1	5	1	4	3	2	5							
Scale 23 of 9 notes on 12-note chromatic																			
Note	1	1♯	2	3	4	5	5♯	6	7	7♯	8	9							
Interval			2^{nd}	$m3^{rd}$	3^{rd}	4^{th}		5^{th}	$m6^{th}$		$m7^{th}$	7^{th}							
Distance	0	3	3	2	2	2	3	2	2	2	3	3							
Scale 7 of 11 notes on 19-note chromatic																			
Note	1	2	2♯	3	3♯	4	5	5♯	6	7	7♯	8	8♯	9	9♯	10	10♯	11	11♯
Interval				2^{nd}		$m3^{rd}$	3^{rd}		4^{th}			5^{th}		$m6^{th}$					
Distance	0	8	3	5	5	4	5	5	4	5	5	4	5	5	4	5	5	3	8
Scale 27 of 11 notes on 19-note chromatic																			
Note	1	1♯	2	3	3♯	4	5	5♯	6	6♯	7	7♯	8	8♯	9	9♯	10	10♯	11
Interval				2^{nd}		$m3^{rd}$	3^{rd}		4^{th}						6^{th}				
Distance	0	8	3	5	4	6	3	6	4	5	5	4	6	3	6	4	5	3	8
Scale 56 of 11 notes on 19-note chromatic																			
Note	1	1♯	2	2♯	3	4	4♯	5	5♯	6	6♯	7	7♯	8	9	9♯	10	10♯	11
Interval						$m3^{rd}$						5^{th}		$m6^{th}$	6^{th}				
Distance	0	8	3	5	6	2	7	3	6	4	4	6	3	7	2	6	5	3	8
Scale 72 of 11 notes on 19-note chromatic																			
Note	1	1♯	2	2♯	3	3♯	4	4♯	5	6	6♯	7	7♯	8	9	9♯	10	10♯	11
Interval							3^{rd}		4^{th}			5^{th}		$m6^{th}$	6^{th}				
Distance	0	8	3	5	6	2	7	3	6	4	4	6	3	7	2	6	5	3	8

Key structure of scales

Classical major scale

Note	1	1♯	2	2♯	3	4	4♯	5	5♯	6	6♯	7
Interval			2^{nd}		3^{rd}	4^{th}		5^{th}		6^{th}		7^{th}
Distance	0	5	2	3	4	1	5	1	4	3	2	5

No key with distance 1. Good or bad?

Scale 23 of 9 notes on 12-note chromatic

Note	1	1♯	2	3	4	5	5♯	6	7	7♯	8	9
Interval			2^{nd}	$m3^{rd}$	3^{rd}	4^{th}		5^{th}	$m6^{th}$		$m7^{th}$	7^{th}
Distance	0	3	3	2	2	2	3	2	2	2	3	3

A limited cycle in scale 72 that skips 2.

Scale 7 of 11 notes on 19-note chromatic

Note	1	2	2♯	3	3♯	4	5	5♯	6	7	7♯	8	8♯	9	9♯	10	10♯	11	11♯
Interval				2^{nd}		$m3^{rd}$	3^{rd}		4^{th}			5^{th}		$m6^{th}$					
Distance	0	8	3	5	5	4	5	5	4	5	5	4	5	5	4	5	5	3	8

Scale 27 of 11 notes on 19-note chromatic

Note	1	1♯	2	3	3♯	4	5	5♯	6	6♯	7	7♯	8	8♯	9	9♯	10	10♯	11
Interval				2^{nd}		$m3^{rd}$	3^{rd}		4^{th}						6^{th}				
Distance	0	8	3	5	4	6	3	6	4	5	5	4	6	3	6	4	5	3	8

Scale 56 of 11 notes on 19-note chromatic

Note	1	1♯	2	2♯	3	4	4♯	5	5♯	6	6♯	7	7♯	8	9	9♯	10	10♯	11
Interval						$m3^{rd}$						5^{th}		$m6^{th}$	6^{th}				
Distance	0	8	3	5	6	2	7	3	6	4	4	6	3	7	2	6	5	3	8

Scale 72 of 11 notes on 19-note chromatic

Note	1	1♯	2	2♯	3	3♯	4	4♯	5	6	6♯	7	7♯	8	9	9♯	10	10♯	11
Interval							3^{rd}		4^{th}			5^{th}		$m6^{th}$	6^{th}				
Distance	0	8	3	5	6	2	7	3	6	4	4	6	3	7	2	6	5	3	8

4 attractive 9-note scales

Scale	Class	Ratios with tonic	Minimal generators
7. 01101011111	E	$\frac{1}{1}\ \frac{25}{24}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{7}{5}\ \frac{3}{2}\ \frac{8}{5}\ \frac{12}{7}\ \frac{25}{18}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{7}{4}\ \frac{3}{2}$
		$\frac{1}{1}\ \frac{36}{35}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{7}{5}\ \frac{3}{2}\ \frac{8}{5}\ \frac{12}{7}\ \frac{13}{17}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{4}{2}\ \frac{7}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{13}{7}$
27. 10101111110	W	$\frac{1}{1}\ \frac{15}{14}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{10}{7}\ \frac{54}{35}\ \frac{5}{3}\ \frac{9}{5}\ \frac{27}{14}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}$
		$\frac{1}{1}\ \frac{16}{15}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{10}{7}\ \frac{14}{9}\ \frac{5}{3}\ \frac{9}{5}\ \frac{35}{18}$	$\frac{3}{2}\ \frac{5}{4}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{7}{4}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}$
56. 11011110110	P	$\frac{1}{1}\ \frac{15}{14}\ \frac{7}{6}\ \frac{6}{5}\ \frac{9}{7}\ \frac{7}{5}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{9}{5}\ \frac{27}{14}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}$
		$\frac{1}{1}\ \frac{13}{12}\ \frac{7}{6}\ \frac{6}{5}\ \frac{9}{7}\ \frac{7}{5}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{9}{5}\ \frac{35}{18}$	$\frac{3}{2}\ \frac{13}{7}\ \frac{5}{3}\ \frac{3}{2}\ \frac{7}{5}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{5}{3}$
72. 11110110110	A	$\frac{1}{1}\ \frac{16}{15}\ \frac{7}{6}\ \frac{5}{4}\ \frac{4}{3}\ \frac{7}{5}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{9}{5}\ \frac{35}{18}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{5}{3}$
		$\frac{1}{1}\ \frac{15}{14}\ \frac{7}{6}\ \frac{5}{4}\ \frac{4}{3}\ \frac{7}{5}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{9}{5}\ \frac{27}{14}$	$\frac{3}{2}\ \frac{7}{5}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{5}{3}\ \frac{9}{5}$

Further focus on scale 72, which has largest number of simple ratios.

Demonstration: 11-note scale

- Software
  - Hex MIDI sequencer for scales satisfying Myhill's property
  - We trick it into generating a 19-tone chromatic
  - Viking synthesizer for use with Hex
  - LoopMIDI virtual MIDI cable

Harmonic Comparison

- Classic major scale
  - Major triad C:E:G = 4:5:6, major 7 chord C:E:G:B = 8:10:12:15
  - Minor triad A:C:E = 10:12:15, minor 7 chord A:C:E:G = 10:12:15:18
  - Dominant 7 chord G:B:D:F = 36:45:54:64
  - Tensions (from jazz) C E G B D F# A
- Scale 72
  - Major triad 1-4-7 = 4:5:6
  - Minor triad 5-8-12 = 10:12:15
  - Minor 7 chord 9-12-15-18 = 10:12:15:18
  - New chord 9-12-14-18 = 5:6:7:9
  - New chord 1-3-5-9 = 6:7:8:10
  - New chord 3-5-9-12 = 7:8:10:12
  - New chord 5-9-12-15 = 4:5:6:7
  - Tensions 1-4-7-10-13-15*b*-16-19-22

11-note Scales with Adjacent Keys

- There are eleven 11-note scales on a 19-note chromatic in which keys can differ by one note.
  - As in classical 7-note scales.
  - One can therefore cycle through all keys.
  - This may be seen as a **desirable property**.
  - The key distances are the same for all these scales.

Scale 9 (class b)																			
Note	1	2	2♯	3	3♯	4	5	5♯	6	6♯	7	7♯	8	9	9♯	10	10♯	11	11♯
Interval				2nd		m3rd	3rd				$\frac{10}{7}$		$\frac{14}{9}$	m6th		$\frac{12}{7}$		7th	
Distance	0	8	3	5	6	2	8	1	7	4	4	7	1	8	2	6	5	3	8
Scale 13 (class e)																			
Note	1	2	2♯	3	3♯	4	4♯	5	6	6♯	7	7♯	8	9	9♯	10	10♯	11	11♯
Interval				2nd		m3rd		$\frac{9}{7}$	4th				$\frac{14}{9}$		m6th	$\frac{12}{7}$		$\frac{13}{7}$	
Scale 14 (class a)																			
Note	1	2	2♯	3	3♯	4	4♯	5	6	6♯	7	7♯	8	8♯	9	10	10♯	11	11♯
Interval				2nd		m3rd		$\frac{9}{7}$	4th		$\frac{10}{7}$		$\frac{14}{9}$		6th	$\frac{12}{7}$		7th	
Scale 30 (class a)																			
Note	1	1♯	2	3	3♯	4	4♯	5	6	6♯	7	7♯	8	8♯	9	10	10♯	11	11♯
Interval				2nd		m3rd		$\frac{9}{7}$	4th		$\frac{10}{7}$			$\frac{14}{9}$	6th	$\frac{12}{7}$		$\frac{13}{7}$	
Scale 34 (class i)																			
Note	1	1♯	2	3	3♯	4	4♯	5	5♯	6	7	7♯	8	8♯	9	10	10♯	11	11♯
Interval				2nd		m3rd		$\frac{9}{7}$	4th		$\frac{10}{7}$			$\frac{14}{9}$	6th	$\frac{9}{5}$		$\frac{13}{7}$	
Scale 35 (class h)																			
Note	1	1♯	2	3	3♯	4	4♯	5	5♯	6	7	7♯	8	8♯	9	9♯	10	11	11♯
Interval				2nd		m3rd		$\frac{9}{7}$	4th		$\frac{10}{7}$			$\frac{14}{9}$	6th	$\frac{9}{5}$		$\frac{13}{7}$	
Scale 50 (class h)																			
Note	1	1♯	2	2♯	3	4	4♯	5	5♯	6	7	7♯	8	8♯	9	9♯	10	11	11♯
Interval				2nd	$\frac{7}{6}$	m3rd		$\frac{9}{7}$		$\frac{7}{5}$	$\frac{10}{7}$		$\frac{14}{9}$		6th		$\frac{9}{5}$	$\frac{13}{7}$	
Scale 53 (class Q)																			
Note	1	1♯	2	2♯	3	4	4♯	5	5♯	6	6♯	7	8	8♯	9	9♯	10	11	11♯
Interval				2nd	$\frac{7}{6}$	m3rd		$\frac{9}{7}$		$\frac{7}{5}$	$\frac{10}{7}$	5th			6th		$\frac{9}{5}$	$\frac{13}{7}$	
Scale 54 (class h)																			
Note	1	1♯	2	2♯	3	4	4♯	5	5♯	6	6♯	7	8	8♯	9	9♯	10	10♯	11
Interval				2nd	$\frac{7}{6}$	m3rd		$\frac{9}{7}$		$\frac{7}{5}$	$\frac{10}{7}$	5th	$\frac{14}{9}$		6th		$\frac{9}{5}$		
Scale 64 (class N)																			
Note	1	1♯	2	2♯	3	3♯	4	5	5♯	6	6♯	7	8	8♯	9	9♯	10	10♯	11
Interval				2nd	$\frac{7}{6}$		3rd	$\frac{9}{7}$		$\frac{7}{5}$		5th	$\frac{14}{9}$		6th		$\frac{9}{5}$		
Scale 66 (class M)																			
Note	1	1♯	2	2♯	3	3♯	4	5	5♯	6	6♯	7	7♯	8	9	9♯	10	10♯	11
Interval				2nd	$\frac{7}{6}$		3rd	$\frac{9}{7}$		$\frac{7}{5}$		5th		$\frac{8}{5}$	6th		$\frac{9}{5}$		

Scales with most attractive intervals

Scale 9 (class b)																			
Note	1	2	2♯	3	3♯	4	5	5♯	6	6♯	7	7♯	8	9	9♯	10	10♯	11	11♯
Interval				2nd		m3rd	3rd				$\frac{10}{7}$		$\frac{14}{9}$	m6th		$\frac{12}{7}$		7th	
Distance	0	8	3	5	6	2	8	1	7	4	4	7	1	8	2	6	5	3	8
Scale 13 (class e)																			
Note	1	2	2♯	3	3♯	4	4♯	5	6	6♯	7	7♯	8	9	9♯	10	10♯	11	11♯
Interval				2nd		m3rd		$\frac{9}{7}$	4th				$\frac{14}{9}$		m6th	$\frac{12}{7}$		$\frac{13}{7}$	
Scale 14 (class a)																			
Note	1	2	2♯	3	3♯	4	4♯	5	6	6♯	7	7♯	8	8♯	9	10	10♯	11	11♯
Interval				2nd		m3rd		$\frac{9}{7}$	4th		$\frac{10}{7}$		$\frac{14}{9}$		6th	$\frac{12}{7}$		7th	
Scale 30 (class a)																			
Note	1	1♯	2	3	3♯	4	4♯	5	6	6♯	7	7♯	8	8♯	9	10	10♯	11	11♯
Interval				2nd		m3rd		$\frac{9}{7}$	4th		$\frac{10}{7}$			$\frac{14}{9}$	6th	$\frac{12}{7}$		$\frac{13}{7}$	
Scale 34 (class i)																			
Note	1	1♯	2	3	3♯	4	4♯	5	5♯	6	7	7♯	8	8♯	9	10	10♯	11	11♯
Interval				2nd		m3rd		$\frac{9}{7}$	4th		$\frac{10}{7}$			$\frac{14}{9}$	6th	$\frac{9}{5}$		$\frac{13}{7}$	
Scale 35 (class h)																			
Note	1	1♯	2	3	3♯	4	4♯	5	5♯	6	7	7♯	8	8♯	9	9♯	10	11	11♯
Interval				2nd		m3rd		$\frac{9}{7}$	4th		$\frac{10}{7}$			$\frac{14}{9}$	6th	$\frac{9}{5}$		$\frac{13}{7}$	
Scale 50 (class h)																			
Note	1	1♯	2	2♯	3	4	4♯	5	5♯	6	7	7♯	8	8♯	9	9♯	10	11	11♯
Interval				2nd	$\frac{7}{6}$	m3rd		$\frac{9}{7}$		$\frac{7}{5}$	$\frac{10}{7}$		$\frac{14}{9}$		6th		$\frac{9}{5}$	$\frac{13}{7}$	
Scale 53 (class Q)																			
Note	1	1♯	2	2♯	3	4	4♯	5	5♯	6	6♯	7	8	8♯	9	9♯	10	11	11♯
Interval				2nd	$\frac{7}{6}$	m3rd		$\frac{9}{7}$		$\frac{7}{5}$	$\frac{10}{7}$	5th			6th		$\frac{9}{5}$	$\frac{13}{7}$	
Scale 54 (class h)																			
Note	1	1♯	2	2♯	3	4	4♯	5	5♯	6	6♯	7	8	8♯	9	9♯	10	10♯	11
Interval				2nd	$\frac{7}{6}$	m3rd		$\frac{9}{7}$		$\frac{7}{5}$	$\frac{10}{7}$	5th	$\frac{14}{9}$		6th		$\frac{9}{5}$		
Scale 64 (class N)																			
Note	1	1♯	2	2♯	3	3♯	4	5	5♯	6	6♯	7	8	8♯	9	9♯	10	10♯	11
Interval				2nd	$\frac{7}{6}$		3rd	$\frac{9}{7}$		$\frac{7}{5}$		5th	$\frac{14}{9}$		6th		$\frac{9}{5}$		
Scale 66 (class M)																			
Note	1	1♯	2	2♯	3	3♯	4	5	5♯	6	6♯	7	7♯	8	9	9♯	10	10♯	11
Interval				2nd	$\frac{7}{6}$		3rd	$\frac{9}{7}$		$\frac{7}{5}$		5th		$\frac{8}{5}$	6th		$\frac{9}{5}$		

That’s it.