Improved Job Sequencing Bounds from Decision Diagrams

John Hooker

CP 2019

University of Connecticut, USA

Motivation

- Job sequencing problems are usually solved by **heuristics**.
  - **Bounds** are needed to judge quality of solutions.
  - It's **really hard** to derive tight bounds for combinatorial problems, except in a branching framework.

Motivation

- Job sequencing problems are usually solved by **heuristics**.
  - **Bounds** are needed to judge quality of solutions.
  - It's **really hard** to derive tight bounds for combinatorial problems, except in a branching framework.
- **Decision diagrams** can provide bounds.
  - But they are **weak** as the problem **scales up**.

Motivation

- Job sequencing problems are usually solved by **heuristics**.
  - **Bounds** are needed to judge quality of solutions.
  - It's **really hard** to derive tight bounds for combinatorial problems, except in a branching framework.
- **Decision diagrams** can provide bounds.
  - But they are **weak** as the problem **scales up**.
- **Lagrangian duality** can provide bounds.
  - But they are usually **weak** because of **duality gap**.

Motivation

- Job sequencing problems are usually solved by **heuristics**.
  - **Bounds** are needed to judge quality of solutions.
  - It's **really hard** to derive tight bounds for combinatorial problems, except in a branching framework.
- **Decision diagrams** can provide bounds.
  - But they are **weak** as the problem **scales up**.
- **Lagrangian duality** can provide bounds.
  - But they are usually **weak** because of **duality gap**.
- How about **DDs + Lagrangian**?
  - **When** can they be combined?

Objectives

- Derive **tight bounds** for job sequencing problems**.**
    - Use **Lagrangian relaxation** to tighten bounds from **decision diagrams**.

Objectives

- Derive **tight bounds** for job sequencing problems**.**
  - Use **Lagrangian relaxation** to tighten bounds from **decision diagrams**.
- Generalize to **dynamic programming.**
  - **General conditions** under which Lagrangian relaxation can **combine** with decision diagrams.

Objectives

- Derive **tight bounds** for job sequencing problems**.**
  - Use **Lagrangian relaxation** to tighten bounds from **decision diagrams**.
- Generalize to **dynamic programming.**
  - **General conditions** under which Lagrangian relaxation can **combine** with decision diagrams.
- Apply to **specific job-sequencing problems.**
  - **Which ones** are suitable for this kind of bounding?
  - **Compute** tight bounds for some well-known benchmarks.

Build on Recent Work

- Tight DD-based bounds for job sequencing with state-dependent processing times**.**
  - Approach **doesn't scale up.**
  - Add **Lagrangian relaxation**.

JH (2017)

Build on Recent Work

- Tight DD-based bounds for job sequencing with state-dependent processing times**.**
  - Approach **doesn't scale up.**
  - Add **Lagrangian relaxation**.

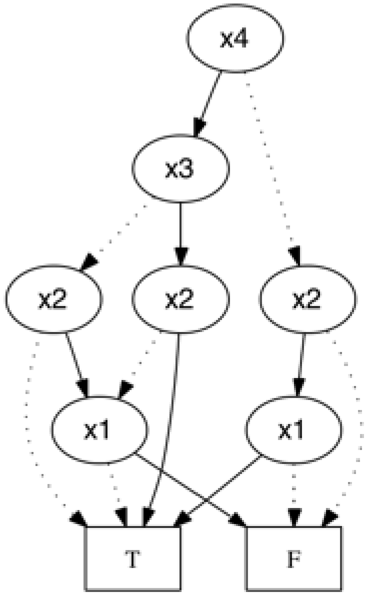
JH (2017)

- Bounds from DDs+Lagrangian for TSP with time windows within CP solver.
  - Use **stand-alone** DD.
  - Extend to **other objectives**, e.g. min tardiness.
  - Find **general conditions** for combining DDs and Lagrangian relaxation.

Bergman, Cire, van Hoeve (2015)

Decision Diagrams

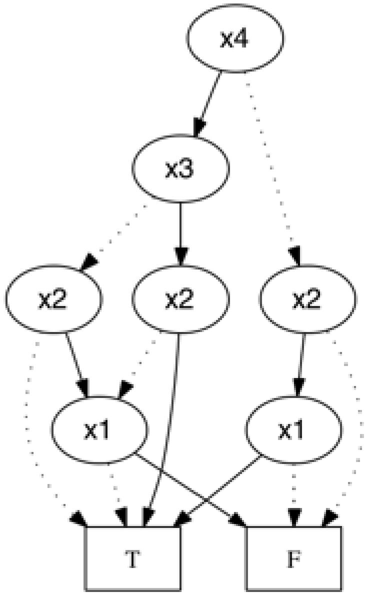
- Graphical encoding of a boolean function
  - Historically used for circuit design & verification
  - **Binary diagrams** easily extended to **multivalued diagrams**.
  - Unique **reduced** diagram for a give variable ordering.



Lee (1959), Bryant (1986)

Decision Diagrams

- Adapt to optimization and constraint programming
  - **Paths** from top to T represent **feasible solutions**
    - Can delete paths to F
  - Path **lengths** represent **costs**.
  - **Shortest** path is **optimal** solution.



Hadžić and JH (2006, 2007)

Job Sequencing Example

- Problem: sequence jobs with given processing times
  - Minimize **tardiness** subject to **time windows.**

j	r_j	p_j	d_j
1	0	3	5
2	1	2	3
3	1	2	5

Job Sequencing Example

- Problem: sequence jobs with given processing times
  - Minimize **tardiness** subject to **time windows**

j	r_j	p_j	d_j
1	0	3	5
2	1	2	3
3	1	2	5

Release time → r_j

Job Sequencing Example

- Problem: sequence jobs with given processing times
  - Minimize **tardiness** subject to **time windows**

j	r_j	p_j	d_j
1	0	3	5
2	1	2	3
3	1	2	5

Release time (r_j)

Processing time (p_j)

Job Sequencing Example

- Problem: sequence jobs with given processing times
  - Minimize **tardiness** subject to **time windows**

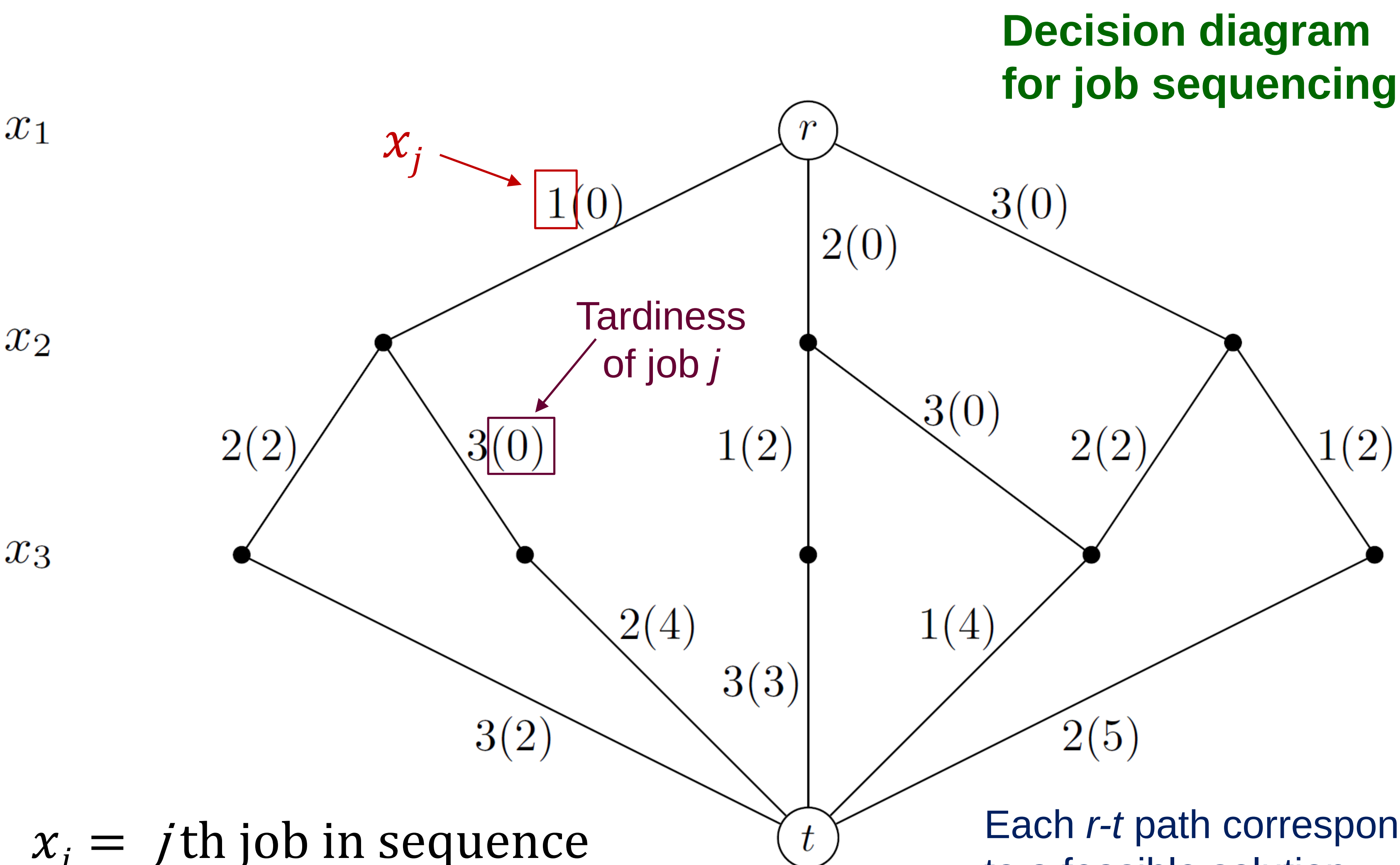
j	r_j	p_j	d_j
1	0	3	5
2	1	2	3
3	1	2	5

Release time → r_j

Processing time → p_j

Due date → d_j

Job Sequencing Example

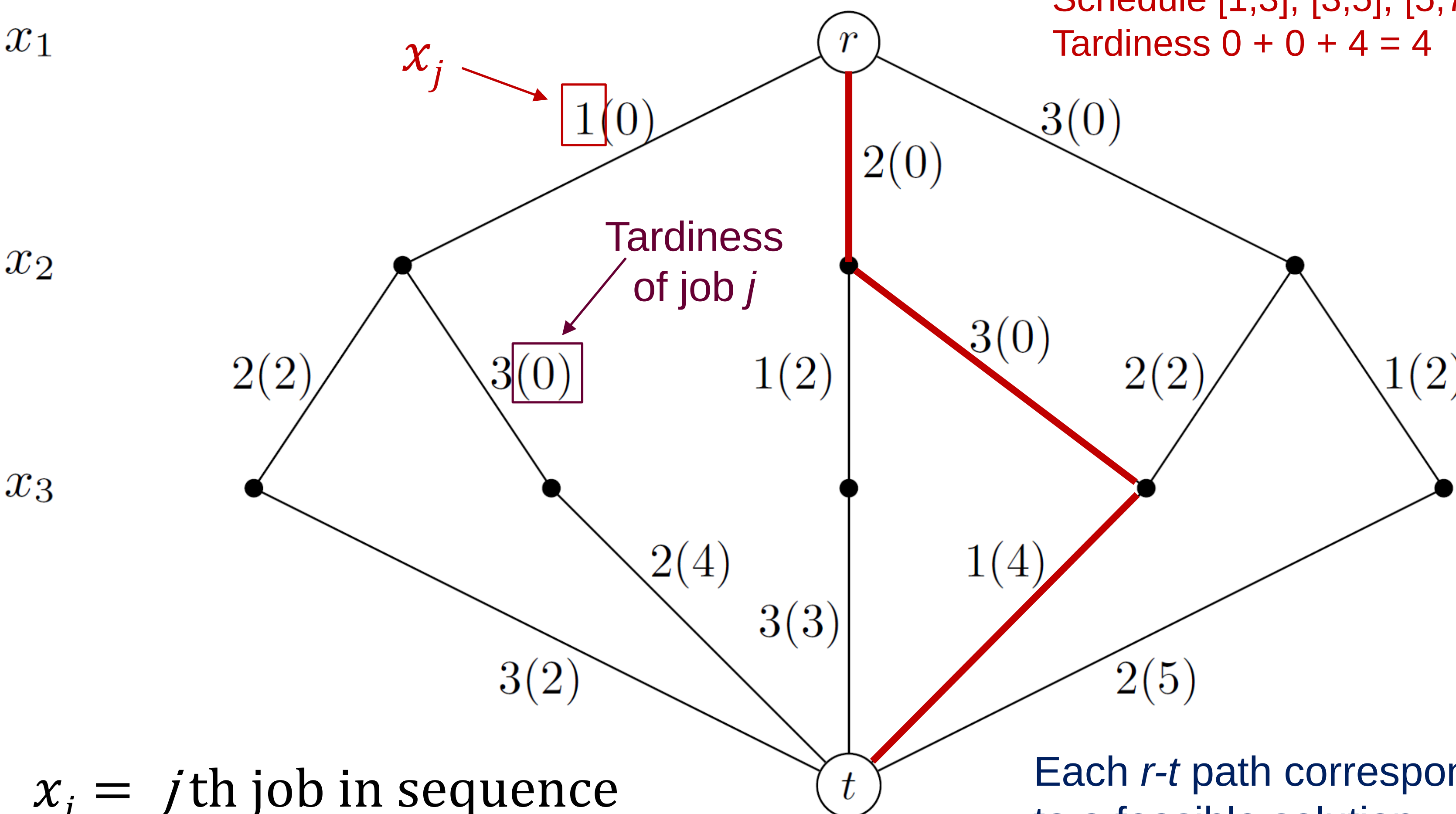


$x_j = j$th job in sequence

Each r-t path corresponds to a feasible solution

Job Sequencing

An optimal solution:
Sequence 2-3-1
Schedule [1,3], [3,5], [5,7]
Tardiness 0 + 0 + 4 = 4



$x_j =$ jth job in sequence

Each r-t path corresponds to a feasible solution

Building a Decision Diagram

- Our approach:
  - Associate dynamic programming **states** with nodes..
  - …as in a state transition graph.

Dynamic Programming Model

- Our approach:
  - Associate dynamic programming **states** with nodes..
  - …as in a state transition graph.

General recursive model

$$h_i(\boldsymbol{S}_i) = \min_{x_i \in X_i(\boldsymbol{S}_i)} \left\{ c_i(\boldsymbol{S}_i, x_i) + h_{i+1}\big((\phi_i(\boldsymbol{S}_i, x_i))\big) \right\}$$

State in stage *i*

$$\boldsymbol{S}_i = \big(S_{i1}, \ldots, S_{ik}\big)$$

Dynamic Programming Model

- Our approach:
  - Associate dynamic programming **states** with nodes..
  - …as in a state transition graph.

General recursive model

$$h_i(\boldsymbol{S}_i) = \min_{x_i \in X_i(\boldsymbol{S}_i)} \left\{ c_i(\boldsymbol{S}_i, x_i) + h_{i+1}\Big((\phi_i(\boldsymbol{S}_i, x_i))\Big) \right\}$$

State in stage *i*

Set of possible controls

Dynamic Programming Model

- Our approach:
  - Associate dynamic programming **states** with nodes..
  - …as in a state transition graph.

General recursive model

$$h_i(\boldsymbol{S}_i) = \min_{x_i \in X_i(\boldsymbol{S}_i)} \left\{ c_i(\boldsymbol{S}_i, x_i) + h_{i+1}\big((\phi_i(\boldsymbol{S}_i, x_i))\big) \right\}$$

State in stage *i*

Set of possible controls

Immediate cost

Dynamic Programming Model

- Our approach:
  - Associate dynamic programming **states** with nodes..
  - …as in a state transition graph.

General recursive model

State transition function

$$h_i(\boldsymbol{S}_i) = \min_{x_i \in X_i(\boldsymbol{S}_i)} \left\{ c_i(\boldsymbol{S}_i, x_i) + h_{i+1}\big(\phi_i(\boldsymbol{S}_i, x_i)\big) \right\}$$

State in stage *i*

Set of possible controls

Immediate cost

Dynamic Programming Model

- Our approach:
  - Associate dynamic programming **states** with nodes..
  - …as in a state transition graph.

General recursive model

State transition function

$$h_i(\boldsymbol{S}_i) = \min_{x_i \in X_i(\boldsymbol{S}_i)} \left\{ c_i(\boldsymbol{S}_i, x_i) + h_{i+1}\left(\left(\phi_i(\boldsymbol{S}_i, x_i)\right)\right) \right\}$$

State in stage *i*

Set of possible controls

Immediate cost

Cost to go

DP Model for Job Sequencing

State: $S_i = (V_i, t_i)$

- V_i: Set of jobs scheduled so far
- t_i: Finish time of last job scheduled

Initial state = $(\emptyset, 0)$

$$h_i(\boldsymbol{S}_i) = \min_{x_i \in X_i(\boldsymbol{S}_i)} \left\{ c_i(\boldsymbol{S}_i, x_i) + h_{i+1}\big((\phi_i(\boldsymbol{S}_i, x_i))\big) \right\}$$

- $\boldsymbol{S}_i$: State in stage *i*
- $X_i(\boldsymbol{S}_i)$: Set of possible controls
- $c_i(\boldsymbol{S}_i, x_i)$: Immediate cost
- $h_{i+1}\big((\phi_i(\boldsymbol{S}_i, x_i))\big)$: Cost to go

DP Model for Job Sequencing

Set of jobs scheduled so far

State: $S_i = (V_i, t_i)$

Initial state = $(\emptyset, 0)$

Finish time of last job scheduled

Controls: $X_i(V_i, t_i) = \{1, \ldots, n\} \setminus V_i$

$$h_i(\boldsymbol{S}_i) = \min_{x_i \in X_i(\boldsymbol{S}_i)} \left\{ c_i(\boldsymbol{S}_i, x_i) + h_{i+1}\big((\phi_i(\boldsymbol{S}_i, x_i)\big) \right\}$$

State in stage *i*

Set of possible controls

Immediate cost

Cost to go

DP Model for Job Sequencing

Set of jobs scheduled so far

State: $S_i = (V_i, t_i)$

Initial state = $(\emptyset, 0)$

Finish time of last job scheduled

Controls: $X_i(V_i, t_i) = \{1, \ldots, n\} \setminus V_i$

Immediate cost: $c_i\big((V_i, t_i), x_j\big) = \big(\max\{r_{x_i}, t_i\} + p_{x_i} - d_{x_i}\big)^+$

$$h_i(\boldsymbol{S}_i) = \min_{x_i \in X_i(\boldsymbol{S}_i)} \Big\{ c_i(\boldsymbol{S}_i, x_i) + h_{i+1}\big((\phi_i(\boldsymbol{S}_i, x_i))\big) \Big\}$$

State in stage *i*

Set of possible controls

Immediate cost

Cost to go

DP Model for Job Sequencing

Set of jobs scheduled so far

Initial state = $(\emptyset, 0)$

State: $S_i = (V_i, t_i)$

Finish time of last job scheduled

Controls: $X_i(V_i, t_i) = \{1, \ldots, n\} \setminus V_i$

Immediate cost: $c_i\big((V_i, t_i), x_j\big) = \big(\max\{r_{x_i}, t_i\} + p_{x_i} - d_{x_i}\big)^+$

Transition: $\phi_i\big((V_i, t_i), x_i\big) = \big(V_i \cup \{x_i\}, \max\{r_{x_i}, t_i\} + p_{x_i}\big)$

$$h_i(\boldsymbol{S}_i) = \min_{x_i \in X_i(\boldsymbol{S}_i)} \Big\{ c_i(\boldsymbol{S}_i, x_i) + h_{i+1}\big((\phi_i(\boldsymbol{S}_i, x_i))\big) \Big\}$$

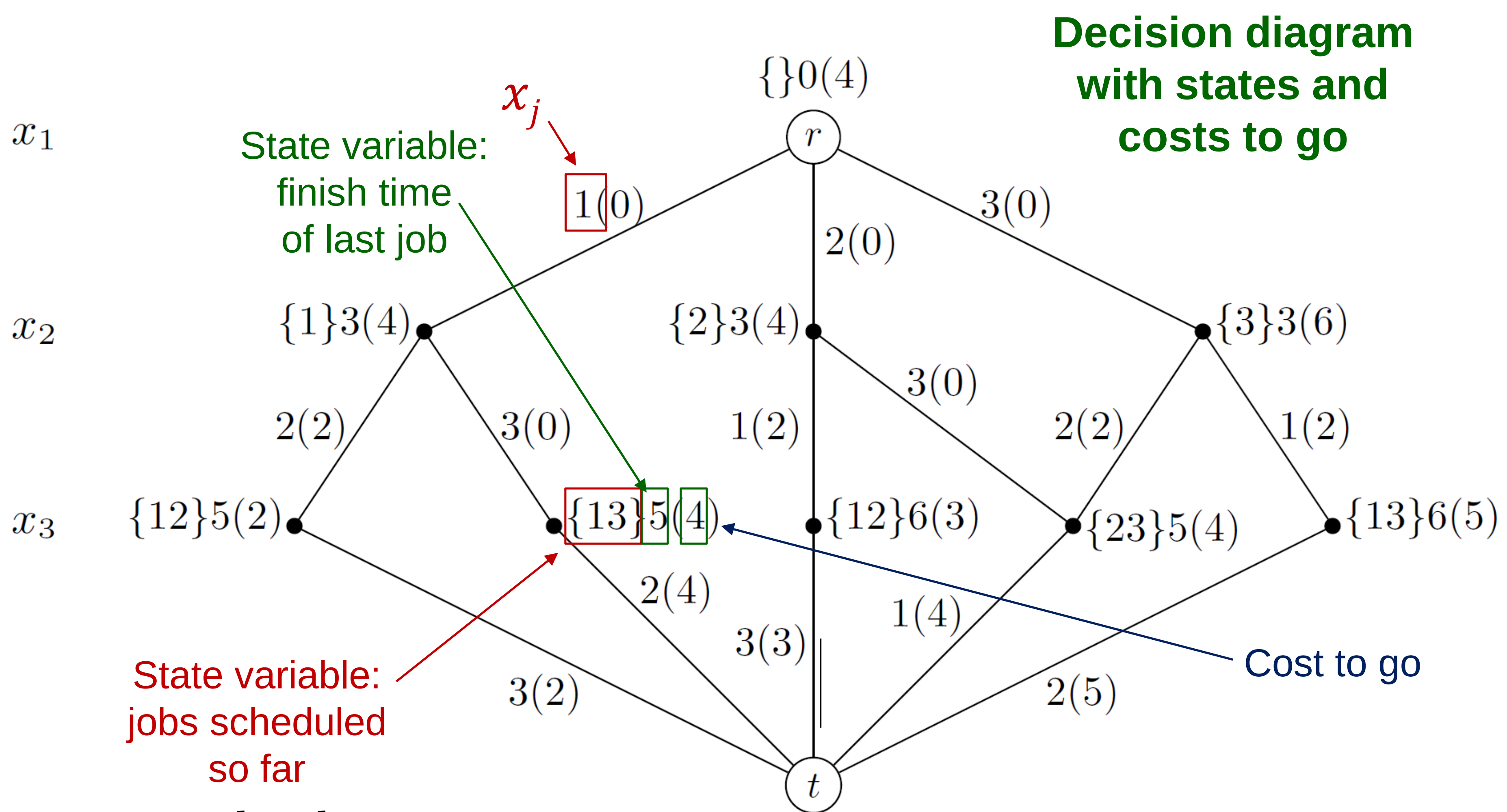
State in stage *i*

Set of possible controls

Immediate cost

Cost to go

Job Sequencing Diagram



$x_j = j$th job in sequence

Relaxed Decision Diagram

- Definition
  - Every *r*-*t* path of the original diagram appears in the relaxed diagram with equal or smaller cost.
  - So a relaxed diagram **may represent some infeasible solutions**.
- Motivation
  - **Shortest path** in the relaxed diagram provides a **lower bound** on the optimal value.

Andersen, Hadžić, JH, Tiedemanmn (2007)

Building a Relaxed Diagram

- Node splitting
  - Start with a diagram that represents **all** solutions (feasible and infeasible) and **refine** it.

Andersen, Hadžić, JH, Tiedemanmn (2007)

Ciré and van Hoeve (2013)

Building a Relaxed Diagram

- Node splitting
  - Start with a diagram that represents **all** solutions (feasible and infeasible) and **refine** it.
- Node merger – used here
  - **Merge** some nodes in the **exact** diagram.
  - …to make the diagram **smaller** while excluding no feasible solutions and introducing some infeasible low-cost solutions.

Andersen, Hadžić, JH, Tiedemanmn (2007)

Ciré and van Hoeve (2013)

Bergman, Ciré, van Hoeve, JH (2013)

Node Merger

- Don’t begin with exact diagram
  – It is too large
- Merge nodes as the diagram is constructed
  – Combine states of the merged nodes in a way that yields a valid relaxation.
  – This may require **additional state variables**.

JH (2017)

Bergman, Ciré, van Hoeve, JH (2013, 2016)

Relaxed DP Model

- In the example, no new states needed
  - Transition function same as before.

Reflects node merger in layer *i* + 1

Recursion:

$$\bar{h}_i(\boldsymbol{S}_i) = \min_{x_i \in X_i(\boldsymbol{S}_i)} \left\{ c_i(\boldsymbol{S}_i, x_i) + \bar{h}_{i+1}\Big(\boxed{\rho_{i+1}} \big(\phi_i(\boldsymbol{S}_i, x_i)\big)\Big) \right\}$$

Relaxed DP Model

$$S_i = (V_i, t_i)$$

Set of jobs scheduled in **all** feasible solutions so far (→ V_i)

Earliest possible finish time of immediately previous job (→ t_i)

Initial state = $(\emptyset, 0)$

Transition:

$$\phi_i\big((V_i, t_i), x_j\big) = \big(V_i \cup \{x_i\}, \max\{r_{x_i}, t_i\} + p_{x_i}\big)$$

Recursion:

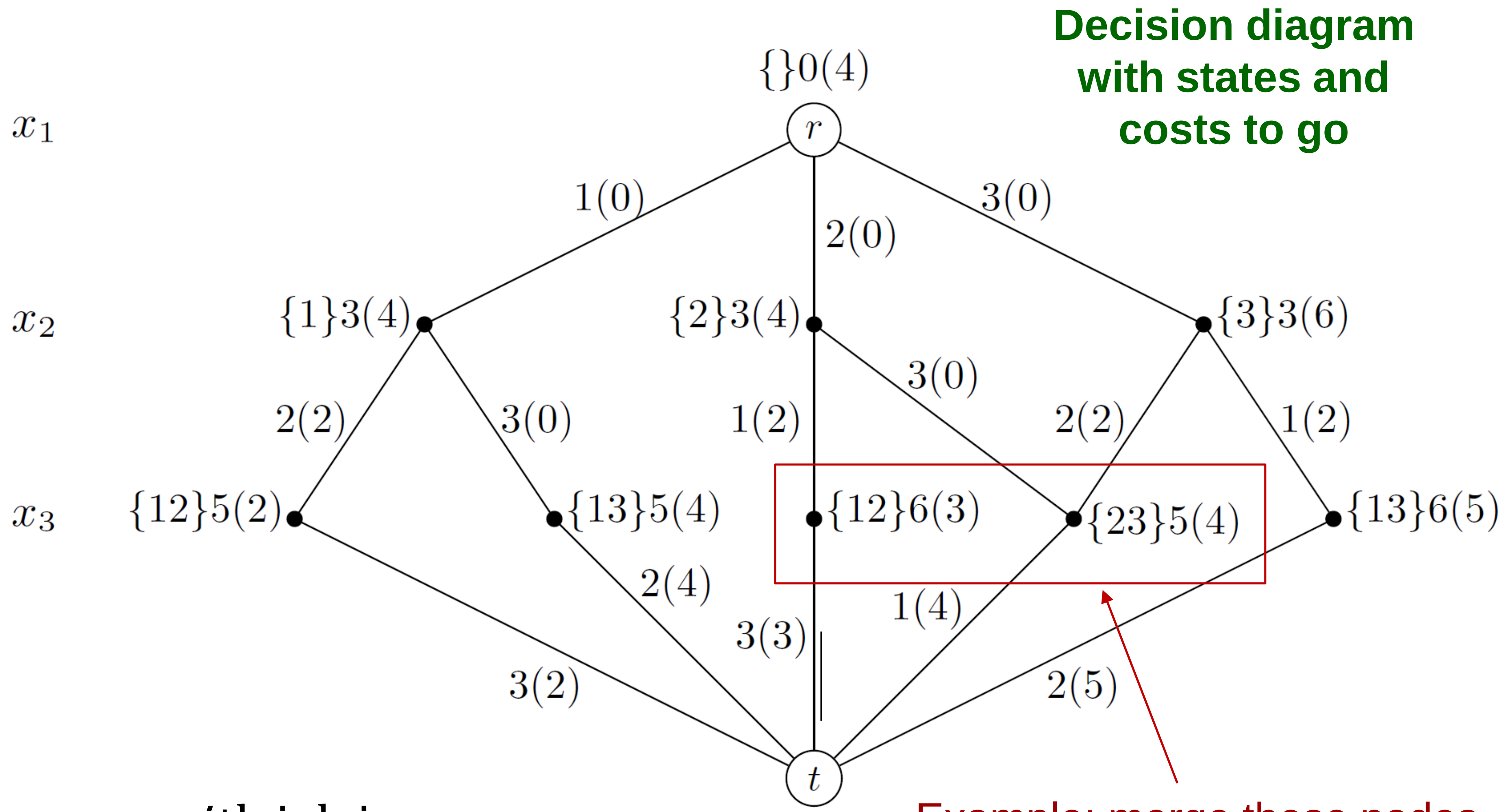
$$\bar{h}_i(\boldsymbol{S}_i) = \min_{x_i \in X_i(\boldsymbol{S}_i)} \Big\{ c_i(\boldsymbol{S}_i, x_i) + \bar{h}_{i+1}\Big(\rho_{i+1}\big(\phi_i(\boldsymbol{S}_i, x_i)\big)\Big) \Big\}$$

Node Merger in Relaxation

- Merge states as the diagram is constructed
  - States *S*, *T* merge to form state $S \oplus T$
- Merger operation must yield valid relaxation
  - There are **sufficient conditions** for this. JH (2017)
  - In state-dependent job sequencing,

$$(V,t) \oplus (V',t') = \big(V \cap V', \min\{t,t'\}\big)$$

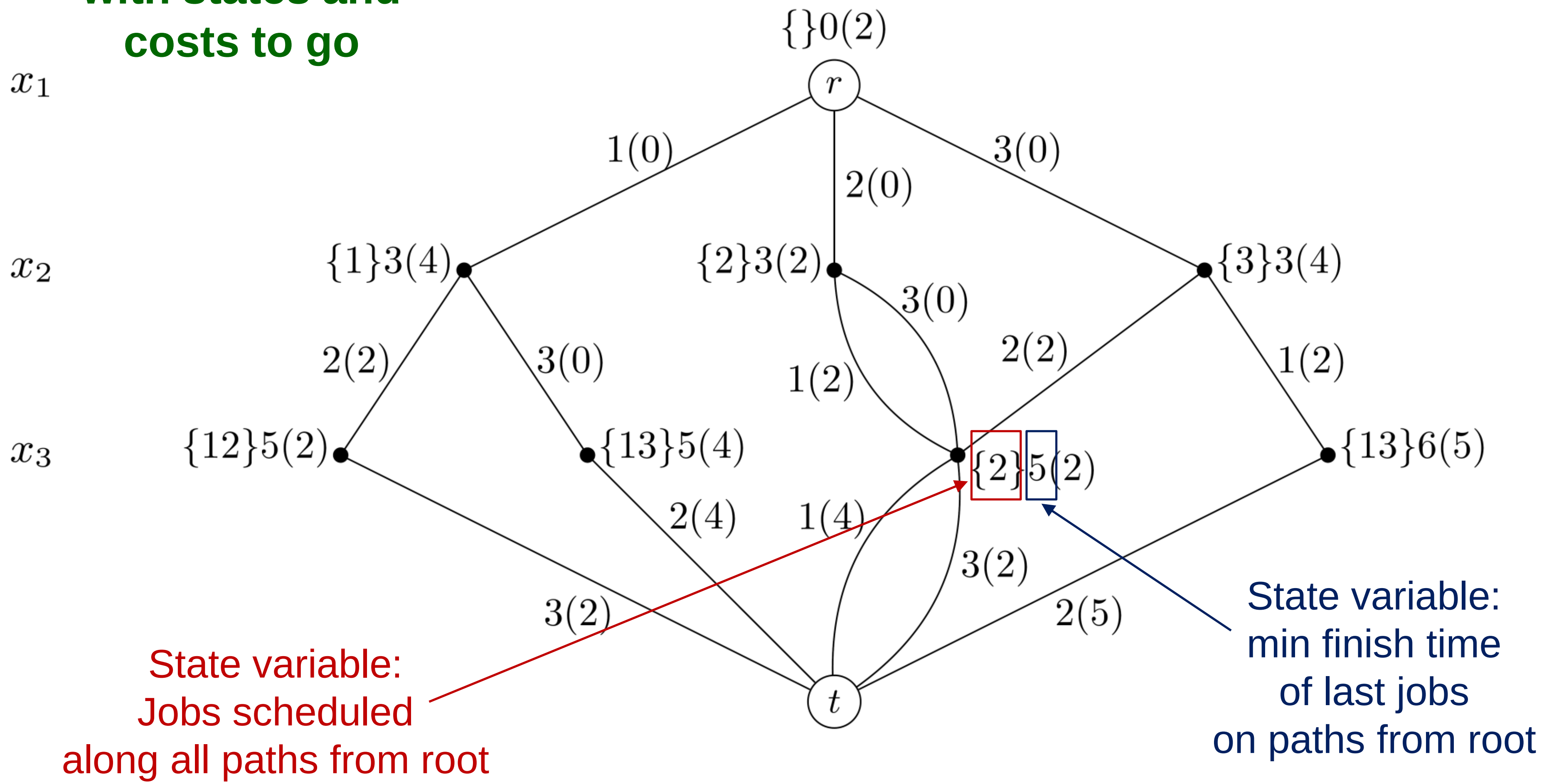
Job Sequencing Diagram



$x_i =$ ith job in sequence

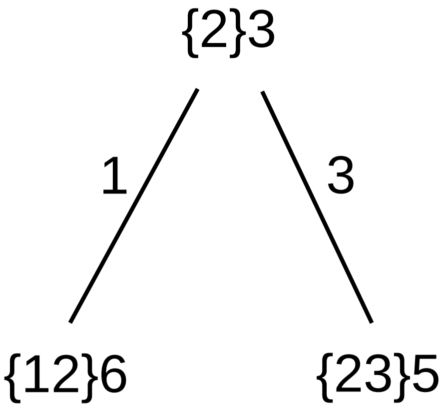
Job Sequencing Relaxed Diagram

Relaxed decision diagram with states and costs to go

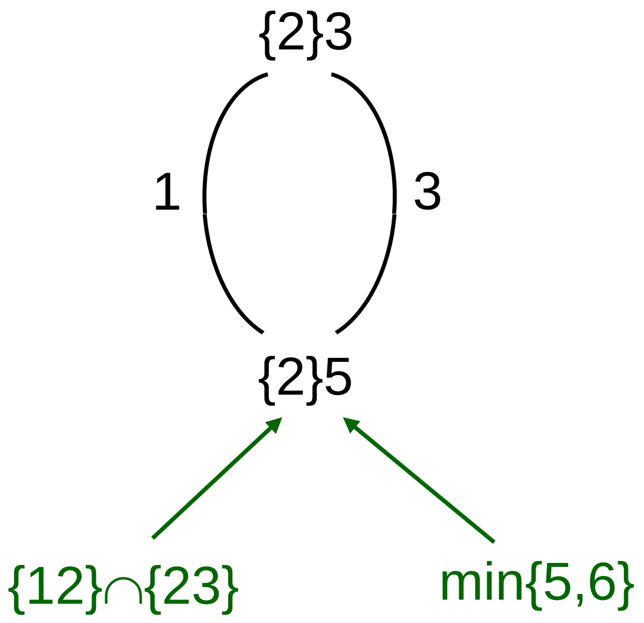


Job Sequencing Node Merger

Without merger



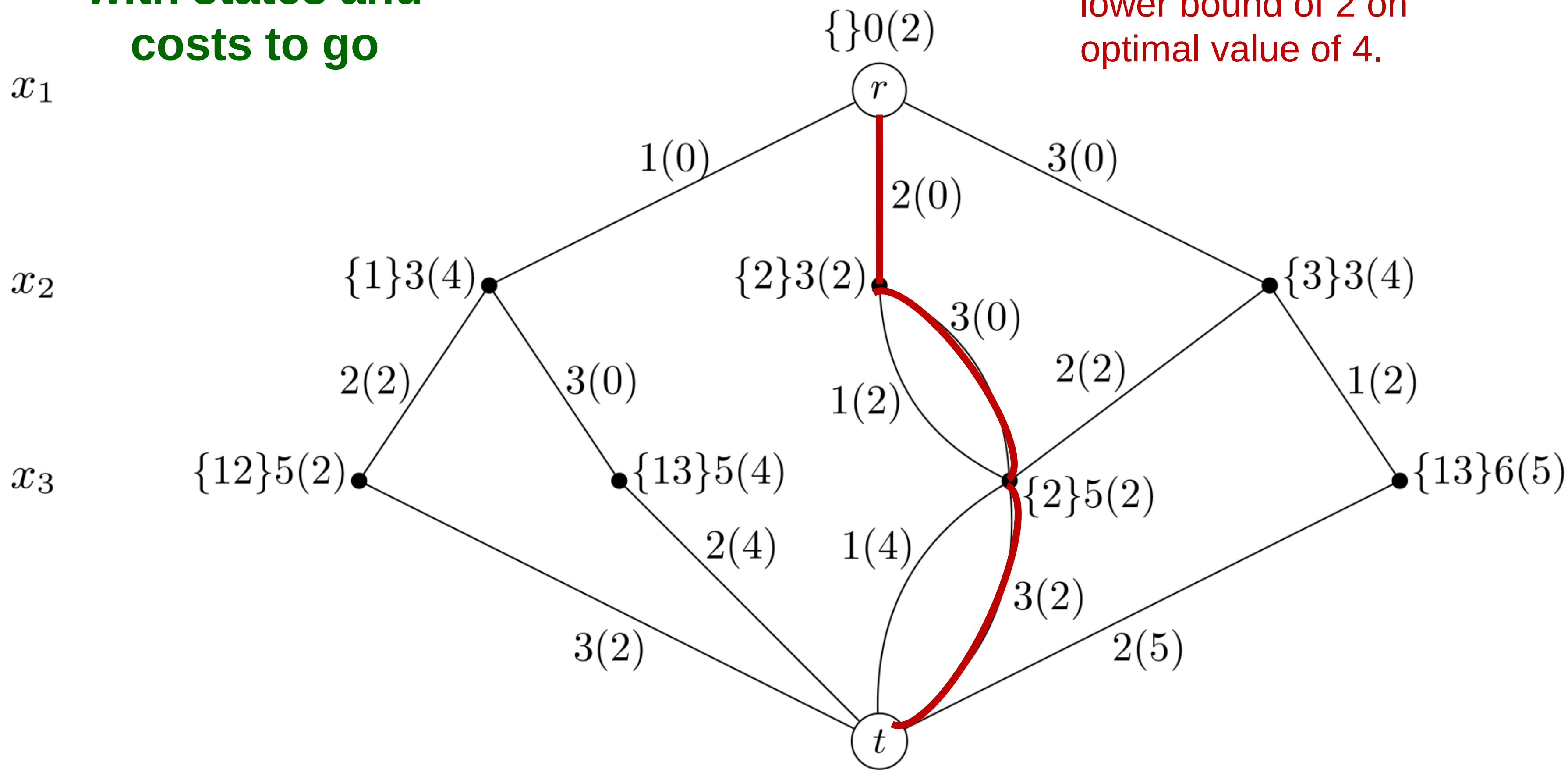
With merger



Job Sequencing Relaxed Diagram

Relaxed decision diagram with states and costs to go

Shortest path yields a lower bound of 2 on optimal value of 4.



Lagrangian Relaxation

- "Dualize" hard constraints.
  - By moving them into the objective functions

Consider a problem:

$$\min_{\boldsymbol{x} \in \boldsymbol{X}} \left\{ f(\boldsymbol{x}) \;\middle|\; \boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{0} \right\}$$

Lagrangian Relaxation

- "Dualize" hard constraints.
  - By moving them into the objective functions

Consider a problem:

$$\min_{\boldsymbol{x} \in \boldsymbol{X}} \left\{ f(\boldsymbol{x}) \;\middle|\; \boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{0} \right\}$$

Lagrangian relaxation:

$$\theta(\boldsymbol{\lambda}) = \min_{\boldsymbol{x} \in \boldsymbol{X}} \left\{ f(\boldsymbol{x}) + \boldsymbol{\lambda}^T \boldsymbol{g}(\boldsymbol{x}) \right\}$$

Lagrangian Relaxation

- "Dualize" hard constraints.
  - By moving them into the objective functions

Consider a problem:

$$\min_{\boldsymbol{x} \in \boldsymbol{X}} \left\{ f(\boldsymbol{x}) \,\middle|\, \boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{0} \right\}$$

Lagrangian relaxation:

$$\theta(\boldsymbol{\lambda}) = \min_{\boldsymbol{x} \in \boldsymbol{X}} \left\{ f(\boldsymbol{x}) + \boldsymbol{\lambda}^T \boldsymbol{g}(\boldsymbol{x}) \right\}$$

Lagrangian dual:

$$\max_{\boldsymbol{\lambda}} \left\{ \theta(\boldsymbol{\lambda}) \right\}$$

Lagrangian Relaxation on DD

- "Dualize" hard constraints.
  - By moving them into the objective functions

In our example:

$$\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{0} \quad \Leftrightarrow \quad \text{alldiff}(x_1, \ldots, .x_n)$$

To formulate this, let

$$g(\boldsymbol{x}) = \big(g_1(\boldsymbol{x}), \ldots, g_n(\boldsymbol{x})\big)$$

$$g_j(\boldsymbol{x}) = -1 + \sum_{i=1}^{n} \boxed{[x_i = j]}$$

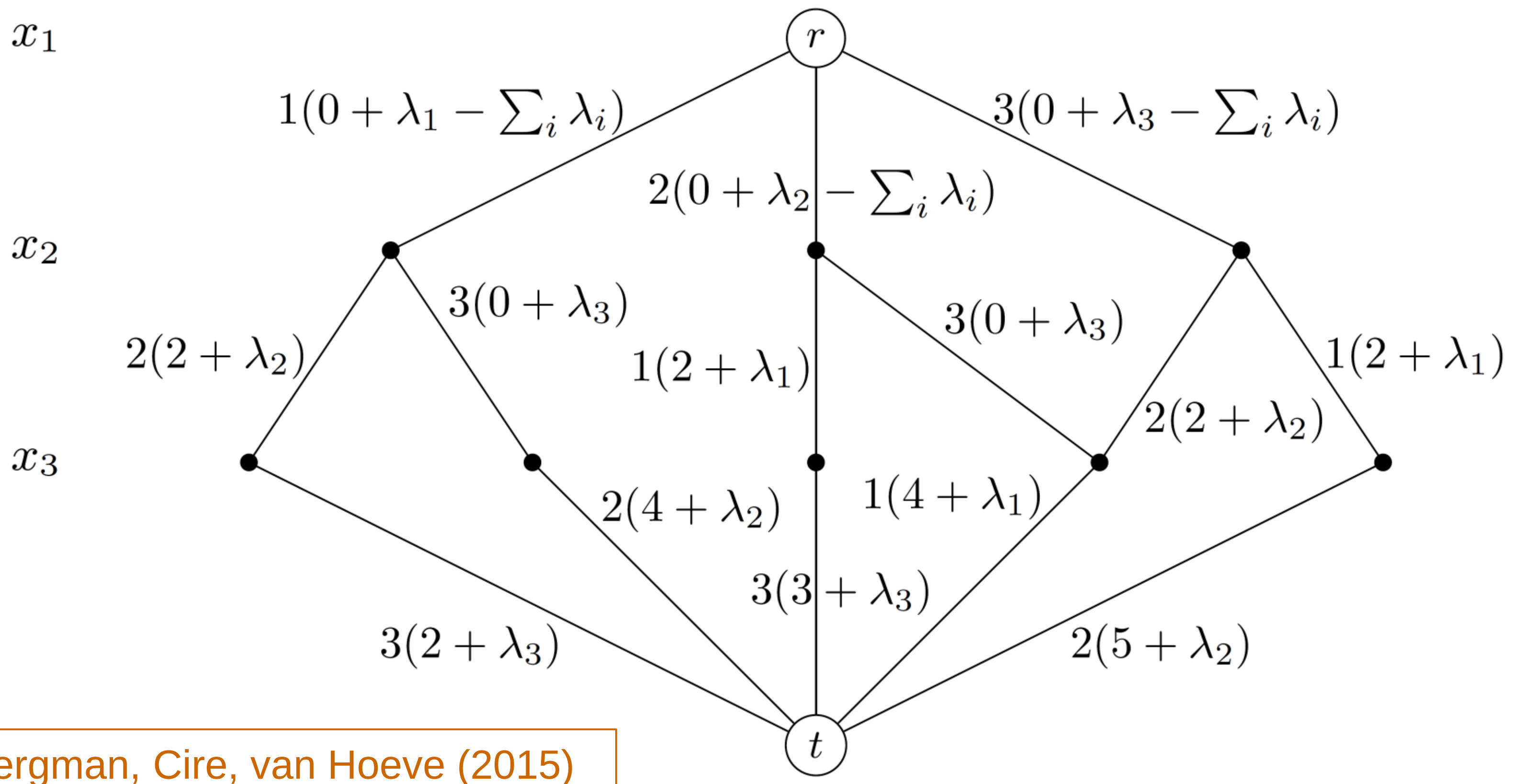
$$= \begin{cases} 1 & \text{if } x_i = j \\ 0 & \text{otherwise} \end{cases}$$

Bergman, Cire, van Hoeve (2015)

Lagrangian Relaxation on DD

Lagrange penalties included in arc costs

Path length now includes total Lagrange penalty



Bergman, Cire, van Hoeve (2015)

Solvilng the Lagrangian Dual

- Solve by subgradient optimization
  - Use **Polyak's method** to determine stepsize

$$\boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k + \sigma_k \, \boldsymbol{g}(\boldsymbol{x}^k)$$

Stepsize, given by

Subgradient, where $\boldsymbol{x}^k$ is value of $\boldsymbol{x}$ obtained when computing $\theta(\boldsymbol{\lambda}^k)$

$$\sigma_k = \frac{\theta^* - \theta(\boldsymbol{\lambda}^k)}{||\boldsymbol{g}(\boldsymbol{x}^k)||_2^2}$$

where θ^* = known upper bound on optimal value.
Let θ^* be value of best known job sequence

Previous DD-based Bounds

JH (2017)

- Job sequencing with state-dependent processing times
  - Processing time depends on which jobs have already been processed.
  - Relaxed DD requires an additional state variable.

Transition:

$$\phi_i\big((V_i, U_i, t_i), x_i\big) = \big(V_i \cup \{x_i\}, \boxed{U_i} \cup \{x_i\}, \max\{r_{x_i}, t_i\} + p_{x_j}(U_i)\big)$$

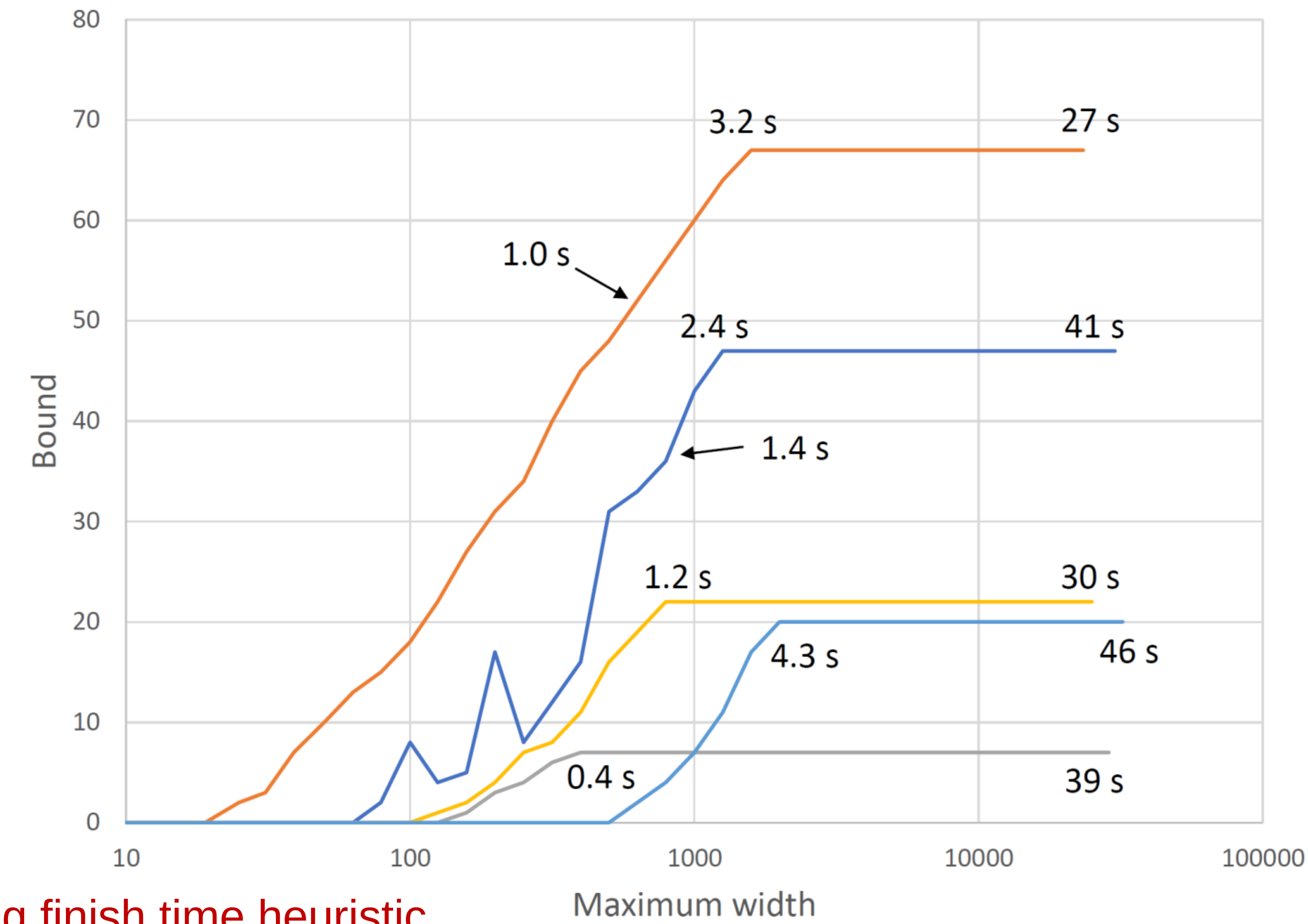
Node merger:

$$(V, U, t) \oplus (V', U', t') = \big(V \cap V', U \cup U', \min\{t, t'\}\big)$$

Previous DD-based Bounds

JH (2017)

12 jobs

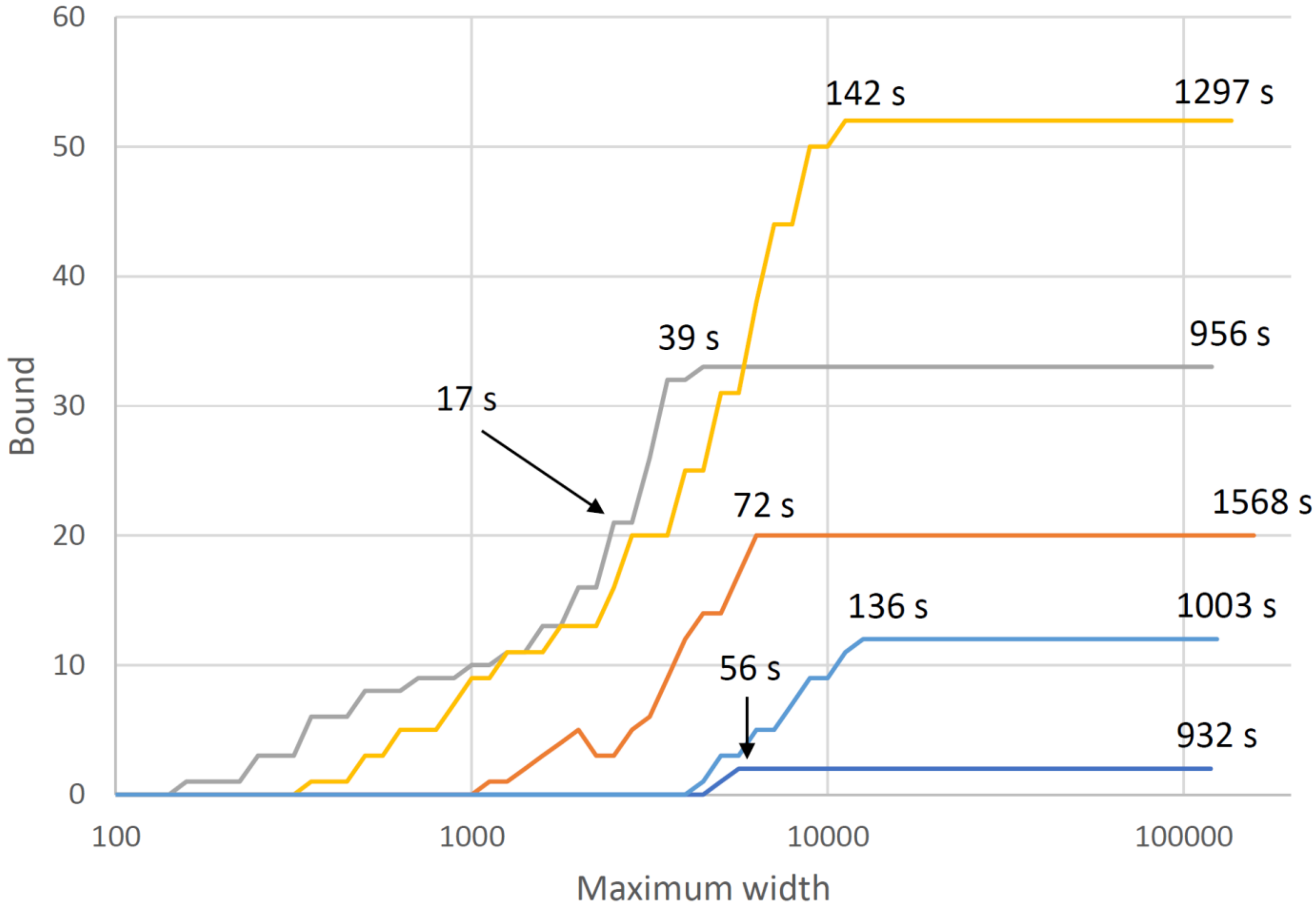


Using finish time heuristic

Previous DD-based Bounds

JH (2017)

14 jobs



Using finish time heuristic

Previous DD-based Bounds

JH (2017)

- Tight bounds, but it doesn't scale
  - Can get optimal value using 10% width of exact DD.
  - But 10% of exact DD grows exponentially.
  - Lower tail is weak.

Transition:

$$\phi_i\big((V_i, U_i, t_i), x_i\big) = \big(V_i \cup \{x_i\}, U_i \cup \{x_i\}, \max\{r_{x_i}, t_i\} + p_{x_j}(U_i)\big)$$

Node merger:

$$(V, U, t) \oplus (V', U', t') = \big(V \cap V', U \cup U', \min\{t, t'\}\big)$$

Previous DD-based Bounds

Bergman, Cire, van Hoeve (2015)

- Traveling salesman with time windows.
  - Objective is total travel time
  - DD represents only alldiff, does not incorporate time windows or measure tardiness.
  - Add Lagrange multipliers to DD
  - Use inside CP solver.

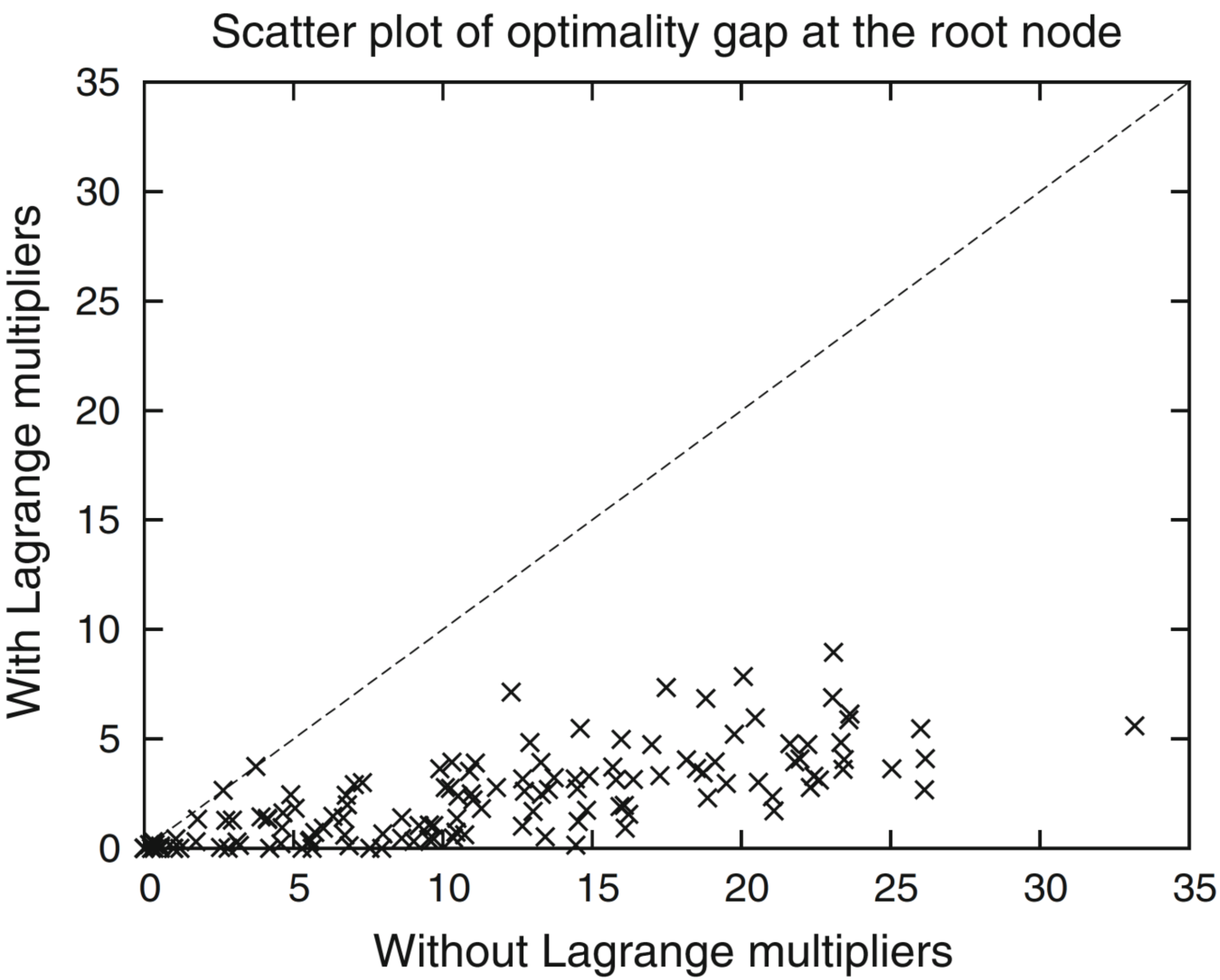
Transition:

$$\phi_i(V_i, x_i) = \big(V_i \cup \{x_i\}\big)$$

Node merger:

$$V \oplus V' = V \cap V'$$

Previous DD-based Bounds



Bergman, Cire, van Hoeve (2015)

Previous DD-based Bounds

Bergman, Cire, van Hoeve (2015)

- Need stand-alone DD that bounds other objectives.
  - Tardiness requires one or more additional state variables
  - How to use more state variables and still implement Lagrangian relaxation in a relaxed DD of practical size?
  - How to get tighter bounds, e.g. 1-2% (without branching)?

Transition:

$$\phi_i(V_i, x_i) = \big(V_i \cup \{x_i\}\big)$$

Node merger:

$$V \oplus V' = V \cap V'$$

Combining DD & Lagrangian Duality

- Express $\boldsymbol{g}(\boldsymbol{x})$ in terms of *immediate penalty functions*

$$\boldsymbol{g}(\boldsymbol{x}) = \sum_{i=1}^{n} \gamma_i(\bar{\boldsymbol{S}}', x_i)$$

Subset of state variables

  - In our example,

$$\boldsymbol{g}(\boldsymbol{x}) = \sum_{i=1}^{n} \big(-[i=1] + [x_i = 1], \ \ldots, \ -[i=1] + [x_i = n] \big)$$

Here, $\bar{\boldsymbol{S}}'_i = \emptyset$

Combining DD & Lagrangian Duality

- Identify state variables on which immediate cost depends.
  - In our example, cost depends on x_i and state variable t_i

$$c_i\big((V_i, t_i), x_j\big) = \big(\max\{r_{x_i}, t_i\} + p_{x_i} - d_{x_i}\big)^+$$

- Identify state variables on which immediate penalty functions depend
  - In our example, they depend only on x_i and no state variables

$$\boldsymbol{\gamma}_i = \big(-[i=1] + [x_i = 1],\ \ldots,\ -[i=1] + [x_i = n]\big)$$

Combining DD & Lagrangian Duality

Theorem. Lagrangian relaxation can be implemented in a relaxed DD if nodes are merged **only when** their states **agree** on the values of the state variables on which the immediate cost functions and the immediate penalty functions depend.

This can be applied to **dynamic programming** models in general.

Survey of Job Sequencing Problems

- Use the theorem to determine for which problems it is practical to implement Lagrangian relaxation on DDs.
  - In all problems we consider, the **immediate Lagrangian penalty** depends only on x_i and **not on any state variables**.
  - So we can merge states whenever they agree on state variables on which the **immediate cost** depends.
  - We will merge **all** such states to keep the relaxed DD as small as possible.

Survey of Job Sequencing Problems

- Minimizing **tardiness** subject to **time windows**
  - In our example, **cost depends** on x_i and **state variable** $\boldsymbol{t_i}$

$$c_i\big((V_i, t_i), x_j\big) = \big(\max\{r_{x_i}, t_i\} + p_{x_i} - d_{x_i}\big)^+$$

  - We can merge states that agree on t_i. The other state variable V_i will lose information, but perhaps retain enough to generate a good bound.
  - **This is practical**, as it results in a relaxed DD of reasonable size.
  - We will experiment with **Crauwells-Potts-Wassenhove (**CPW) instances.

Survey of Job Sequencing Problems

- Minimizing **earliness + tardiness** wrt time windows
  - Measure lateness by due date d_j and earliness by desired release date e_j.
  - **Cost now depends on** x_i and **2 state variables** s_i**,** t_i

$$(V_i, t_i), x_j) = \alpha_{x_i}\left(s_i - p_{x_i} + e_{x_i}\right)^+ + \beta_{x_i}\left(t_i + p_{x_i} - d_{x_i}\right)^+$$

  - We only can merge states that agree on s_i and t_i. But these states are initially equal. So they remain equal throughout the relaxed DD. So **in effect, cost depends on only one state variable**.
  - **This is practical**, as it results in a relaxed DD of reasonable size.
  - We will experiment with **Biskup-Feldman** instances.

Survey of Job Sequencing Problems

- Minimizing tardiness with **time-dependent** costs or processing times
  - Two senses:
    - Dependent on **position** of each job in the sequence.
    - Dependent on **clock time** when job is processed.
  - Easy to check that in either case, costs depends only on current stage (not a state variable) and state variable t_i
  - **This is practical**, and similar to previous problems.

Survey of Job Sequencing Problems

- **Traveling salesman** problem
  - …**without** time windows.
  - **Cost depends only on a state variable *y_i*** representing previous job.

$$c_i\big((V_i, y_i), x_j\big) = p_{y_i x_i}$$

  - **This is practical** and used in

Bergman, Cire, van Hoeve (2015)

Survey of Job Sequencing Problems

- **Traveling salesman** problem with **time windows**
  - **Cost depends on state variables t_i and y_i.**

$$c_i\big((V_i, y_i, t_i), x_j\big) = (r_{x_i} - t_i)^+ + p_{y_i x_i}$$

  - Mergers must agree on **two state variables** and can result in **huge** relaxed DD.
  - This is confirmed by experiments on Dumas instances.
  - **Not practical.**
  - So problem addressed by Bergman, Cire, van Hoeve (2015) cannot be bounded by DD + Lagrangian that incorporates time windows.
  - Also DD + Lagrangian is impractical for TSPTW that minimizes **total tardiness**.

Survey of Job Sequencing Problems

- Minimizing stardiness with state-dependent processing times.
  - Cost depends on state variables t_i and U_i.

$$c_i\big((V_i, U_i, t_i), x_j\big) = \Big(\max\{r_{x_i}, t_i\} + p_{x_i}(U_i) - d_{x_i}\Big)^+$$

  - Mergers must agree on **two state variables** and can result in huge relaxed DD.
  - **Not practical.**
  - So problem addressed by JH (2017) cannot be bounded by DD + Lagrangian.

Computational Results

- To test quality of bound…
  - We need instances with **known optimal solutions** or **very good heuristic solutions**.
  - Instances large enough to be interesting are very hard to solve exactly.

Computational Results

- 50 Crauwels-Potts-Wassenhove (CPW) instances
  - Only a handful solved to optimality in 1998
  - **Most have been solved** to proven optimality since then.

Computational Results

- 50 Crauwels-Potts-Wassenhove (CPW) instances
  - Only a handful solved to optimality in 1998
  - **Most have been solved** to proven optimality since then.
- 60 Biskup-Feldman instances
  - Intensely studied problem since Ow and Morton (1989).
  - Highly refined heuristics developed for these instances since their introduction in 2001
  - **None solved** to proven optimality
  - **No useful bounds** known
  - Compare with best known solutions (Ying, Lin, Lu 2017)

Computational Results

- 50 Crauwels-Potts-Wassenhove (CPW) instances
  - Only a handful solved to optimality in 1998
  - **Most have been solved** to proven optimality since then.
- 60 Biskup-Feldman instances
  - Intensely studied problem since Ow and Morton (1989).
  - Highly refined heuristics developed for these instances since their introduction in 2001
  - **None solved** to proven optimality
  - **No useful bounds** known
  - Compare with best known solutions (Ying, Lin, Lu 2017)

– We need a gap < 1% or 2% to be really useful

Implementation

- Code written in C++
  – Run on my laptop.
- Solving the Lagrangean dual
  – Convergence typically slow for Lagrangian duality.
    – Let it run for 50,000 iterations
    – Iterations are fast since each is an easy shortest-path problem.
    – Bound almost as good if truncated much earlier.
    – **Almost all reported computation time is due to solving Lagrangian dual.**
    – Computation time is worth it to get a good bound on a hard combinatorial problem.

Computational Results

CPW instances, 40 jobs

40 jobs				
Instance	Target	Bound	Gap	Percent gap
1	913	883	30	3.29%
2	1225	1179	46	3.76%
3	537	483	54	10.06%
4	2094	2047	47	2.24%
5	990	980	10	1.01%
6	6955	6939	16	0.23%
7	6324	6299	25	0.40%
8	6865	6743	122	1.78%
9	16225	16049	176	1.08%
10	9737	9591	146	1.50%
11	17465	17417	48	0.27%
12	19312	19245	67	0.35%
13	29256	29003	253	0.86%

*Best known solution

40 jobs				
Instance	Target	Bound	Gap	Percent gap
14	*14377	14100	277	1.93%
15	26914	26755	159	0.59%
16	72317	72120	197	0.27%
17	78623	78501	122	0.16%
18	74310	74131	179	0.24%
19	77122	77083	39	0.05%
20	63229	63217	12	0.02%
21	77774	77754	20	0.03%
22	100484	100456	28	0.03%
23	135618	135617	1	0.001%
24	119947	119914	33	0.03%
25	128747	128705	42	0.03%

*Best known solution

Time = about 20 minutes per instance

Computational Results

CPW instances, 50 jobs

Instance	Target	Bound	Gap	Percent gap
1	2134	2100	34	1.59%
2	1996	1864	132	6.61%
3	2583	2552	31	1.20%
4	2691	2673	18	0.67%
5	1518	1342	176	11.59%
6	26276	26054	222	0.84%
7	11403	11128	275	2.41%
8	8499	8490	9	0.11%
9	9884	9507	377	3.81%
10	10655	10594	61	0.57%
11	*43504	43472	32	0.07%
12	*36378	36303	75	0.21%
13	45383	45310	73	0.16%

*Best known solution

Instance	Target	Bound	Gap	Percent gap
14	*51785	51702	83	0.16%
15	38934	38910	47	0.12%
16	87902	87512	390	0.44%
17	84260	84066	194	0.23%
18	104795	104633	162	0.15%
19	*89299	89163	136	0.15%
20	72316	72222	94	0.13%
21	214546	214476	70	0.03%
22	150800	150800	0	0%
23	224025	223922	103	0.05%
24	116015	115990	25	0.02%
25	240179	240172	7	0.003%

*Best known solution

Time = about 40 minutes per instance

Computational Results

- **CPW results**
  - Bounds are reasonably tight.
  - 42 of 50 bounds < 2%
  - 35 of 50 bounds < 1%.
  - 13 of 50 bounds < 0.1%
  - 3 bounds really bad
  - Optimality proved for 1 instance.

Computational Results

Biskup-Feldman instances, 20 jobs

Instance	$(h_1, h_2) = (0.1, 0.2)$			
	Target	Bound	Gap	Percent gap
20 jobs				
1	4089	4089	0	0%
2	8251	8244	7	0.08%
3	5881	5877	4	0.07%
4	8977	8971	6	0.07%
5	4028	4024	4	0.10%
6	6306	6288	18	0.29%
7	10204	10204	0	0%
8	3742	3739	3	0.08%
9	3317	3310	7	0.21%
10	4673	4669	4	0.09%

Instance	$(h_1, h_2) = (0.2, 0.5)$			
	Target	Bound	Gap	Percent gap
20 jobs				
1	1162	1162	0	0%
2	2770	2766	4	0.14%
3	1675	1669	6	0.36%
4	3113	3108	5	0.16%
5	1192	1187	5	0.42%
6	1557	1557	0	0%
7	13573	3569	4	0.11%
8	990	979	11	1.11%
9	1056	1055	1	0.09%
10	1355	1349	6	0.44%

Time = about 30 seconds per instance

Computational Results

Biskup-Feldman instances, 50 jobs

Instance	$(h_1, h_2) = (0.1, 0.2)$ Target	Bound	Gap	Percent gap
50 jobs				
1	39250	39250	0	0%
2	29043	29043	0	0%
3	33180	33180	0	0%
4	25856	25847	9	0.03%
5	31456	31439	17	0.05%
6	33452	33444	8	0.02%
7	42234	42228	6	0.01%
8	42218	42203	15	0.04%
9	33222	33218	4	0.01%
10	31492	31481	11	0.03%

Instance	$(h_1, h_2) = (0.2, 0.5)$ Target	Bound	Gap	Percent gap
50 jobs				
1	12754	12752	2	0.02%
2	8468	8463	5	0.06%
3	9935	9935	0	0%
4	7373	7335	38	0.52%
5	8947	8938	9	0.10%
6	10221	10213	8	0.08%
7	12002	11981	21	0.17%
8	11154	11141	13	0.12%
9	10968	10965	3	0.03%
10	9652	9650	3	0.03%

Time = about 8 minutes per instance

Computational Results

Biskup-Feldman instances, 100 jobs

Instance	Target	Bound	Gap	Percent gap
	$(h_1, h_2) = (0.1, 0.2)$			
100 jobs				
1	139573	139556	17	0.01%
2	120484	120465	19	0.02%
3	124325	124289	36	0.03%
4	122901	122876	25	0.02%
5	119115	119101	14	0.01%
6	133545	133536	9	0.007%
7	129849	129830	19	0.01%
8	153965	153958	7	0.005%
9	111474	111466	8	0.007%
10	112799	112792	7	0.006%

Instance	Target	Bound	Gap	Percent gap
	$(h_1, h_2) = (0.2, 0.5)$			
100 jobs				
1	39495	39467	28	0.07%
2	35293	35266	27	0.08%
3	38174	38150	24	0.06%
4	35498	35467	31	0.09%
5	34860	34826	34	0.10%
6	35146	35123	23	0.07%
7	39336	39303	33	0.08%
8	44963	44927	36	0.08%
9	31270	31231	39	0.12%
10	34068	34048	20	0.06%

Time = about 65 minutes per instance

Computational Results

- **Biskup-Feldman results**
  - Bounds are very tight
    - perhaps even tighter wrt optimal values
  - 60 of 60 bounds < 2%
  - 59 of 60 bounds < 1%.
  - 44 of 60 bounds < 0.1%
  - 12 of 50 bounds < 0.01%
  - Optimality proved for 8 instances (closing these instances)

Future Work

- Explore DP models for job shop scheduling, etc.
  - Check if DD + Lagrangian relaxation is practical

- Extend to other DP models.

- Extend Lagrangian relaxation to stochastic DDs.
  - They currently provide weak bounds.

Future Work

- Problem: diagrams of a **fixed size** lose their ability to generate bounds as instances scale up.
  - Bound does not rise above zero until relaxed diagram width is 1/1000 to 1/25 that of exact diagram
- This suggests a combination with other bounding techniques
  - …that can yield a nonzero bound in smaller relaxed diagrams.
  - Such as **Lagrangean relaxation** obtained by modifying costs in the diagram..

Bergman, Ciré, van Hoeve (2015)

Future Work

- Bounds for **stochastic dynamic programming**
  - From **stochastic diagrams**.
  - Node merger can again provide a valid relaxation.
  - A theoretical result is available.
  - Awaiting good merger heuristics and computational tests.