Solving Fixed-Charge Network Flow Problems with a Hybrid Optimization and Constraint Programming Approach

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An Approach that Combines Optimization and Constraint Programming

- *Fixed charge network flow* problems illustrate a general approach to modeling and solution that combines elements of *optimization* and *constraint programming*.
- *Logical expressions* capture the discrete element (whether to install an arc) without integer variables.

A Combined Approach

- A relaxation is formed using the *continuous variables only* (“projected” relaxation).
- The relaxation solves rapidly because it is a *min-cost network flow* problem.
- The constraints are gathered up into a *global constraint*.
- The global constraint triggers generation of the relaxation as well as constraint propagation.

Modeling Advantage

- In general the modeling advantage is a more *natural and succinct* model.
- The *structure* of the model indicates how procedures should interact to solve it.
- In this case, the model is not much simpler than an IP model.
- But the global constraint *identifies structure* that speeds solution.

Algorithmic Advantage

- In this case, *constraint propagation* at every node tends to shrink the search tree.
- The *absence of flow cuts* in the relaxation tends to make the search tree larger.
  - Flow cuts would destroy the network structure of the projected relaxation.
- But *fast solution of network flow* problems speeds the processing of each node.
- The method appears to work well for *fixed-charge transportation* problems but not *warehouse location problems*, both special cases.

Fixed-Charge Network Flow: 0-1 Formulation

$$
\begin{aligned}
\min \quad & \sum_{ij} z_{ij} \\
\text{s.t.} \quad & z_{ij} \geq c_{ij} x_{ij} + f_{ij} y_{ij}, \quad \text{all } i, j \quad \text{(cost)} \\
& \sum_i x_{ij} - \sum_i x_{ji} = s_j, \quad \text{all } j \quad \text{(flow balance)} \\
& 0 \leq x_{ij} \leq M_{ij} y_{ij}, \quad \text{all } i, j \quad \text{(big } M\text{)} \\
& y_{ij} \in \{0,1\}
\end{aligned}
$$

Total cost

Variable arc cost

Fixed arc cost

Net supply at node

Flow on arc

Arc capacity

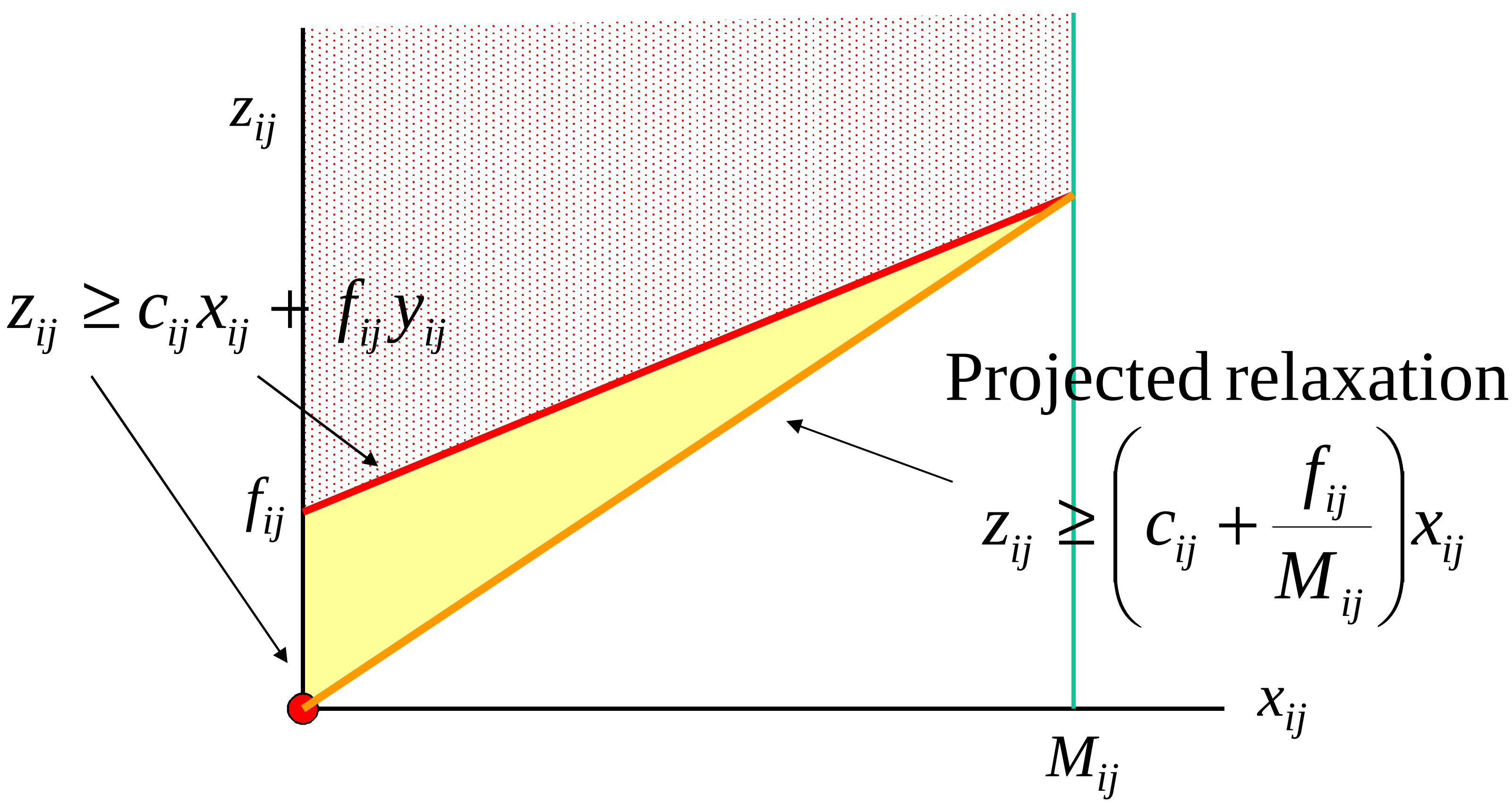
Fixed-Charge Network Flow: Logic-Based Formulation

Total cost

$$
\begin{aligned}
\min \quad & \sum_{ij} z_{ij} \\
\text{s.t.} \quad & y_{ij} \rightarrow \begin{pmatrix} z_{ij} \geq c_{ij} x_{ij} + f_{ij} \\ 0 \leq x_{ij} \leq M_{ij} \end{pmatrix}, \quad \text{all } i, j \quad \text{(arc included)} \\
& \neg y_{ij} \rightarrow (z_{ij} = x_{ij} = 0), \quad \text{all } i, j \quad \text{(arc omitted)} \\
& \sum_i x_{ij} - \sum_i x_{ji} = s_j, \quad \text{all } j \quad \text{(flow balance)}
\end{aligned}
$$

Propositional variable (true or false)

Projected Relaxation for Arc (*i*,*j*)



Continuous Relaxations

Lifted

$$
\begin{aligned}
\min \quad & \sum_{ij} z_{ij} \\
\text{s.t.} \quad & z_{ij} \geq c_{ij} x_{ij} + f_{ij} y_{ij} \\
& \sum_i x_{ij} - \sum_i x_{ji} = s_j \\
& 0 \leq x_{ij} \leq M_{ij} y_{ij} \\
& 0 \leq y_{ij} \leq 1
\end{aligned}
$$

Projected
(min-cost network flow)

$$
\begin{aligned}
\min \quad & \sum_{ij} z_{ij} \\
\text{s.t.} \quad & z_{ij} \geq \left(c_{ij} + \frac{f_{ij}}{M_{ij}} \right) x_{ij} \\
& \sum_i x_{ij} - \sum_i x_{ji} = s_j \\
& 0 \leq x_{ij} \leq M_{ij}
\end{aligned}
$$

The relaxations are equivalent

Branch-and-bound Solution

- Branch on propositional variables y_{ij}.
- Apply constraint propagation to fix y_{ij}'s when possible.
- Solve projected relaxation with x_{ij} fixed to zero whenever y_{ij} = false.
- Solution at current node is feasible if

  $$z_{ij} \geq c_{ij}x_{ij} + f_{ij}$$

  whenever $x_{ij} > 0$ (analogous to integral solution in 0-1 programming).

Constraint Propagation

- Use arc capacities to fix some y_{ij}’s to 1.
- Use reduced costs to fix some y_{ij}’s to 0.
- Use nogoods in the form of Benders cuts to fix some y_{ij}’s on which the search has branched.
- Cycle through these steps at each node until no further variables can be fixed.

Constraint Propagation on Arc Capacities

For each node *i* use the valid inequality

$$\sum_{\substack{j \\ y_{ij} \text{ not fixed to } 0}} M_{ij} y_{ij} \geq s_i$$

to fix some y_{ij}'s to 1 (true) if possible
(i.e., make sure there is enough capacity in outgoing arcs to carry net supply),
and similarly for incoming arcs.

Constraint Propagation Using Reduced Costs

- Can use reduced costs to fix logical variables y_{ij}.
- Fix y_{ij} to false if the reduced cost of y_{ij} in the 0-1 model implies that it can be fixed to 0.
- The reduced cost of y_{ij} in the 0-1 model is easily recoverable, even though the 0-1 model is not used. It is M_{ij} times the reduced cost of x_{ij} in the projected relaxation.

Constraint Propagation Using Benders Cuts as Nogoods

- When some discrete variables have been fixed, nogoods identify why this partial solution is unsatisfactory and rule out all solutions that are unsatisfactory for the same reason.
- Nogoods contain variables currently fixed by branching, whereas cutting planes contain variables not yet fixed.
- Benders cuts can serve as nogoods, where the master problem contains the variables currently fixed.
- Nogoods are used here for propagation, not as part of the relaxation.

Suppose y_{ij} is currently fixed to $\bar{y}_{ij}$ for $ij \in F$. The subproblem is:

$$
\begin{aligned}
\min\quad & z \\
\text{s.t.}\quad & z \ge \sum_{ij} c_{ij} x_{ij} + \sum_{ij \notin F} f_{ij} y_{ij} + \sum_{ij \in F} f_{ij} \bar{y}_{ij} \\
& \sum_i x_{ij} - \sum_i x_{ji} = s_j & (u_j) \\
& 0 \le x_{ij} \le M_{ij} y_{ij}, \quad ij \notin F & (v_{ij}) \\
& 0 \le x_{ij} \le M_{ij} \bar{y}_{ij}, \quad ij \in F & (v_{ij}) \\
& 0 \le y_{ij} \le 1, \quad ij \notin F & (w_{ij})
\end{aligned}
$$

Dual variables

Generate Benders cut:

$$z \ge \sum_{ij \in F} \left(f_{ij} + v_{ij} M_{ij} \right) y_{ij} + \sum_j u_j s_j + \sum_{ij \notin F} w_{ij}$$ when feasible

$$\sum_{ij \in F} v_{ij} M_{ij} y_{ij} + \sum_j u_j s_j + \sum_{ij \notin F} w_{ij} \le 0$$ when infeasible

Although 0-1 relaxation is not solved, Benders cuts can be recovered from the subproblem for the projected relaxation.

$$
\begin{aligned}
\min \quad & z \\
\text{s.t.} \quad & z \ge \sum_{ij \notin F} \left(c_{ij} + \frac{f_{ij}}{M_{ij}} \right) x_{ij} + \sum_{ij \in F_1} \left(c_{ij} x_{ij} + f_{ij} \right) \\
& \sum_i x_{ij} - \sum_i x_{ji} = s_j && (u_j) \\
& 0 \le x_{ij} \le M_{ij}, \quad ij \notin F && (t_{ij}) \\
& 0 \le x_{ij} \le M_{ij}, \quad ij \in F_1 && (t_{ij}) \\
& x_{ij} = 0, \quad ij \in F_0 && (t_{ij})
\end{aligned}
$$

y_{ij} fixed to true ($ij \in F_1$), to false ($ij \in F_0$)

The Benders cut is one of the following:

$$
z \ge \sum_{ij \in F} t_{ij} M_{ij} y_{ij} + \sum_j u_j s_j + \sum_{ij \notin F} t_{ij} M_{ij}
$$

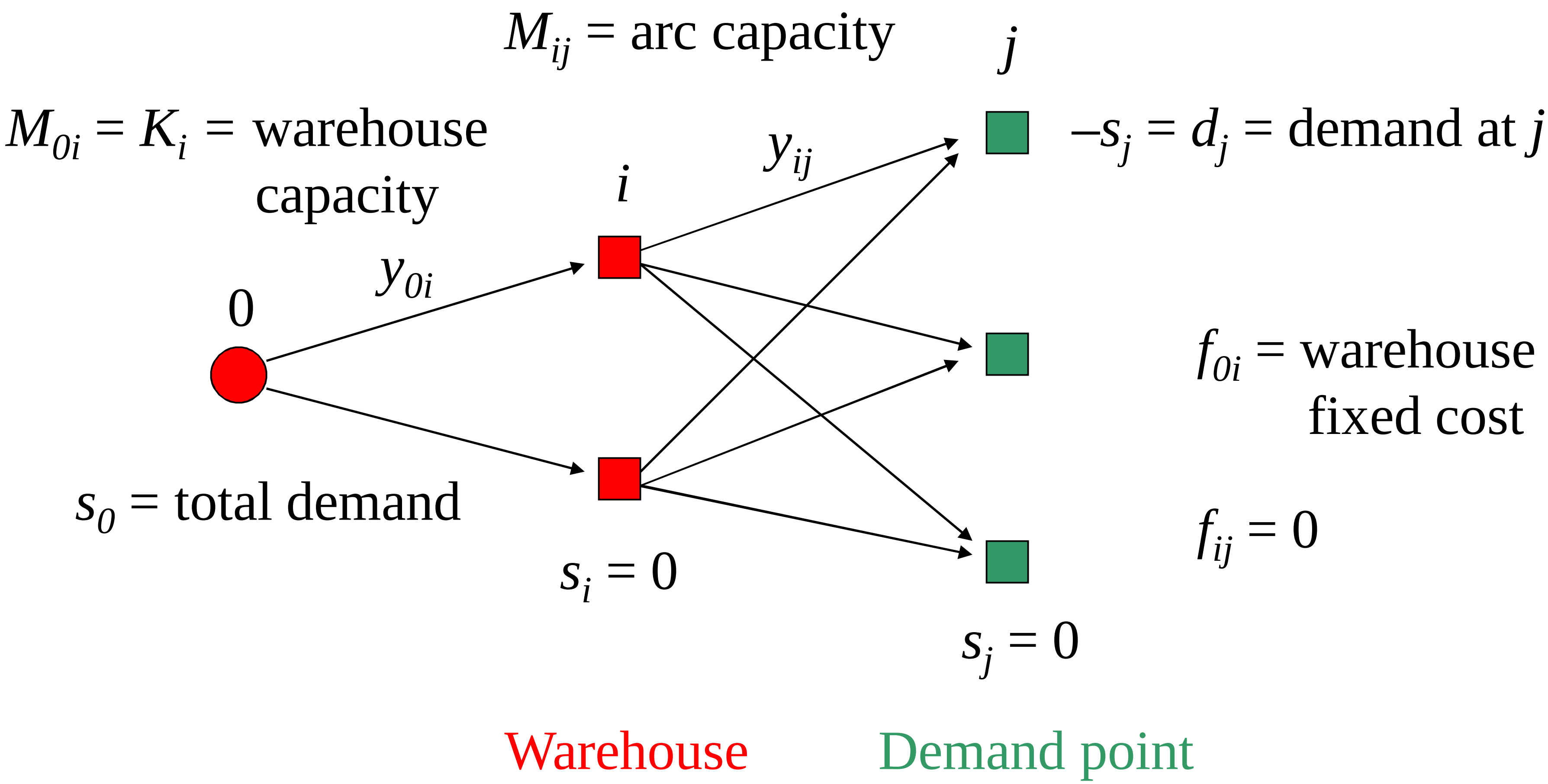
$$
\sum_{ij \in F} \left(t_{ij} M_{ij} - f_{ij} \right) y_{ij} + \sum_j u_j s_j + \sum_{ij \notin F} t_{ij} M_{ij} \le 0
$$

Global Constraint Formulation

Formulate using global constraint, to trigger specialized relaxation and constraint propagation.

$$\begin{array}{ll} \min & \sum_{ij} z_{ij} \\ \text{s.t.} & \text{FixedCostNetworkFlow}(z, x, s, M) \end{array}$$

Special Case: Warehouse Location



Lifted relaxation of min cost flow problem

$$
\begin{aligned}
\min\quad & \sum_{ij} c_{ij} x_{ij} + \sum_i y_{0i} f_i \\
\text{s.t.}\quad & x_{0i} = \sum_j x_{ij} \\
& \sum_i x_{ij} = d_j \\
& 0 \le x_{ij} \le M_{ij} y_{ij} \\
& 0 \le x_{0i} \le K_i y_{0i} \\
& 0 \le y_{ij} \le 1
\end{aligned}
$$

$$\equiv$$

Projected relaxation

$$
\begin{aligned}
\min\quad & \sum_{ij} c_{ij} x_{ij} + \sum_i \frac{f_i}{K_i} x_{0i} \\
\text{s.t.}\quad & x_{0i} = \sum_j x_{ij} \\
& \sum_i x_{ij} = d_j \\
& 0 \le x_{ij} \le M_{ij} \\
& 0 \le x_{0i} \le K_i
\end{aligned}
$$

These are equivalent.

Lifted relaxation of min cost flow problem

$$
\begin{aligned}
\min \quad & \sum_{ij} c_{ij} x_{ij} + \sum_i y_{0i} f_i \\
\text{s.t.} \quad & x_{0i} = \sum_j x_{ij} \\
& \sum_i x_{ij} = d_j \\
& 0 \le x_{ij} \le M_{ij} y_{ij} \\
& 0 \le x_{0i} \le K_i y_{0i} \\
& 0 \le y_{ij} \le 1 \\
& y_{ij} \le y_i
\end{aligned}
$$

Stronger lifted relaxation used for 0-1 model

$$
\begin{aligned}
\min \quad & \sum_{ij} c_{ij} x_{ij} + \sum_i y_{0i} f_i \\
\text{s.t.} \quad & x_{0i} = \sum_j x_{ij} \\
& \sum_i x_{ij} = d_j \\
& 0 \le x_{ij} \le M_{ij} y_{0i} \\
& 0 \le x_{0i} \le K_i y_{0i} \\
& 0 \le y_{ij} \le 1
\end{aligned}
$$

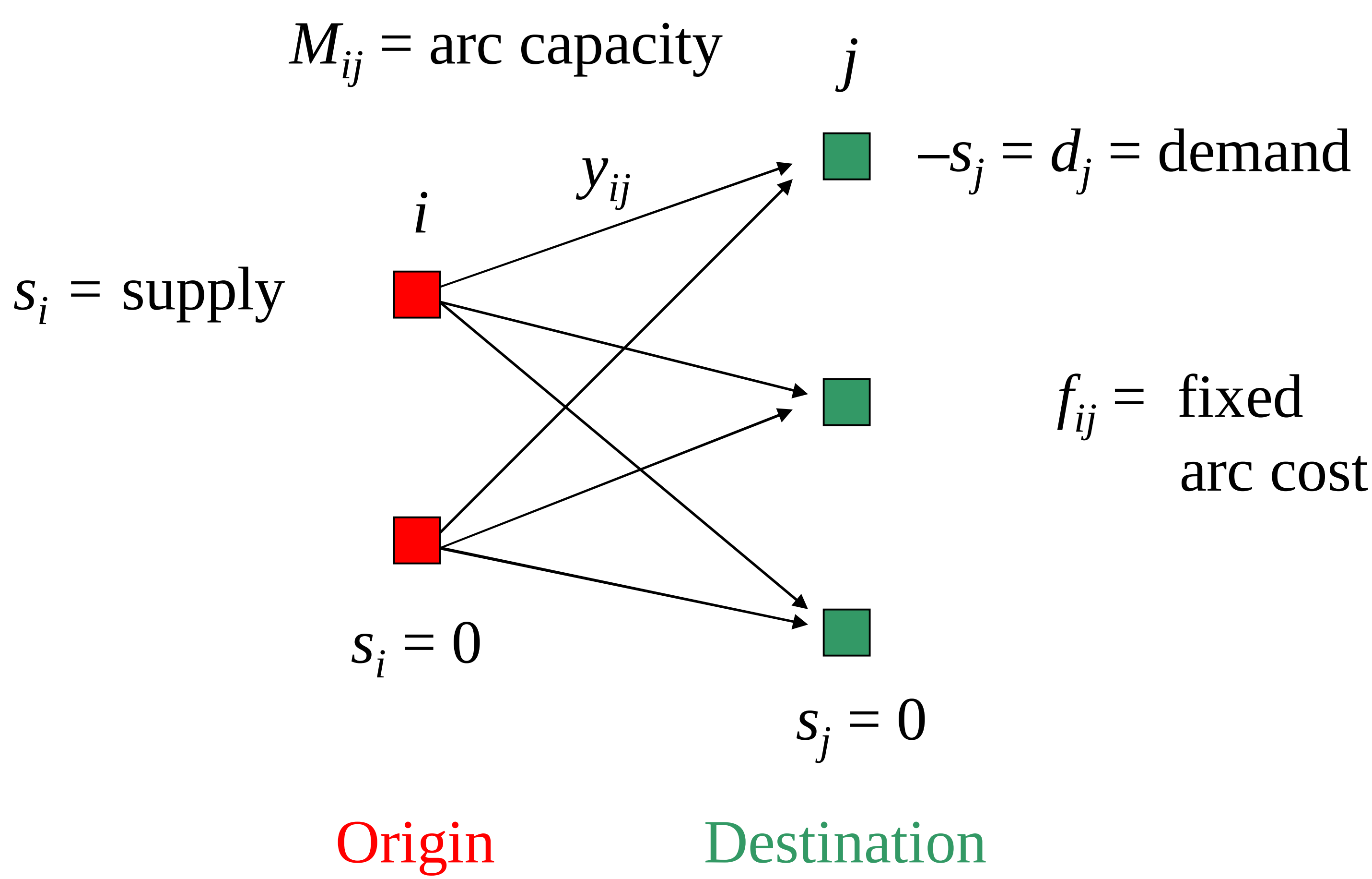
$y_{ij} \le y_i$ ← Must add this to obtain stronger relaxation. Its projection loses network structure.

- The projected relaxation of the warehouse location problem is weaker than the usual relaxation of 0-1 model.
- The difference may be negligible if the arc capacities M_{ij} are large.
- If the arc capacities are tight, traditional methods may be better.
- Also, the warehouse location problem has relatively few 0-1 variables relative to continuous variables.

Special Case: Fixed Charge Transportation Problem

- The fixed charge transportation problem is a more promising application of the projected relaxation.
- The projected relaxation is equivalent to the relaxation of the standard 0-1 model.
- There are many 0-1 variables.

Fixed Charge Transportation Problem

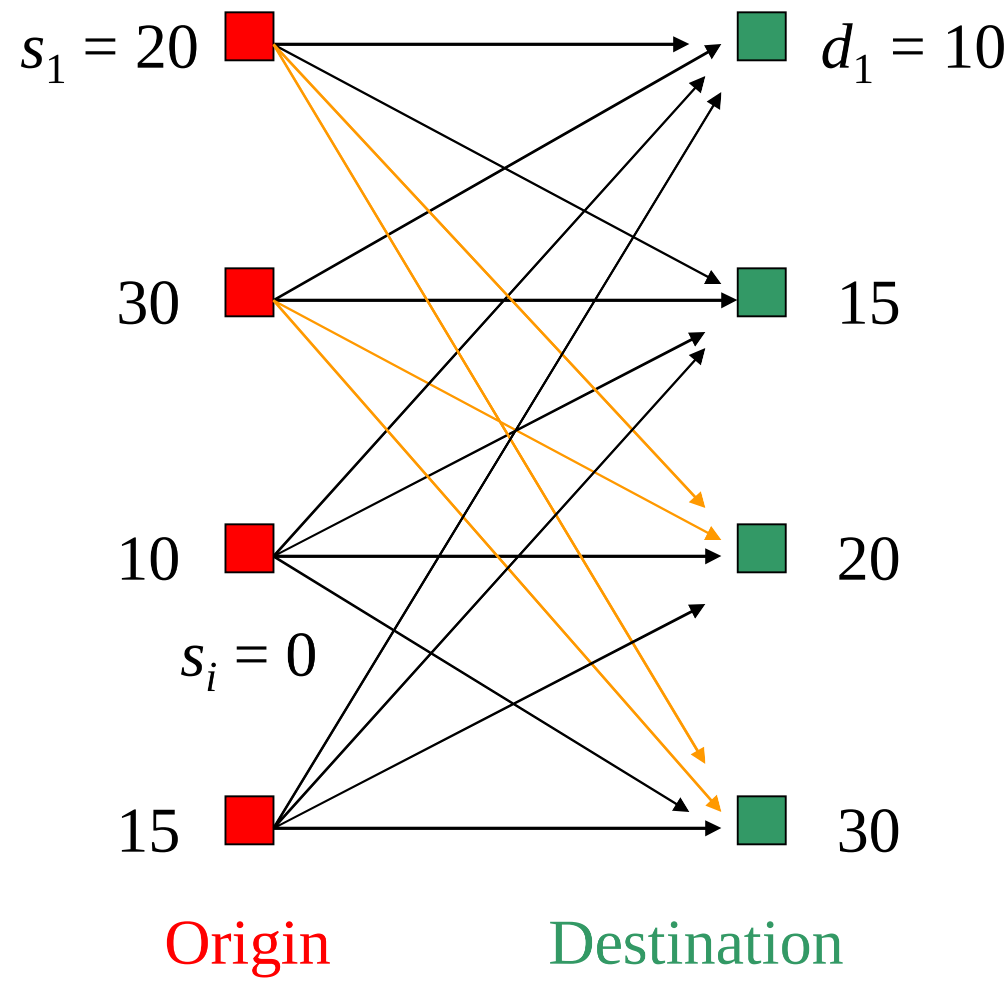


Generation of Benders Cuts

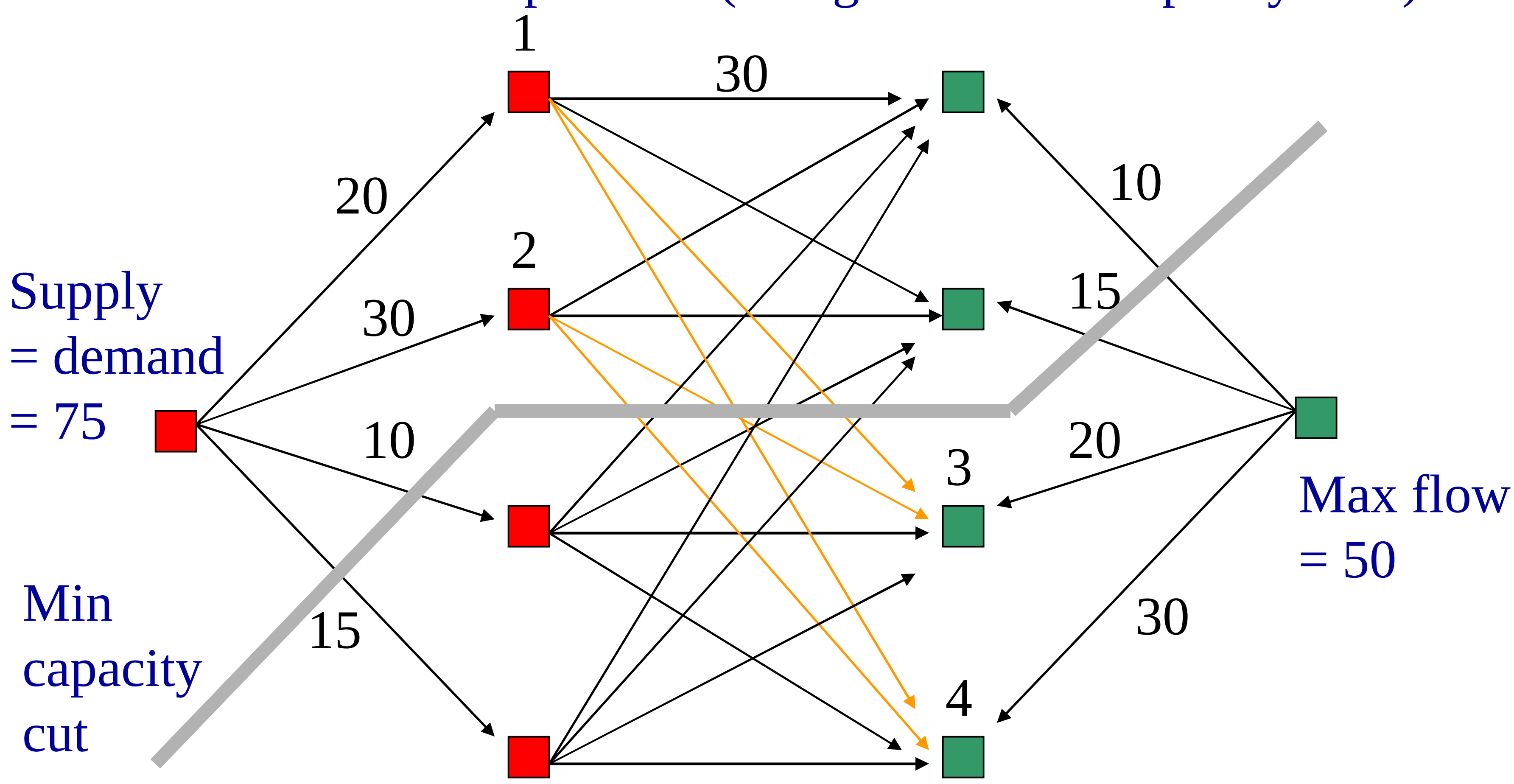
- Benders cuts at infeasible nodes correspond to the solution of max flow/min cut problems on the transportation network.

Suppose y_{ij} has been fixed to false for all orange arcs ij.
There is no feasible flow.

$M_{ij} = 30$, all i,j



Convert to a max flow problem (orange arcs have capacity zero).



Benders cut is $30y_{13} + 30y_{14} + 30y_{23} + 30y_{24} \geq 75 - 50$