Optimal Solutions with Bounded Inequality

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**Presenter*

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Modeling Fairness

- A growing interest* in incorporating **fairness** into models
  - Health care resources.
  - Facility location (e.g., emergency services, infrastructure).
  - Telecommunications.
  - Traffic signal timing
  - Disaster recovery (e.g., power restoration)

*51 fairness-related talks **today alone**.

Modeling Fairness

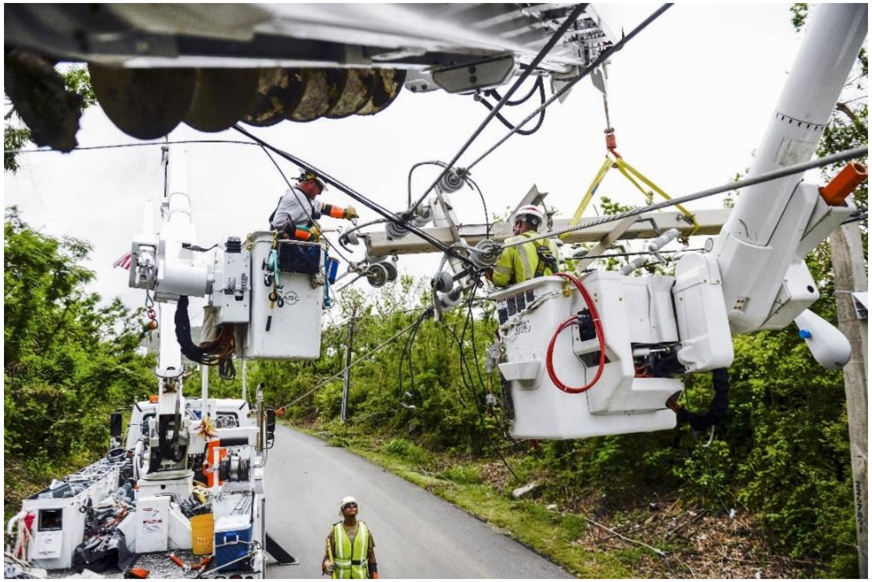
- Example: **Emergency facility location**
  - Locations in densely populated zone minimize **average response time,** but are unfair to those in outlying areas
  - Locations that minimize **worst-case response time** result in poor service for most of the population
- A more **equitable** solution
  - …would compromise between equity and efficiency.

Modeling Fairness

- Example: **Traffic signal timing**
  - **Throughput** is maximized by giving constant green light to the major street, red light to cross street.
  - Then motorists on the cross street **wait forever**.
- A more **equitable** solution would find a compromise.
  - Similar issues in **telecommunications**, an early adopter of fairness modeling.

Modeling Fairness

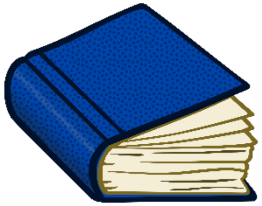
- Example: **Disaster relief**
  - Power restoration can focus on **urban** areas first (**efficiency**).
  - This can leave rural areas without power for weeks/months.
  - This happened in Puerto Rico after Hurricane Maria (2017).
- A more **equitable** solution
  - …would give some priority to rural areas without overly sacrificing efficiency.



Modeling Fairness

- The problem: How to incorporate **fairness** into an **optimization model**?

- Our approach: determine how various fairness models **affect the structure of optimal solutions**.



Özgün Elçi, John Hooker, Peter Zhang
The Structure of Fair Solutions:
Achieving Fairness in an Optimization Model
Springer, forthcoming.

Modeling Fairness

- We focus on two models that **maximize total utility** subject to a **bound on inequality**:
    - Inequality as measured by **range** of utilities.
    - Inequality as measured by **Gini coefficient**.

Modeling Fairness

- We focus on two models that **maximize total utility** subject to a **bound on inequality**:
  - Inequality as measured by **range** of utilities.
  - Inequality as measured by **Gini coefficient.**
  - Structural results may help explain why societies historically tend to have **2 or 3 fairly homogeneous social classes**.

Modeling Fairness

- Optimization models are normally formulated to **maximize total utility**.
  - where utility = wealth, health, negative cost, etc.
  - This can lead to **very unfair** resource distribution.

- For example...

Maximize Utility?

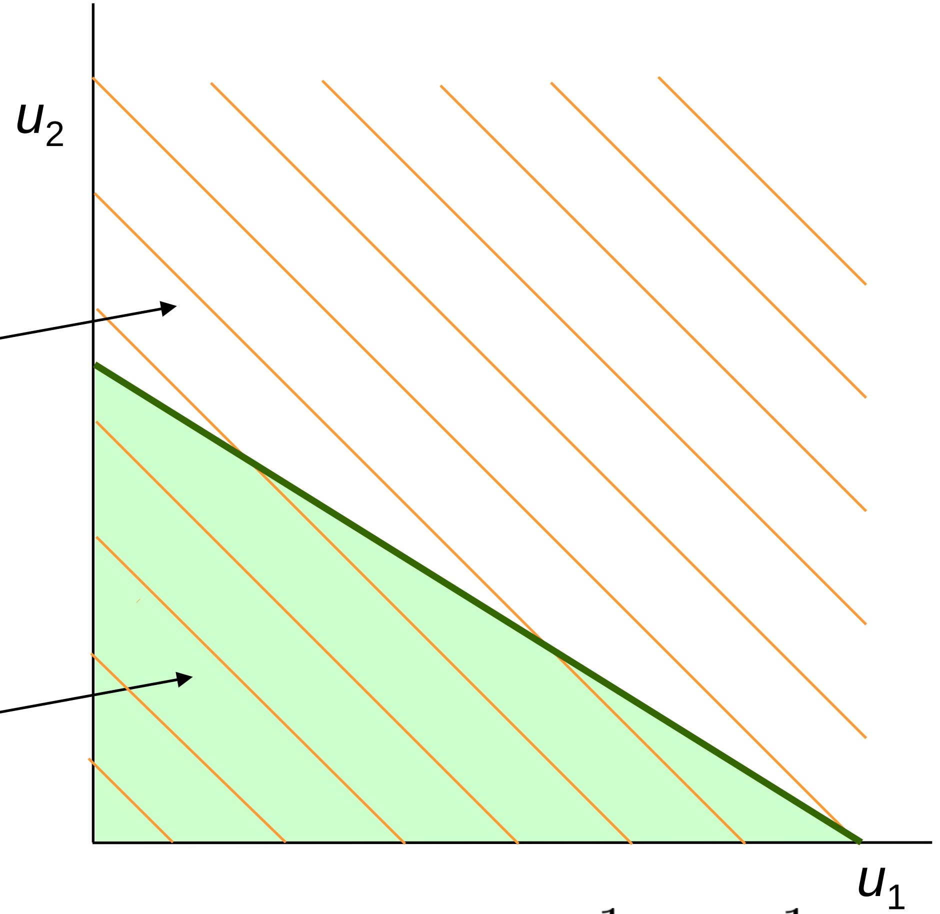
Utility maximizing
distribution
for 2 persons,
subject to
budget constraint

Utility contours

$$u_1 + u_2$$

Feasible
region

$$a_1 u_1 + a_2 u_2 \leq B$$



Person 1 has greater **conversion efficiency**: $\frac{1}{a_1} > \frac{1}{a_2}$

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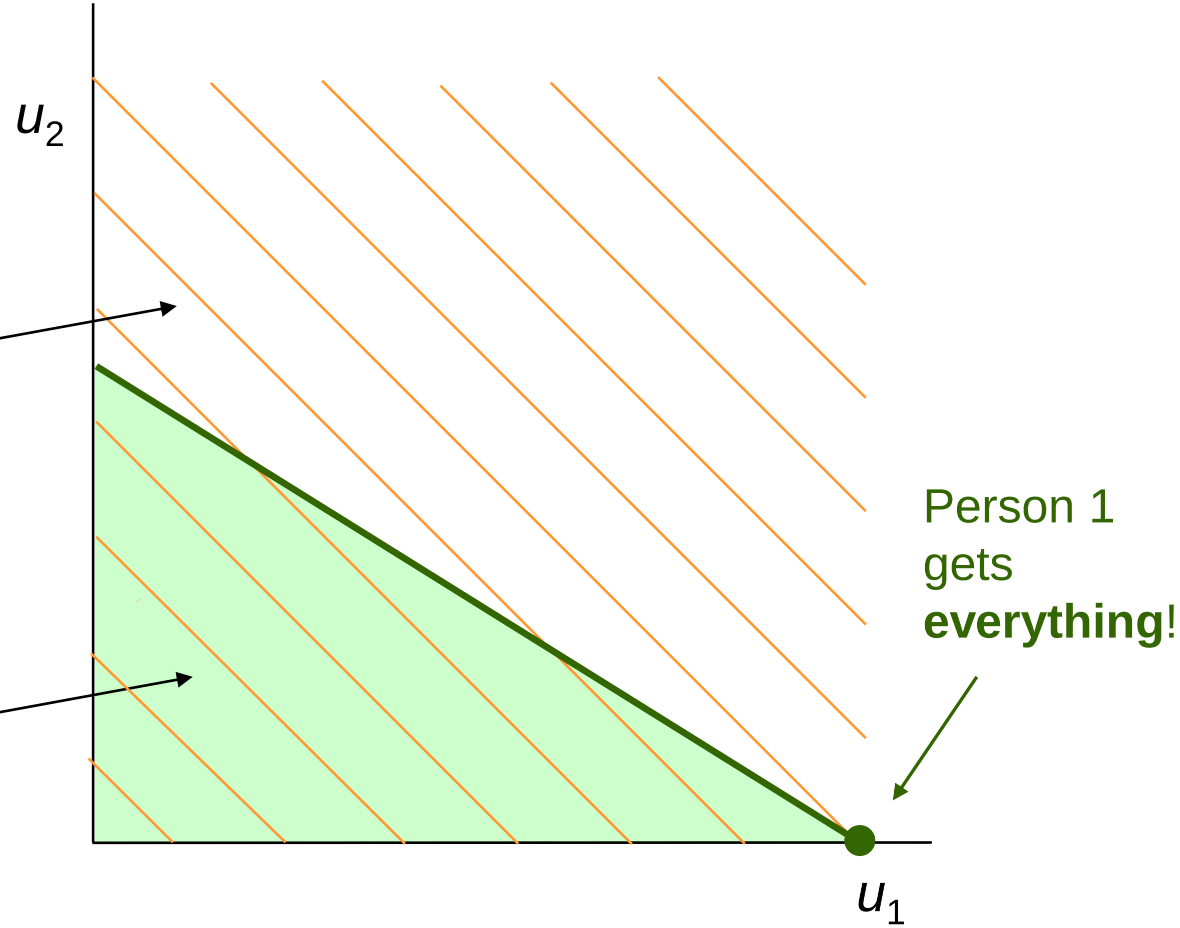


Person 2 makes **less efficient use of resources**
(e.g., has a more serious disease)

Modeling Equity/Efficiency Trade-off

- Maximize **utility** subject to an **inequality bound**
  - Coefficient of variation (Jain's index)
  - **Range**
  - **Gini coefficient**
  - Hoover index
  - McLoone index

Modeling Equity/Efficiency Trade-off

- Maximize **utility** subject to an **inequality bound**
  - Coefficient of variation (Jain's index)
  - **Range**
  - **Gini coefficient**
  - Hoover index
  - McLoone index
- Maximize a **social welfare function**
  - Alpha fairness (or Nash bargaining solution as special case)
  - Beta fairness
  - Kalai-Smorodinsky bargaining solution
  - Utility threshold function
  - **Equity threshold function**

Modeling Equity/Efficiency Trade-off

- **Theorem:** the **optimal trade-off** subject to a budget constraint divides stakeholders into **2 or 3 homogeneous classes…**
  - …when **maximizing utility** subject to **range** constraint, which is same as maximizing **equity threshold** function.
  - …most surprisingly, when **maximizing utility** subject to constraint on **Gini coefficient**.

Maximizing Utility with Range Bound

Utility maximizing distribution for 2 persons, subject to budget constraint and range bound Δ

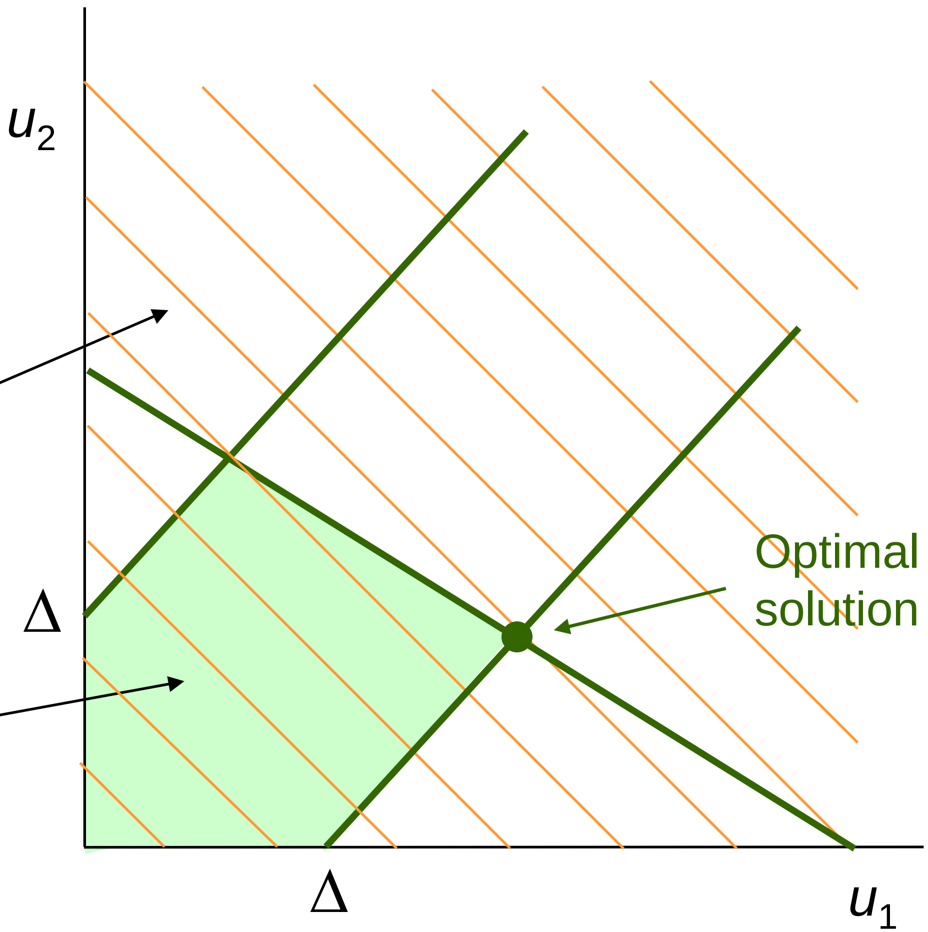
Utility contours

$$u_1 + u_2$$

Feasible region

$$a_1 u_1 + a_2 u_2 \leq B$$

$$|u_1 - u_2| \leq \Delta$$



Maximizing Utility with Range Bound

In general, the problem is

$$\max \left\{ \sum_i u_i \;\middle|\; \sum_i a_i u_i \le B;\;\; |u_i - u_j| \le \Delta,\; \text{all } i, j \right\}$$

- We **do not normalize** the range by the mean utility.
  - Otherwise, the problem is infeasible or has a purely utilitarian solution.
- For the same reason, we use an **unnormalized Gini** coefficient.

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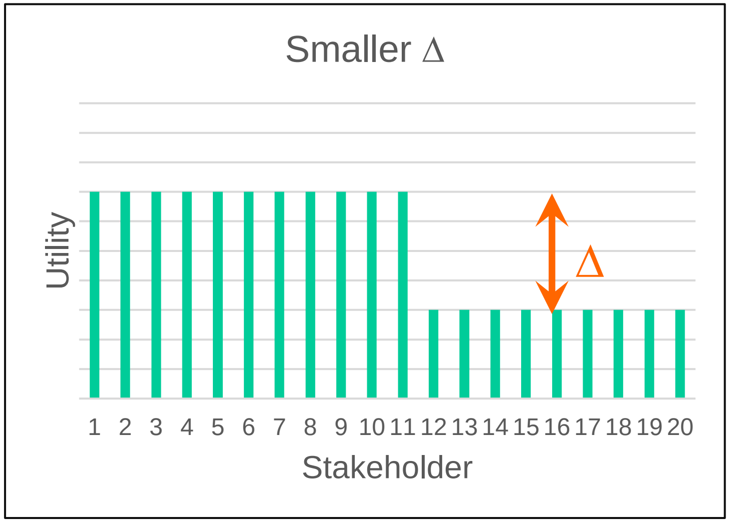
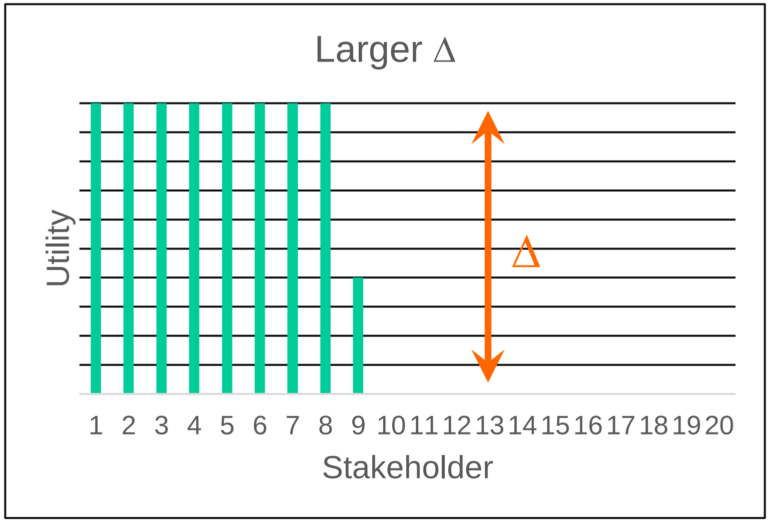
Maximizing Utility with Range Bound

In general, the problem is

$$\max \left\{ \sum_i u_i \;\middle|\; \sum_i a_i u_i \le B; \;\; |u_i - u_j| \le \Delta, \;\; \text{all } i, j \right\}$$

Solutions have two possible patterns $(a_1 \le \cdots \le a_n)$

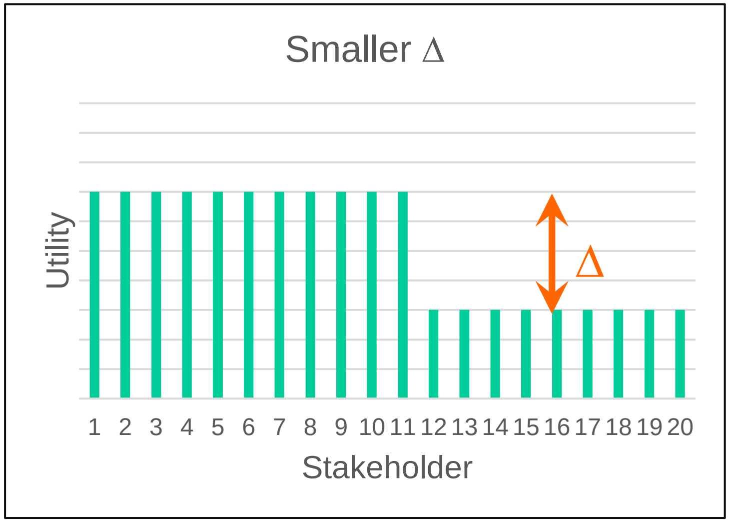


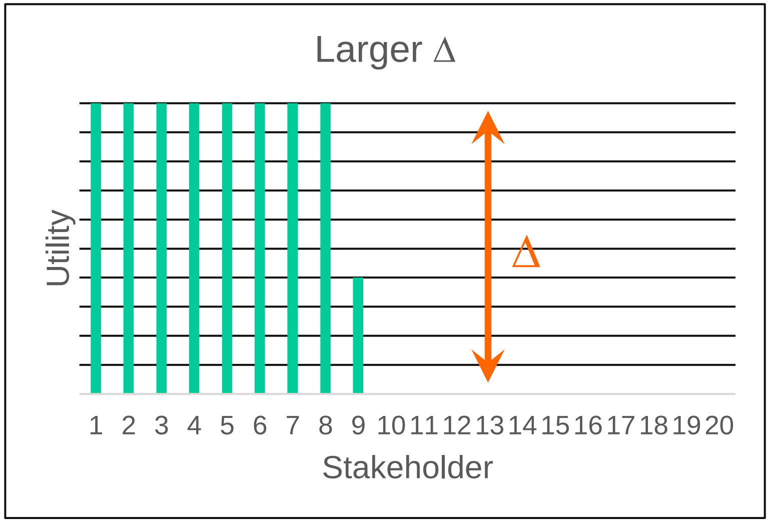


Maximizing Utility with Range Bound

- There are 2 or 3 groups of stakeholders
  - If 3, the middle group contains **only 1** stakeholder, and the 3rd group has **zero** utility.
  - Everyone a group receives the **same** utility, even when the conversion efficiencies a_i differ.

$$(a_1 \leq \cdots \leq a_n)$$



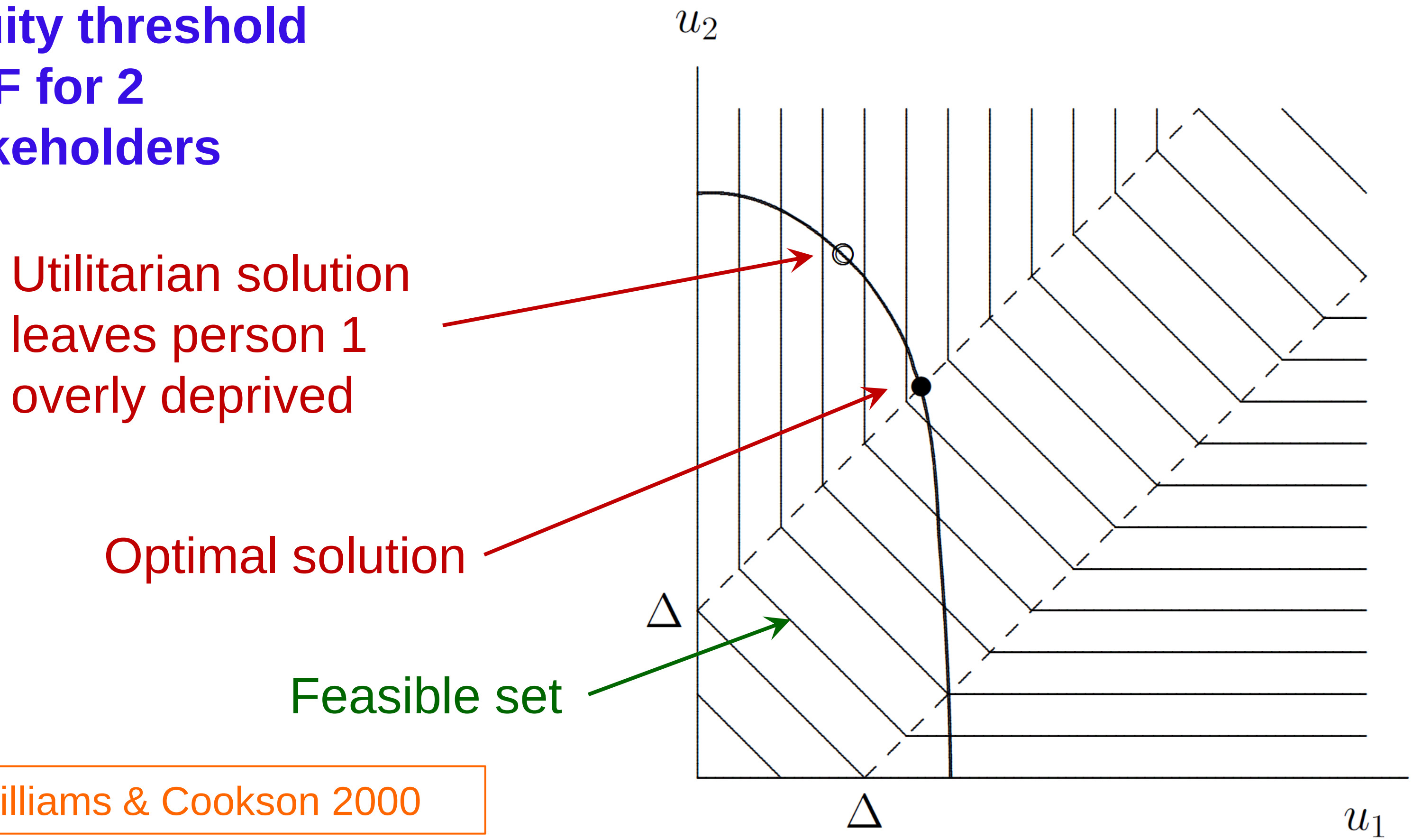


Equity Threshold

- Maximizing utility with a range bound is equivalent to maximizing the **equity threshold** function
  - …which **combines utilitarian and maximin criteria**

Equity Threshold

Equity threshold SWF for 2 stakeholders



Williams & Cookson 2000

$$W(u_1, u_2) = \begin{cases} 2\min\{u_1, u_2\} + \Delta, & \text{if } |u_1 - u_2| \geq \Delta \\ u_1 + u_2, & \text{otherwise} \end{cases}$$

Equity Threshold

Generalization to *n* persons

$$W(\boldsymbol{u}) = n\Delta + \sum_{i=1}^{n} \min\{u_i - \Delta, u_{min}\}$$

- Δ is chosen so that well-off individuals **do not deserve more utility** unless utilities within Δ of smallest are also increased.
- $\Delta = \infty$ corresponds to utilitarian, $\Delta = 0$ to maximin.

Equivalence theorem

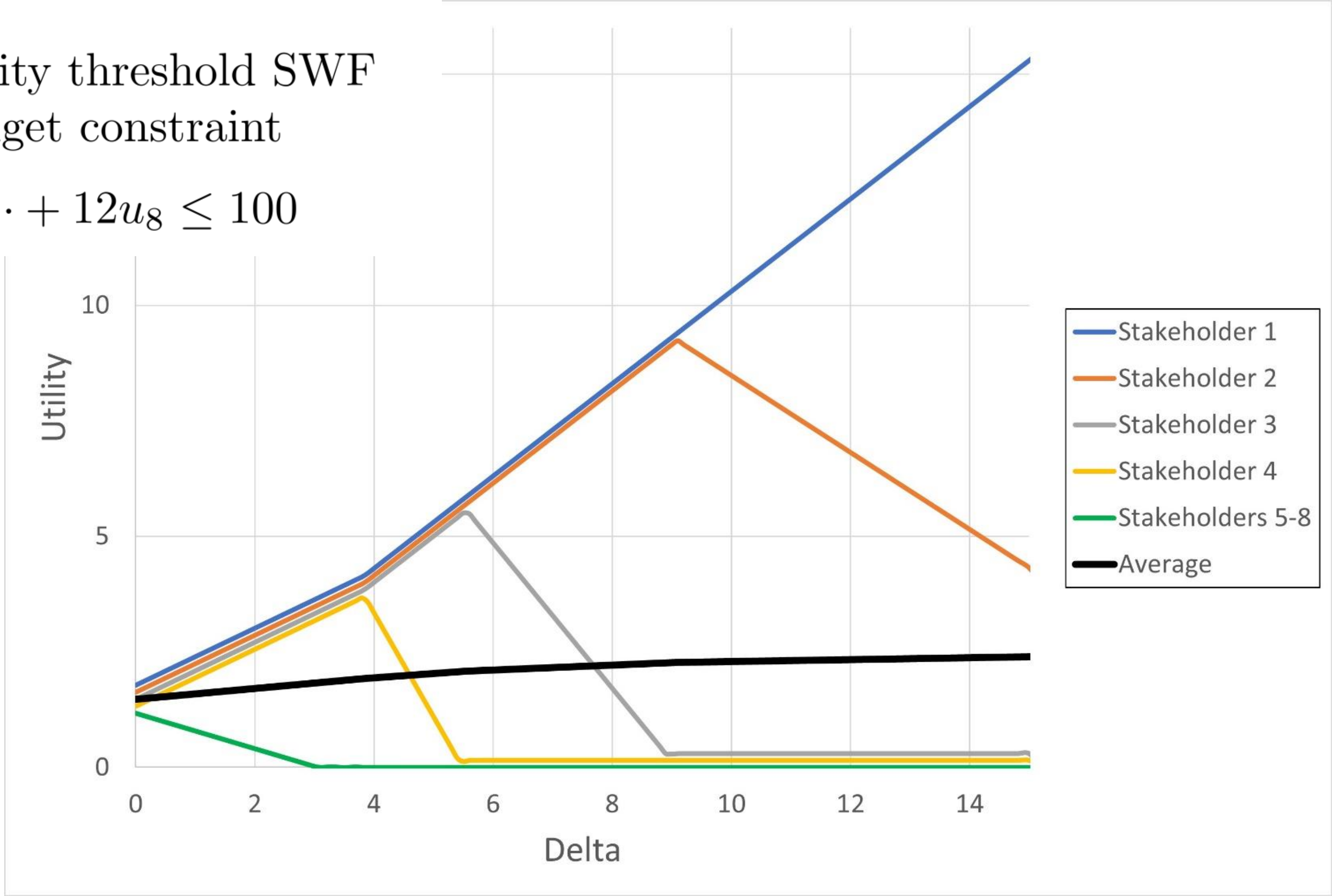
- Maximizing $W(\boldsymbol{u})$ subject to a budget constraint gives the **same solution** as maximizing total utility subject to a budget constraint **and range bound Δ.**

Equity Threshold

Example:

Maximum equity threshold SWF subject to budget constraint

$$5u_1 + 6u_2 + \cdots + 12u_8 \leq 100$$

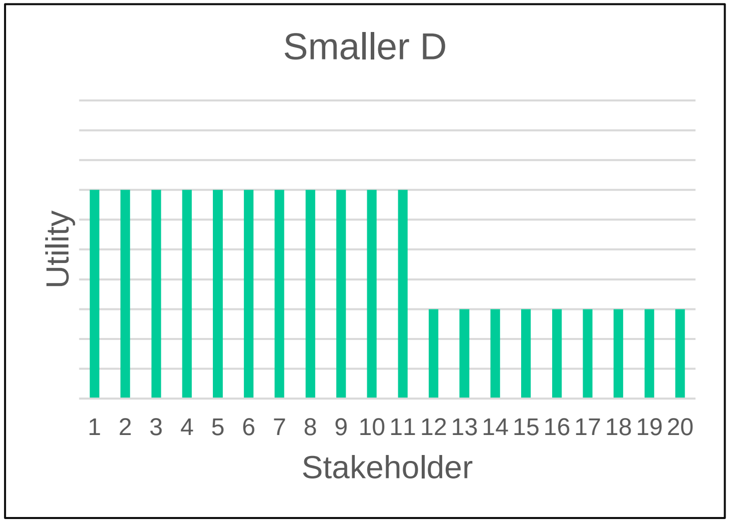


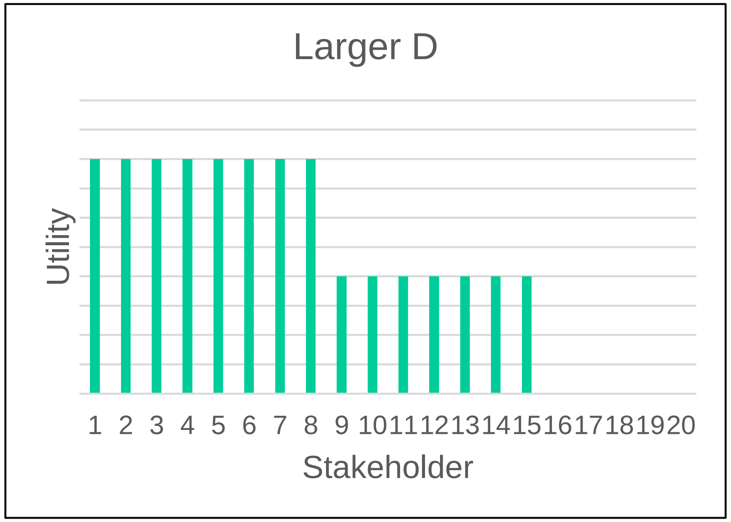
Maximizing Utility with Gini Bound

In general, the problem is

$$\max \left\{ \sum_i u_i \;\middle|\; \sum_i a_i u_i \leq B; \;\; \frac{1}{2n^2} \sum_{ij} |u_i - u_j| \leq D \right\}$$

Solutions have two possible patterns $(a_1 \leq \cdots \leq a_n)$

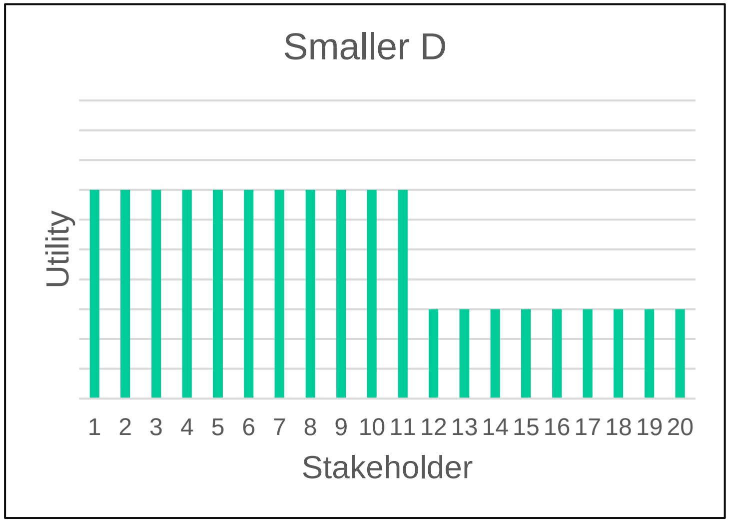


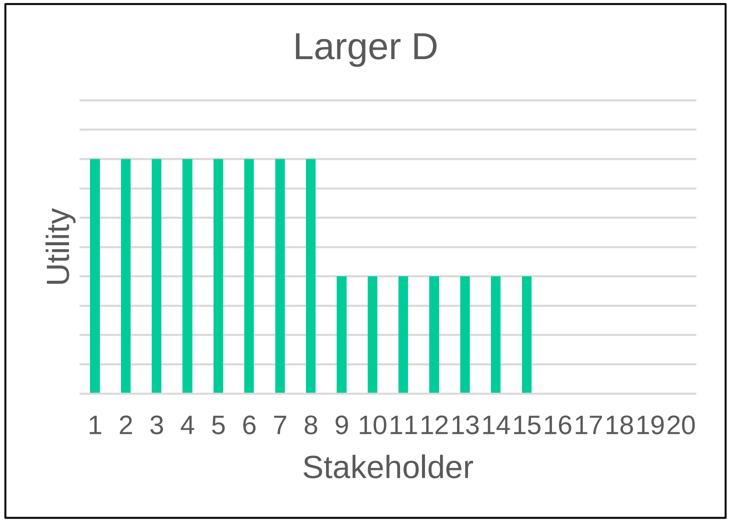


Maximizing Utility with Gini Bound

- There are 2 or 3 groups of stakeholders
  - If 3, one group has **zero** utility.
  - The middle group can contain **multiple** stakeholders.

$$(a_1 \leq \cdots \leq a_n)$$





Maximizing Utility with Gini Bound

Why? If $(a_1 \le \cdots \le a_n)$, the problem can be written as an LP:

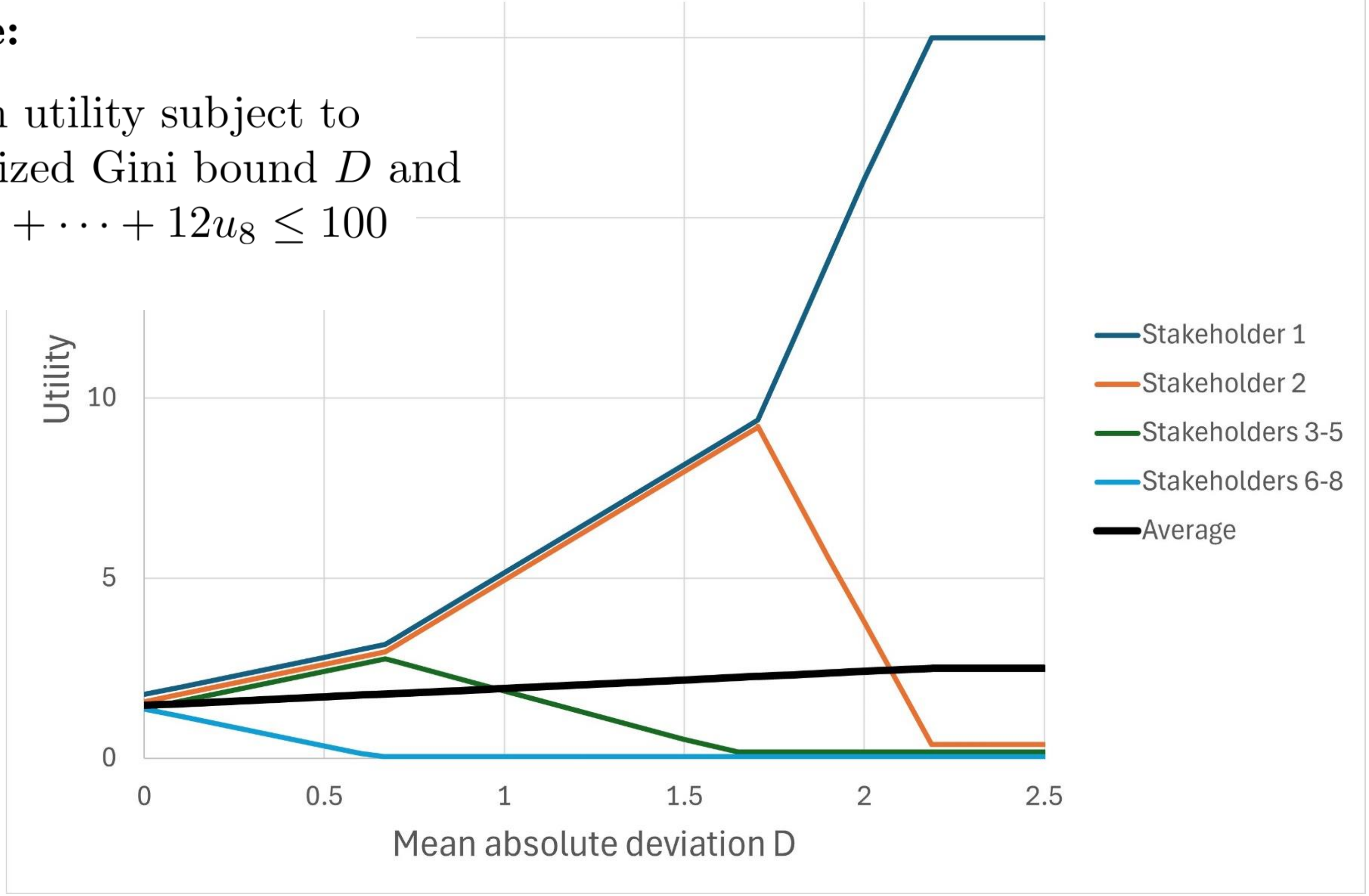
$$\max \left\{ \sum_i u_i \;\middle|\; \begin{array}{l} \sum_i a_i u_i \le B \\ \sum_i (n-2i+1) u_i \le n^2 D \\ u_i \ge u_{i+1},\ i = 1, \ldots, n-1 \end{array} \right\}$$

Then the proof is based on duality and structure of a feasible basis.

Maximizing utility with Gini Bound

Example:

Maximum utility subject to unnormalized Gini bound D and $5u_1 + 6u_2 + \cdots + 12u_8 \leq 100$



Conclusions

- Maximizing **utility** subject to an **inequality bound creates 2 or 3 homogeneous classes**.
    - In particular, when maximizing subject to a bound on **range** or **Gini** coefficient.

- This may help explain why societies historically tend to consist of **2 or 3 major social classes**.
    - And closer study of the optimality conditions may suggest how to **minimize class differences**.


Questions or
comments?