Robust Scheduling with Logic-Based Benders Decomposition

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Abstract We study project scheduling at a large IT services delivery center in which there are unpredictable delays. We apply robust optimization to minimize tardiness while informing the customer of a reasonable worst-case completion time, based on empirically determined uncertainty sets. We introduce a new solution method based on logic-based Benders decomposition. We show that when the uncertainty set is polyhedral, the decomposition simplifies substantially, leading to a model of tractable size. Preliminary computational experience indicates that this approach is superior to a mixed integer programming model solved by state-of-the-art software.

1 Introduction

We analyze a project scheduling problem at a large IT services delivery center in which there are unpredictable delays in start times. This study is motivated by a real problem at a global IT services delivery organization. To design a schedule that is not unduly disrupted by contingencies, we formulate a robust optimization problem. We minimize tardiness cost while informing the customer of a reasonable worst-case completion time.

Due to the impracticality of quantifying joint probability distributions for delays, we apply robust optimization with uncertainty sets rather than probabilistic information [4][1]. An uncertainty set is an empirically determined space of possible outcomes for which one should realistically plan, without encompassing theoretically

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worst-case scenarios [5]. To our knowledge, uncertainty sets have not previously been applied to service scheduling. We propose a new solution method based on logic-based Benders decomposition [12][10]. We show that when the uncertainty set is polyhedral, the problem has convexity properties that result in a simplified decomposition. In addition, the Benders subproblem decouples into many smaller subproblems, each corresponding to an agent or small group of agents.

Robust optimization with uncertainty sets was introduced by [15] and polyhedral undertainty sets by [2][3]. Detailed reviews of robust optimization with uncertainty sets can be found in [5][4][1].

2 Modeling the Problem

Several hundred agents with different skill sets process thousands of incoming customer projects each year. A project may consist of several ordered tasks, each of which requires certain skills. Late deliveries result primarily from interruptions that require a task to be set aside for an unpredictable period of time. Because processing times are short, we treat an interrupted task as having a delayed start time. We re-compute the schedule on a rolling basis as projects arrive. The notation is summarized in Table 1.

Table 1 Notation

Sets	
(J,E)	precedence graph with task set $J=\{1,2,\ldots,n\}$
S_i, S'_j	skill set of agent i and required for task j
I_α, J_α	αth agent class, jobs assigned to agents in I_α
R	uncertainty set for release time delays
Parameters	
r_j, d_j	release time and due date of task j
p_j, c_j	processing time and unit tardiness cost of task j
$\text{pr}(j,\sigma,y)$	task performed by agent y_j immediately before task j in sequence σ
$\Delta\bar{r}_j^k$	task j release time delay in subproblem solution of Benders iteration k
Δr_j^ℓ	task j release time delay at extreme point ℓ of R
Variables	
y_j (x_{ij})	agent assigned to task j ($x_{ij}=1$ if $y_j=i$)
s_j	start time of task j
s_j^k	task j start time in kth Benders subproblem
s_j^ℓ	task j start time for extreme point ℓ of R
σ_j	position of task j in sequence
Δr_j	task j release time delay in uncertainty subproblem

In the robust model, we require that the tuple of uncertain release time delays $\Delta r=(\Delta r_1,\ldots,\Delta r_n)$ belong to uncertainty set R. We minimize worst-case tardiness cost subject to R as follows, where $\alpha^+ = \max\{0, \alpha\}$:

$$\min_{y,\sigma} \left\{ \max_{\Delta r \in R} \{ f(\sigma, y, r + \Delta r, p\} \;\middle|\; S'_j \subset S_{y_j} \right\} \tag{1}$$

Here $f(\sigma, y, r + \Delta r, p)$ is the cost that results from a greedy schedule in which each agent performs assigned tasks, without overlap, in the order given by σ. Thus we have $f(\sigma, y, r + \Delta r, p) = \sum_j c_j (s_j + p_j - d_j)^+$, where s_j is recursively defined for all $j = 1, \ldots, n$ by

$$s_j = \max\left\{ r_j + \Delta r_j,\; s_{\mathrm{pr}(j,\sigma,y)} + p_{\mathrm{pr}(j,\sigma,y)},\; \max_{(j',j) \in E} \{ s_{j'} + p_{j'} \} \right\} \tag{2}$$

3 Logic-Based Benders Decomposition

Logic-based Benders decomposition (LBBD) is a generalization of Benders decomposition in which the subproblem can in principle be any combinatorial problem, not necessarily a linear or nonlinear programming problem [8][13][12]. This approach can reduce solution times by several orders of magnitude relative to conventional methods [14][7][10][11][6].

We apply LBBD as follows. The master problem determines agent assignments y and the task sequence σ:

$$\begin{array}{l} \min z \\ S'_j \subset S_{y_j}, \text{ all } j;\;\; \sigma_j < \sigma_{j'}, \text{ all } (j, j') \in E \\ \text{Benders cuts} \end{array} \tag{3}$$

where each $y_j \in \{1, \ldots, m\}$. The subproblem is

$$\begin{array}{l} \displaystyle\max_{s, \Delta r, \Delta p} \sum_j (s_j + p_j - d_j)^+ \\ s_j = \max\{ r_j + \Delta r_j, s_{\mathrm{pr}(j,\bar{\sigma},\bar{y})} + p_{\mathrm{pr}(j,\bar{\sigma},\bar{y})} \}, \text{ all } j;\;\; \Delta r \in R \end{array} \tag{4}$$

where $(\bar{\sigma}, \bar{y})$ is the solution of the master problem. It is straightforward to formulate this as an MILP, but the problem becomes very large as Benders cuts accumulate due to the addition of new variables with each cut.

To overcome this, (3) can be solved by a second Benders decomposition scheme in which the subproblem decouples by agent classes, resulting in the three-stage decomposition of Figure 1(a). Agent classes are the smallest sets of agents whose assigned tasks are not coupled by precedence relations with tasks assigned to other sets of agents.

When the uncertainty set R is polyhedral, the three-stage decomposition simplifies to the more tractable two-stage decomposition in Fig. 1(b). The proof of the

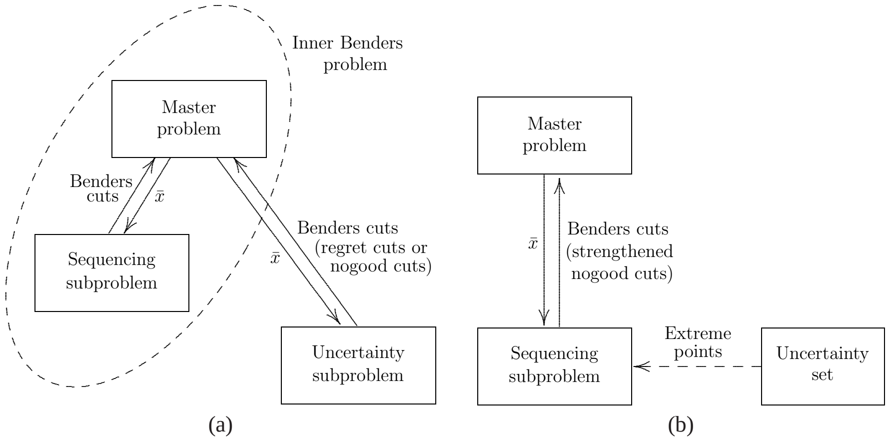

Fig. 1 (a) Three-stage decomposition. (b) Two-stage decomposition.

following theorem is based on that fact that (4) can be viewed as maximizing a convex function over the polyhedron R.

Theorem 1. *If the uncertainy set R is a bounded polyhedron, then at least one of its extreme points is an optimal solution of the uncertainty subproblem (4).*

We can now minimize worst-case tardiness for a given assignment by finding a sequence that minimizes the maximum tardiness over all extreme points ℓ.

The master problem computes an optimal assignment of tasks to agents:

$$\begin{array}{l} \min\ z \\ S_j' \subset S_i,\ \text{all } i,j \text{ with } x_{ij}=1;\ \sum_i x_{ij}=1,\ \text{all } j \\ \text{relaxation and Benders cuts} \end{array} \tag{5}$$

where $x_{ij} \in \{0,1\}$ for all i, j. The subproblem minimizes worst-case tardiness T:

$$\begin{array}{l} \min\ T \\ T \geq \sum_j (s_j^\ell + p_j - d_j)^+,\ \text{all } \ell \\ \text{noOverlap}\left(\sigma(\bar{x},i), s^\ell(\bar{x},i), p(\bar{x},i)\right),\ \text{all } i,\ell \\ s_j^\ell \geq r_j + \Delta r_j^\ell,\ \text{all } j,\ell;\ s_j^\ell \geq s_{j'}^\ell + p_{j'}\ \ \text{all } (j',j)\in E,\ \text{all } \ell \end{array} \tag{6}$$

The minimum is taken over variables s_j^ℓ and T.

Theorem 2. *If the uncertainty set R is a bounded polyhedron, the three-stage and two-stage decompositions are equivalent optimization problems.*

The number of constraints in the subproblem (6) grows linearly with the number of extreme points, and therefore only linearly with the number of tasks when the polyhedron is a simplex. In addition, (6) decouples by agent class I_α. We can strengthen the master problem (5) with relaxations of the subproblem, which allow the master problem to select more reasonable assignments before many Benders cuts have been accumulated. We use the two relaxations described in [10].

As Benders cuts, we use *strengthened nogood cuts* in the master problem (5). If T^* is the optimal value of the subproblem when $x = \bar{x}$, the simplest nogood cut is

$$z \geq \begin{cases} T^* & \text{if } x = \bar{x} \\ -\infty & \text{otherwise} \end{cases} \tag{7}$$

The cut can be strengthened by heuristically removing task assigments and re-solving the subproblem until the minimum tardiness is less than z^*.

4 Computational Results

We compared a rudimentary implementation of the two-stage Benders model with an MILP model solved by commercial software (IBM ILOG CPLEX 12.5.0). The MILP model uses a discrete-time formulation, which previous experience shows to be best for MILP [9][10], and takes advantage of Theorem 1. The time horizon is 5 times the maximum due date in one formulation (MILP1), and 40 plus the maximum due date in a second formulation (MILP2), based on a maximum total tardiness of 40 obtained from the Benders solution. The instances, which have 13 tasks and 2 agents, are based on actual data obtained from an IT services company. The Benders master problem is solved by the CPLEX MILP solver and the subproblems by IBM ILOG CP Optimizer 12.5.0. Table 2 shows that the Benders method is significantly faster than MILP. The advantage is less for MILP2, but MILP2 relies on prior information about the optimal solution.

Table 2 Computational Results

Instance	Optimal value	Computation times (sec) Benders	MILP1	MILP2
1	39	5.4	32.1	7.5
2	20	7.8	122.0	14.5
3	39	5.6	26.4	7.2
4	22	6.1	216.0	29.9
5	33	8.2	68.8	14.7
6	36	8.3	68.8	10.9
7	34	5.6	40.0	10.9
8	25	11.2	61.4	13.7
9	31	5.8	29.5	5.8
10	23	5.8	61.0	14.9
11	30	5.6	54.1	27.5
12	11	8.4	59.9	15.4
13	21	8.1	7.8	6.0
14	33	8.2	79.7	27.9
15	34	6.2	52.1	14.6
16	36	8.9	19.6	17.0
17	38	24.1	78.7	34.1
18	7	5.6	18.5	5.3
19	40	11.0	57.8	13.3
20	40	11.32	27.8	7.2

5 Conclusion

We introduced a novel robust scheduling method for IT services delivery based on logic-based Benders decomposition. We obtained solutions for small but realistic instances in significantly less time that a state-of-the-art mixed integer solver. The advantage of Benders is likely to be much greater as the instances scale up, because the decoupled Benders subproblems remain about the same size as the number of agents increases. In addition, the MILP model grows with the length and granularity of the time horizon, which does not occur in the Benders model. Finally, the Benders model is suitable for distributed computation due to the decoupling of the Benders subproblems.

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