# Integrating Optimization and Constraint Satisfaction

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*Some work joint with…*

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# Integration of optimization and constraint satisfaction

- Optimization and constraint satisfaction have complementary strengths.
- There is much interest in combining them, but their different origins have impeded unification.
  - Optimization -- operations research, mathematics
  - Constraint satisfaction -- artificial intelligence, computer science
- This barrier is now being overcome, but there is no generally accepted scheme for unification.

# Complementary strengths

- Problem types.
  - Optimization excels at loosely-constrained problems in which find the best solution is the primary task.
  - Constraint satisfaction is more effective on tightly-constrained problems in which finding a feasible solution is paramount.

# Complementary strengths

- Exploiting structure
  - Optimization relies on deep analysis of the mathematical structure of specific classes of problems, particularly polyhedral analysis, which yields strong cutting planes.
  - Constraint satisfaction identifies subsets of problem constraints that have special structure (e.g., `all-different, cumulative`) and apply tailor-made domain-reduction algorithms.

# Complementary strengths

- Relaxation and inference.
  - Optimization creates strong relaxations with cutting planes, Lagrangean relaxation, etc. These provide bounds on the optimal value.
  - Constraint satisfaction exploits the power of inference, especially in domain reduction algorithms. This reduces the search space.

# Complementary strengths

- Modeling style
  - Optimization uses declarative models that can be solved with a variety of algorithms. But the language is highly restricted (e.g, inequality constraints).
  - Constraint satisfaction models are formulated in a procedural or quasi-procedural manner that gives the user more opportunity to direct the solution algorithm. But the model is tied to the solution method.

# Procedural vs. declarative

- The issue of procedural vs. declarative modeling is orthogonal to the issue of how solution methods can be combined.
  - Constraint (logic) programming generally implements constraint satisfaction techniques in a quasi-procedural manner. But these techniques can equally well be applied to a declarative model, and the flexible modeling language of C(L)P can be used declaratively (OPL moves in this direction).
  - Optimization methods are generally applied to declarative models, but they can be implemented in a high-level programming language as well.

# Constraint programming vs. constraint satisfaction

- Constraint programming is sometimes identified with different techniques than constraint satisfaction.
  - For example, domain reduction (arc or hyperarc consistency) rather than minimum-width search orders, adaptive consistency, etc.
- Here, constraint programming is viewed as a modeling approach. Constraint satisfaction techniques include all solution techniques implemented by CP or CLP.

# Scheme for unifying optimization and constraint satisfaction methods

- Both optimization and constraint satisfaction rely on two fundamental dualities.
- Search/inference
    - Search = branching, local search
    - Inference = constraint propagation, cutting planes
- Strengthening/relaxation
    - Strengthening = fix variables or restrict domains
    - Relaxation = weaken constraints, bound objective function

# Scheme for unifying optimization and constraint satisfaction methods

- Rather than use optimization or constraint satisfaction methods exclusively, focus on how these two dualities can be exploited in a given problem.
- The resulting algorithm is likely to contain elements from both optimization and constraint satisfaction, and perhaps new methods that belong to neither.
- In particular, optimization can benefit from inference methods of constraint satisfaction.
- Constraint satisfaction can benefit from relaxation methods of optimization.

# Outline

- A motivating example
  - Constraint satisfaction approach
  - Integer programming approach
  - Combined approach
- The search/inference duality
  - Complementary solution methods
  - A formal duality
- The strengthening/relaxation duality
  - Complementary solution methods
  - Relaxations for global constraints
  - Formal relaxation duality
- Relaxation duality
  - An integer programming example
  - Continuous relaxation
  - Discrete relaxation
  - *dependency graph, nonserial dynamic programming*
  - Relaxation duality
  - Lagrangean & surrogate duals
  - Discrete relax + Lagrangean dual
  - Discrete relaxation dual
  - Discrete + Lagrangean duals
  - Summary of relaxations
- Research agenda

# A motivating example

$$\begin{aligned} \min \quad & 4x_1 + 3x_2 + 5x_3 \\ \text{subject to} \quad & 4x_1 + 2x_2 + 4x_3 \geq 17 \\ & \text{all-different}\{x_1, x_2, x_3\} \\ & x_j \in \{1, \ldots, 5\} \end{aligned}$$

Formulate and solve 3 ways:

- a constraint satisfaction problem
- an integer programming problem
- a combined approach

# Solve as a constraint satisfaction problem

$$
\begin{aligned}
&4x_1 + 3x_2 + 5x_3 \le z \\
&4x_1 + 2x_2 + 4x_3 \ge 17 \\
&\text{all-different}\{x_1, x_2, x_3\} \\
&x_j \in \{1, \ldots, 5\}
\end{aligned}
$$

Start with z = ∞.
Will decrease as feasible solutions are found.

# Domain reduction used

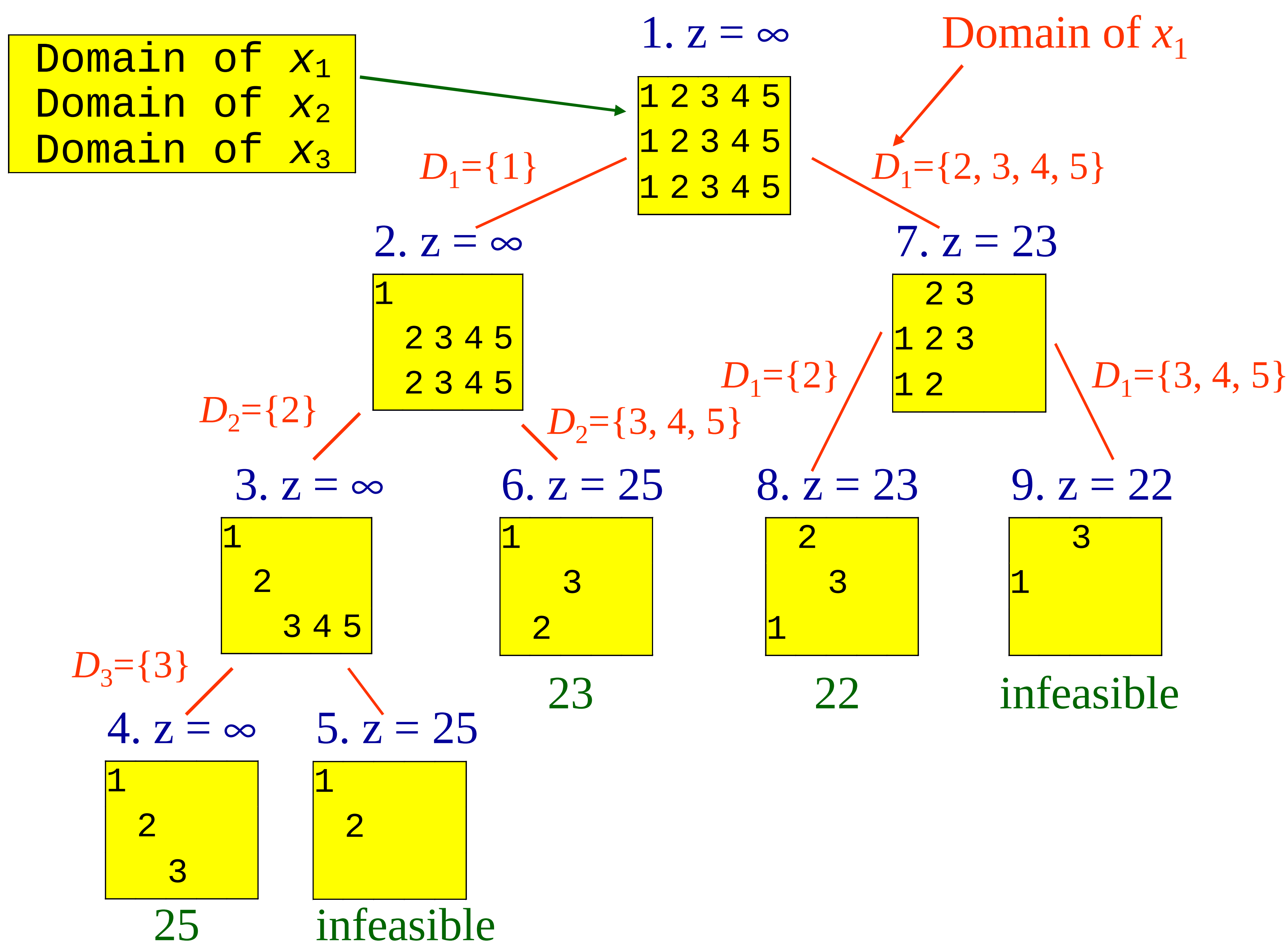
- Bounds propagation on

$$4x_1 + 3x_2 + 5x_3 \le z$$
$$4x_1 + 2x_2 + 4x_3 \ge 17$$

- Maintain hyperarc consistency on

$$\text{all-different}\{x_1, x_2, x_3\}$$

- Cycle through domain reductions until a fixed point is obtained



Domain of $x_1$
Domain of $x_2$
Domain of $x_3$
1. z = ∞
1 2 3 4 5
1 2 3 4 5
1 2 3 4 5
Domain of $x_1$
$D_1$={1}
$D_1$={2, 3, 4, 5}
2. z = ∞
7. z = 23
$D_2$={2}
$D_2$={3, 4, 5}
$D_1$={2}
$D_1$={3, 4, 5}
3. z = ∞
6. z = 25
8. z = 23
9. z = 22
23
22
infeasible
$D_3$={3}
4. z = ∞
5. z = 25
25
infeasible

# Solve as an integer programming problem

$$
\begin{aligned}
\text{min} \quad & 4x_1 + 3x_2 + 5x_3 \\
\text{subject to} \quad & 4x_1 + 2x_2 + 4x_3 \geq 17 \\
& x_j \leq (x_k - 1) + 5(1 - y_{jk}), \quad \text{all } j,k \text{ with } j < k \\
& x_k \leq (x_j - 1) + 5y_{jk}, \quad \text{all } j,k \text{ with } j < k \\
& 1 \leq x_j \leq 5, \quad x_j \text{ integer}, \ \ j = 1,2,3 \\
& y_{jk} \in \{0,1\}, \quad \text{all } j,k \text{ with } j < k
\end{aligned}
$$

Big-M constraints (the two constraints above)

$x_j < x_k$ if $y_{jk} = 1$

# Linear relaxation

Use a linear programming algorithm to solve a continuous relaxation of the problem at each node of the search tree to obtain a lower bound on the optimal value of the problem at that node.

$$
\begin{aligned}
\text{min} \quad & 4x_1 + 3x_2 + 5x_3 \\
\text{subject to} \quad & 4x_1 + 2x_2 + 4x_3 \geq 17 \\
& x_j \leq (x_k - 1) + 5(1 - y_{jk}), \quad \text{all } j,k \text{ with } j < k \\
& x_k \leq (x_j - 1) + 5y_{jk}, \quad \text{all } j,k \text{ with } j < k \\
& \left.\begin{aligned} & 1 \leq x_j \leq 5, \quad j = 1,2,3 \\ & 0 \leq y_{jk} \leq 1, \quad \text{all } j,k \text{ with } j < k \end{aligned}\right\} \text{Relax integrality}
\end{aligned}
$$

# Alternate model

The following model has a better relaxation and would be used for this problem in practice. The big-*M* construction is used here to illustrate a popular and general technique.

$$
\begin{aligned}
\text{min} \quad & 4x_1 + 3x_2 + 5x_3 \\
\text{subject to} \quad & 4x_1 + 2x_2 + 4x_3 \geq 17 \\
& x_i = \sum_{j=1}^{5} j y_{ij}, \quad i = 1,2,3 \\
& \sum_{j=1}^{5} y_{ij} = 1, \quad i = 1,2,3 \\
& y_{jk} \in \{0,1\}, \quad \text{all } j,k
\end{aligned}
$$

# Cutting planes

Infer the cutting planes

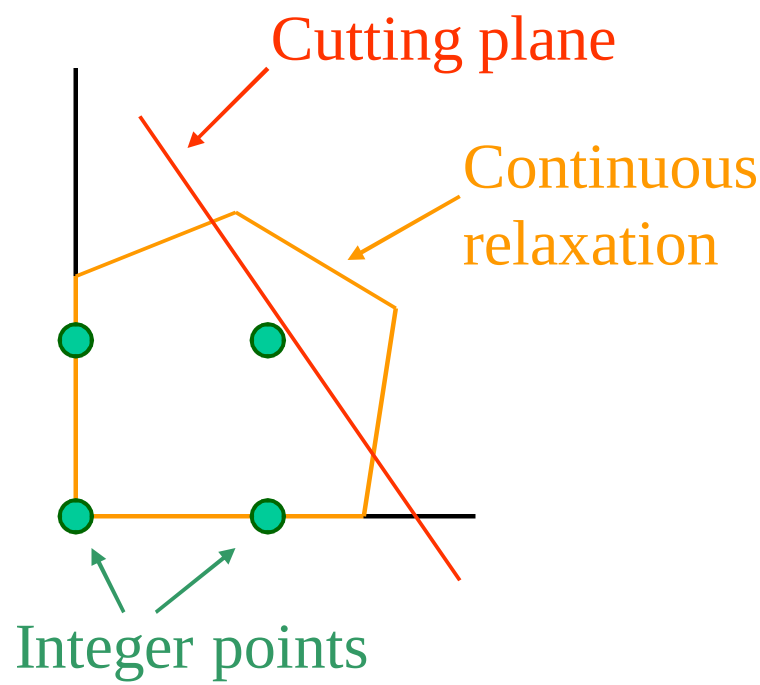
$$x_1 + x_2 + x_3 \geq 5$$

$$2x_1 + x_2 + 2x_3 \geq 9$$

From the inequality

$$4x_1 + 2x_2 + 4x_3 \geq 17$$

The cutting plane is implied by the inequality but strengthens the continuous relaxation




(One could also use the all-different constraint to obtain the stronger cutting plane $x_1 + x_2 + x_3 \geq 6$ )

# Branch and bound

The *incumbent solution* is the best feasible solution found so far.

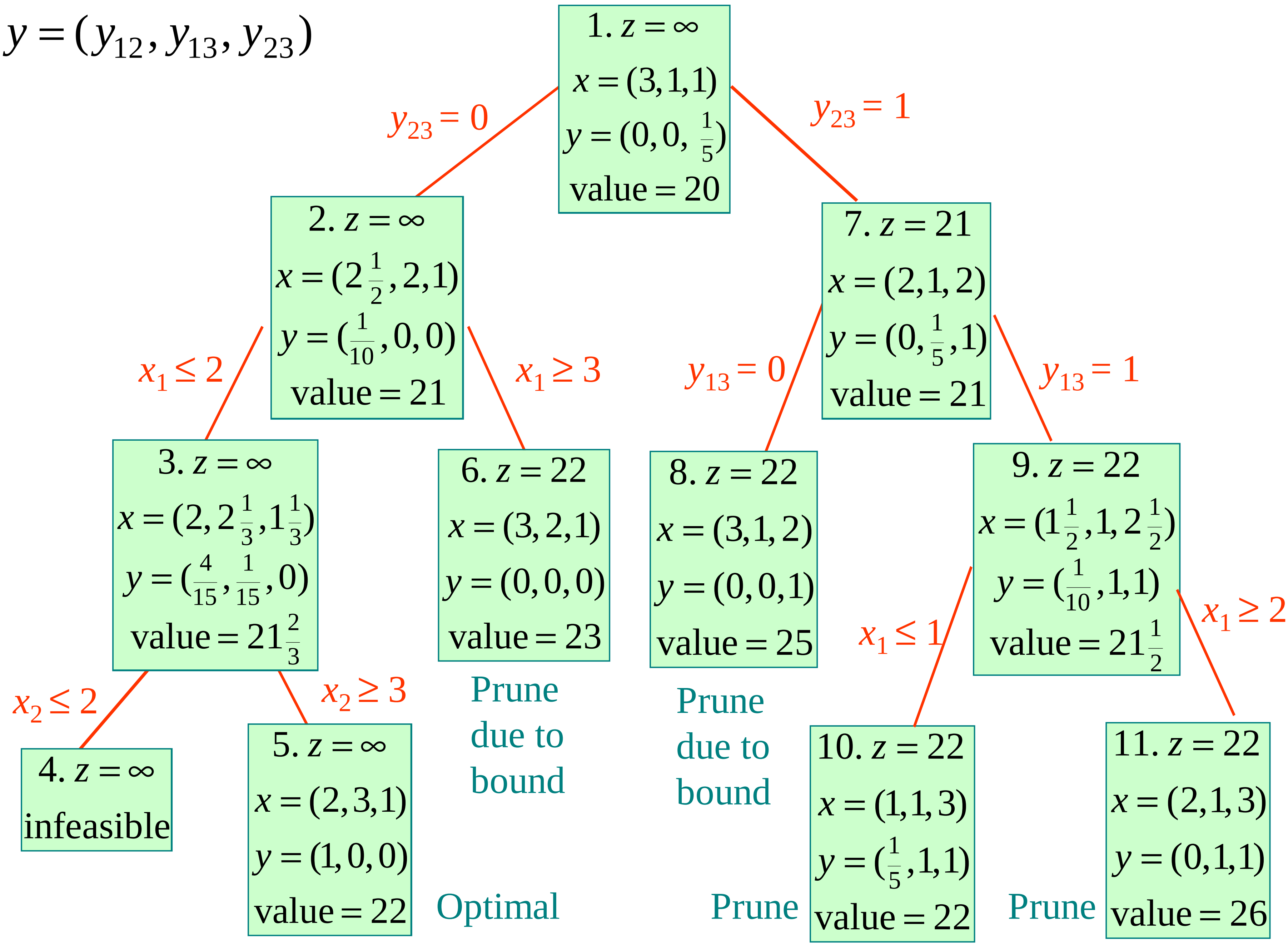
At each node of the branching tree:

- If Optimal value of relaxation $\geq$ Value of incumbent solution

There is no need to branch further.

- No feasible solution in that subtree can be better than the incumbent solution.

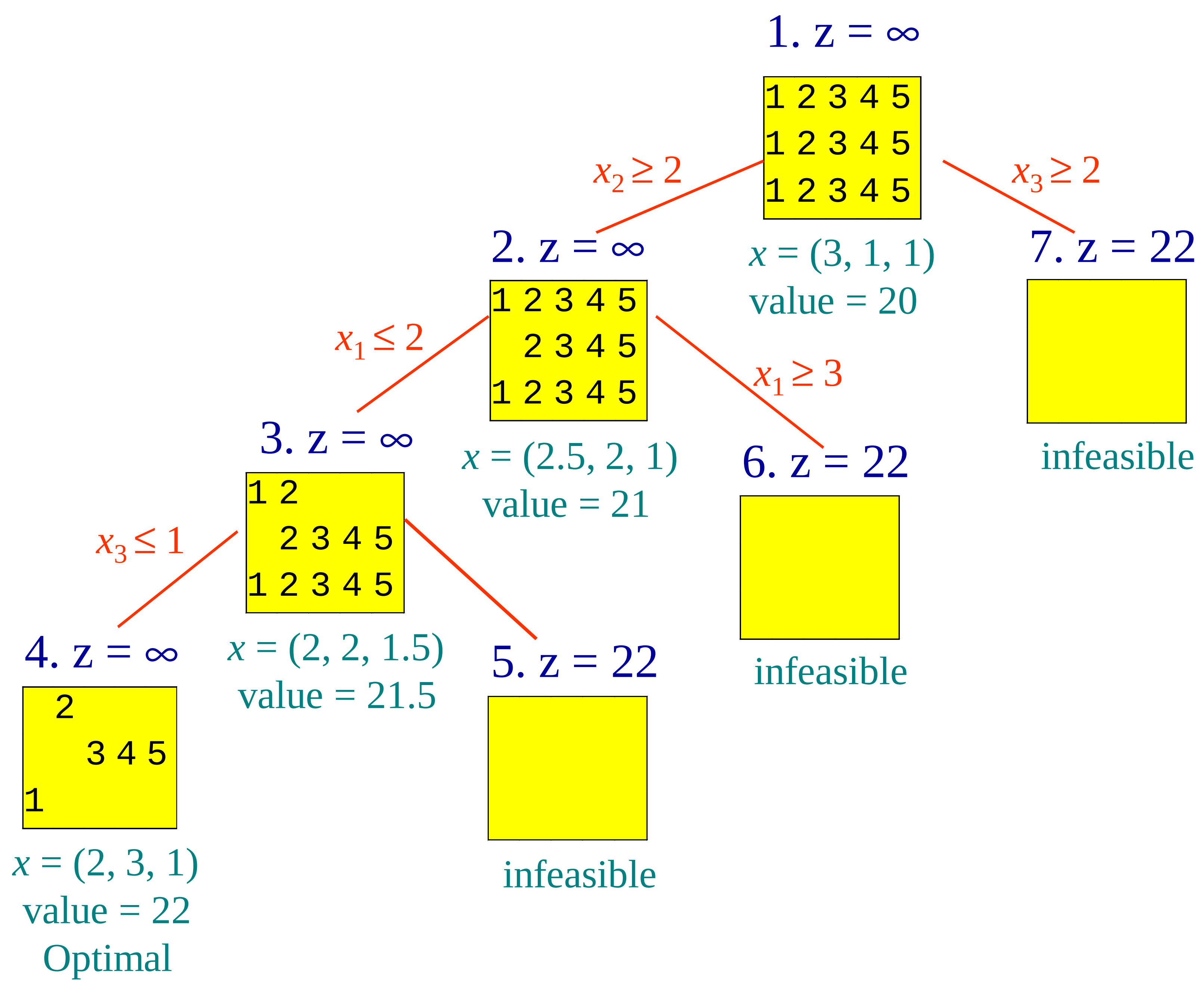
$$y = (y_{12}, y_{13}, y_{23})$$



1. z = ∞
x = (3,1,1)
y = (0,0,1/5)
value = 20
y23 = 0
y23 = 1
2. z = ∞
x = (2 1/2, 2, 1)
y = (1/10, 0, 0)
value = 21
7. z = 21
x = (2,1,2)
y = (0, 1/5, 1)
value = 21
x1 ≤ 2
x1 ≥ 3
y13 = 0
y13 = 1
3. z = ∞
x = (2, 2 1/3, 1 1/3)
y = (4/15, 1/15, 0)
value = 21 2/3
6. z = 22
x = (3,2,1)
y = (0,0,0)
value = 23
8. z = 22
x = (3,1,2)
y = (0,0,1)
value = 25
9. z = 22
x = (1 1/2, 1, 2 1/2)
y = (1/10, 1, 1)
value = 21 1/2
x2 ≤ 2
x2 ≥ 3
x1 ≤ 1
x1 ≥ 2
4. z = ∞
infeasible
5. z = ∞
x = (2,3,1)
y = (1,0,0)
value = 22
Optimal
Prune due to bound
Prune due to bound
10. z = 22
x = (1,1,3)
y = (1/5, 1, 1)
value = 22
Prune
11. z = 22
x = (2,1,3)
y = (0,1,1)
value = 26
Prune

# Combined approach

- Use continuous relaxation with cutting planes but without big-M constraints
  - Because relaxation is distinguished from model, both are succinct.
- Use bounds propagation on cutting planes as well as original inequality constraints.
- Maintain hyperarc consistency for all-different.
- Branch on nonintegral variable when possible; otherwise branch by splitting domain.


1. z = ∞
1 2 3 4 5
1 2 3 4 5
1 2 3 4 5
x = (3, 1, 1)
value = 20
x2 ≥ 2
x3 ≥ 2
2. z = ∞
1 2 3 4 5
2 3 4 5
1 2 3 4 5
x = (2.5, 2, 1)
value = 21
7. z = 22
infeasible
x1 ≤ 2
x1 ≥ 3
3. z = ∞
1 2
2 3 4 5
1 2 3 4 5
x = (2, 2, 1.5)
value = 21.5
6. z = 22
infeasible
x3 ≤ 1
4. z = ∞
2
3 4 5
1
x = (2, 3, 1)
value = 22
Optimal
5. z = 22
infeasible

# Summary of dualities

| | *Search* | *Inference* | *Strength-ening* | *Relaxation* |
|---|---|---|---|---|
| *Constraint satisfac.* | Domain splitting | Bounds prop. On ineq. cons. Domain red. on all-diff | Domain splitting | Reduced domains |
| *Integer program-ing* | Branching on fractions | Cutting planes | Branching on fractions | Continuous relaxation of IP model |
| *Combined* | Both of the above | Bounds prop. on ineq. cons. & cutting planes Domain red. on all-diff | Both of the above | Reduced domains Continuous relaxation of part of IP model |

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  - A formal duality
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Relaxation duality

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- Continuous relaxation
- Discrete relaxation
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Research agenda

# The search/inference duality

- Two interpretations:
  - Complementary solution methods that can work together.
  - A formal mathematical duality that can lead to new methods.

# The search/inference duality

- Complementary solution methods:
  - Search alone may find a good solution early, but it must examine many other solutions to determine that it is good.
  - Inference can rule out families of inferior solutions, but this is not the same as finding a good solution.
  - Working together, search & inference can find and verify good solutions more quickly.

# The search/inference duality

- A formal duality:
  - Search and inference are related by a formal optimization duality
  - Linear programming duality is a special case.
  - This provides a general method for sensitivity analysis.
  - It also provides a general form of Benders decomposition, which is closely related to the use of nogoods.

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# The strengthening/relaxation duality

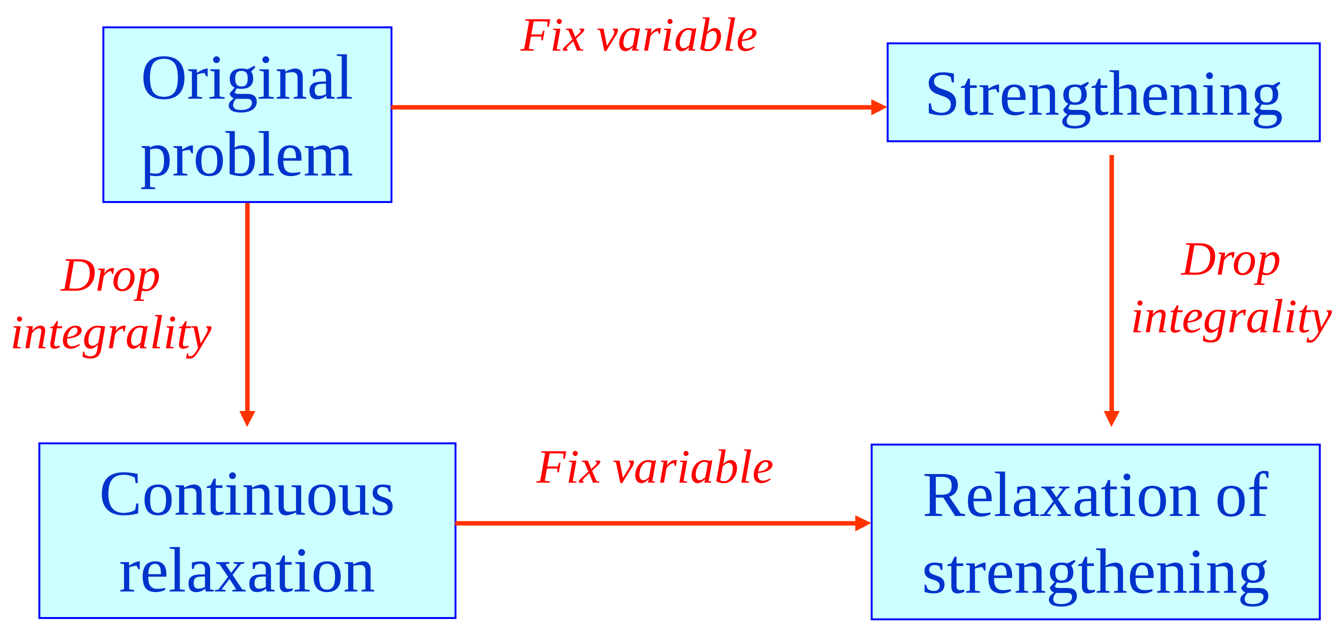
- Three interpretations:
  - Complementary solution methods that can work together.
  - Creation of relaxations as well as domain reduction algorithms to exploit structure of subsets of constraints (e.g., element constraints).
  - A formal mathematical duality that can lead to new relaxations, particularly for constraint satisfaction models.

# The strengthening/relaxation duality

- Complementary solution methods
  - Branch-and-bound solves relaxations of strengthenings. Branching creates strengthenings, and one solves a relaxation of each to obtain bounds.
  - There are other ways strengthening and relaxation can relate. One can solve strengthenings of a relaxation. Branching creates strengthenings of an initial relaxation.

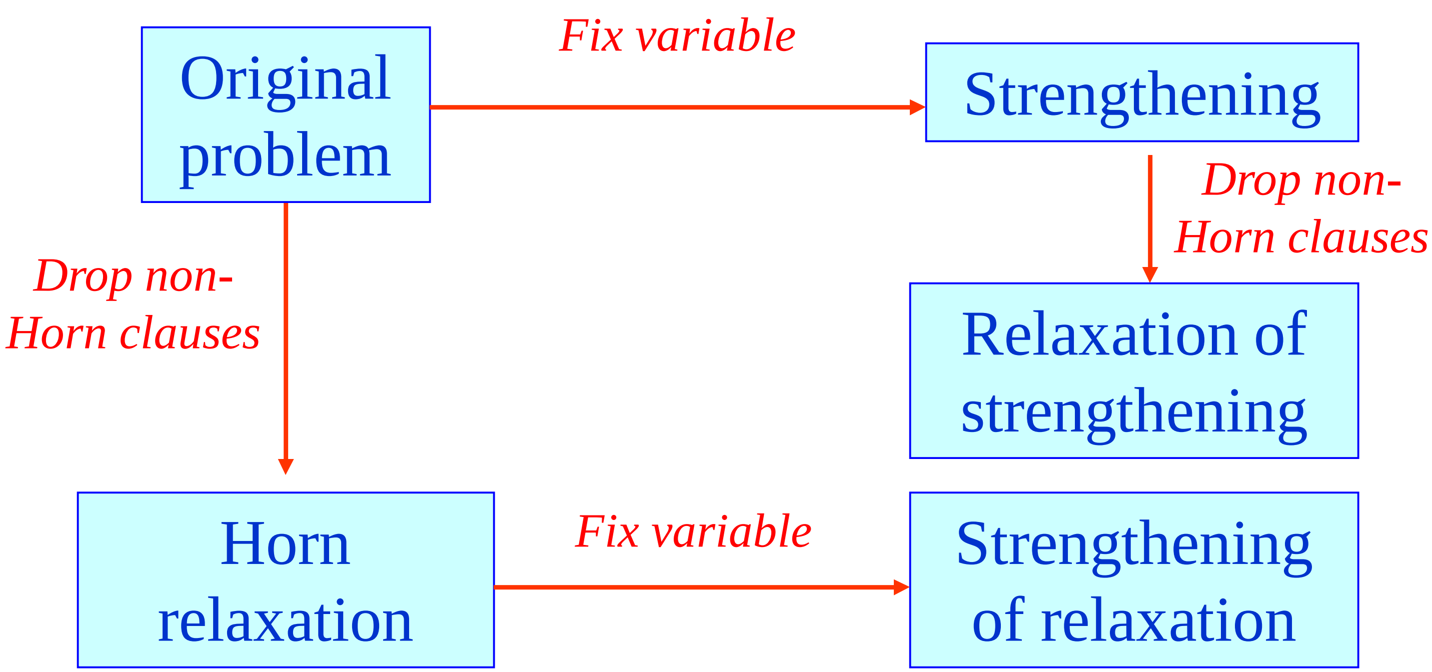
# Relaxations of strengthenings vs. strengthenings of a relaxation

They are the same in integer programming because the strengthening and relaxation functions commute:

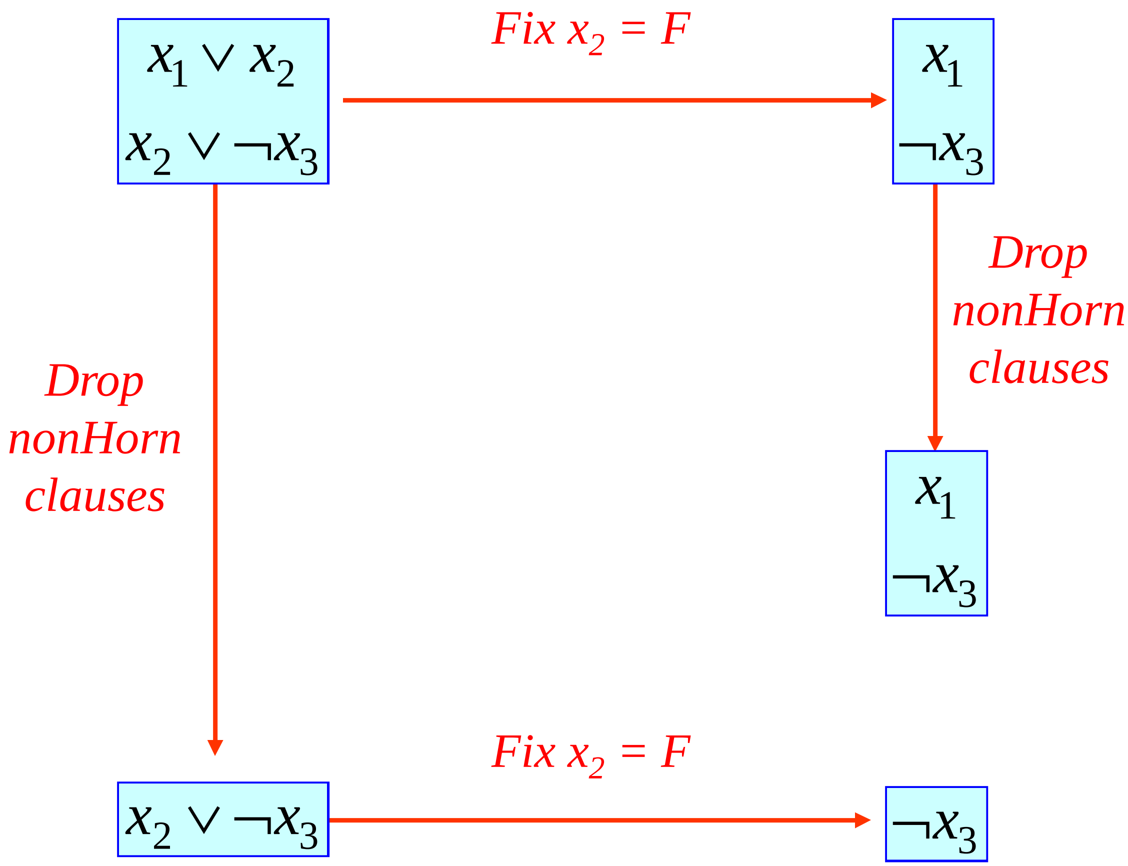


# Relaxations of strengthenings vs. strengthenings of a relaxation

They are not the same in general; for example, when solving a propositional satisfiability problem with the help of Horn relaxation.  Diagram does not commute.



# Example...



$x_1 \vee x_2$
$x_2 \vee \neg x_3$
Fix $x_2 = F$
$x_1$
$\neg x_3$
Drop nonHorn clauses
$x_1$
$\neg x_3$
Drop nonHorn clauses
$x_2 \vee \neg x_3$
Fix $x_2 = F$
$\neg x_3$

# The strengthening/relaxation duality

- Another interpretation: create relaxations as well as domain reduction algorithms for specially structured "global" constraints.
    - This will be illustrated with the element constraint, which can be used to implement variable subscripts (indices).

# Discrete variable with variable index

The constraint $x_y \leq \beta$ where $x_j \in D_{xj}$ , $y \in D_y$ are discrete variables, can be implemented:

$$z \leq \beta$$
$$\text{element}(y, (x_1, \ldots, x_n), z)$$

Here, element is processed with a discrete domain reduction algorithm that maintains hvperarc consistency.

$$D_z \leftarrow D_z \cap \bigcup_{j \in D_y} D_{x_j}$$
$$D_y \leftarrow D_y \cap \{ j \mid D_x \cap D_{x_j} \neq \emptyset \}$$
$$D_{x_j} \leftarrow \begin{cases} D_z & \text{if } D_y = \{j\} \\ D_{x_j} & \text{otherwise} \end{cases}$$

Example... $\text{element}(y, (x_1, x_2, x_3, x_4), z)$

The initial domains are:

$$D_z = \{20,30,60,80,90\}$$
$$D_y = \{1,3,4\}$$
$$D_{x_1} = \{10,50\}$$
$$D_{x_2} = \{10,20\}$$
$$D_{x_3} = \{40,50,80,90\}$$
$$D_{x_4} = \{40,50,70\}$$

The reduced domains are:

$$D_z = \{80,90\}$$
$$D_y = \{3\}$$
$$D_{x_1} = \{10,50\}$$
$$D_{x_2} = \{10,20\}$$
$$D_{x_3} = \{80,90\}$$
$$D_{x_4} = \{40,50,70\}$$

# Continuous variable with variable index

The constraint $x_y \leq \beta$ where each $x_j$ $(0 \leq x_j \leq m_j)$ is a continuous variable, can be implemented: $z \leq \beta$

$$\text{element}(y, (x_1, \ldots, x_n), z)$$

Here, element generates a continuous relaxation that is added to the linear programming subproblem:

$$\sum_{i \in D_y} \frac{x_i}{m_i} - \left( \sum_{i \in D_y} \frac{1}{m_i} \right) z \geq -|D_y| + 1$$

$$-\sum_{i \in D_y} \frac{x_i}{m_i} + \left( \sum_{i \in D_y} \frac{1}{m_i} \right) z \geq -|D_y| + 1$$

Example...

$$\text{element}(y, (x_1, x_2), z)$$
$$0 \le x_1 \le 4$$
$$0 \le x_2 \le 5$$

The relaxation is:

$$5x_1 + 4x_2 - 9z \le 20$$
$$5x_1 + 4x_2 - 9z \ge -20$$
$$0 \le x_1 \le 4$$
$$0 \le x_2 \le 4$$

# The strengthening/relaxation duality

- Can be interpreted as a formal relaxation duality.
  - Linear programming duality, Lagrangean duality, surrogate duality are special cases.
  - These classical dualities apply only to numeric equality and inequality constraints.
  - General relaxation duality can be used to create new relaxations for other constraints.
  - One approach is to use the concept of induced width of a dependency graph, along with nonserial dynamic programming.

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# Integer programming example

$$
\begin{array}{ll}
\text{min} & 35x_1 + 20x_2 + 15x_3 + 40x_4 + 30x_5 \\
\text{subject to} & 2x_1 + 3x_2 + x_3 \geq 4 \\
 & x_2 + 4x_3 + 3x_4 \geq 3 \\
 & 2x_3 + 3x_4 + x_5 \geq 4 \\
 & 3x_1 + 4x_3 + 5x_5 \geq 5 \\
 & x_j \in \{0,1\}, \quad \text{all } j
\end{array}
$$

Optimal value = 105

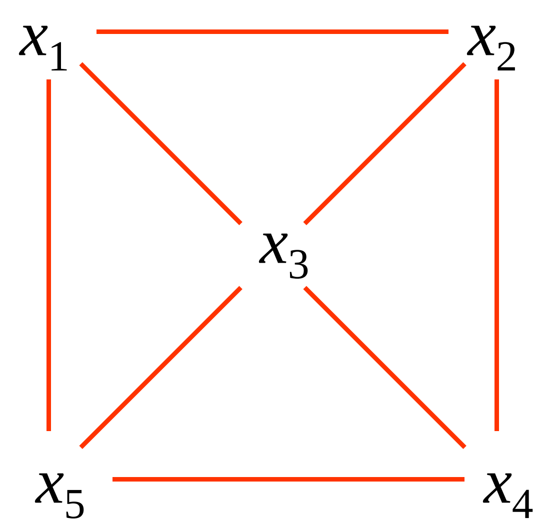
# Continuous relaxation

$$\begin{aligned}
\text{min} \quad & 35x_1 + 20x_2 + 15x_3 + 40x_4 + 30x_5 \\
\text{subject to} \quad & 2x_1 + 3x_2 + x_3 \geq 4 \\
& x_2 + 4x_3 + 3x_4 \geq 3 \\
& 2x_3 + 3x_4 + x_5 \geq 4 \\
& 3x_1 + 4x_3 + 5x_5 \geq 5 \\
& 0 \leq x_j \leq 1, \quad \text{all } j
\end{aligned}$$

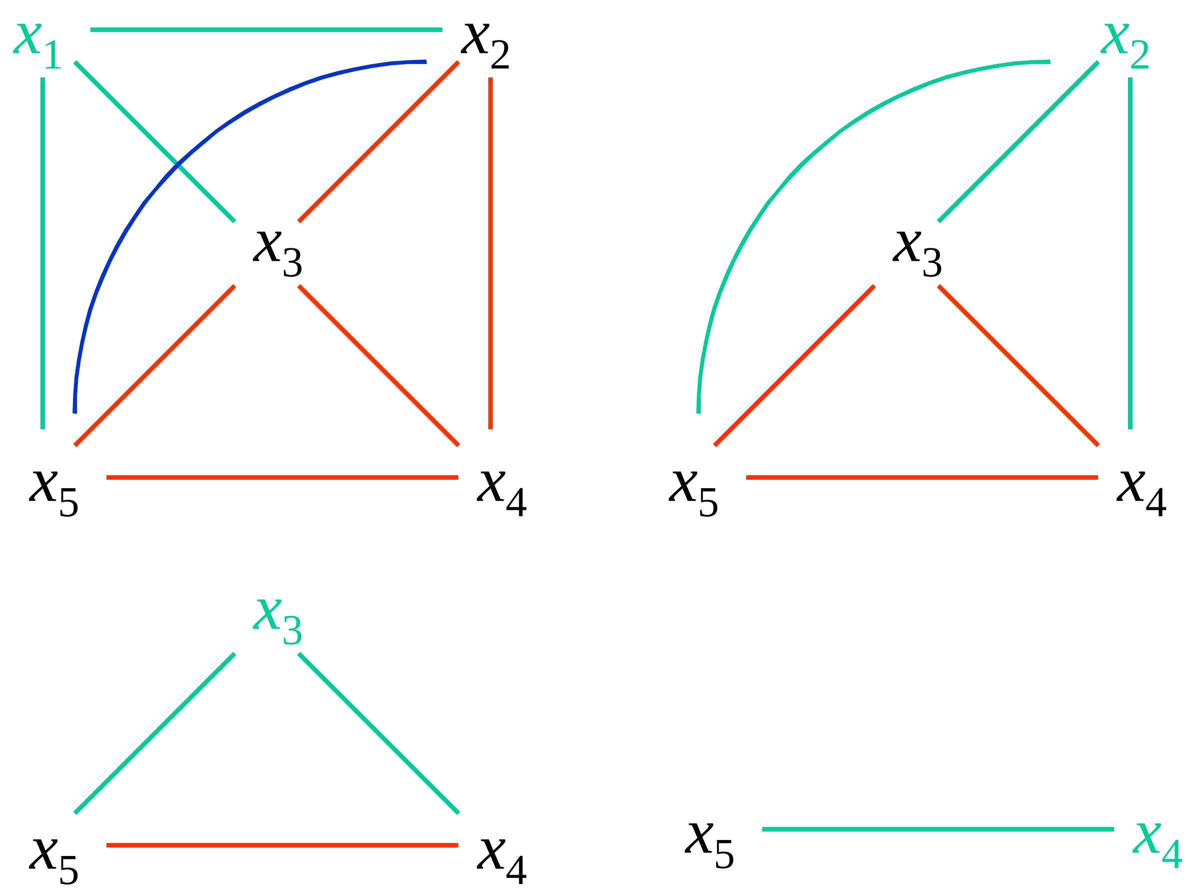
Optimal value = $43\frac{1}{3}$

# Dependency graph

$$2x_1 + 3x_2 - x_3 \geq 4$$
$$x_2 + 4x_3 + 3x_4 \geq 3$$
$$2x_3 + 3x_4 + x_5 \geq 4$$
$$3x_1 + 4x_3 + 5x_5 \geq 5$$



# Induced width = 3

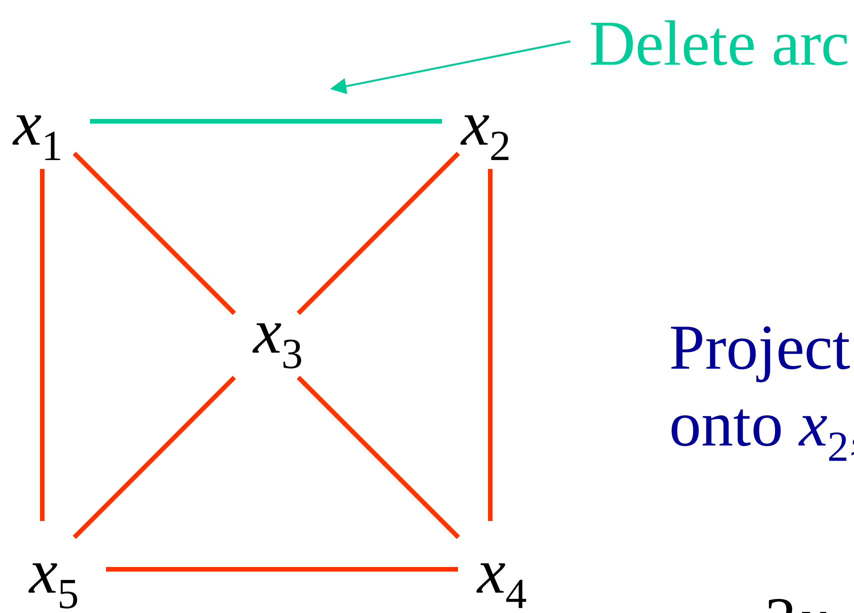


# Discrete relaxation

To create a discrete relaxation:

- Thin out the dependency graph so that it has a smaller induced width.
- Use projection to remove variable couplings that correspond to deleted arcs.
- Solve resulting problem by nonserial dynamic programming, whose complexity varies exponentially with induced width.
    - The idea of nonserial dynamic programming has surfaced in several contexts: Markov trees, solution of Bayesian networks, etc.

# Reduce induced width to 2




$$2x_1 + 3x_2 + x_3 \geq 4$$

Project onto $x_2, x_3$

$$3x_2 + x_3 \geq 1$$

Project onto $x_1, x_3$

$$2x_1 + x_3 \geq 1$$

This removes coupling between $x_1, x_2$ in constraint 1

# Discrete relaxation

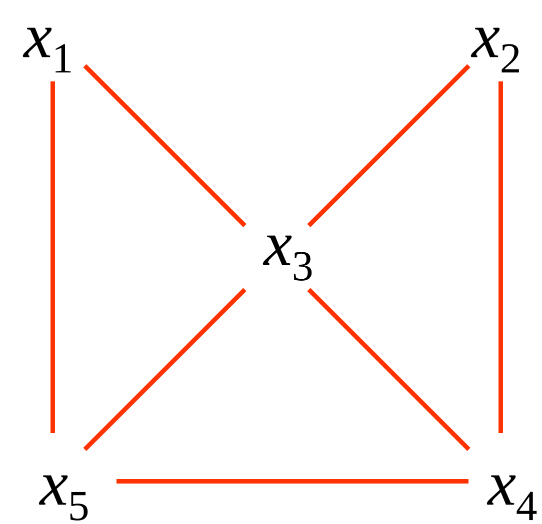
$$\begin{aligned}
\text{min} \quad & 35x_1 + 20x_2 + 15x_3 + 40x_4 + 30x_5 \\
\text{subject to} \quad & 3x_2 + x_3 \ge 2 \\
& 2x_1 + x_3 \ge 1 \\
& x_2 + 4x_3 + 3x_4 \ge 3 \\
& 2x_3 + 3x_4 + x_5 \ge 4 \\
& 3x_1 + 4x_3 + 5x_5 \ge 5 \\
& x_j \in \{0,1\}, \quad \text{all } j
\end{aligned}$$

Optimal value = 105 (same as original problem)

# Nonserial dynamic programming

$$g_1(x_3, x_5) = \min_{x_1}\{35x_1 + 15x_3 + 30x_5 + M(1 - 2x_1 - x_3)^+ + M(5 - 3x_1 - 4x_3 - 5x_5)^+\}$$

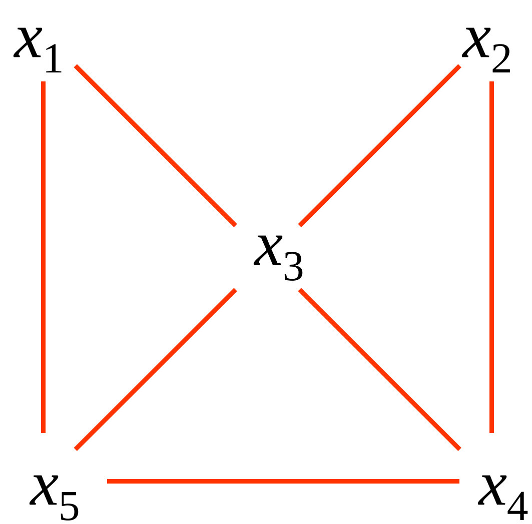
$$g_2(x_3, x_4) = \min_{x_2}\{20x_2 + 40x_4 + M(2 - 3x_2 - x_3)^+ + M(3 - x_2 - 4x_3 - 3x_4)^+\}$$



# NSDP, continued

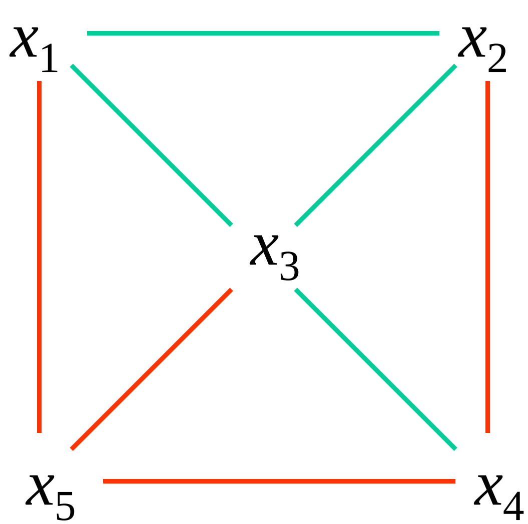
$$g_3(x_4, x_5) = \min_{x_3}\{g_1(x_3, x_5) + g_2(x_3, x_4) + M(4 - 2x_3 - 3x_4 - x_5)^+\}$$

$$\text{solution} = \min_{x_4, x_5} g_3(x_4, x_5) = 105$$



# Reduce induced width to 1

$$
\begin{array}{ll}
\min & 35x_1 + 20x_2 + 15x_3 + 40x_4 + 30x_5 \\
\text{subject to} & 3x_2 \geq 1 \\
& 3x_4 + x_5 \geq 2 \\
& 2x_3 + x_5 \geq 1 \\
& 4x_3 + 5x_5 \geq 2 \\
& 3x_1 + 5x_5 \geq 1 \\
& x_j \in \{0,1\}, \quad \text{all } j
\end{array}
$$




Optimal value = 90

# Continuous & discrete relaxations

$$43\tfrac{1}{3} < 90 < 105$$

Value of continuous relaxation

Value of discrete relaxation

Optimal value

# Parameterized relaxation

$$\left.\begin{array}{ll} \min & f(x) \\ \text{subject to} & x \in S \end{array}\right\} \text{Generic optimization problem}$$

$$\left.\begin{array}{lll} \theta(\lambda) = & \min & f(x,\lambda) \\ & \text{subject to} & x \in S(\lambda) \end{array}\right\} \text{Parameterized relaxation}$$

$$\left.\max_{\lambda \in \Lambda}\{\theta(\lambda)\}\right\} \text{Relaxation dual}$$

General conditions for a relaxation:

$$f(x,\lambda) \le f(x), \quad \text{all } x \in S, \lambda \in \Lambda$$

$$S(\lambda) \supset S, \quad \text{all } \lambda \in \Lambda$$

# Lagrangean relaxation

$$\left.\begin{array}{ll} \min & f(x) \\ \text{subject to} & g_i(x) \leq 0, \quad i \in I \\ & x \in S \end{array}\right\} \text{Optimization problem}$$

$$\left.\begin{array}{lll} \theta(\lambda) = & \min & f(x) + \sum_{i \in I} g_i(x) \\ & \text{subject to} & x \in S \end{array}\right\} \text{Lagrangean relaxation}$$

$$\left.\max_{\lambda \geq 0}\{\theta(\lambda)\}\right\} \text{Lagrangean dual}$$

# Surrogate dual

$$\left.\begin{array}{lll} \min & f(x) & \\ \text{subject to} & g_i(x) \le 0, & i \in I \\ & x \in S & \end{array}\right\} \text{Optimization problem}$$

$$\left.\begin{array}{lll} \theta(\lambda) = & \min & f(x) \\ & \text{subject to} & \sum_{i \in I} \lambda_i g_i(x) \le 0 \\ & & x \in S \end{array}\right\} \text{Surrogate relaxation}$$

$$\max_{\lambda \ge 0} \{\theta(\lambda)\} \quad \Big\} \text{ Surrogate dual}$$

# Combine discrete relaxation with Lagrangean duality

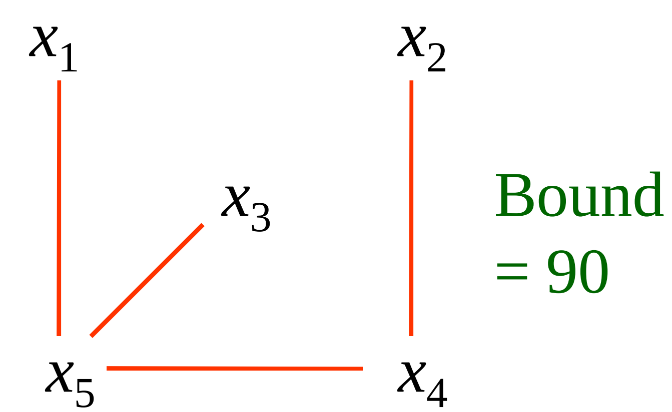
$$
\begin{aligned}
\min \quad & 35x_1 + 20x_2 + 15x_3 + 40x_4 + 30x_5 \\
& \left.\begin{aligned}
& + \lambda_1(4 - 2x_1 - 3x_2 - x_3) \\
& + \lambda_2(3 - x_2 - 4x_3 - 3x_4) \\
& + \lambda_3(4 - 2x_3 - 3x_4 - x_5) \\
& + \lambda_4(5 - 3x_1 - 4x_3 - 5x_5)
\end{aligned}\right\} \text{Lagrangean} \\
\text{subject to} \quad & 3x_2 \ge 1 \\
& \left.\begin{aligned}
& 3x_4 + x_5 \ge 2 \\
& 2x_3 + x_5 \ge 1 \\
& 4x_3 + 5x_5 \ge 2
\end{aligned}\right\} \text{Discrete relaxation} \\
& 3x_1 + 5x_5 \ge 1 \\
& x_j \in \{0,1\}, \quad \text{all } j
\end{aligned}
$$

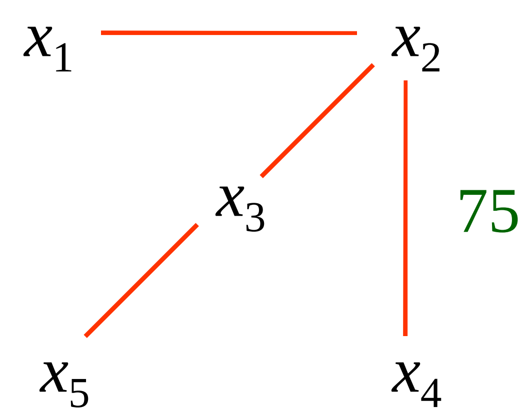
# Solve by subgradient optimization

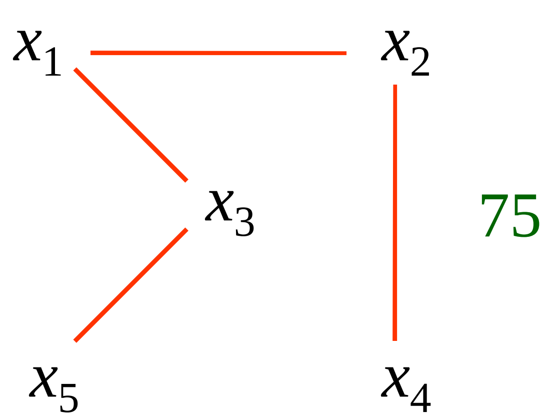
| Step | $\lambda$ | Value | Subgradient |
|---|---|---|---|
| | 0 0 0 0 | 90 | 1 -1 -4 0 |
| 6 | 6 0 0 0 | 96 | 1 -1 -4 0 |
| 3 | 9 0 0 0 | 92 | -2 -5 -1 0 |
| 2 | 5 0 0 2 | 95 | 1 -1 -4 0 |
| 1.5 | 6.5 0 0 2 | 96.5 | 1 -1 -4 0 |
| 1.2 | 7.7 0 0 2 | 96.6 | -2 -5 -1 0 |
| 1 | 5.7 0 0 3 | 95.7 | 1 -1 -4 0 |
| .86 | 6.56 0 0 3 | 96 | 0 -5 -6 -3 |
| .75 | 6.56 0 0 .75 | 96.56 | 1 -1 -4 0 |
| .67 | 7.23 0 0 .75 | 96.29 | -2 -5 -1 1 |
| .6 | 7.23 0 0 1.35 | 96.89 | -2 -5 -1 1 |

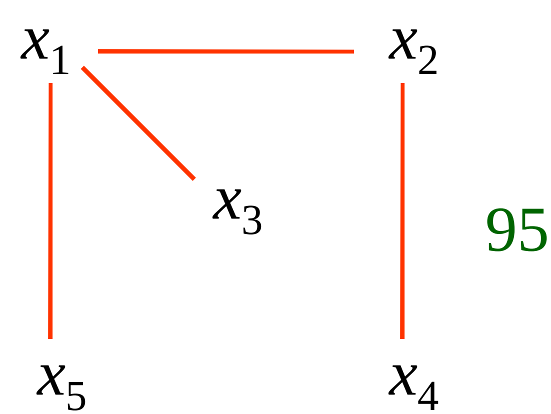
Use NSDP at each iteration.

# Discrete relaxation dual



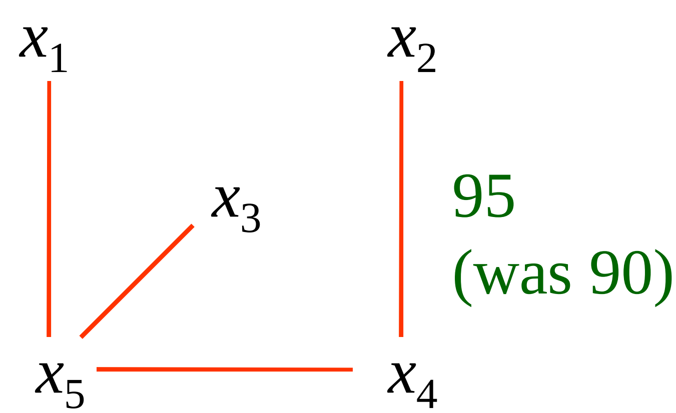



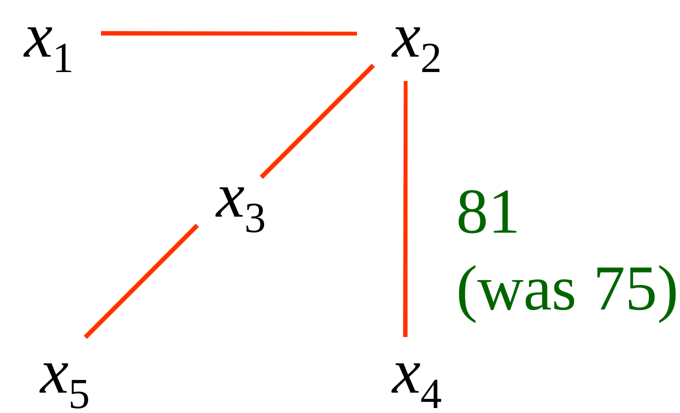



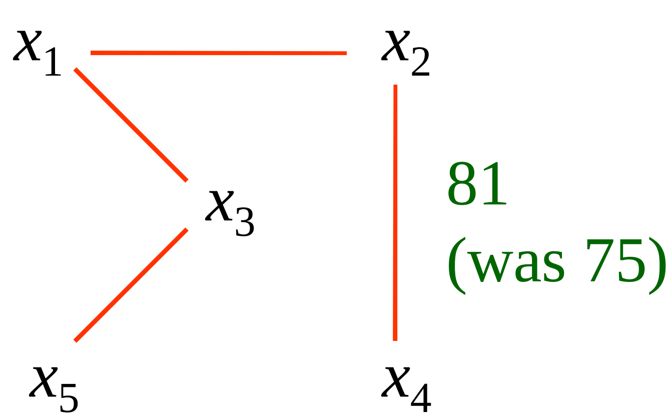



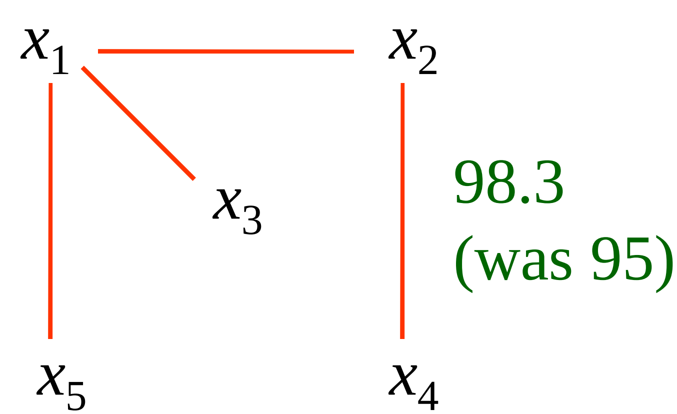

Enumerate relaxations with induced width of 1

# Combine Lagrangean & discrete relaxation duals



# Summary of relaxations

Value of discrete relaxation

Value of discrete relaxation with Lagrangean dual

Value of discrete and Lagrangean dual

$$43\tfrac{1}{3} < 90 < 95 < 96.9 < 98.3 < 105$$

Value of continuous relaxation

Value of discrete relaxation dual

Optimal value

# Outline

- A motivating example
  - Constraint satisfaction approach
  - Integer programming approach
  - Combined approach
- The search/inference duality
  - Complementary solution methods
  - A formal duality
- The strengthening/relaxation duality
  - Complementary solution methods
  - Relaxations for global constraints
  - Formal relaxation duality
- Relaxation duality
  - An integer programming example
  - Continuous relaxation
  - Discrete relaxation
  - *dependency graph, nonserial dynamic programming*
  - Relaxation duality
  - Lagrangean & surrogate duals
  - Discrete relax + Lagrangean dual
  - Discrete relaxation dual
  - Discrete + Lagrangean duals
  - Summary of relaxations
- Research agenda

# Research Agenda

- Identify cutting planes that propagate well.
- Learn how to choose constraints that have a useful continuous relaxation.
- Find continuous relaxations for global constraints not in inequality form (e.g., element, piecewise linear costs).
- Implement variable index sets as well as variable indices.
- Use relaxation duals to discover new relaxations common constraints in constraint satisfaction languages.

# Research Agenda

- Identify inference techniques (other than cutting planes) that obtain relaxations that are easy to solve.
- Develop inference-based sensitivity analysis for problem classes.
- Investigate the possibility of using nogoods in branch-and-bound search along with cutting planes and domain reduction.
- Use generalized Benders decomposition to obtain useful nogoods.

# Research Agenda

- Experiment with new ways to combine strengthening and relaxation.
- Solve a wide variety of problems with a view to how the search/inference and strengthening/relaxation dualities may be exploited.
- Build a solution technology that unifies and goes beyond classical optimization and constraint satisfaction methods.