



Integer Programming as Projection

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Abstract

We generalise polyhedral projection (Fourier-Motzkin elimination) to integer programming (IP) and derive from this an alternative perspective on IP that parallels the classical theory. We first observe that projection of an IP yields an IP augmented with linear congruence relations and finite-domain variables, which we term a *generalised IP*. The projection algorithm can be converted to a branch-and-bound algorithm for generalised IP in which the search tree has bounded depth (as opposed to conventional branching, in which there is no bound). It also leads to valid inequalities that are analogous to Chvátal-Gomory cuts but are derived from congruences rather than rounding, and whose rank is bounded by the number of variables. Finally, projection provides an alternative approach to IP duality. It yields a value function that consists of nested roundings as in the classical case, but in which ordinary rounding is replaced by rounding to the nearest multiple of an appropriate modulus, and the depth of nesting is again bounded by the number of variables. For large perturbations in right-hand sides, the value function is shift periodic and can be interpreted economically as yielding “average” shadow prices.

Keywords:

integer programming, projection, duality, value function

1. Introduction

We propose an alternative perspective on integer programming that is based on projection. It begins with the observation that the projection of an integer programming (IP) problem is not an IP problem. More precisely, the projection of an IP problem’s feasible set onto a subset of variables is not the feasible set of an IP. It is the feasible set of a system of linear integer inequalities and congruence relations, where the congruence relations define sublattices of the integer lattice. This suggests that an IP problem can be viewed more generally as an inequality constrained problem over sublattices of the integer lattice, rather than exclusively over the entire integer lattice as in conventional IP. We will call this a *generalised IP problem*.

The projection problem for generalised IPs can be solved by introducing integer auxiliary variables with finite domains, and taking advantage of a generalised Chinese Remainder Theorem. The function of the auxiliary variables is to help define sublattices. Projecting out all the original variables transforms the optimization problem to one that minimises over a system of congruence relations that involve only the auxiliary variables. A problem of optimising over possibly infinite domains is therefore transformed to one of optimising over finite domains.

This perspective leads to an alternative theory of cutting planes, branching algorithms, and IP duality. We introduce “congruence cuts,” which are analogous to Chvátal-Gomory cuts, except that they are derived from a linear combination strengthened by a congruence relation, rather than a linear combination strengthened by rounding. We

use the projection algorithm to show that their rank is bounded by the number of variables. This contrasts with the classical Chvátal rank, which has no bound related only to the number of variables [11].

In addition, we show that the projection algorithm can be converted to a branching algorithm that branches on integer auxiliary variables rather than the original integer variables, and in which the possible branches are defined by congruence relations. The depth of the tree is again bounded by the number of variables, whereas a conventional branching tree has unbounded depth.

Finally, by applying the projection algorithm to an IP problem with general right-hand sides, we obtain a value function that is analogous to a Gomory function [1, 2] in that it contains nested rounding operations. However, rather than rounding to the nearest integer, one rounds to the nearest multiple of an appropriate modulus. In contrast to a value function obtained by Gomory’s method, the depth of nesting (which is analogous to cutting plane rank) is bounded by the number of variables, and the function can be obtained by one pass through the model. We show that this value function is shift-periodic for large perturbations of the right-hand sides, which leads to an economic interpretation and “average” shadow prices.

We begin with brief discussion of previous work. We then review projection and duality in linear programming (LP), to clarify how it is generalised for the IP case. At this point we proceed to establish the results just described.

2. Previous Work

The idea of extending Fourier-Motzkin elimination to project the feasible set of an IP, so as to produce congruence relations as well as inequalities, originally appeared in [12]. Here we formally state the procedure and prove its correctness. We also show how it forms the basis for an alternative theory of IP that includes cutting planes, branching and duality, as well as suggesting a generalization of IP to sublattices of the integer lattice.

The concept of the value function of a mathematical programme is due to Blair and Jeroslow [1]. Here we show that the value function can be expressed in a different form with bounded nesting depth. A forerunner of this form appears in an unpublished manuscript [14]. We state the procedure in general, show how it can be expressed as a minimum over certain Gomory functions, and prove its correctness.

The idea of carrying out the projection procedure in [12] for perturbed inequalities was suggested by Ryan [10], who showed in this fashion that a finitely generated integer monoid can be described with finitely many “disjunctive Chvátal-Gomory constraints.” This result can be seen as a corollary of our analysis. In addition, we reveal the structure of the value function and how it provides economic information.

Some related results for the more general case of mixed integer/linear programming (MILP) appear in [17]. These results lead to an analytic solution of the MILP when applied only to the constraints binding in the LP relaxation; that is, when applied to an MILP over a cone.

The economic interpretation of shadow prices in integer programming has been discussed for some time, as for example in [5]; see [8] for a survey. Our results differ in that we exhibit a value function of bounded depth and use properties of shift-periodic functions derived in [9] to show that the value function is eventually shift-periodic and yields “average” shadow prices.

3. LP Projection

A polyhedron can be projected onto a subspace using Fourier-Motzkin elimination [3, 13]. We will suppose the polyhedron is described by the constraint set of an LP in the following form, where A is an $m \times n$ integral matrix and b is integral:

$$\begin{aligned} \min \quad & z \\ \text{subject to} \quad & -cx \geq -z \\ & Ax \geq b \\ & x \in \mathbb{R}^n \end{aligned} \tag{1}$$

We assume that any nonnegativity constraints on the variables are represented in the above constraints. Fourier-Motzkin elimination relies on the following elementary lemma, which we prove to allow comparison with a parallel result (Theorem 2) that we will prove for IP projection.

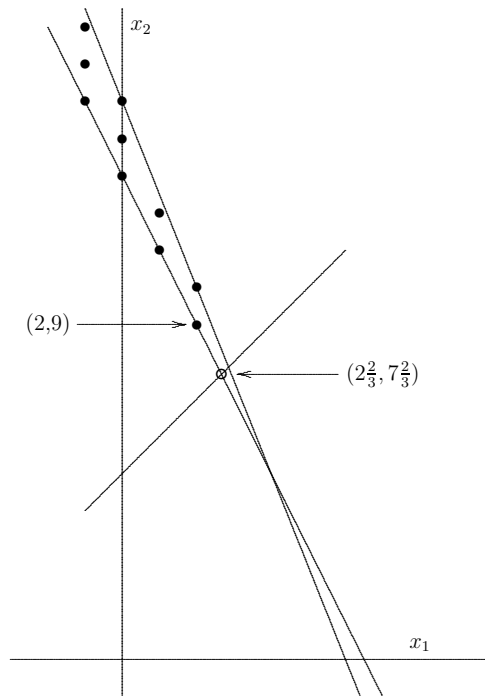


Figure 1. Illustration of a linear (integer) programming problem. Black dots are integer feasible solutions, with $(x_1, x_2) = (2, 9)$ optimal. The small open circle is the optimal solution of the LP.

Lemma 1. Suppose $a_{ij}, a_{kj} > 0$ for all $i \in I, k \in K$. Then

(a) There exists $x_j \in \mathbb{R}$ such that $a_{ij}x_j \geq f_i$ and $-a_{kj}x_j \geq g_k$ for all $i \in I, k \in K$

if and only if.

(b) $a_{kj}f_i + a_{ij}g_k \leq 0$ for all $i \in I, k \in K$.

Proof. (a) \Rightarrow (b). This is obtained by taking a linear combination of each pair of inequalities $a_{ij}x_j \geq f_i, -a_{kj}x_j \geq g_k$, using multipliers $1/a_{ij}$ and $1/a_{kj}$, respectively.

(a) \Leftarrow (b). The inequalities in (a) can be written $f_i/a_{ij} \leq x_j \leq -g_k/a_{kj}$ for all i, k . But from (b) we have that $f_i/a_{ij} \leq -g_k/a_{kj}$ for all i, k . We can therefore let $x_j = \max_i\{f_i/a_{ij}\}$ (or $\min_k\{-g_k/a_{kj}\}$), and the inequalities in (a) are satisfied. \square

Note that if I or K is empty then (b) is vacuous (they cannot both be empty, since variable x_j would then not be in the model).

The lemma implies that any variable x_j can be eliminated from (1) by removing each pair of inequalities that have the form $a_{ij}x_j \geq f_i, -a_{kj}x_j \geq g_k$ with $a_{ij}, a_{kj} > 0$, and replacing each pair with the inequality $a_{kj}f_i + a_{ij}g_k \leq 0$. (If I or K is empty, i.e. (b) is vacuous, then x_j and all the inequalities in which it occurs can be removed.) The variables x_j can be successively eliminated, in any order, until the constraints of (1) are replaced by inequalities of the form $z \geq \ell$. The minimum value of z can be immediately read from these. If the final inequalities contain no lower bound on z , then the model is unbounded. If z is eliminated from the final inequalities (by the above method) and at least one “false” inequality (e.g., $0 \geq 1$) results, then the original model is infeasible. It can be shown [6, 15] that after the elimination of r variables, any resulting inequality that depends on more than $r + 1$ of the original inequalities is redundant (implied by the other inequalities).

We can illustrate projection with a small example (Fig. 1).

$$\begin{array}{ll}
 \min & z \\
 \text{subject to} & -x_2 \geq -z \quad \text{C0} \\
 & 2x_1 + x_2 \geq 13 \quad \text{C1} \\
 & -5x_1 - 2x_2 \geq -30 \quad \text{C2} \\
 & -x_1 + x_2 \geq 5 \quad \text{C3} \\
 & x_1, x_2 \in \mathbb{R}
 \end{array} \tag{2}$$

The optimal solution is $(x_1, x_2, z) = (2\frac{2}{3}, 7\frac{2}{3}, 7\frac{2}{3})$, with binding constraints C1 and C3. Eliminating x_1 yields $z \geq x_2$, $x_2 \geq 5$, and $x_2 \geq 7\frac{2}{3}$. Eliminating x_2 from this yields $z \geq 5$ and $z \geq 7\frac{2}{3}$. This confirms the optimal value $7\frac{2}{3}$.

Suppose now that we perturb the right-hand sides of (2) as follows:

$$\begin{array}{ll}
 \min & z \\
 & -x_2 \geq -z \quad \text{C0} \\
 & 2x_1 + x_2 \geq 13 + \Delta_1 \quad \text{C1}_\Delta \\
 & -5x_1 - 2x_2 \geq -30 + \Delta_2 \quad \text{C2}_\Delta \\
 & -x_1 + x_2 \geq 5 + \Delta_3 \quad \text{C3}_\Delta \\
 & x_1, x_2 \in \mathbb{Z}
 \end{array} \tag{3}$$

We can perform the same projection operations while carrying through the perturbations. This yields $z \geq 5 + 5\Delta_1 + 2\Delta_2$ and $z \geq 7\frac{2}{3} + \frac{1}{3}\Delta_1 + \frac{10}{3}\Delta_3$. From this we can write a *value function*

$$v(\Delta_1, \Delta_2, \Delta_3) = \max \left\{ 5 + 5\Delta_1 + 2\Delta_2, 7\frac{2}{3} + \frac{1}{3}\Delta_1 + \frac{2}{3}\Delta_3 \right\} \tag{4}$$

that gives the optimal value as a function of the perturbations, provided the problem remains feasible after perturbation. In general, Fourier-Motzkin elimination yields inequalities that contain perturbations Δ_i but not z , and the perturbations yield a feasible problem if and only if these inequalities are feasible. In the present case, there are no such inequalities, and the problem is feasible for any perturbation.

The value function provides the basis for what might be called *marginal* and *eventual* shadow prices. The best known are marginal shadow prices, which indicate the sensitivity of cost to small perturbations in the right-hand side. The marginal shadow price for constraint i is the coefficient of Δ_i in the largest argument of the max when each Δ_i is set to zero. In the example, the second term is larger when $(\Delta_1, \Delta_2, \Delta_3) = (0, 0, 0)$, and so the shadow prices are $\frac{1}{3}$, 0, and $\frac{2}{3}$ for the three constraints, respectively.

As a given right-hand side is increased, one of the terms of the max eventually dominates, and its coefficient can be regarded as an eventual shadow price in the positive direction. An eventual shadow price in the negative direction is similarly obtained. In the example, the first term of the max dominates as Δ_1 increases, and so the eventual shadow price in the positive direction is 5. The eventual shadow price in the negative direction is $1/3$. Economically, the eventual shadow price can be used to compute the approximate cost of large perturbations in a right-hand side.

We will find that eventual shadow prices can be derived for generalized IP problems, although the concept of a marginal shadow price does not extend to IP in an obvious way.

4. IP Projection

In analogy with the LP case, we consider an IP in the following form:

$$\begin{array}{ll}
 \min & z \\
 \text{subject to} & -cx \geq -z \\
 & Ax \geq b \\
 & x \in \mathbb{Z}^p
 \end{array} \tag{5}$$

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A generalised IP can be written

$$\begin{aligned}
& \min && z \\
& \text{subject to} && -cx - hu \geq -z \\
& && Ax + Bu \geq b \\
& && r^i x + s^i u \equiv \rho_i \pmod{m_i}, \quad i \in I \\
& && x \in \mathbb{Z}^n \\
& && u_j \in D_j \subset \mathbb{Z}_{\geq 0}, \quad j = 1, \dots, p
\end{aligned} \tag{6}$$

where $u = (u_1, \dots, u_p)$ are auxiliary variables restricted to finite domains D_1, \dots, D_p .

When projecting out an integer variable x_j , we can no longer infer $f_i/a_{ij} \leq x_j \leq -g_k/a_{kj}$ as in the proof of Lemma 1. However, we can project out integer variables by strengthening the resultant inequalities. The idea can be illustrated using the example (2) with integer variables x_1, x_2 . This is a classical IP with no congruence relations, but we will see that the same method applies to generalised IPs.

Step 1. We first project out x_1 . We obtain the following from the constraint pairs shown:

$$\begin{aligned}
5(-x_2 + 13) \leq 5 \cdot 2x_1 \leq 2(-2x_2 + 30) & \quad \text{from C1,C2} \\
-x_2 + 13 \leq 2x_1 \leq 2(x_2 - 5) & \quad \text{from C1,C3}
\end{aligned} \tag{7}$$

Because the middle term of the first line is divisible by $5 \cdot 2$, we can increase the term $-x_2 + 13$ on the left to the nearest multiple of 2 (unless it is already a multiple of 2) without violating the inequality. We do this by introducing an integer auxiliary variable $u_1 \in \{0, 1\}$. This yields the system on the left below, which implies the system on the right:

$$\begin{aligned}
5(-x_2 + 13 + u_1) \leq 5 \cdot 2x_1 \leq 2(-2x_2 + 30) & \quad \Rightarrow \quad x_2 \geq 5 + 5u_1 \\
-x_2 + 13 + u_1 \equiv 0 \pmod{2}, \quad u_1 \in \{0, 1\} & \quad \Rightarrow \quad x_2 \equiv u_1 + 1 \pmod{2}, \quad u_1 \in \{0, 1\}
\end{aligned}$$

The congruence relation $-x_2 + 13 + u_1 \equiv 0 \pmod{2}$ reflects the fact that $-x_2 + 13 + u_1$ is a multiple of 2. (We could have just as well have introduced a surplus variable on the right.) We similarly strengthen the second line of (7) to obtain:

$$\begin{aligned}
-x_2 + 13 + u_1 \leq 2x_1 \leq 2(x_2 - 5) & \quad \Rightarrow \quad 3x_2 \geq 23 + u_1 \\
-x_2 + 13 + u_1 \equiv 0 \pmod{2}, \quad u_1 \in \{0, 1\} & \quad \Rightarrow \quad x_2 \equiv u_1 + 1 \pmod{2}, \quad u_1 \in \{0, 1\}
\end{aligned}$$

Putting these together, we have the projected system

$$\begin{aligned}
-x_2 & \geq -z && \text{C0} \\
x_2 & \geq 5 + 5u_1 && \text{C12} \\
3x_2 & \geq 23 + u_1 && \text{C13} \\
x_2 & \equiv u_1 + 1 \pmod{2}, \quad u_1 \in \{0, 1\} &&
\end{aligned} \tag{8}$$

Note that the problem of minimising z subject to this projected constraint set is a generalised IP. Its feasible set is the union of two regions, each of which is described by a linear system applied to a sublattice of \mathbb{Z}^2 defined by the congruences $x_2 \equiv u_1 + 1 \pmod{2}$ and $z \equiv 0 \pmod{1}$. The two regions correspond to the two values of u_1 (namely, $u_1 = 0, 1$) that are feasible in the congruence. In general, we will refer to the feasible values of the u_i s as defining *scenarios*. A generalised IP can be solved by solving the problem in each scenario, and taking the best solution across the scenarios. In this case, the optimal solution is $(x_2, z) = (9, 9)$ in scenario $u_1 = 0$ and $(x_2, z) = (10, 10)$ in scenario $u_1 = 1$, with an overall optimum of $(x_2, z) = (9, 9)$.

Step 2. We now wish to project out x_2 from the system (8). Because the system is now a generalised IP, we must extend the above elimination procedure. We first obtain the following by pairing inequalities, as before:

$$\begin{aligned}
5 + 5u_1 \leq x_2 \leq z & \quad \text{from C0, C12} \\
23 + u_1 \leq 3x_2 \leq 3z & \quad \text{from C0, C13}
\end{aligned} \tag{9}$$

Table 1. Solution of the projected system.

u_1	u_{13}	$5 + 5u_1$	$\frac{1}{3}(23 + u_1 + u_{13})$
0	4	5	9
1	0	10	8

Because $x_2 \equiv u_1 + 1 \pmod{2}$, we can increase the left-hand term in the first line until it is congruent to $u_1 + 1 \pmod{2}$. Introducing an auxiliary variable u_{12} , we obtain the system on the left below:

$$\begin{aligned} 5 + 5u_1 + u_{12} \leq x_2 \leq z & \Rightarrow z \geq 5 + 5u_1 + u_{12} \\ 5 + 5u_1 + u_{12} \equiv u_1 + 1 \pmod{2}, u_{12} \in \{0, 1\} & \Rightarrow u_{12} \equiv 0 \pmod{2}, u_{12} \in \{0, 1\} \end{aligned}$$

It is clearly desirable that only one congruence in the system (8) contain x_2 , so that we can use this kind of reasoning. We indicate below how this can be achieved in general. The second line of (9) gives

$$\begin{aligned} 23 + u_1 + u_{13} \leq 3x_2 \leq 3z & \Rightarrow z \geq \frac{1}{3}(23 + u_1 + u_{13}) \\ 23 + u_1 + u_{13} \equiv 3u_1 + 3 \pmod{6} & \Rightarrow 4u_1 + u_{13} \equiv 4 \pmod{6}, u_{13} \in \{0, \dots, 5\} \end{aligned}$$

Note that u_{12} can be fixed to zero and dropped from the problem. We therefore have the projected system

$$\begin{aligned} z & \geq 5 + 5u_1 \\ z & \geq \frac{1}{3}(23 + u_1 + u_{13}) \\ 4u_1 + u_{13} & \equiv 4 \pmod{6}, u_1 \in \{0, 1\}, u_{13} \in \{0, \dots, 5\} \end{aligned} \tag{10}$$

Step 3. We have reduced the original IP to the problem of minimising z subject to a system (10) of inequalities and congruences that involve only z and the auxiliary variables u_1, u_{13} . As before, we can solve the problem by taking note of the optimal value of z in the two scenarios corresponding to the solutions $(u_1, u_{13}) = (0, 4), (1, 0)$ of the congruences. Because x_1, x_2 are eliminated, computing the optimal value in each scenario is now trivial. The two scenarios are listed in Table 1, where the tightest bound on z in each scenario is shown in boldface. The minimum of these is the optimal value of z , namely $z = 9$, corresponding to $(u_1, u_{13}) = (0, 4)$. Since the bound of 9 comes from C0 and C13, we have $23 + u_1 + u_{13} = 3x_2$ from C13, or $x_2 = 9$. Since C13 comes from C1 and C3, we have $5(-x_2 + 13 + u_1) = 5 \cdot 2x_1$ from C1, or $x_1 = 2$. The optimal solution is therefore $(x_1, x_2, z) = (2, 9, 9)$.

When the variable x_j to be projected out, occurs in several congruences, we wish to replace the congruences with an equivalent single congruence containing x_j . This can be accomplished as follows using a generalised Chinese Remainder Theorem (GCRT). Without loss of generality, suppose the congruences have the form $\alpha x_j \equiv d_s \pmod{m_s}$ for $s \in S$. The GCRT can then be stated as follows.

Theorem 1 (Generalised Chinese Remainder). *Let $C = \{\alpha x_j \equiv d_s \pmod{m_s} \mid s \in S\}$ be a system of congruences, and let $M = \text{lcm}\{m_s \mid s \in S\}$ and $m'_s = M/m_s$. Then we have: (i) $d_s \equiv d_t \pmod{\text{gcd}(m_s, m_t)}$ for all $s, t \in S$ when C is feasible, (ii) there is a set of integers λ_s satisfying $\sum_s \lambda_s m'_s = 1$, and (iii) integer x_j solves C if and only if it solves*

$$\alpha x_j \equiv \sum_{s \in S} \lambda_s m'_s d_s \pmod{M} \tag{11}$$

The multipliers λ_s can be obtained using the well-known Euclidean algorithm.

Proof. Claim (i) can be obtained by subtracting the congruences of C in pairs. Claim (ii) is a well-known consequence of the Euclidean algorithm. To show (iii), suppose first that integer x_j satisfies the congruences in C . Taking a linear combination of the congruences in C with multipliers $\lambda_s m'_s$, we obtain (11). Conversely, suppose x_j satisfies (11). Because $d_s \equiv d_t \pmod{\text{gcd}(m_s, m_t)}$ for all $s, t \in S$, we have

$$\sum_{s \in S} \lambda_s m'_s d_s \equiv \sum_s \lambda_s m'_s d_t \pmod{\text{gcd}\{\lambda_s m'_s \text{gcd}(m_s, m_t)\}}$$

for any $t \in S$, which implies

$$\sum_{s \in S} \lambda_s m'_s d_s \equiv \sum_s \lambda_s m'_s d_t \pmod{\gcd\{m'_s \gcd(m_s, m_t)\}} \quad (12)$$

But $\gcd_{s \in S}\{m'_s \gcd(m_s, m_t)\} = m_t$ because $m'_s = M/m_s$. Given this and (ii), (12) simplifies to

$$\sum_{s \in S} \lambda_s m'_s d_s \equiv d_t \pmod{m_t} \quad (13)$$

Also (11) implies

$$\alpha x_j \equiv \sum_{s \in S} \lambda_s m'_s d_s \pmod{m_t}$$

which, together with (13), implies $\alpha x_j \equiv d_t \pmod{m_t}$. Since $t \in S$ is arbitrary, x_j satisfies the congruences in C . \square

The general procedure for projecting out a variable x_j is as follows. We suppose that the current system (S, C) consists of a set S of inequalities and a set C of congruences in variables z , x_j , and u_i , and finite domains $u \in D$.

1. Apply the GCRT to all congruences in C containing x_j to obtain a single congruence relation that can be written $\alpha x_j \equiv d \pmod{m}$.
2. Consider all pairs of inequalities in S of the form $a_{ij}x_j \geq f_i$, $-a_{kj}x_j \geq g_k$ for which $a_{ij}, a_{kj} > 0$. Introduce an auxiliary variable u_i for each i , and for each pair generate the inequality $a_{kj}(f_i + u_i) + a_{ij}g_k \leq 0$ along with the congruence $f_i + u_i \equiv \lambda_m a_{ij}d/\beta \pmod{a_{ij}m/\beta}$. The multiplier λ_m can be obtained by using the Euclidean algorithm to find multipliers $\lambda_\alpha, \lambda_m$ for which $\lambda_\alpha \text{lcm}(\alpha, m)/\alpha + \lambda_m \text{lcm}(\alpha, m)/m = 1$. If all inequalities are of the form $a_{ij}x_j \geq f_i$, or all are of the form $-a_{kj}x_j \geq g_k$, then no new inequalities are generated (as in the LP case).
3. Update the system (S, C) by removing from S all inequalities containing x_j , adding to S all generated inequalities, adding to C all the associated congruence relations, and adding $u_i \in \{0, \dots, a_{ij}m/\beta - 1\}$ to the domains.

The correctness of the procedure is due to the following theorem.

Theorem 2. *Suppose $a_{ij}, a_{kj} > 0$ for all $i \in I, k \in K$. Then*

(a) *There exists $x_j \in \mathbb{Z}$ such that $a_{ij}x_j \geq f_i$ and $-a_{kj}x_j \geq g_k$ for all $i \in I, k \in K$, and such that $\alpha x_j \equiv d \pmod{m}$,*

if and only if

(b) *$d \equiv 0 \pmod{\beta}$, where $\beta = \gcd(\alpha, m)$; there exist $\lambda_\alpha, \lambda_m \in \mathbb{Z}$ satisfying $\lambda_\alpha m + \lambda_m \alpha = \beta$; and there exists $u_i \in \{0, 1, \dots, a_{ij}m/\beta - 1\}$ such that $a_{kj}(f_i + u_i) + a_{ij}g_k \leq 0$ for all $i \in I, k \in K$, and $f_i + u_i \equiv \lambda_m a_{ij}d/\beta \pmod{a_{ij}m/\beta}$ for all $i \in I$.*

Proof. (a) \Rightarrow (b). We can write the inequalities in (a) as

$$a_{kj}\alpha f_i \leq a_{ij}a_{kj}\alpha x_j \leq -a_{ij}\alpha g_k \quad (14)$$

for all i, k . From the congruence in (a), $a_{ij}a_{kj}\alpha x_j \equiv a_{ij}a_{kj}d \pmod{a_{ij}a_{kj}m}$. Thus if we let $y = a_{ij}a_{kj}\alpha x_j$, we obtain $y \equiv 0 \pmod{a_{ij}a_{kj}\alpha}$ and $y \equiv a_{ij}a_{kj}d \pmod{a_{ij}a_{kj}m}$. Applying part (i) of the GCRT to these two congruences, we get $d \equiv 0 \pmod{\beta}$. From part (ii), there are integers $\lambda_\alpha, \lambda_s$ for which $\lambda_\alpha \text{lcm}(\alpha, m)/\alpha + \lambda_m \text{lcm}(\alpha, m)/m = 1$. Since $\text{lcm}(\alpha, m)/m = \alpha/\beta$, this is equivalent to $\lambda_\alpha m + \lambda_m \alpha = \beta$, as claimed in (b). From part (iii), we have $y \equiv \lambda_m a_{ij}a_{kj} \text{lcm}(\alpha, m) d/m \pmod{a_{ij}a_{kj} \text{lcm}(\alpha, m)}$, which implies the congruence

$$y \equiv \lambda_m a_{ij}a_{kj} \alpha d/\beta \pmod{a_{ij}a_{kj} \text{lcm}(\alpha, m)}$$

again because $\text{lcm}(\alpha, m)/m = \alpha/\beta$. So from (14) we have

$$a_{kj}\alpha f_i - \lambda_m a_{ij}a_{kj} \alpha d/\beta \leq y \leq -a_{ij}\alpha g_k - \lambda_m a_{ij}a_{kj} \alpha d/\beta \quad (15)$$

where γ is an integer multiple of $a_{ij}a_{kj}\text{lcm}(\alpha, m)$. Since $d \equiv 0 \pmod{\beta}$, β divides d , and the leftmost expression in (15) is an integer multiple of $a_{kj}\alpha$. So we can add $a_{kj}\alpha u_i$ to the left-hand side of (15), and we have

$$a_{kj}\alpha(f_i + u_i) \leq \gamma + \lambda_m a_{ij} a_{kj} \alpha d / \beta \leq -a_{ij} \alpha g_k \quad (16)$$

and

$$a_{kj}\alpha(f_i + u_i) - \lambda_m a_{ij} a_{kj} \alpha d / \beta \equiv 0 \pmod{a_{ij} a_{kj} \text{lcm}(\alpha, m)} \quad (17)$$

Inequality (16) implies the inequality in (b). Congruence (17) simplifies to

$$f_i + u_i - \lambda_m a_{ij} d / \beta \equiv 0 \pmod{a_{ij} m / \beta}$$

which implies the congruence in (b). We can also restrict u_i to $\{0, 1, \dots, a_{ij}m/\beta - 1\}$. For if u_i were greater than $a_{ij}m/\beta - 1$ then the original inequalities and congruences would still be valid if $a_{ij}m/\beta - 1$ were subtracted from u_i .

(a) \Leftarrow (b). The inequalities in (b) can be written

$$-\frac{g_k}{a_{kj}} \geq \frac{f_i + u_i}{a_{ij}} \quad (18)$$

for all i, k . From (b) we have that $d \equiv 0 \pmod{\beta}$, so that d/β is integral. Also from (b),

$$f_i + u_i \equiv \lambda_m a_{ij} d / \beta \pmod{a_{ij} a_{kj} m / \beta} \quad (19)$$

Because d/β and m/β are integral, this implies $f_i + u_i$ is an integer multiple of a_{ij} . We can therefore let

$$x_j = \max_i \left\{ \frac{f_i + u_i}{a_{ij}} \right\} \quad (20)$$

and x_j is integral. This and (18) imply $-g_k/a_{kj} \geq x_j$, or $-g_k \geq a_{kj}x_j$. To show $a_{ij}x_j \geq f_i$, we note that

$$a_{ij}x_j \geq a_{ij} \frac{f_i + u_i}{a_{ij}} \geq f_i$$

because $u_i \geq 0$. Finally, we show $\alpha x_j \equiv d \pmod{m}$. From (20), we have that $x_j = (f_i + u_i)/a_{ij}$ for some i . So (19) implies that $x_j \equiv \lambda_m d / \beta \pmod{m/\beta}$, and therefore $\alpha x_j \equiv \lambda_m \alpha d / \beta \pmod{\alpha m / \beta}$. This implies the following due to $\lambda_\alpha m + \lambda_m \alpha = \beta$ in (b):

$$\alpha x_j \equiv (d - \lambda_\alpha m d / \beta) \pmod{\alpha m / \beta}$$

which implies $\alpha x_j \equiv d \pmod{\alpha m / \beta}$, or $\alpha x_j \equiv d \pmod{\text{lcm}(\alpha, m)}$. But this implies $\alpha x_j \equiv d \pmod{m}$. \square

To solve a generalised IP problem, we suppose the problem is given in the form $(\mathcal{S}, \mathcal{C})$ with domains $u \in D$, as above. It can be viewed as an optimization problem subject to the inequalities \mathcal{S} in variables x_j , over the integer sublattices defined by the congruence relations in \mathcal{C} . In a conventional IP problem, the congruences in \mathcal{C} are simply $x_j \equiv 0 \pmod{1}$, which require integrality, and there are no variables u_i . We sequentially project out variables x_1, \dots, x_n , which yields a system $(\mathcal{S}', \mathcal{C}')$ in which \mathcal{S}' contains only z and variables u_i , and \mathcal{C}' contains only u_i s. The inequalities in \mathcal{S}' have the form $z \geq v_t(u)$, and the optimal value of the problem is $\min_u \{\max_t \{v_t(u)\} \mid \mathcal{C}', u \in D\}$. The original problem is therefore transformed to one in which the variables u_i have finite domains.

Note that unboundedness is revealed by the final inequalities in the same way as for the LP case. Infeasibility is revealed through either false final inequalities, as in the LP case, or final inequalities that have no solution in the auxiliary variables which satisfy the generated congruences. These cases correspond, respectively, to the case of an IP which is infeasible because its LP relaxation is infeasible and an IP which has a feasible LP relaxation.

5. Congruence Cuts

Given a generalized IP, one can define valid *congruence cuts* that consist of an inequality and a congruence relation. Congruence cuts are analogous to Chvátal–Gomory cuts in that they are derived by a linear combination and strengthening operation. However, whereas Chvátal–Gomory cuts are strengthened by rounding, congruence cuts are strengthened by adding an auxiliary variable to the right-hand side and imposing an additional congruence relation on the expression that results. The inequality portion of a congruence cut can be interpreted as a separating cut in the sense that it cuts off infeasible solutions in one or more scenarios defined by the current auxiliary variables.

As with Chvátal–Gomory cuts, one can generate a finite sequence of separating congruence cuts that solve a given optimization problem. The cuts are surrogates that are strengthened by congruences, rather than by rounding as in Gomory’s method. We use projection to show that cuts of rank at most n can deduce the optimal value, while no such bound is available for Gomory cuts. As in the case of Gomory cuts, these cuts allow one to obtain the optimal solution (in addition to the optimal value) by solving the linear relaxation, provided the relaxation is dual nondegenerate. The cuts may not remove all fractional values, however, when the relaxation is dual degenerate.

To define a congruence cut, let \mathcal{S} be a system of linear inequalities in variables $x = (x_1, \dots, x_n) \in \mathbb{Z}^n$, and let \mathcal{C} be a system of congruences in variables x and $u = (u_1, \dots, u_t) \in D \subset \mathbb{Z}_{\geq 0}^t$. A *rank 1 congruence cut* for $(\mathcal{S}, \mathcal{C})$ is any nonnegative linear combination of $a_{ij}x_j \geq f_i + u_i$ and an inequality in \mathcal{S} , where $a_{ij}x_j \geq f_i$ belongs to \mathcal{S} and a congruence of the form $\alpha x_j \equiv d \pmod{m}$ belongs to \mathcal{C} . The rank 1 cut is associated with the congruence relation

$$\alpha(f_i + u_i) \equiv a_{ij}d \pmod{a_{ij}m} \quad (21)$$

and domain $u_i \in D_i = \{0, \dots, a_{ij}m - 1\}$. The cut is *valid* when all (x, u) satisfying $(\mathcal{S}, \mathcal{C})$, congruence (21), $u \in D$, and $u_i \in \{0, \dots, a_{ij}m - 1\}$ also satisfy the cut.

A *rank k congruence cut* for $(\mathcal{S}, \mathcal{C})$ is a rank 1 cut for some system $(\mathcal{S}', \mathcal{C}')$ consisting of cuts of rank $k - 1$ or less for $(\mathcal{S}, \mathcal{C})$ and their associated congruences and domains, provided it is not a rank 1 cut for any such system of cuts with rank less than $k - 1$. A *congruence cut* is any rank k projection cut for finite k .

Theorem 3. *Any congruence cut for $(\mathcal{S}, \mathcal{C})$ is valid for $(\mathcal{S}, \mathcal{C})$.*

Proof. It is enough to show that any rank 1 cut for $(\mathcal{S}, \mathcal{C})$ is valid, because then it follows by induction that any rank k cut is valid. Because a nonnegative linear combination of valid inequalities is valid, we can show that a rank 1 cut is valid by showing that $a_{ij}x_j \geq f_i + u_i$ is valid for $(\mathcal{S}, \mathcal{C})$ when $a_{ij}x_j \geq f_i$ is in \mathcal{S} , (21) holds, $u \in D$, and $u_i \in \{0, \dots, a_{ij}m - 1\}$. Equivalently, we wish to show

$$\alpha a_{ij}x_{ij} - a_{ij}d \geq \alpha(f_i + u_i) - a_{ij}d \quad (22)$$

is valid under these conditions. However, we know that $\alpha a_{ij}x_{ij} - a_{ij}d \geq \alpha f_i - a_{ij}d$ is valid, because $a_{ij}x_{ij} \geq f_i$ belongs to \mathcal{S} . Also the congruence $\alpha x_j \equiv d \pmod{m}$ implies that the left-hand side of (22) is a multiple of $a_{ij}m$. The inequality (22) is therefore valid if u_i is the smallest nonnegative integer for which the right-hand side is a multiple of $a_{ij}m$. For this, it suffices that (21) hold and $u_i \in \{0, \dots, a_{ij}m - 1\}$. \square

As an example, we will generate a congruence cut from inequalities C1 and C3 and the congruence $x_1 \equiv 0 \pmod{1}$. We first strengthen C1 by adding an auxiliary variable $u_1 \in \{0, 1\}$, to obtain

$$2x_1 \geq 13 - x_2 + u_1 \quad (\text{C1}')$$

We now take a linear combination of C1' with C3 using unit multipliers (for example) to obtain the cut

$$x_1 \geq 18 - 2x_2 + u_1 \quad (\text{C13}')$$

Because $\alpha = 1$, $a_{ij} = 2$, and $m = 1$, the associated congruence (21) is

$$13 - x_2 + u_1 \equiv 0 \pmod{2}, \quad \text{or} \quad x_2 \equiv u_1 + 1 \pmod{2}$$

The auxiliary variable u_1 defines two scenarios, corresponding to $u_1 = 0, 1$. The cut C13' cuts off the solution $(x_1, x_2, z) = (2\frac{2}{3}, 7\frac{2}{3}, 7\frac{2}{3})$ of the linear relaxation (2) in the second scenario, where C13' is $x_1 \geq 19 - 2x_2$. It can

therefore be regarded as a separating cut in the sense that it cuts off the solution of the current relaxation in at least one scenario.

Projection steps provide one means of generating congruence cuts. As an example, consider again (2). In step 1 of the projection, C1 and C3 were combined to yield C13 and the congruence $x_2 \equiv u_1 + 1 \pmod{2}$. But C13 is the linear combination of C1' and C3 using the same multipliers (given on the left below) that were used to combine C1 and C3 in the projection step (7).

$$\begin{array}{rcl} (1) & 2x_1 \geq 13 - x_2 + u_1 & C1' \\ (2) & -x_1 \geq 5 - x_2 & C3 \\ \hline & 0 \geq 23 - 3x_2 + u_1 & C13 \end{array}$$

C13 and the associated relation $13 - x_2 + u_1 \equiv 0 \pmod{2}$ therefore comprise a congruence cut that is obtained from C1, C3, and $x_1 \equiv 0 \pmod{1}$. The congruence relation can be simplified to $x_2 \equiv u_1 + 1 \pmod{2}$, as before. In general, we have the following.

Theorem 4. *Each step of the integer projection method produces rank 1 projection cuts for the system (S, C) from which the cuts are derived.*

Proof. Each inequality generated by projection has the form

$$a_{kj}(f_i + u_i) + a_{ij}g_k \leq 0 \quad (23)$$

and is derived from $a_{ij}x_j \geq f_i$, $-a_{kj}x_j \geq g_k \in S$. We wish to show that (23) is a rank 1 projection cut for (S, C) . We first note that (23) is a linear combination of $a_{ij}x_j \geq f_i + u_i$ and $-a_{kj}x_j \geq g_k$, using multipliers $a_{kj}, a_{ij} > 0$, respectively. Because $\alpha x_j \equiv d \pmod{m}$ is in C , it remains only to show that $u_i \in \{0, \dots, a_{ij}m - 1\}$ and that (21) holds. The projection step yields the congruence relation

$$\alpha(f_i + u_i) \equiv \alpha \lambda_m a_{ij} d / \beta \pmod{a_{ij} \alpha m / \beta} \quad (24)$$

where $\lambda_\alpha m + \lambda_m \alpha = \beta$. Substituting $\beta - \lambda_\alpha m$ for $\lambda_m \alpha$, this becomes

$$\alpha(f_i + u_i) \equiv a_{ij} d \pmod{a_{ij} \alpha m / \beta}$$

This implies (21) because $\alpha m / \beta = \text{lcm}(\alpha, m)$. Also, the projection step yields $u_i \in \{0, \dots, a_{ij}m / \beta - 1\}$, which implies $u_i \in \{0, \dots, a_{ij}m - 1\}$. \square

One can deduce the optimal value of a problem by generating a series of congruence cuts corresponding to integer projection steps. The projection steps illustrated in the previous section generate the following congruence cuts:

$$\begin{array}{llll} x_2 \geq 5 + 5u_1 & C12 & x_1 \equiv 0 \pmod{1} & \\ 3x_2 \geq 23 + u_1 & C13 & x_2 \equiv u_1 + 1 \pmod{2}, \quad u_1 \in \{0, 1\} & \\ z \geq 5 + 5u_1 & C012 & 4u_1 + u_{13} \equiv 4 \pmod{6}, \quad u_{13} \in \{0, \dots, 5\} & (25) \\ 3z \geq 23 + u_1 + u_{13} & C013 & & \end{array}$$

There are again two scenarios, corresponding to $(u_1, u_{13}) = (1, 0), (0, 4)$. In the first scenario, the minimum of z subject to the inequalities in (25) is $(x_1, x_2, z) = (2, 10, 10)$, which satisfies all congruences. (The optimal value $z = 10$ can be immediately deduced from inequalities C012 and C013.) In the second scenario, the linear system is dual degenerate, and there are two minima: $(x_1, x_2, z) = (2, 9, 9), (2\frac{2}{5}, 9, 9)$. The optimal value of the IP is the minimum value of z in the two scenarios, namely $z = 9$.

In general, the congruence cuts generated by projecting a generalised IP onto z prove optimality. Since each of the n projection steps obtains congruence cuts by combining inequalities from previous steps, the necessary congruence cuts have rank at most n . The congruence cuts prove optimality in the following sense. Let S' be the set of cut inequalities of the form $z \geq v_t(u)$ and C' the set of associated congruences containing only the variables u_i , where $u \in D$. Then as noted in the previous section, the optimal value of the problem is $\min_u \{\max_t \{v_t(u)\} \mid C', u \in D\}$. We therefore have the following.

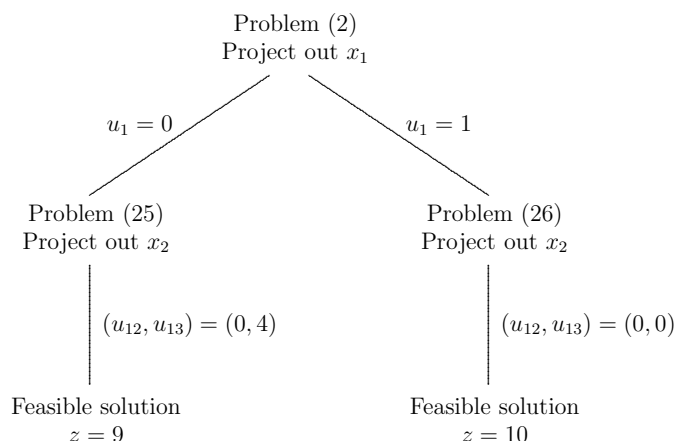


Figure 2. Projection-based branching tree for example (2).

Corollary 1. *Some system of congruence cuts for the IP problem (5) with rank at most n proves the optimal value of (5).*

Unlike Chvátal-Gomory cuts, congruence cuts do not necessarily allow one to identify an optimal solution (as opposed to the optimal value) by solving linear relaxations. Consider again the above example. If solving the linear relaxation in scenario $(u_1, u_{13}) = (0, 4)$ yields $(x_1, x_2, z) = (2, 9, 9)$, then we know it is an optimal solution of the IP because it satisfies the congruences. However, if we obtain the alternate optimal solution $(x_1, x_2, z) = (2\frac{2}{5}, 9, 9)$, we do not have an optimal solution of the IP because $x_1 = 2\frac{2}{5}$ violates the congruence $x_1 \equiv 0 \pmod{1}$. Since no congruence cut excludes this solution, we cannot identify an optimal solution by solving a linear relaxation with cuts. This problem does not arise if the linear relaxation is dual nondegenerate, but otherwise it may be necessary to construct an optimal solution from a sequence of projection-based cuts as in the previous section.

6. Solution by Branching

The above analysis of integer projection leads to a branching algorithm for the generalised IP problem (S, C) , $u \in D$. Each time a variable x_j is projected out, we branch on the possible values of auxiliary variables u_i created during the projection step. This means that auxiliary variables do not appear in the branches. If the original problem contains variables u_i , we branch on them as well at the root node. Branching continues until all the variables x_j are eliminated.

We therefore branch on scenarios, or equivalently, on disjunctions of constraint sets that result from projection. This was the original way that the method was presented in [12].

An advantage of branching is that auxiliary variables do not accumulate as in sequential projection. In addition, the branching tree has depth of at most n , although the width can grow exponentially. A bounding mechanism, explained below, may limit the width.

Branching can be illustrated using the example (2), for which the branching tree appears in Fig. 2. At the root node of the tree, we carry out step 1 above, which yields the projected system (8). Now, rather than branch on x_1 , we branch on $u_1 \in \{0, 1\}$.

Left branch, $u_1 = 0$. Here (8) simplifies to

$$\begin{aligned}
 -x_2 &\geq -z \\
 x_2 &\geq 5 \\
 3x_2 &\geq 23 \\
 x_2 &\equiv 1 \pmod{2}
 \end{aligned} \tag{26}$$

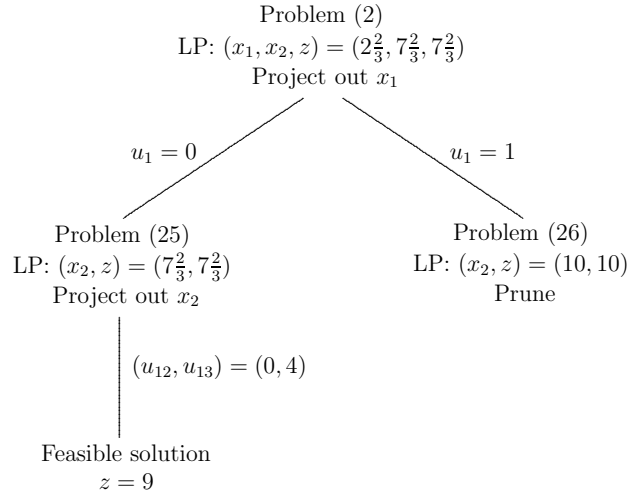


Figure 3. Projection-based branch-and-bound tree for example (2).

We now project out x_2 , which yields

$$\begin{array}{ll}
 5 + u_{12} \leq x_2 \leq z & \Rightarrow \quad z \geq 5 + u_{12} \\
 23 + u_{13} \leq 3x_2 \leq 3z & z \geq \frac{1}{3}(23 + u_{13}) \\
 5 + u_{12} \equiv 1 \pmod{2}, \quad u_{12} \in \{0, 1\} & u_{12} \equiv 0 \pmod{2}, \quad u_{12} \in \{0, 1\} \\
 23 + u_{13} \equiv 3 \pmod{6}, \quad u_{13} \in \{0, \dots, 5\} & u_{13} \equiv 4 \pmod{6}, \quad u_{13} \in \{0, \dots, 5\}
 \end{array}$$

Only one branch $(u_{12}, u_{13}) = (0, 4)$ satisfies the congruence. In this branch, the problem is to minimise z subject to $z \geq 5$ and $z \geq 9$, yielding the bound $z \geq 9$.

Right branch, $u_1 = 1$. Here (8) simplifies to

$$\begin{array}{l}
 -x_2 \geq -z \\
 x_2 \geq 10 \\
 3x_2 \geq 24 \\
 x_2 \equiv 0 \pmod{2}
 \end{array} \tag{27}$$

Projecting out x_2 , we get

$$\begin{array}{ll}
 10 + u_{12} \leq x_2 \leq z & \Rightarrow \quad z \geq 10 + u_{12} \\
 24 + u_{13} \leq 3x_2 \leq 3z & z \geq 8 + \frac{1}{3}u_{13} \\
 10 + u_{12} \equiv 0 \pmod{2}, \quad u_{12} \in \{0, 1\} & u_{12} \equiv 0 \pmod{2}, \quad u_{12} \in \{0, 1\} \\
 24 + u_{13} \equiv 0 \pmod{6}, \quad u_{13} \in \{0, \dots, 5\} & u_{13} \equiv 0 \pmod{6}, \quad u_{13} \in \{0, \dots, 5\}
 \end{array}$$

Only one branch $(u_{12}, u_{13}) = (0, 0)$ is possible, at which the problem is to minimise z subject to $z \geq 10$ and $z \geq 8$, yielding the bound $z \geq 10$.

The optimal solution occurs at the left leaf node, with $z = 9$ and $(u_1, u_{12}, u_{13}) = (0, 0, 4)$.

We can introduce a branch-and-bound mechanism by solving a relaxation at each node. The solution of the relaxation can also indicate how to branch, as in traditional branch and bound, because we can branch on a variable x_j that violates its associated congruence $x_j \equiv d \pmod{m}$. The simplest relaxation is an LP relaxation obtained by dropping the congruences.

For example, the LP relaxation of (2) at the root node has solution $(x_1, x_2, z) = (2\frac{2}{3}, 7\frac{2}{3}, 7\frac{2}{3})$ (Fig. 3). Because x_1 and x_2 must satisfy the implicit congruence $x_j \equiv 0 \pmod{1}$ for $j = 1, 2$, we can project out either variable and branch

on the corresponding auxiliary variable. We choose to project out x_1 and branch on u_1 . Solving the LP relaxation of (26) in the left branch yields $(x_2, z) = (7\frac{2}{3}, 7\frac{2}{3})$. Because x_2 violates $x_2 \equiv 1 \pmod{2}$, we must project out x_2 . The LP relaxation of (27) in the right branch has solution $(x_2, z) = (10, 10)$. Because 10 is greater than the incumbent value of 9, it is unnecessary to project out x_2 and branch further. In addition, x_2 satisfies $x_2 \equiv 0 \pmod{2}$, which in itself obviates the necessity of further branching.

Note that it may be necessary to branch even when all the variables x_j are integral in the LP solution. The relevant criterion is whether they satisfy their respective congruences.

7. A Value Function and Dual Solution

We can obtain a value function by applying the projection algorithm to inequalities with perturbed right-hand sides. It can be regarded as a dual solution of the generalised IP.

To illustrate the idea, consider the constraint C1 in example (2), which is $2x_1 + x_2 \geq 13$. While projecting out x_1 we used the strengthened inequality

$$-x_2 + 13 + u_1 \leq 2x_1 \quad (28)$$

where

$$-x_2 + 13 + u_1 \equiv 0 \pmod{2} \quad (29)$$

and $u \in \{0, 1\}$. Suppose we now perturb the right-hand side of C1 to obtain the constraint $2x_1 + x_2 \geq 13 + \Delta$, so that (28) becomes $-x_2 + 13 + \Delta + u_1 \leq 2x_1$. This inequality is not generally valid, given congruence (29). However, we can strengthen C1 in a different way by adding $\Delta + \text{mod}_2(u_1 - \Delta)$ rather than u_1 :

$$-x_2 + 13 + \Delta + \text{mod}_2(u_1 - \Delta) \leq 2x_1 \quad (30)$$

where $\text{mod}_m(a)$ is the remainder after dividing a by m . This has the same effect as (28) when $\Delta = 0$. To ensure validity, we need the congruence

$$-x_2 + 13 + \Delta + \text{mod}_2(u_1 - \Delta) \equiv 0 \pmod{2} \quad (31)$$

However, this is equivalent to congruence (29), because $u_1 \equiv \Delta + \text{mod}_m(u_1 - \Delta) \pmod{m}$ due to the obvious fact that $u_1 - \Delta \equiv \text{mod}_m(u_1 - \Delta) \pmod{m}$. It is easy to show that

$$\Delta + \text{mod}_m(u_1 - \Delta) = u_1 + \lceil \Delta - u_1 \rceil_m \quad (32)$$

where $\lceil a \rceil_m = m\lceil a/m \rceil$ is a rounded up to the nearest multiple of m . So (30) can be written

$$-x_2 + 13 + u_1 + \lceil \Delta - u_1 \rceil_2 \leq 2x_1$$

By incorporating this idea into the projection algorithm, we can derive a value function. Consider again the perturbed example (3).

Step 1. To project out x_1 , we combine $C1_\Delta$ and $C2_\Delta$ to obtain

$$5(-x_2 + 13 + u_1 + \lceil \Delta_1 - u_1 \rceil_2) \leq 5 \cdot 2x_1 \leq 2(-2x_2 + 30)$$

This yields

$$x_2 \geq 5 + 5u_1 + 5\lceil \Delta_1 - u_1 \rceil_2 + 2\Delta_2 \quad C12_\Delta$$

where $x_2 \equiv u_1 + 1 \pmod{2}$ as before. We combine $C\Delta 1$ and $C\Delta 3$ to obtain

$$3x_2 \geq 23 + u_1 + \lceil \Delta_1 - u_1 \rceil_2 + 2\Delta_3 \quad C13_\Delta$$

Step 2. To eliminate x_2 , we combine $C0$ and $C12_\Delta$ to obtain

$$5 + 5u_1 + u_{12} + 5\lceil \Delta_1 - u_1 \rceil_2 + 2\Delta_2 - u_{12} \leq x_2 \leq z$$

Table 2. Lower bounds for perturbations in individual constraints i .

u_1	u_{13}	Bound from C012 $_{\Delta}$			Bound from C013 $_{\Delta}$		
		$i = 1$	2	3	$i = 1$	2	3
0	4	$5 + 5\lceil\Delta_1\rceil_2$	$5 + \lceil 2\Delta_2\rceil_2$	5	$9 + \frac{1}{3}\lceil\Delta_1 - 4\rceil_6$	9	$9 + \frac{2}{3}\lceil\Delta_3 - 2\rceil_3$
1	0	$10 + 5\lceil\Delta_1 - 1\rceil_2$	$10 + \lceil 2\Delta_2\rceil_2$	10	$8 + \frac{1}{3}\lceil\Delta_1 - 1\rceil_6$	8	$8 + \frac{2}{3}\lceil\Delta_3\rceil_3$

This yields

$$z \geq 5 + 5u_1 + u_{12} + \lceil 5\lceil\Delta_1 - u_1\rceil_2 + 2\Delta_2 - u_{12}\rceil_2 \quad (33)$$

where $u_{12} \equiv 0 \pmod{2}$ and $u_{12} \in \{0, 1\}$. Note the nesting of functions $\lceil \cdot \rceil_m$, which is analogous to the nesting of rounding operations in a Chvátal function. Because $u_{12} = 0$ and $\lceil\Delta_1 - u_1\rceil_2$ is even, the bound (33) simplifies to

$$z \geq 5 + 5u_1 + 5\lceil\Delta_1 - u_1\rceil_2 + \lceil 2\Delta_2\rceil_2 \quad \text{C012}_{\Delta}$$

We similarly combine C0 and C13 $_{\Delta}$ to obtain

$$3z \geq 23 + u_1 + u_{13} + \lceil \lceil\Delta_1 - u_1\rceil_2 + 2\Delta_3 - u_{13}\rceil_6 \quad \text{C013}_{\Delta}$$

where $4u_1 + u_{13} \equiv 4 \pmod{6}$ and $u_{13} \in \{0, \dots, 5\}$.

Step 3. We now have a value function from C012 $_{\Delta}$ and C013 $_{\Delta}$:

$$v(\Delta_1, \Delta_2, \Delta_3) = \min_{u_1, u_{13}} \left\{ \max \left\{ \begin{array}{l} 5 + 5u_1 + 5\lceil\Delta_1 - u_1\rceil_2 + \lceil 2\Delta_2\rceil_2, \\ \frac{1}{3}(23 + u_1 + u_{13} + \lceil \lceil\Delta_1 - u_1\rceil_2 + 2\Delta_3 - u_{13}\rceil_6) \end{array} \right\} \right\} \quad (34)$$

where the minimum is taken over u_1, u_{13} satisfying $u_{12} \equiv 0 \pmod{2}$, $4u_1 + u_{13} \equiv 4 \pmod{6}$, $u_1 \in \{0, 1\}$, and $u_{13} \in \{0, \dots, 5\}$. In this case, the congruences have only two solutions $(u_1, u_{13}) = (0, 4), (1, 0)$.

The value function can be regarded as a function of each perturbation Δ_i individually. The functions $v(\Delta_1, 0, 0)$, $v(0, \Delta_2, 0)$ and $v(0, 0, \Delta_3)$ for this problem instance are graphed in Figs. 4–6. As in the case of LP, the value function is valid only for feasible perturbations. A perturbation is infeasible if the congruences, together with the inequalities containing only Δ_i 's, have no feasible solution.

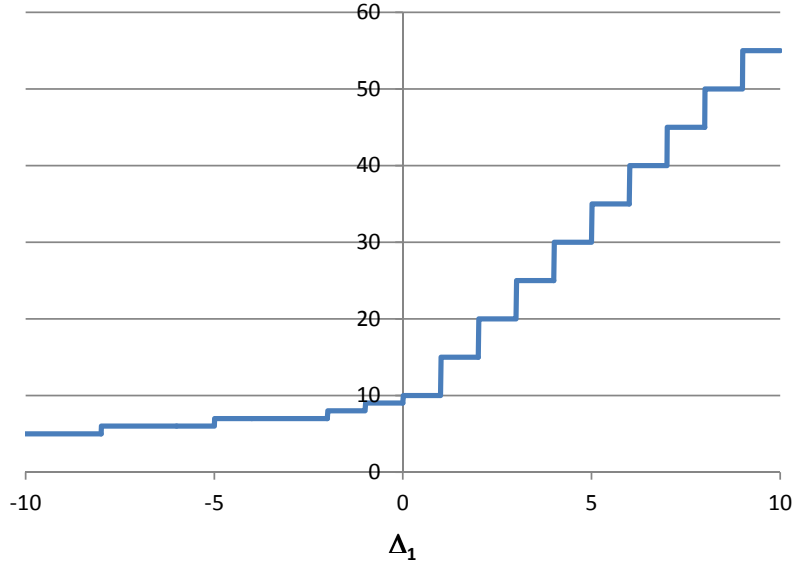
Because $\lceil a \rceil_m$ can be replaced by $m\lceil a/m \rceil$, the maximum in the value function can be viewed as a maximum over Chvátal functions and is therefore a Gomory function, as is the value function generated by Gomory's method. However, expressions of the form $\lceil a \rceil_m$ reveal the structure of the value function more clearly. The value function presented here differs from that obtained from Gomory's method in two additional ways: it takes a minimum over multiple Gomory functions, each corresponding to a scenario, and the nesting depth of rounding in the Chvátal functions is bounded by n .

We now describe the general procedure for projecting out a variable x_j in a perturbed system. The expressions $\bar{\Delta}_i$ are defined when the previous variable is eliminated. If x_j is the first variable to be eliminated, then $\bar{\Delta}_i = \Delta_i$ for each i .

1. Apply the GCRT to the congruences in C containing x_j to obtain a single congruence $\alpha x_j \equiv d \pmod{m}$.
2. Consider all pairs of inequalities in S of the form $a_{ij}x_j \geq f_i + \bar{\Delta}_i$, $-a_{kj}x_j \geq g_k + \bar{\Delta}_k$ for which $a_{ij}, a_{kj} > 0$. Generate the inequality (35) and associate it with the congruence $f_i + u_i \equiv \lambda_m a_{ij}d/\beta \pmod{a_{ij}m/\beta}$ and the domain $u_i \in \{0, \dots, a_{ij}m/d - 1\}$. The multiplier λ_m can be obtained by using the Euclidean algorithm as before.
3. Update the system (S, C) by removing from S all inequalities containing x_j , adding to S all generated inequalities, adding to C all the associated congruence relations, and adding $u_i \in \{0, \dots, a_{ij}m/\beta - 1\}$ to the domains.
4. If x_ℓ is the next variable to be eliminated, write (35) as $a_{ij}x_\ell \geq f_\ell + \bar{\Delta}_\ell$, where $\bar{\Delta}_\ell = a_{kj}\lceil\bar{\Delta}_i - u_i\rceil_{a_{ij}m/\beta} + a_{ij}\bar{\Delta}_k$.

When all variables x_j have been eliminated, the result is a system (S', C') and domains $u \in D$ such that S contains only z and u_i 's, and C contains only u_i 's. The inequalities in S provide bounds of the form $z \geq v_i(u, \Delta)$. Due to Theorem 5, this describes the projection onto z , and the function

$$v(\Delta) = \min_u \left\{ \max_i \{v_i(u, \Delta)\} \mid C', u \in D \right\}$$

Figure 4. Value function $v(\Delta_1)$ for constraint 1.

is therefore the optimal value of the perturbed IP problem (5). In other words, $v(\Delta)$ is a value function for (5). It is clear from the form of (35) that $v(\Delta)$ contains nested roundings $\lceil \cdot \rceil_m$. Because n variables are eliminated, the depth of the nesting is at most n .

To show that projection creates a value function for an general IP problem, we must extend Theorem 2 to deal with perturbed right-hand sides. Interestingly, the perturbations do not affect the congruences, and the perturbation terms Δ_i appear only in the generated inequalities. The inequalities $a_{ij}x_j \geq f_i$ and $-a_{kj}x_j \geq g_k$ in Theorem 2 are replaced with $a_{ij}x_j \geq f_i + \bar{\Delta}_i$ and $-a_{kj}x_j \geq g_k + \bar{\Delta}_k$ to account for the effect of perturbations on generated inequalities. Thus $\bar{\Delta}_i = \bar{\Delta}_k = 0$ when all the perturbations are zero.

Theorem 5. Suppose $a_{ij}, a_{kj} > 0$ for all $i \in I, k \in K$. Then

- (a) There exists $x_j \in \mathbb{Z}$ such that $a_{ij}x_j \geq f_i + \bar{\Delta}_i$ and $-a_{kj}x_j \geq g_k + \bar{\Delta}_k$ for all $i \in I, k \in K$, and such that $\alpha x_j \equiv d \pmod{m}$,

if and only if

- (b) $d \equiv 0 \pmod{\beta}$, where $\beta = \gcd(\alpha, m)$; there exist $\lambda_\alpha, \lambda_m \in \mathbb{Z}$ satisfying $\lambda_\alpha m + \lambda_m \alpha = \beta$; and there exists $u_i \in \{0, 1, \dots, a_{ij}m/\beta - 1\}$ such that

$$a_{kj}(f_i + u_i + \lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta}) + a_{ij}(g_k + \bar{\Delta}_k) \leq 0 \quad (35)$$

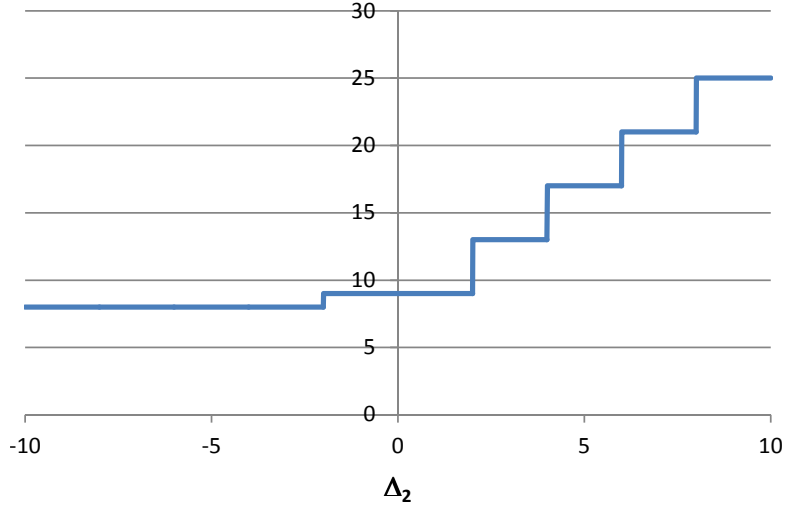
for all $i \in I, k \in K$, and $f_i + u_i \equiv \lambda_m a_{ij} d / \beta \pmod{a_{ij}m/\beta}$ for all $i \in I$.

Furthermore, if $\bar{\Delta}_i = \bar{\Delta}_k = 0$, then inequality (35) reduces to $a_{kj}(f_i + u_i) + a_{ij}g_k \leq 0$.

Proof. We first note that if $\bar{\Delta}_i = \bar{\Delta}_k = 0$, then in (35) we round $-u_i$ up to the nearest multiple of $a_{ij}m/\beta$, which is zero because $0 \leq u_i < a_{ij}m/\beta$. Thus (35) reduces to $a_{kj}(f_i + u_i) + a_{ij}g_k \leq 0$.

(a) \Rightarrow (b). We can write the inequalities in (a) as

$$a_{kj}\alpha(f_i + \bar{\Delta}_i) \leq a_{ij}a_{kj}\alpha x_j \leq -a_{ij}\alpha(g_k + \bar{\Delta}_k) \quad (36)$$

Figure 5. Value function $v(\Delta_2)$ for constraint 2.

for all i, k . If we let $y = a_{ij}a_{kj}\alpha x_j$, then we can show as in the proof of Theorem 2 that $d \equiv 0 \pmod{\beta}$, $\lambda_\alpha m + \lambda_m \alpha = \beta$ for some $\lambda_\alpha, \lambda_m \in \mathbb{Z}$, and the congruence relation $y \equiv \lambda_m a_{ij} a_{kj} \alpha d / \beta \pmod{a_{ij} a_{kj} \text{lcm}(\alpha, m)}$ holds. From the congruence relation and (36), we have

$$a_{kj}\alpha(f_i + \bar{\Delta}_i) - \lambda_m a_{ij} a_{kj} \alpha d / \beta \leq \gamma \leq -a_{ij}\alpha(g_k + \bar{\Delta}_k) - \lambda_m a_{ij} a_{kj} \alpha d / \beta \quad (37)$$

where γ is an integer multiple of $a_{ij} a_{kj} \text{lcm}(\alpha, m)$. Since β divides d , the leftmost expression in (37) is an integer multiple of $a_{kj}\alpha$. So we can add the term

$$a_{kj}\alpha(-\bar{\Delta}_i + u_i + \lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta}) \quad (38)$$

to the left-hand side of (37), where the expression $s_i = -\bar{\Delta}_i + u_i + \lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta}$ takes a value in

$$\{0, \dots, a_{ij}a_{kj}\text{lcm}(\alpha, m)/(a_{kj}\alpha) - 1\} = \{0, \dots, a_{ij}m/\beta - 1\}$$

We therefore have

$$a_{kj}\alpha(f_i + u_i + \lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta}) \leq \gamma + \lambda_m a_{ij} a_{kj} \alpha d / \beta \leq -a_{ij}\alpha(g_k + \bar{\Delta}_k) \quad (39)$$

where

$$a_{kj}\alpha(f_i + u_i + \lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta}) - \lambda_m a_{ij} a_{kj} \alpha d / \beta \equiv 0 \pmod{a_{ij} a_{kj} \text{lcm}(\alpha, m)} \quad (40)$$

Inequality (39) implies (35). Congruence (40) simplifies to

$$f_i + u_i + \lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta} - \lambda_m a_{ij} d / \beta \equiv 0 \pmod{a_{ij}m/\beta}$$

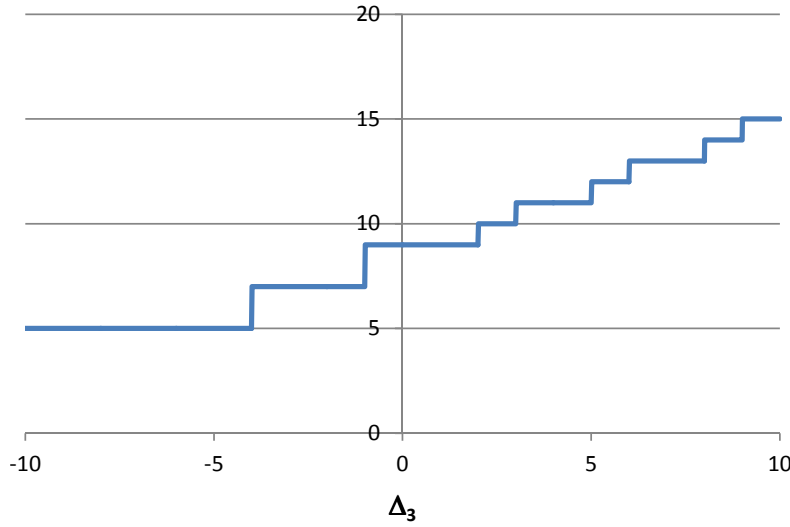
which implies the congruence in (b) because $\lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta}$ is a multiple of $a_{ij}m/\beta$. Finally, $s_i = \text{mod}_{a_{ij}m/\beta}(u_i - \bar{\Delta}_i)$ due to (32). Because we need only consider values $0, \dots, a_{ij}m/\beta - 1$ for s_i , we generate the required values by restricting u_i to $\{0, \dots, a_{ij}m/\beta - 1\}$.

(a) \Leftarrow (b). The inequalities in (b) can be written

$$-\frac{g_k + \bar{\Delta}_k}{a_{kj}} \geq \frac{f_i + u_i + \lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta}}{a_{ij}} \quad (41)$$

for all i, k . From (b) we have that $d \equiv 0 \pmod{\beta}$, so that d/β is integral. Also from (b),

$$f_i + u_i \equiv \lambda_m a_{ij} d / \beta \pmod{a_{ij} a_{kj} m / \beta}$$

Figure 6. Value function $v(\Delta_3)$ for constraint 3.

Because d/β and m/β are integral, this implies $f_i + u_i$ is an integer multiple of a_{ij} . We also have that $\lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta}$ is a multiple of $a_{ij}m/\beta$ and therefore a_{ij} . So we can let

$$x_j = \max_i \left\{ \frac{f_i + u_i + \lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta}}{a_{ij}} \right\}$$

and x_j is integral. This and (41) imply $-(g_k + \bar{\Delta}_k)/a_{kj} \geq x_j$, or $-g_k \geq a_{kj}x_j + \bar{\Delta}_k$. To show that $a_{ij}x_j \geq f_i + \bar{\Delta}_i$, we note that

$$a_{ij}x_j \geq a_{ij} \frac{f_i + u_i + \lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta}}{a_{ij}} \geq f_i + \bar{\Delta}_i$$

because $u_i + \lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta} = \bar{\Delta}_i + \text{mod}_{a_{ij}m/\beta}(u_i - \bar{\Delta}_i) \geq \bar{\Delta}_i$ due to (32). Finally, it can be shown as in the proof of Theorem 2 that $\alpha x_j \equiv d \pmod{m}$. \square

We note that the above procedure for computing an IP value function computes an LP value function for the same set of inequalities when the u_i s and roundings are removed. For example, the IP value function (34) becomes the LP value function (4).

8. Economic Interpretation

It is unclear how to derive marginal shadow prices from an IP value function, because it is a step function. However, the value function assumes a regular pattern for sufficiently large perturbations in a given right-hand side. In particular, it becomes a shift periodic function whose ‘‘average’’ slope can be interpreted as an eventual shadow price. In fact, this price is equal to the eventual shadow price in the linear relaxation. Fortunately, the periodicity of the value function is the same in all scenarios and can therefore be obtained without solving congruence relations to identify the possible scenarios.

This analysis begins with the fact that Chvátal functions are shift-periodic functions [9]. A function $f : \mathbb{Q} \rightarrow \mathbb{Q}$ is (p, q) -shift-periodic on a domain $D \subseteq \mathbb{Q}$, where p and q are positive integers, if for each $x \in D$ and each integer λ , $f(x + \lambda p) = f(x) + \lambda q$. The pair (p, q) is the *periodicity* of f . Thus one might view q/p as the ‘‘average slope’’ of f in domain D . It is shown in [9] that Chvátal functions are shift-periodic on \mathbb{Q} in each argument.

The periodicity of a Chvátal function can be computed using a few simple rules. Let a and b ($\neq 0$) be integers. It is easy to show that if f is (p, q) -shift-periodic, then $(a/b)f$ is (bp, aq) -shift-periodic, and $\lceil f \rceil_m$ is (mp, mq) -shift-periodic. Obviously, $f + \alpha$ is (p, q) -shift-periodic for any constant α . It is shown in [9] that if f_1 and f_2 are shift-periodic with periodicities (p_1, q_1) and (p_2, q_2) , respectively, then the rational linear combination $(a_1/b_1)f_1 + (a_2/b_2)f_2$ is shift-periodic with periodicity $(b_1b_2p_1p_2, a_1b_2p_2q_1 + a_2b_1p_1q_2)$. Finally, note that if f has periodicity (p, q) , then it also has periodicity (mp, mq) for any integer $m > 0$. So a shift-periodic function has infinitely many periodicities with the same average slope.

Gomory functions are not shift periodic in general but become shift-periodic for sufficiently large arguments. A function $f : \mathbb{Q} \rightarrow \mathbb{Q}$ is *eventually (p, q) -shift-periodic* in a positive direction on domain D if there exists positive integer r such that for each $x \in D$ and each integer λ , $f(x + rp + \lambda p) = f(x + rp) + \lambda q$. It is eventually (p, q) -shift-periodic in a negative direction if there exists positive integer r such that for each $x \in D$ and each integer λ , $f(x - rp - \lambda p) = f(x - rp) - \lambda q$.

It is also shown in [9] that if a finite family of functions f_i are (p_i, q_i) -shift-periodic on D for $i \in I$, then $\max_{i \in I}\{f_i\}$ is eventually (p_j, q_j) -shift-periodic in a positive direction on $[d, \infty) \cap D$ for any finite d , where $p_j/q_j = \max_{i \in I}\{p_i/q_i\}$. In addition, $\max_{i \in I}\{f_i\}$ is eventually (p_j, q_j) -shift-periodic in a negative direction on $(-\infty, d'] \cap D$ for any d' , where $p_j/q_j = \min_{i \in I}\{p_i/q_i\}$. It follows that if the functions f_i are Chvátal functions, then the Gomory function $\max_{i \in I}\{f_i\}$ is eventually (p_j, q_j) -shift-periodic on $[d, \infty)$ in a positive direction, and eventually (p_j, q_j) -shift-periodic on $(-\infty, d']$ in a negative direction.

As an example, we can use these rules to deduce the periodicity of the value function (34) for any scenario (u_1, u_{13}) . We first note that the periodicity is the same for any scenario, because u_1 and u_{13} appear only as additive constants in the Chvátal functions. Consider first the value function $v(\Delta_1, 0, 0)$ for constraint C1. For any given scenario, it is a Gomory function equal to the maximum of two Chvátal functions, which have periodicities $(2, 10)$ and $(12, 4)$, respectively. This Gomory function is eventually $(2, 10)$ -shift-periodic in the positive direction and $(12, 4)$ -shift-periodic in the negative direction. This means that the “average” shadow price is $10/2 = 5$ for sufficiently large right-hand side, and $4/12 = 1/3$ for sufficiently small right-hand side. Similarly, the average shadow price for C2 is $2/1 = 2$ in the positive direction and $0/1 = 1$ in the negative direction, and for C3 is $2/3$ in the positive direction and 0 in the negative direction.

Finally, we show that the minimum of finitely many Gomory functions with the same periodicity is eventually shift periodic, which implies that the value function v is eventually shift periodic.

Lemma 2. *Let f_i be eventually (p, q) -shift-periodic in the positive direction on $[d, \infty)$ for $i \in I$ and finite set I , and eventually (p', q') -shift-periodic in the negative direction on $(-\infty, d']$ for $i \in I$. Then $\min_{i \in I}\{f_i\}$ is eventually (p, q) -shift-periodic in the positive direction on $[d, \infty)$ and eventually (p', q') -shift-periodic in the negative direction on $(-\infty, d']$.*

Proof. Because f_i is eventually (p, q) -shift periodic on $[d, \infty)$ in the positive direction, there exists a positive integer r_i such that for any $x \in [d, \infty)$,

$$f_i(x + \lambda r_i + \lambda p) = f_i(x + \lambda r_i) + \lambda q \quad (42)$$

Let $r_{\max} = \max_{i \in I}\{r_i\}$. To show that $\min_{i \in I}\{f_i\}$ is eventually (p, q) -shift-periodic on $[d, \infty)$ in the positive direction, it suffices to show that for any $x \in [d, \infty)$,

$$\min_{i \in I}\{f_i(x + r_{\max}p + \lambda p)\} = \min_{i \in I}\{f_i(x + r_{\max}p)\} + \lambda q$$

For this it suffices to show that for any $i \in I$ and any $x \in [d, \infty)$,

$$f_i(x + r_{\max}p + \lambda p) = f_i(x + r_{\max}p) + \lambda q \quad (43)$$

But we have

$$\begin{aligned} f_i(x + r_{\max}p + \lambda p) &= f_i(x + r_i p + (r_{\max} - r_i)p + \lambda p) \\ &= f_i(x + r_i p) + (r_{\max} - r_i)q + \lambda q = f_i(x + r_i p + (r_{\max} - r_i)p) + \lambda q \end{aligned}$$

where the second and third equations are due to (42). This implies (43), as desired. One can show similarly that $\min_{i \in I}\{f_i\}$ is eventually (p', q') -shift-periodic in the negative direction. \square

Thus the value function $v(\Delta_1, 0, 0)$ is $(2, 10)$ -shift-periodic in the positive direction and $(12, 4)$ -shift-periodic in the negative direction. The “average” slope eventually becomes 5 in the positive direction and $1/3$ in the negative direction, as one can verify by examining Fig. 4. The situation is similar for value functions $v(0, \Delta_2, 0)$ and $v(0, 0, \Delta_3)$.

These eventual shadow prices are the same as those derived earlier for the linear case. This follows from the fact that the LP value function has a shift-periodicity equal to that of the IP value function for each perturbation Δ_i . To see this, recall that the LP value function can be obtained from the IP value function by removing u_i s and roundings. Thus we can compute a periodicity of the LP value function following the same steps as for the IP value function, except that constants u_i are not added, and roundings $\lceil f \rceil_m$ are not taken. But additive constants do not affect periodicity. Also, $\lceil f \rceil_m$ is (mp, mq) -shift-periodic if f is (p, q) -shift-periodic. But f is also (mp, mq) -shift-periodic and therefore has a shift-periodicity equal to that of $\lceil f \rceil_m$. The Chvátal functions that make up the IP value function therefore have shift-periodicities equal to those of the corresponding components of the LP value function. But the Chvátal function that dominates as Δ_i increases (or decreases) depends only on the coefficients of Δ_i , and these coefficients are not affected by removing u_i s and roundings. The LP value function therefore has an eventual shift-periodicity equal to that of the IP value function, resulting in the same eventual shadow prices.

The equivalence of IP and LP eventual shadow prices also follows from the analyses in [4, 5]. For sufficiently large right-hand sides, one of the component Chvátal functions in the classical IP value function becomes dominant, and the relaxation of this same Chvátal function becomes dominant in the LP value function. The IP becomes an IP over the cone defined by the binding constraints in the LP relaxation. This is sometimes referred to as an “asymptotic IP.” As a consequence, the eventual shadow prices for the IP are the same as those for the LP relaxation.

9. Conclusion

We presented an alternative perspective on integer programming (IP) that is suggested by projection and parallels the classical theory. We showed that if IP is generalised to include optimisation over sublattices of the integer lattice, then the projection of a generalised IP is itself a generalised IP. This perspective inspires the definition of a family of congruence cuts that are parallel to Chvátal-Gomory cuts in some respects, but whose maximum rank is bounded by the number of variables in the original IP. It also leads to a new branching algorithm in which the depth of the tree is likewise bounded by the number of variables, in contrast to conventional IP branch-and-bound methods, where there is no bound. Finally, it produces a value function for generalised IP that differs in structure from value functions generated by Gomory’s method, and in which the maximum nesting depth of rounding is bounded by the number of variables. The value function becomes shift periodic and provides “average” shadow prices for large perturbations in the right-hand sides.

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