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NETWORKLIKE METRIC SPACES

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Two distinctive and fundamental metrical properties of a network (conceived as an infinite point set with a metric) are: (a) it is a tree if and only if its metric is convex, and (b) it decomposes into finitely many treelike segments on which the metric is convex. We show that (a) and (b) are intimately related by proving that equivalence (a) holds in a class of networklike or 'reticular' metric spaces that are characterized by decomposability into treelike segments and whose finite models exhibit many of the important metrical properties of networks.

1. Introduction

Let us consider a *network* (rigorously defined below) to be the infinite collection of points on finitely many rectifiable *arcs*, whose endpoints are *nodes*. Two arcs may intersect, if at all, only at nodes. The *metric* of the network is the function that assigns to every pair of points the shortest-path distance between them. We therefore speak of a network, a continuous structure, in opposition to a *graph*, the familiar discrete structure.

Recent work on network location problems has uncovered two distinctive and fundamental properties of a network's metric. One, established in [2], is that the metric is convex (in a natural sense to be defined shortly) if and only if the network is a tree. The other, which we observed in [3, 4], is that any network is made up of finitely many "treelike" subsets on which the metric is convex.

We intend to show that these two properties are intimately related, in the following sense. We generalize the notion of treelikeness to metric spaces and define a class of metric spaces, "reticular spaces", that have certain points distinguished as "nodes" and that decompose into treelike pieces in much the way that networks do. We show that if a metric space is reticular, the entire space is treelike if and only if it has a convex metric, whereas this equivalence breaks down if the metric space is not reticular.

This result suggests that decomposability into treelike pieces is the characteristic of networks that in some sense explains the equivalence of treelikeness and convexity of metric. Furthermore, since most of the properties of reticular spaces we discuss occur in finite spaces, we are led to believe that many of the key metrical properties of a network can be captured in a discrete structure.

A network's metric is "piecewise convex" in that a network decomposes into subsets on which the metric is convex. In fact, it is convex on treelike subsets. But a reticular space need not be piecewise convex (i.e., have a piecewise convex metric), and a piecewise convex space need not be reticular. Piecewise convexity is a weaker property than reticularity, in that (a) treelikeness and convexity of metric are not equivalent in piecewise convex spaces, as they are in reticular spaces, and (b) we prove that finite reticular spaces are piecewise convex, whereas the converse does not hold.

2. Convexity in metric spaces

Let (S, d) be a metric space with metric or distance function d. A natural definition of betweenness, introduced by Pasch [7, 8], is that given distinct points $x, y, z \in S$, z is *between* x and y if $d(x, y) = d(x, z) + d(z, y)$. Following [2], we say that $z \in L_\lambda(x, y)$ if $d(x, z) = \lambda d(x, y)$, $\lambda \in (0, 1)$, and z is between x and y. Also $L_0(x, y) = x$ and $L_1(x, y) = y$. The *line segment* $L(x, y)$ connecting x and y is the set containing x, y and the points between them.

We note that possibly $L(x, y) = L(x', y')$ even when $\{x, y\} \neq \{x', y'\}$. For instance, if $S = R^2$ and d is the rectilinear metric whereby $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$, then $L((x_1, x_2), (y_1, y_2)) = L((x_1, y_2), (y_1, x_2))$.

The definitions in [2] for convexity in networks adapt readily to metric spaces. Set $D \subset S$ is a *convex set* if $L(x, y) \subset D$ for all $x, y \in D$. A function $f: D \to R$ is *convex* on a convex set D if for all $\lambda \in [0, 1]$ and all $z \in L_\lambda(x, y)$, $f(z) \leq (1 - \lambda) f(x) + \lambda f(y)$.

3. Convexity in networks

We define a network essentially as in [1]. Let $\underline{V} = \{v_1, \ldots, v_m\}$ be a finite set of *nodes*. With each pair v_i, v_j $(1 \leq i < j \leq m)$ associate a bijection $t_{ij}: [0, 1] \to S_{ij}$ for which $t_{ij}(0) = v_i$ and $t_{ij}(1) = v_j$, an *arc* $[v_i, v_j] = [v_j, v_i] = S_{ij}$, and a positive *arc length* e_{ij}. We require that the sets $[v_i, v_j] \setminus \{v_i, v_j\}$ be pairwise disjoint for all i, j. Then if $\underline{S} = \bigcup \{S_{ij}\}$, $E = \{e_{ij}\}$, and $T = \{t_{ij}\}$, the quadruple $(\underline{S}, \underline{V}, E, T)$ is a *network*.

The *metric* $\underline{d}: \underline{S} \to R$ of $(\underline{S}, \underline{V}, E, T)$ is defined as follows. Let any sequence of arcs $[v_{i(1)}, v_{i(2)}], [v_{i(2)}, v_{i(3)}], \ldots, [v_{i(k-1)}, v_{i(k)}]$, where $v_{i(1)}, \ldots, v_{i(k)}$ are distinct, be a (simple) *path* from $v_{i(1)}$ to $v_{i(k)}$, $1 \leq k \leq m$; the *length* of the path is the sum of the arc lengths. For $x, y \in [v_i, v_j]$ we denote by $[x, y] = [y, x]$ the path $t_{ij}([t_{ij}^{-1}(x), t_{ij}^{-1}(y)])$ if $t_{ij}^{-1}(x) \leq t_{ij}^{-1}(y)$ or the set $t_{ij}([t_{ij}^{-1}(y), t_{ij}^{-1}(x)])$ otherwise, and we assign it length $\underline{d}(x, y) = \underline{d}(y, x) = e_{ij} |t_{ij}^{-1}(x) - t_{ij}^{-1}(y)|$. Given $x \in [v_i, v_j]$ and $y \in [v_k, v_l]$, we say that $[x, v_j] \cup P \cup [v_k, y]$ is a path from x to y if P is a path from v_j to v_k; the length of the path is the sum of $\underline{d}(x, v_j) + \underline{d}(v_k, y)$ and the length of P. Finally, for any $x, y \in S$ let $\underset{\sim}{d}(x, y) = \underset{\sim}{d}(y, x)$ be the length of any shortest path from x to y; $\underset{\sim}{d}(x, y) = \infty$ if no path connects x and y. A *connected* network is one in which every distance is finite.

It will be convenient to denote $(\underset{\sim}{S}, \underset{\sim}{V}, E, T)$ with the triple $(\underset{\sim}{S}, \underset{\sim}{V}, \underset{\sim}{d})$. It follows immediately from the definition of $\underset{\sim}{d}$ that if $(\underset{\sim}{S}, \underset{\sim}{V}, \underset{\sim}{d})$ is connected, $(\underset{\sim}{S}, \underset{\sim}{d})$ is a metric space. From here out we suppose that all networks under discussion are connected.

Given points x, y in a network, $L(x, y)$ contains the points on all shortest paths from x to y. We say that a network. $(\underset{\sim}{S}, \underset{\sim}{V}, \underset{\sim}{d})$ *has a convex metric* if $\underset{\sim}{d}(v_i, \cdot): \underset{\sim}{S} \to R$ is convex for all $v_i \in V$.

A *tree* is a network in which exactly one path connects any given pair of nodes. Dearing, Francis and Lowe prove in [2] that $\underset{\sim}{d}(x, \cdot): \underset{\sim}{S} \to R$ is convex for all $x \in \underset{\sim}{S}$ if and only if $(\underset{\sim}{S}, \underset{\sim}{V}, \underset{\sim}{d})$ is a *tree*. A slight modification of their proof establishes that $\underset{\sim}{d}$ is a convex metric if and only if it $(\underset{\sim}{S}, \underset{\sim}{V}, \underset{\sim}{d})$ is a tree.

We show in [3] that any network $(\underset{\sim}{S}, \underset{\sim}{V}, \underset{\sim}{d})$ decomposes into finitely many treelike pieces on which the metric is convex, in the following sense. Let us say that $x \in \underset{\sim}{S}$ is a *boundary point* if there are at least two shortest paths from x to some node v_i, where x is said to be *generated* by v_i. If a connected set $D \subset \underset{\sim}{S}$ contains no boundary points, then D and its closure (in the metric topology) are *treelike sets*. A network is clearly the union of finitely many treelike sets (a tree is itself a treelike set). In particular, any line segment $L(x, y)$ is a union of finitely many treelike line segments (*treelike segments*, for short).

It is not hard to show [3] that $\underset{\sim}{d}(v_i, \cdot): D \to R$ is convex for each convex treelike set $D \subset \underset{\sim}{S}$. This is useful in solving network location problems, because one can decompose the problem into subproblems on treelike segments, where the convexity of the metric makes the subproblems relatively easy to solve.

4. Convexity and treelike spaces

If $L(z_0, \ldots, z_k)$ denotes $L(z_0, z_1) \cup \cdots \cup L(z_{k-1}, z_k)$, we say that $L(z_0, \ldots, z_k)$ is a *chain* (in particular, a *k-chain*) connecting z_0 and z_k if $d(z_0, z_k) = d(z_0, z_1) + \cdots + d(z_{k-1}, z_k)$. It will be convenient to adopt the convention that $C(y_0, \ldots, y_m)$ denotes any chain of the form $L(z_0, \ldots, z_k)$, where $y_0 = z_0$, $y_m = z_k$, and $\{y_1, \ldots, y_{m-1}\} \subset \{z_1, \ldots, z_{k-1}\}$.

If (S, d) is a metric space and $V \subset S$ is a set of points arbitrarily distinguished as *nodes*, we refer to (S, V, d) as a *metric space with nodes*. The major role of nodes is to serve as points with respect to which treelikeness is defined. We first define a point u_v in a set $D \subset S$ to be a *collection point* of D with respect to $v \in V$ if any $z \in D$ satisfies $d(z, v) = d(z, u_v) + d(u_v, v)$. Then D is *treelike* if it contains a collection point u_v with respect to each $v \in V$. Finally (S, V, d) is a *treelike space* if every 2-*chain* in S is treelike, a definition we will justify shortly. A tree network is clearly a treelike space.

Theorem 1. *If (S, V, d) is treelike, then d is a convex metric.*

Proof. Let $x, y \in S$ and $v \in V$. Pick an arbitrary $\lambda \in [0, 1]$ and $z \in L_\lambda(x, y)$. Since (S, V, d) is treelike, $L(x, z, y)$ contains a collection point u_v with respect to v. If $u_v \in L(z, y)$, then $d(z, v) = d(x, v) - d(x, z) = d(x, v) - \lambda d(x, y) \leq d(x, v) - \lambda[d(x, v) - d(y, v)] = (1 - \lambda)d(x, v) + \lambda d(y, v)$, and similarly if $u_v \in L(x, z)$. The convexity of the metric follows. □

We note that it is not enough to ensure convexity of metric that every line segment be treelike. Consider the following

Example 1. $V = S = \{w, x, y, z\}$, $d(w, x) = d(x, y) = d(y, z) = d(z, w) = 1$, and $d(w, y) = d(x, z) = 2$. Here every line segment is treelike. For $L(w, x)$, $u_w = u_z = w$ and $u_x = u_y = x$, and similarly for $L(x, y)$, $L(y, z)$ and $L(z, w)$. Also $L(w, y) = L(x, z) = S$, so that $L(w, y)$ and $L(x, z)$ are trivially treelike. But $d(\cdot, w)$ is not convex on $L(x, z)$. We can observe, however, that neither of the 2-chains $L(x, w, z)$ and $L(x, y, z)$ is treelike, as required for a treelike space.

We also note that $d(\cdot, u)$ need not be convex for *all* points u in a treelike space. If we let $V = \emptyset$ in Example 1, the space is trivially treelike, but $d(\cdot, w)$ is again not convex. We therefore require of a convex metrix d only that $d(\cdot, v)$ be convex for all $v \in V$.

Although treelikeness is sufficient for convexity of metric, the converse is not true.

Example 2. (Fig. 1). Let $S = V = \{w, x, y, z\}$, $d(w, x) = d(x, y) = d(y, z) = d(z, w) = 2$, $d(w, y) = 4$, and $d(x, z) = 1$. (The arrows in Fig. 1 are merely distance markers.) We see that $d(\cdot, v)$ is convex for any node v, but neither $L(w, x, y)$ nor $L(w, z, y)$ nor even $L(x, z)$ is treelike. We can observe here that $L(w, y)$ fails to contain a chain $C(w, y)$ of treelike pieces, since $L(w, x)$, $L(x, y)$, $L(w, z)$ and $L(z, y)$ are not treelike. Similarly, $L(x, z)$ fails to contain such a chain, since $L(x, z)$ is not treelike. This suggests the definition of a reticular space in the next section.

We conclude this section with a discussion of paths. One might expect that a treelike space would contain only one "path", in some sense, between any two points. Let us say that $L(x, y)$ is a *path* if for any $z_0, \ldots, z_k \in L(x, y)$ with $z_0 = x$, $z_k = y$, and $d(z_0, z_i) \leq d(z_0, z_{i+1})$ for $i = 0, \ldots, k-1$, the set $L(z_0, \ldots, z_k)$ is a chain. We can observe that not all line segments in a treelike space need be paths.

Example 3. Let $S = \{v, x, y, z\}$ and $V = \{v\}$, where $d(v, x) = d(x, y) = 2$, $d(y, z) = d(x, z) = 3$, $d(z, v) = 1$, and $d(v, y) = 4$. The space is treelike, but

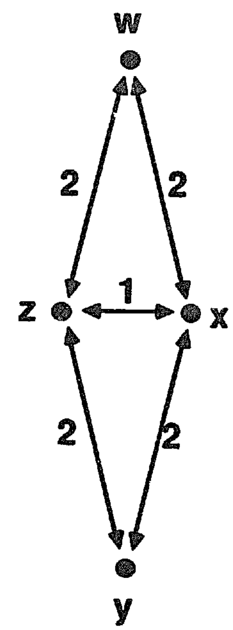

Fig. 1.

$L(v, y)$ is not a path because $L(v, z, x, y)$ is not a chain. We can, however, show the following.

Theorem 2. *Every line segment connecting two nodes in a treelike space is a path.*

Proof. Suppose to the contrary that $L(x, y)$ is not a path. Then for some $z_1, \ldots, z_{k-1} \in L(x, y)$, with $d(x, z_i) \leqslant d(x, z_{i+1})$ for $i = 1, \ldots, k-1$ and $z_k = y$, $L(x, z_1, \ldots, z_{k-1}, y)$ is not a chain. Thus $d(x, y) < d(x, z_1) + \cdots + d(z_{k-1}, y)$. Let j be the smallest integer such that $d(x, z_j) < d(x, z_1) + \cdots + d(z_{j-1}, z_j)$. Clearly, j exists, $j > 1$, and $L(x, z_{j-1}, z_j)$ is not a chain. Let u_x and u_y be collection points of $L(z_{j-1}, z_j)$ with respect to x and y, respectively. Clearly we cannot have $u_x = z_{j-1}$, since this would imply that $L(x, z_{j-1}, z_j)$ is a chain. Also since $d(z_{j-1}, y) \geqslant d(z_j, y)$, we observe that $u_y \neq z_{j-1}$. Then we have $d(x, z_{j-1}) + d(z_{j-1}, y) = d(x, u_x) + d(u_x, z_{j-1}) + d(z_{j-1}, u_y) + d(u_y, y) > d(x, u_x) + d(u_x, u_y) + d(u_y, y) \geqslant d(x, y)$, which violates the assumption that $z_{j-1} \in L(x, y)$. □

5. Reticular spaces

We noted in the previous section that convexity of metric does not imply treelikeness when a line segment $L(x, y)$ fails to contain a chain $C(x, y)$ of treelike pieces. But the implication can fail even when every segment contains such a chain.

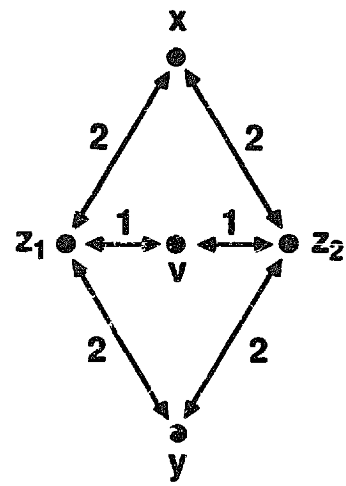

Fig. 2.

Example 4. (Fig. 2). Let $S=\{v, x, y, z_1, z_2\}$ and $V=\{v\}$, where $d(x, z_1)=d(x, z_2)=d(z_1, y)=d(z_2, y)=d(z_1, z_2)=2$, $d(v, z_1)=d(v, z_2)=1$, $d(v, x)=d(v, y)=3$, and $d(x, y)=4$. The metric is convex, and every line segment contains a treelike chain. But the space is not treelike because the 2-chain $L(x, x, y)=L(x, y)$ is not treelike.

The missing condition is analogous to the requirement that intersections of three or more arcs in a network must be nodes. We first say that $z \in D$ is an *exit point* of $D \subset S$ if z is a collection point u_v of D with respect to some node $v \notin D$. Also $z \in C(x, y)$ is *interior* to a chain $C(x, y)$ if for all $w \in S$, $C(z, w) \neq C(x, y)$. If (S, V, d) is a metric space with nodes, $z \in S$ is an *interior exit point* of (S, V, d) if it is an interior exit point of some chain in S.

A metric space with nodes (S, V, d) is a *reticular space* if it satisfies two conditions: (a) every line segment $L(x, y)$ in S contains a chain $C(x, y)$ of finitely many treelike segments, and (b) V contains all the interior exit points of (S, V, d). We note that Example 4 is not a reticular space because interior exit points z_1 and z_2 are not nodes. If they were, the space would still fail to be reticular because $L(x, z_1)$, for instance, would not contain a chain $C(x, z_1)$ of treelike segments.

The most obvious examples of reticular spaces are networks and certain subsets of networks. Given network $(\underline{S}, \underline{V}, \underline{d})$, suppose that $\underline{V}' \subset \underline{S}' \subset \underline{S}$, $\underline{V}' \subset \underline{V}$, and $\underline{d}'$ is the restriction of $\underline{d}$ to $\underline{S}' \times \underline{S}'$. Then clearly $(\underline{S}', \underline{V}', \underline{d}')$ is a reticular space, provided $\underline{V}'$ contains all of the interior exit points of $\underline{S}'$ with respect to nodes in $\underline{V}'$, and $\underline{S}'$ contains all of the boundary points generated by the nodes in $\underline{V}'$. This is because for any $L(x, y)$, there is a chain $C(x, y)=(T_1 \cup \cdots \cup T_k) \cap \underline{S}'$ of treelike segments, where $T_1, \ldots, T_k$ are the treelike segments of $\underline{S}$ along any shortest path connecting x and y. Since there are finitely many treelike segment boundaries, we have here a large class of finite reticular spaces.

A *subnetwork* $(\underline{S}', \underline{V}', \underline{d}')$ of a network $(\underline{S}, \underline{V}, \underline{d})$ must satisfy $\underline{V}'=\underline{V} \cap \underline{S}'$. Let

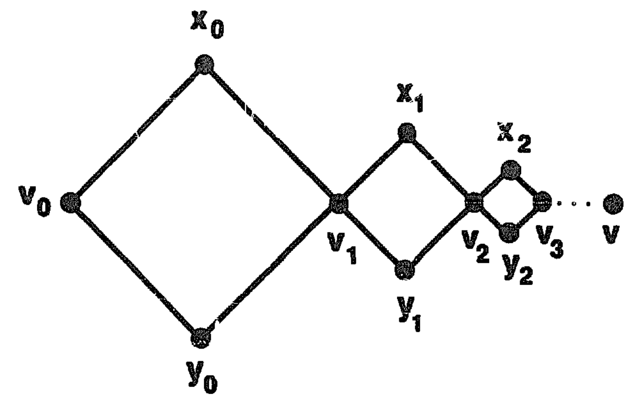

Fig. 3.

us say that a reticular space (S, V, d) can be *embedded* in a network $(\underline{S}, \underline{V}, \underline{d})$ if there is a distance-preserving bijection $\phi: S \to \underline{S}'$, where $(\underline{S}', \underline{V}', \underline{d}')$ is a subnetwork of $(\underline{S}, \underline{V}, \underline{d})$ and $\phi(V) = \underline{V}'$. A large variety of infinite reticular spaces cannot be embedded in networks.

Example 5. Let S be the unit square $[0, 1] \times [0, 1]$ in R^2, let $V = \{(0, 0), (1,0), (0, 1), (1, 1)\}$, and let d be the rectilinear metric. Then every line segment $L(x, y)$ is itself treelike, because its closest corner to $v \in V$ is a collection point with respect to v. Also there are no interior exit points. Thus (S, V, d) is a reticular space, but it is embeddable in no network. (Note that if $S = [0, 1 + \varepsilon] \times [0, 1]$, then for all $x_2 \in [0, 1]$, $(1, x_2)$ is a node because it is an interior exit point of $L((0, 0), (1 + \varepsilon, x_2))$, from which it similarly follows that all points in S are nodes. Thus (S, V, d) is not reticular because $L(x, y)$ contains no chain $C(x, y)$ of finitely many treelike segments.)

Example 6. Consider the networks $(\underline{N}_i, \underline{V}_i, \underline{d}_i)$ of Fig. 3, $i = 0, 1, \ldots$, where $\underline{N}_i$ is the set of points and $\underline{V}_i$ the set of nodes between and including v_0 and v_{i+1}, and all arcs between v_i and v_{i+1} have length 2^{-i}. Let S contain all the nodes in the union of the $\underline{N}_i$'s, plus v, and let $V = \{v, v_0, v_1, \ldots\}$. For $w, z \in \underline{V}_i$ define $d(w, z) = \underline{d}_i(w, z)$ and $d(w, z) = \underline{d}_i(w, v_{i+1}) + d(v_{i+1}, v)$, where $d(v_{i+1}, v) = 2^{-i}$. (S, V, d) is reticular because the interior exit points $v_1, v_2, \ldots$ are nodes, and because $L(x_i, v_i)$, $L(x_i, v_{i+1})$, $L(v_i, y_i)$, $L(v_{i+1}, y_i)$, $L(v_i, y_j)$ and $L(v_i, v)$ are treelike for $i, j \in \{0, 1, \ldots\}$. But (S, V, d) is not embeddable.

Finite reticular spaces can likewise fail to be embeddable, since it is possible that every distance-preserving bijection maps a non-node to a node.

Example 7. Consider the space of Example 4 (Fig. 2), except that $V = \{x, y\}$. Again we have a reticular space, but it is embeddable in no network. To see this, suppose to the contrary that there is a distance-preserving bijection ϕ from (S, V, d) to some subnetwork $(\underline{S}', \underline{V}', \underline{d}')$ of a network $(\underline{S}, \underline{V}, \underline{d})$, with $\phi(V) = \underline{V}'$.

Consider a shortest path L_x in S from $\phi(x)$ to $\phi(z_1)$, a shortest path L_y from $\phi(y)$ to $\phi(z_1)$, and a shortest path L_z from $\phi(z_2)$ to $\phi(z_1)$. Let w_{xy} be the point in $L_x \cap L_y$ that is nearest z_2, and similarly for w_{xz} and w_{yz}. If $w_{xy} = w_{xz} = w_{yz} = \phi(z_1)$, then $\phi(z_1) \in \underline{V}'$, contrary to the condition that $\phi(V) = \underline{V}'$. Thus w_{xy}, w_{xz} or w_{yz} must be distinct from $\phi(z_1)$; we may suppose without loss of generality that $w_{xy} \neq \phi(z_1)$. Then $d(x, y) = \underline{d}'(\phi(x), \phi(y)) \leq \underline{d}(\phi(x), w_{xy}) + \underline{d}(w_{xy}, \phi(y)) < \underline{d}'(\phi(x), \phi(z_1)) + \underline{d}'(\phi(z_1), \phi(y)) = d(x, z_1) + d(z_1, y) = d(x, y)$, a contradiction.

6. Convexity and reticular spaces

To prove our main result, that treelikeness and convexity of metric are equivalent in reticular spaces, we must first establish some properties of chains.

Lemma 1. *If $L(x, y)$ contains a treelike chain $C(x, y)$, then $C(x, y)$ contains all the nodes of $L(x, y)$.*

Proof. Take any node $v \in L(x, y)$, and let u_v be a collection point of $C(x, y)$ with respect to v. Since $u_v, v \in L(x, y)$, we have $d(x, u_v) + d(u_v, y) = d(x, v) + d(v, y) = [d(x, u_v) + d(u_v, v)] + [d(v, u_v) + d(u_v, y)]$. This implies $d(u_v, v) = 0$, or $u_v = v$. Thus $v \in C(x, y)$. □

Lemma 2. *If $L(x, z, y)$, $L(x, u, z)$ and $L(z, v, y)$ are chains, then $L(u, z, v)$ is a chain.*

Proof. By definition of a chain and the triangle inequality, $d(x, y) = d(x, u) + d(u, z) + d(z, v) + d(v, y) \geq d(x, u) + d(u, v) + d(v, y) \geq d(x, y)$, which implies $d(u, z) + d(z, v) = d(u, v)$. □

Lemmas 3–5 follow from the definition of a chain and the triangle inequality.

Lemma 3. *If $C(x, z)$ and $C(z, y)$ are chains and $z \in L(x, y)$, then $C(x, z) \cup C(z, y)$ is a chain.*

Lemma 4. *If $L(z_0, \ldots, z_k)$ is a chain, then $L(z_0, \ldots, z_j)$ is a chain for $j = 0, \ldots, k$.*

Lemma 5. *If $L(x, x', y)$ and $L(x', x, y)$ are chains, then $x = x'$.*

Lemma 6. *If $C(x, y) = C(x', y')$, then $L(x, y) = L(x', y')$.*

Proof. Let $C(x, y) = C(x, z, y)$. Then since any point in $C(x', y')$ belongs to $L(x, z)$ or $L(z, y)$, we may say without loss of generality that $x' \in L(x, z)$ and that $C(x', y') = C(x', z, y')$. We may also assume that $z \notin \{x, y\}$ or $z \notin \{x', y'\}$, since otherwise the lemma is trivial; we assume the latter without loss of generality. Furthermore, $x \in L(x', z)$ or $x \in L(z, y')$. In the latter case $x' \in L(z, y')$ since $x' \in L(z, x)$, so that $L(z, x', y')$ and $L(x', z, y')$ are chains. This implies by Lemma 5 that $x' = z$, contrary to assumption. Thus $x \in L(x', z)$, so that $L(x, x', z)$ and $L(x', x, z)$ are chains, and $x = x'$ by Lemma 5. Also $L(y, y', x)$ and $L(y', y, x') = L(y', y, x)$ are chains, so that $y = y'$ by Lemma 5. The claim follows. □

Lemma 7. *Every 2-chain $L(x, z, y)$ in a reticular space contains a chain $C(x, z, y)$ of finitely many treelike segments.*

Proof. We know that chains $C(x, z)$ and $C(z, y)$ of finitely many treelike segments exist. Since $z \in L(x, y)$, we have by Lemma 3 that $C(x, z, y) \equiv C(x, z) \cup C(z, y)$ is a chain of finitely many treelike segments. □

It is convenient to break the equivalence proof of convexity and treelikeness down into lemmas.

Lemma 8. *If (S, V, d) has a convex metric, then any chain $C(x, w, y)$ of two treelike chains $C(x, w)$ and $C(w, y)$ is itself a treelike chain.*

Proof. Suppose to the contrary that $C(x, w, y)$ is not treelike so that it lacks a collection point with respect to some node v. Let u_1, u_2 be collection points of $C(x, w)$ and $C(w, y)$, respectively, with respect to v. We note first that if $u_1 = w$, then for any $z \in C(x, w)$ we have $d(z, v) = d(z, w) + d(w, v) = d(z, w) + d(w, u_2) + d(u_2, v) = d(z, u_2) + d(u_2, v)$, so that u_2 serves as a collection point of $C(x, w, y)$ with respect to v, contrary to hypothesis. Thus $u_1 \neq w$, and similarly $u_2 \neq w$. By Lemma 2 $w \in L_\lambda(u_1, u_2)$ for some $\lambda \in (0, 1)$. Also $d(w, v) = (1 - \lambda)d(w, v) + \lambda d(w, v) > (1 - \lambda)d(u_1, v) + \lambda d(u_2, v)$, which contradicts the convexity of the metric. □

Lemma 9. *If all chains of two treelike chains in (S, V, d) are treelike, then any chain $C(x, y)$ of finitely many treelike chains is treelike.*

Proof. Let $C(x, y) = C(z_0, \ldots, z_k)$, where $z_0 = x$, $z_k = y$, and each $C(z_i, z_{i+1})$ is treelike, $i = 1, \ldots, k - 1$. By hypothesis $C(z_0, z_1)$ is treelike. If we assume $C(z_0, z_j)$ is a treelike chain, then by Lemma 4 $C(z_0, z_{j+1}) = C(z_0, z_j) \cup C(z_j, z_{j+1})$ is a chain, which by hypothesis is treelike. The lemma follows by induction. □

Theorem 3. *If (S, V, d) is reticular, then it is treelike if and only if d is a convex metric.*

Proof. If (S, V, d) is treelike, then by Theorem 1 d is a convex metric. Suppose, then, that d is convex, and show that any 2-chain $L(x, z, y)$ is treelike. By Lemma 7, $L(x, z, y)$ contains a chain $C(x, z, y)$ of finitely many treelike segments, which by Lemmas 8 and 9 is treelike and therefore contains a collection point u_v with respect to any given $v \in V$. We will show that u_v is also a collection point for $L(x, z, y)$, so that $L(x, z, y)$ is treelike.

Case 1. u_v is not interior to $C(x, z, y)$, so that $C(x, z, y) = C(u_v, y')$ for some y'. Lemma 6 implies $L(x, y) = L(u_v, y')$. So, given any $t \in L(x, z, y)$, we have $t \in L(u_v, y')$ and $d(y', v) = d(y', u_v) + d(u_v, v) = d(y', t) + d(t, u_v) + d(u_v, v) \geqslant d(y', t) + d(t, v) \geqslant d(y', v)$. Thus $d(t, v) = d(t, u_v) + d(u_v, v)$, which implies that u_v is a collection point for $L(x, z, y)$.

Case 2. u_v is interior to $C(x, z, y)$, so that $u_v \neq x, y$. We may suppose that $v \notin C(x, z, y)$, since otherwise v is a collection point of $L(x, z, y)$ with respect to itself. Thus u_v is an interior exit point and therefore a node. Take any $t \in L(x, z, y)$ and suppose without loss of generality that $t \in L(x, z)$. Then $L(x, t, z)$ contains a chain $C(x, t, z)$ of finitely many treelike segments, and $L(z, y)$ contains a chain $C(z, y)$ of finitely many treelike segments; let $C(x, t, y) = C(x, t, z) \cup C(z, y)$. By Lemmas 8 and 9, $C(x, t, y)$ is treelike and contains a collection point u'_v with respect to v. If u'_v is not interior to $C(x, t, y)$, then by the argument of Case 1 u'_v is a collection point for $L(x, z, y)$ and the theorem is proved. Thus we assume that u'_v is interior to $C(x, t, y)$. As before we may suppose that $v \notin C(x, t, y)$, so that u'_v is an interior exit point and therefore a node. Now by Lemma 1 $u'_v \in C(x, z, y)$ and $u_v \in C(x, t, y)$, so that $d(u'_v, v) = d(u'_v, u_v) + d(u_v, v)$ and $d(u_v, v) = d(u_v, u'_v) + d(u'_v, v)$. These imply by Lemma 5 that $u_v = u'_v$. Thus u_v is a collection point for $C(x, t, y)$, and $d(t, v) = d(t, u_v) + d(u_v, v)$. Since t is an arbitrary element of $L(x, z, y)$, u_v is a collection point of $L(x, z, y)$. □

7. Piecewise convex metrics

Given set $D \subset S$ in a metric space with nodes (S, V, d), let us say that d is *convex on D* if $d(v, \cdot): D \to R$ is convex for all $v \in V$. Let d be *piecewise convex* if every line segment $L(x, y)$ contains a chain $C(x, y)$ of finitely many line segments on which d is convex. For brevity we will say that (S, V, d) is piecewise convex if its metric is.

A network is piecewise convex as well as reticular, because its metric is convex on treelike segments. But treelikeness and convexity of metric are not equivalent on piecewise convex spaces, even those in which every interior exit point is a node, as they are on reticular spaces. We have already seen this in Example 2. Thus a piecewise convex space need not be reticular.

One might conjecture that all reticular spaces are piecewise convex, and we prove this below for finite spaces. In Example 1, for instance, we see that although there is one decomposition of $L(w, y)$ into treelike segments, namely into the single segment $L(w, y)$, on which d is not convex, there is another decomposition, into $L(w, x)$ and $L(x, y)$, on which d is convex. We see in Example 6, however, that an infinite reticular space need not be piecewise convex, since $L(v_0, v)$ contains no chain $C(v_0, v)$ of finitely many segments on which d is convex. The results to follow can be extended to infinite spaces that satisfy certain additional properties, but these greatly complicate the proofs [5, 6].

To show that finite reticular spaces are piecewise convex, we show that the metric is convex on any "treelike subspace" and that any $L(x, y)$ contains a chain $C(x, y)$ of line segments that are treelike subspaces.

Let a *treelike subspace* T of a metric space with nodes (S, V, d) be a subset of (S, V, d) that, in isolation, is a treelike space. More precisely, $T \subset S$ must be a convex treelike set, and (T, V_T, d') must be a treelike space, where V_T consists of $V \cap T$ plus all exit points of T in (S, V, d), and where d' is d restricted to $T \times T$. In Example 1, $L(w, y)$ contains a chain $L(w, x, y)$ of two treelike subspaces.

Lemma 10. *The metric d is convex on any treelike subspace of a reticular space (S, V, d).*

Proof. To show that $d(v, \cdot)$ is convex on treelike subspace (T, V_T, d') for any $v \in V$, let u_v be a collection point on T with respect to v. Since u_v is either a node of (S, V, d) or an exit point of T, it is a node of (T, V_T, d'), so that by Theorem 1 $d'(u_v, \cdot)$ is convex on T. Thus $d(v, \cdot) = d(v, u_v) + d'(u_v, \cdot)$ is convex on T. □

Lemma 11. *Every line segment $L(x, y)$ of a finite reticular space (S, V, d) contains a chain $C(x, y)$ of convex treelike segments.*

Proof. We construct $C(x, y)$ using a recursive procedure P, whose argument is a line segment to be replaced with a chain of treelike segments. We begin by taking $C(x, y) = C(x_0, y_0)$ to be a chain of treelike segments, which we know to exist, and by calling $P(L(x_1, y_1))$ for each nonconvex segment $L(x_1, y_1)$ of $C(x_0, y_0)$. The Procedure P is as follows.

Procedure $P(L(x_n, y_n))$. Since $L(x_n, y_n)$ is nonconvex, for some pair $x'_n, y'_n \in L(x_n, y_n)$, $L(x'_n, y'_n)$ contains a point $z_n \notin L(x_n, y_n)$. Thus $\{x'_n, y'_n\} \neq \{x_n, y_n\}$, and by Lemma 3 either $L(x_n, x'_n, y_n)$ or $L(x_n, y'_n, y_n)$ contains a chain $C(x_n, y_n)$ of at least two distinct treelike segments. Let $C(x_n, y_n)$ replace $L(x_n, y_n)$ in $C(x, y)$. For each nonconvex treelike segment $L(x_{n+1}, y_{n+1})$ of $C(x_n, y_n)$ call $P(L(x_{n+1}, y_{n+1}))$. This completes Procedure P.

It is clear that since S is finite, n is bounded, and the procedure terminates with a chain $C(x, y)$ of convex treelike segments. □

Theorem 4. *If a reticular space (S, V, d) is finite, then every line segment $L(x, y)$ contains a chain $C(x, y)$ of treelike subspaces.*

Proof. We construct $C(x, y)$ with a recursive procedure Q. We begin with a chain $C(x, y) = C(x_0, y_0)$ of convex treelike segments, which we know by Lemma 12 to exist, and by calling $Q(L(x_1, y_1))$ for each segment $L(x_1, y_1)$ of $C(x_0, y_0)$ that is not a treelike space. The Procedure Q is as follows.

Procedure $Q(L(x_n, y_n))$. Since $L(x_n, y_n)$ is a convex treelike segment but not a treelike subspace, there is a node or exit point w of $L(x_n, y_n)$ with respect to which some 2-chain $L(x'_n, z'_n, y'_n)$ in $L(x_n, y_n)$ has no collection point. We can suppose that $\{x'_n, z'_n, y'_n\} \neq \{x_n, y_n\}$, since otherwise $L(x'_n, z'_n, y'_n) = L(x_n, y_n)$ and w would be a collection point of $L(x'_n, z'_n, y'_n)$ with respect to itself. Thus by Lemmas 3 and 11, $L(x_n, x'_n, y_n)$, $L(x_n, z'_n, y_n)$ or $L(x_n, y'_n, y_n)$ contains a chain $C(x_n, y_n)$ of at least two distinct convex treelike segments. Let $C(x_n, y_n)$ replace $L(x_n, y_n)$ in $C(x, y)$. Call $Q(L(x_{n+1}, y_{n+1}))$ for each segment $L(x_{n+1}, y_{n+1})$ of $C(x_n, y_n)$ that is not a treelike subspace. This completes Procedure Q.

Since S is finite, the procedure must terminate with a chain $C(x, y)$ of treelike subspaces. □

From Lemma 10 and Theorem 4 we have,

Corollary. *Any finite reticular space has a piecewise convex metric.*

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