Equitable and Efficient Healthcare Resource Distribution

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The Problem

- Healthcare resources are normally allocated to **maximize utility**.
    - As measured by health outcomes (QALYs, etc.)

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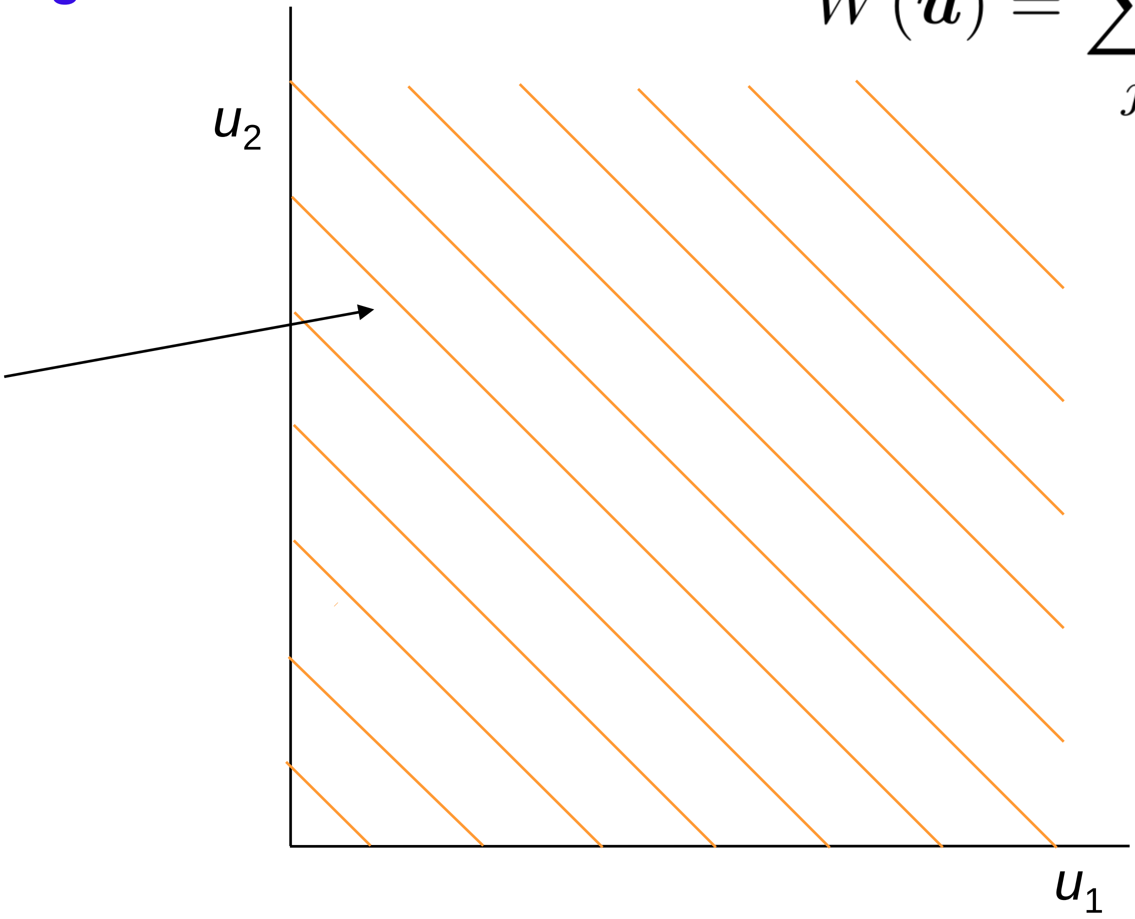
- Healthcare resources are normally allocated to **maximize utility**.
    - As measured by health outcomes (QALYs, etc.)
    - This can lead to **very unfair** resource distribution.

- For example...

The Problem

Utility maximizing
distribution
for 2 patients

$$W(\boldsymbol{u}) = \sum_j u_j$$

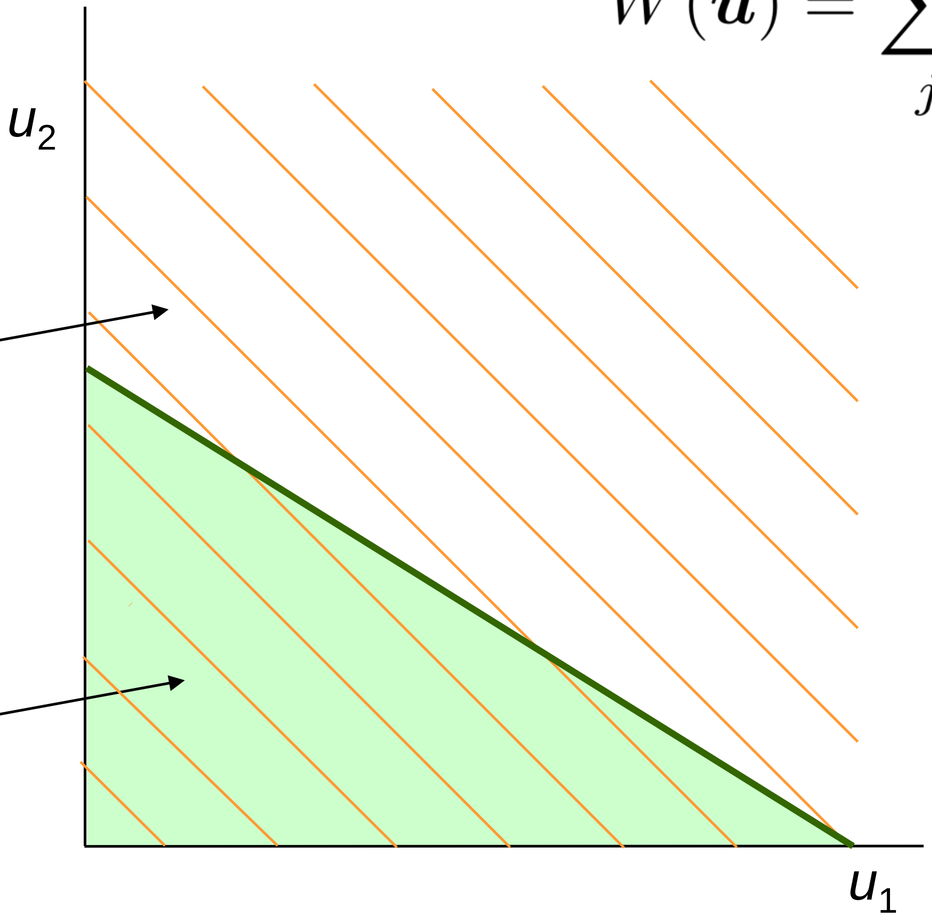


Utility contours
$u_1 + u_2$

The Problem

Utility maximizing distribution for 2 patients, subject to budget constraint

$$W(\boldsymbol{u}) = \sum_j u_j$$



Utility contours
$u_1 + u_2$

Feasible region
$a_1 u_1 + a_2 u_2 \leq B$

Patient 2 is more expensive to treat

The Problem

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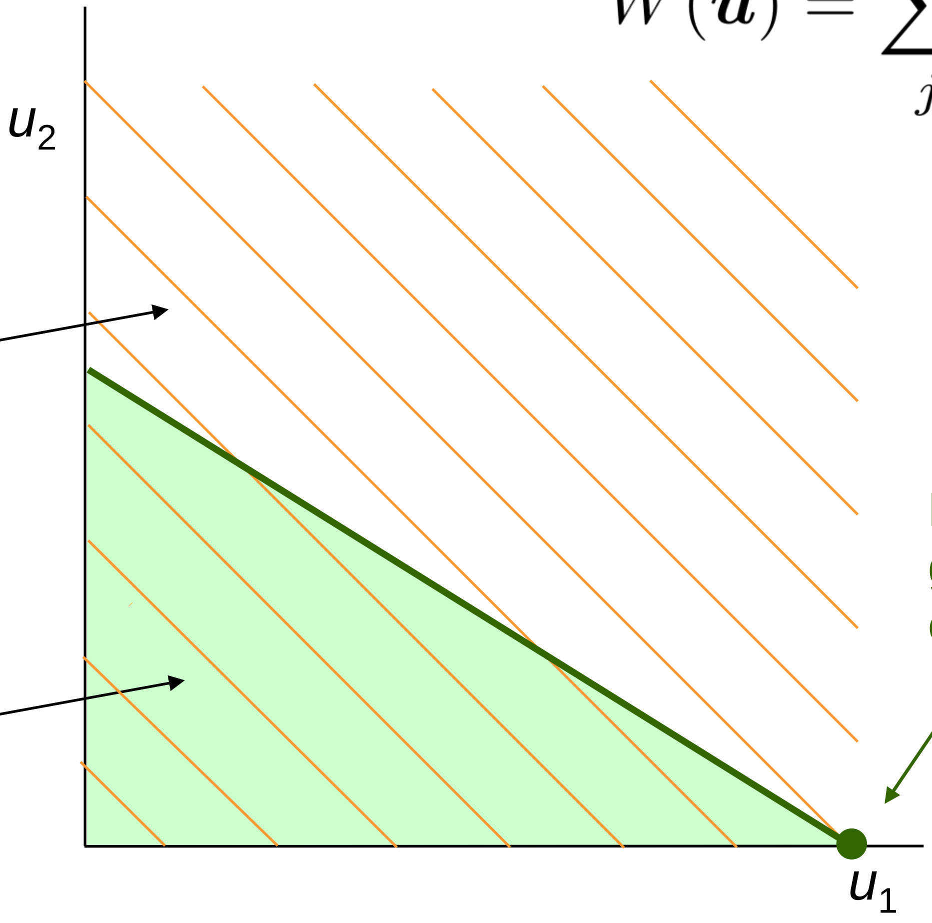


Utility contours
$u_1 + u_2$

Feasible
region
$a_1 u_1 + a_2 u_2 \leq B$

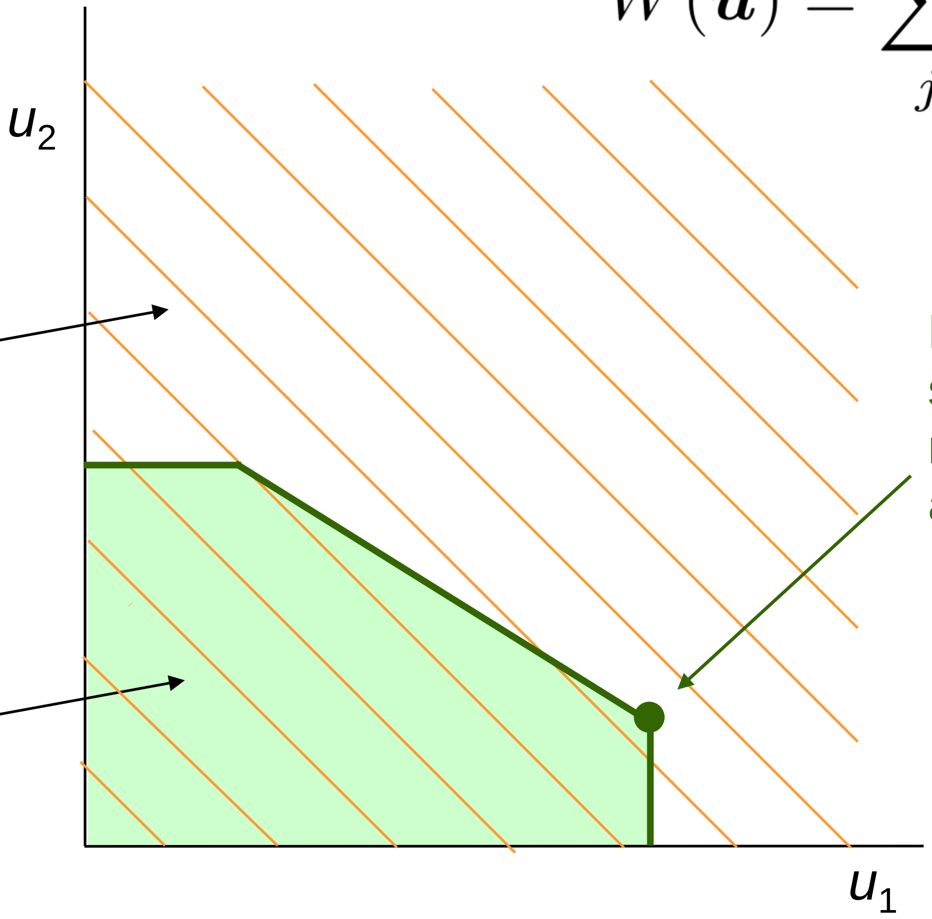
Patient 1
gets
everything!

Patient 2 is more expensive to treat

The Problem

Add **bounds** on individual utilities…

$$W(\boldsymbol{u}) = \sum_j u_j$$



Utility contours
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Feasible region

Patient 1 still gets maximum allotment

The Problem

- True, these constraints are simplistic...
    - ...and such extreme solutions rarely occur in practice.

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    - ...and such extreme solutions rarely occur in practice.
    - Yet the complexity of the constraints only **conceals the basic inadequacy** of the objective function!

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- True, these constraints are simplistic...
  - ...and such extreme solutions rarely occur in practice.
  - Yet the complexity of the constraints only **conceals the basic inadequacy** of the objective function!
  - We need an objective function that **incorporates equity** as well as efficiency.

The Problem

- Several **social welfare functions** have been designed for this purpose...

The Problem

- Several **social welfare functions** have been designed for this purpose...
- Yet many of these also lead to **extreme solutions**…
  - …when maximized subject to **simple constraints**.
  - Extreme solutions may not often occur **in practice**…
  - …but only **by accident**, not due to some **underlying concept of fairness** in the constraint set.

Research Program

- This suggests a research program.
  - Study the **structure of optimal solutions** using various social welfare functions (SWFs), subject to **simple constraints**.
  - Identify SWFs that yield **reasonable solutions**.

Research Program

- This suggests a research program.
    - Study the **structure of optimal solutions** using various social welfare functions (SWFs), subject to **simple constraints**.
    - Identify SWFs that yield **reasonable solutions**.
    - The results apply to **resource distribution in general**.

Generic Welfare Optimization Problem

$$\max_{\boldsymbol{x}} \left\{ W\big(\boldsymbol{U}(\boldsymbol{x})\big) \;\middle|\; \sum_j x_j \leq B,\; \bar{\boldsymbol{c}} \leq \boldsymbol{x} \leq \bar{\boldsymbol{d}} \right\}$$

Generic Welfare Optimization Problem

$$\max_{\boxed{\boldsymbol{x}}} \left\{ W\left(\boldsymbol{U}(\boldsymbol{x})\right) \;\middle|\; \sum_{j} x_j \leq B,\ \bar{\boldsymbol{c}} \leq \boldsymbol{x} \leq \bar{\boldsymbol{d}} \right\}$$

Resource allocation vector $(x_1,\ldots,x_n)$ to patient groups

Generic Welfare Optimization Problem

Budget constraint on resource distribution

$$\max_{\boxed{\boldsymbol{x}}} \left\{ W\left(\boldsymbol{U}(\boldsymbol{x})\right) \;\middle|\; \boxed{\sum_j x_j \leq B,} \; \bar{\boldsymbol{c}} \leq \boldsymbol{x} \leq \bar{\boldsymbol{d}} \right\}$$

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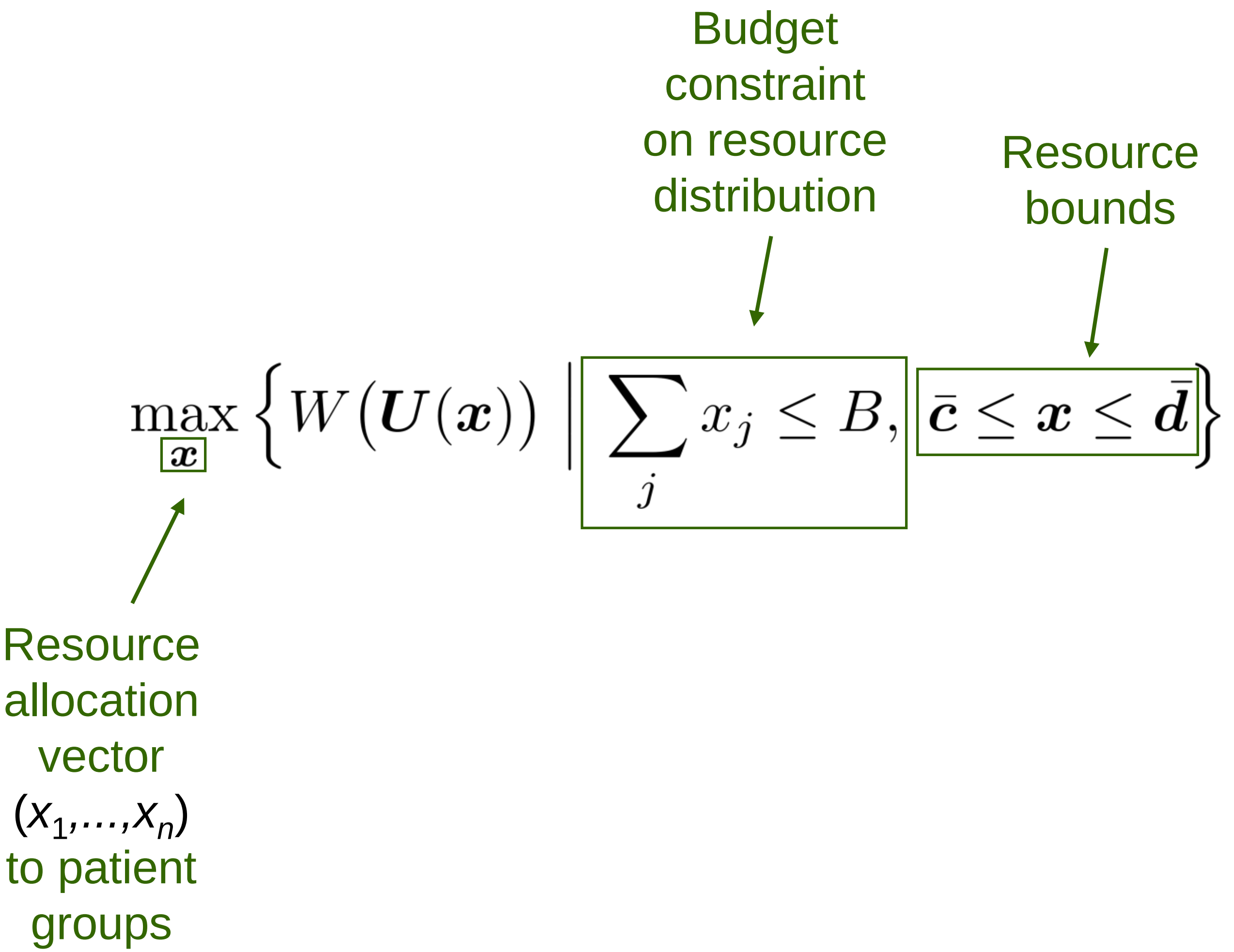

Budget constraint on resource distribution

Resource bounds

$$\max_{\boxed{\boldsymbol{x}}} \left\{ W\left(\boldsymbol{U}(\boldsymbol{x})\right) \;\middle|\; \boxed{\sum_j x_j \leq B,} \boxed{\bar{\boldsymbol{c}} \leq \boldsymbol{x} \leq \bar{\boldsymbol{d}}} \right\}$$

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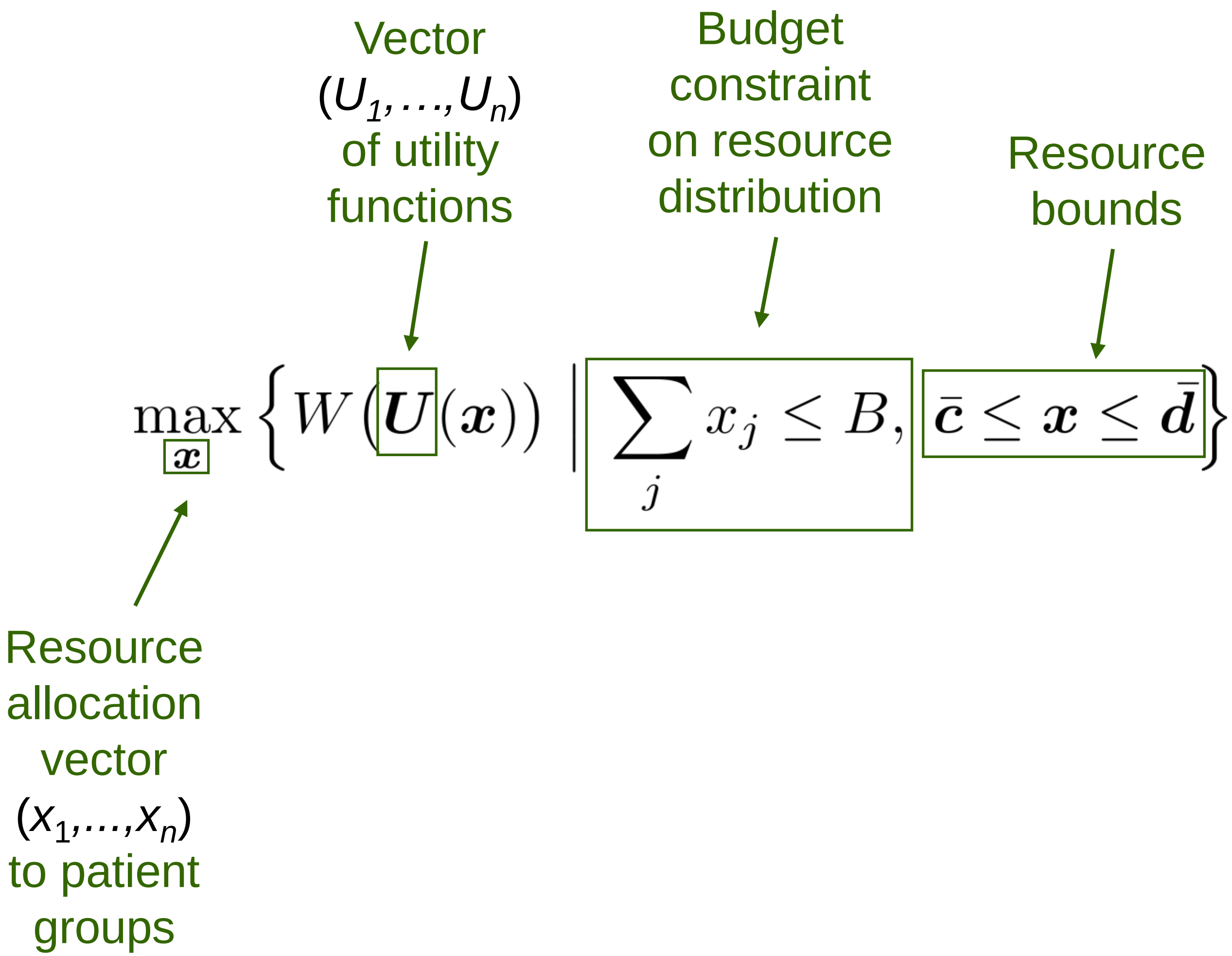

Vector $(U_1,\ldots,U_n)$ of utility functions

Budget constraint on resource distribution

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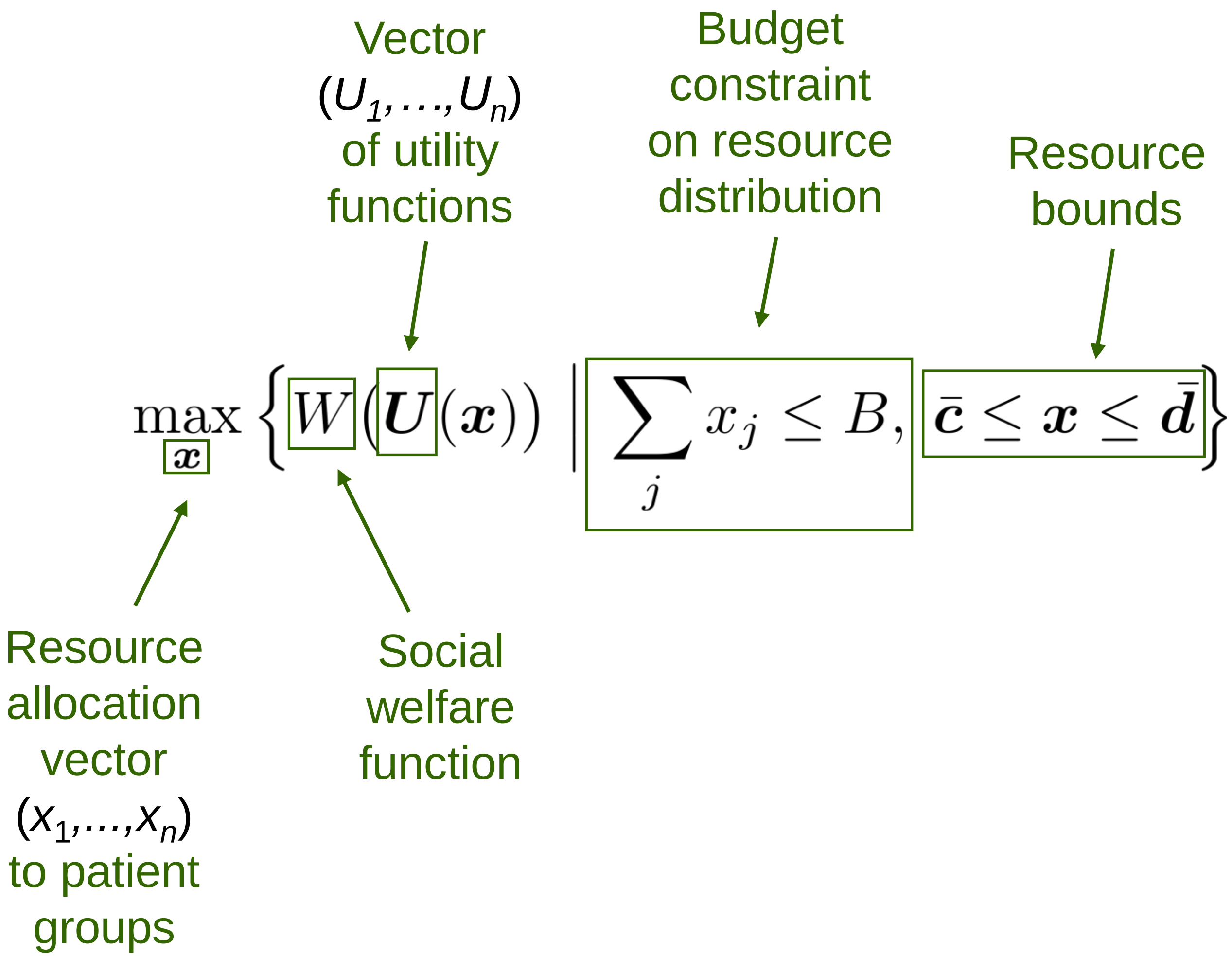

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Resource allocation vector $(x_1,\dots,x_n)$ to patient groups

Social welfare function

Generic Welfare Optimization Problem

We suppose ***U***(***x***) has the form

$$\boldsymbol{U}(\boldsymbol{x}) = \big(x_1/a_1, \ldots, x_n/a_n\big), \quad \text{with } a_j > 0, \; \text{all } j$$

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This allows us to get rid of ***x***:

$$\max_{\boxed{\boldsymbol{u}}} \left\{ W(\boldsymbol{u}) \;\middle|\; \boldsymbol{a}^{\intercal}\boldsymbol{u} \le B,\ \boldsymbol{c} \le \boldsymbol{u} \le \boldsymbol{d} \right\}$$

where $c_j = \bar{c}_j/a_j, \quad d_j = \bar{d}_j/a_j$

Utility allocation vector $(u_1,\ldots,u_n)$ to patient groups

Generic Welfare Optimization Problem

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where $c_j = \bar{c}_j/a_j, \quad d_j = \bar{d}_j/a_j$

A large a_j means that patient group *j* is expensive to treat.

We assume $a_1 \le \cdots \le a_n$

Healthcare Example

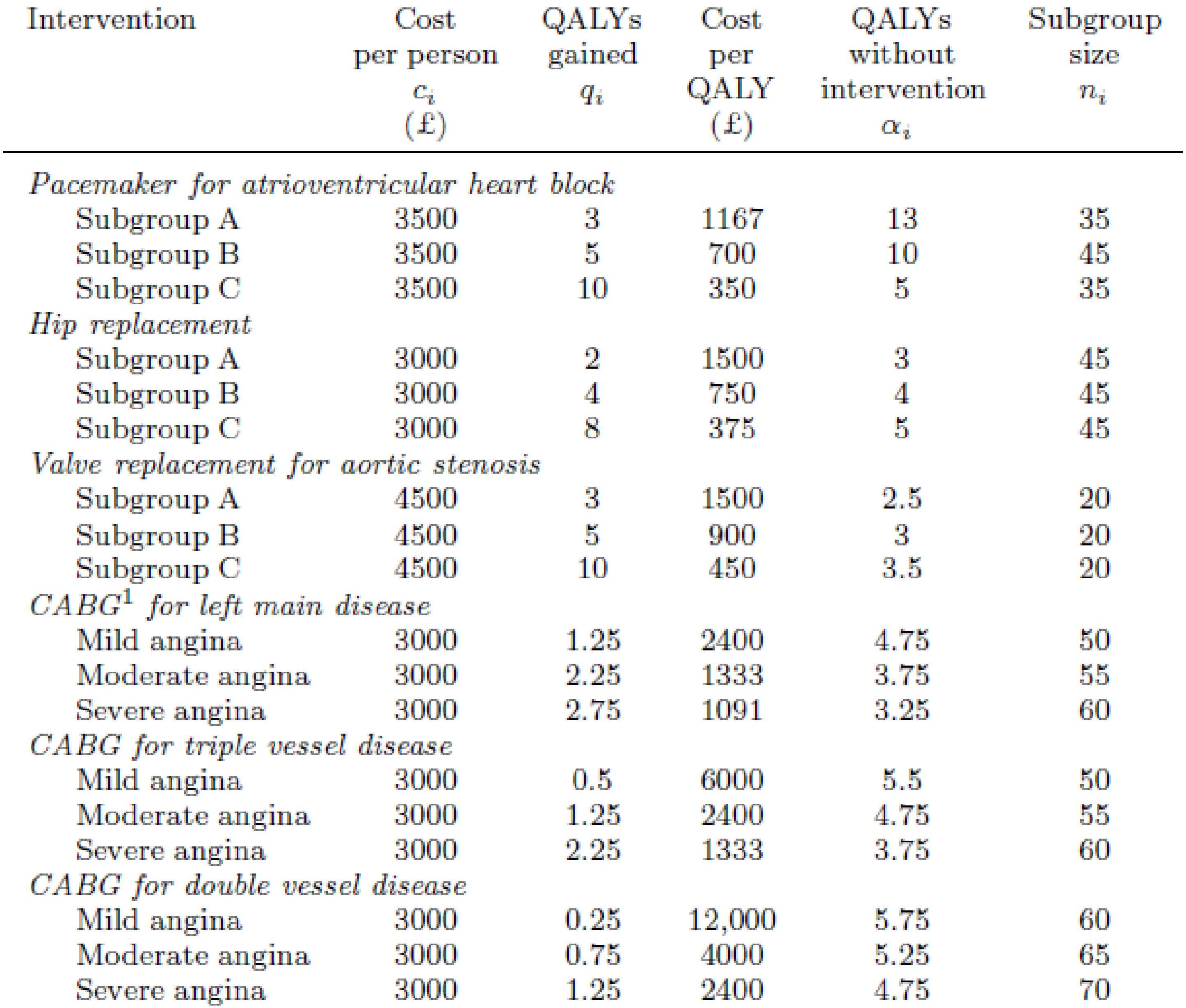
- Based on budget decisions in UK National Health Service
- Allocate treatment resources to patient groups
  - Groups characterized by disease and prognosis.
  - Based on cost and estimated QALY estimates with and without treatment*
  - We will solve this example later.**

*Data reflect a particular situation and are not valid in general.

**Solutions presented here should not be taken as a general recommendation for healthcare resource allocation.

QALY & cost data

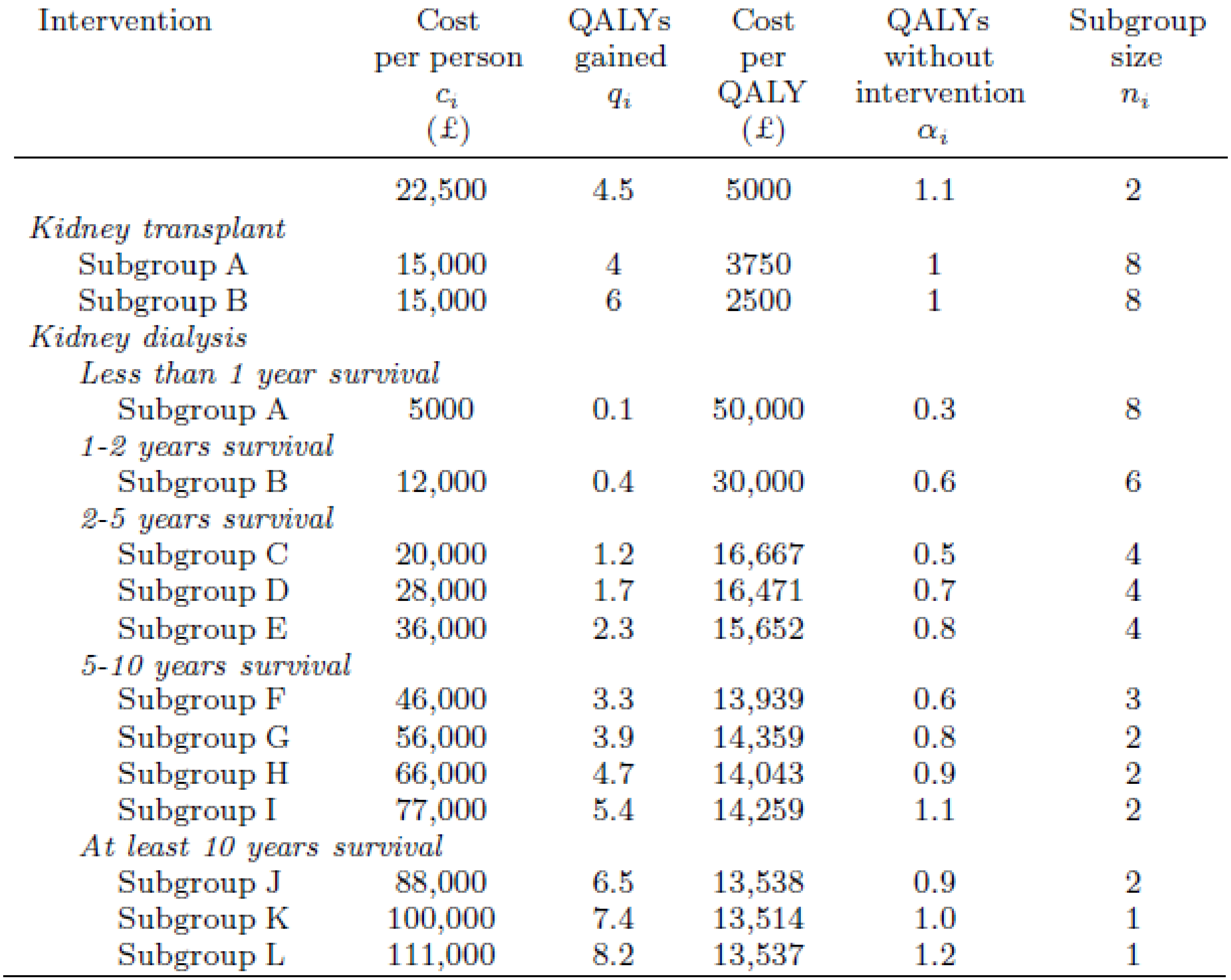
Part 1



Intervention	Cost per person c_i (£)	QALYs gained q_i	Cost per QALY (£)	QALYs without intervention α_i	Subgroup size n_i
Pacemaker for atrioventricular heart block					
Subgroup A	3500	3	1167	13	35
Subgroup B	3500	5	700	10	45
Subgroup C	3500	10	350	5	35
Hip replacement					
Subgroup A	3000	2	1500	3	45
Subgroup B	3000	4	750	4	45
Subgroup C	3000	8	375	5	45
Valve replacement for aortic stenosis					
Subgroup A	4500	3	1500	2.5	20
Subgroup B	4500	5	900	3	20
Subgroup C	4500	10	450	3.5	20
CABG[1] for left main disease					
Mild angina	3000	1.25	2400	4.75	50
Moderate angina	3000	2.25	1333	3.75	55
Severe angina	3000	2.75	1091	3.25	60
CABG for triple vessel disease					
Mild angina	3000	0.5	6000	5.5	50
Moderate angina	3000	1.25	2400	4.75	55
Severe angina	3000	2.25	1333	3.75	60
CABG for double vessel disease					
Mild angina	3000	0.25	12,000	5.75	60
Moderate angina	3000	0.75	4000	5.25	65
Severe angina	3000	1.25	2400	4.75	70

QALY & cost data

Part 2



Intervention	Cost per person c_i (£)	QALYs gained q_i	Cost per QALY (£)	QALYs without intervention α_i	Subgroup size n_i
	22,500	4.5	5000	1.1	2
Kidney transplant					
Subgroup A	15,000	4	3750	1	8
Subgroup B	15,000	6	2500	1	8
Kidney dialysis					
Less than 1 year survival					
Subgroup A	5000	0.1	50,000	0.3	8
1-2 years survival					
Subgroup B	12,000	0.4	30,000	0.6	6
2-5 years survival					
Subgroup C	20,000	1.2	16,667	0.5	4
Subgroup D	28,000	1.7	16,471	0.7	4
Subgroup E	36,000	2.3	15,652	0.8	4
5-10 years survival					
Subgroup F	46,000	3.3	13,939	0.6	3
Subgroup G	56,000	3.9	14,359	0.8	2
Subgroup H	66,000	4.7	14,043	0.9	2
Subgroup I	77,000	5.4	14,259	1.1	2
At least 10 years survival					
Subgroup J	88,000	6.5	13,538	0.9	2
Subgroup K	100,000	7.4	13,514	1.0	1
Subgroup L	111,000	8.2	13,537	1.2	1

Healthcare Example

Budget constraint

$$\sum_j n_j c_j y_j \leq \bar{B}$$

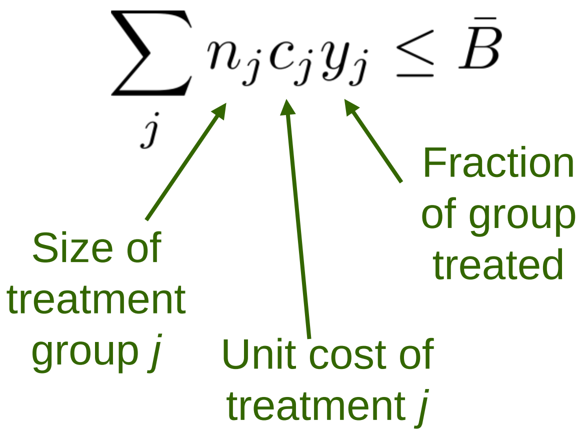
Size of treatment group *j*

Unit cost of treatment *j*

Fraction of group treated

Healthcare Example

Budget constraint



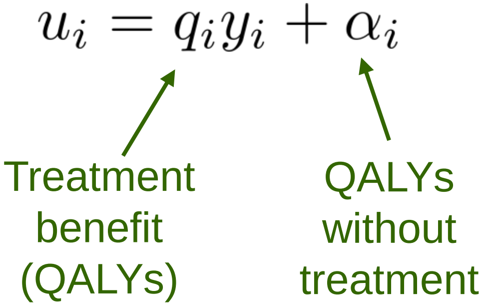
$$\sum_j n_j c_j y_j \leq \bar{B}$$

Size of treatment group *j*

Unit cost of treatment *j*

Fraction of group treated

Utility function



$$u_i = q_i y_i + \alpha_i$$

Treatment benefit (QALYs)

QALYs without treatment

which implies $y_i = (u_i - \alpha_i)/q_i$

Healthcare Example

Budget constraint

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$$u_i = q_i y_i + \alpha_i$$

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So the optimization problem $\max_{\boldsymbol{u}} \left\{ W(\boldsymbol{u}) \;\middle|\; \boldsymbol{a}^\intercal \boldsymbol{u} \leq B,\; \boldsymbol{c} \leq \boldsymbol{u} \leq \boldsymbol{d} \right\}$

becomes

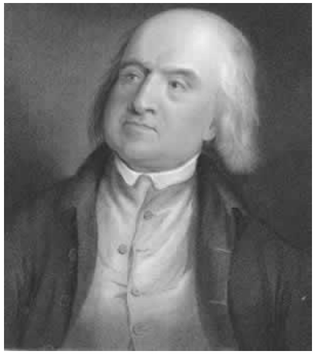
$$\max_{\boldsymbol{u}} \left\{ W(\boldsymbol{u}) \;\middle|\; \sum_j \frac{n_j c_j}{q_j} u_j \leq \bar{B} + \sum_j \frac{n_j c_j \alpha_j}{q_j};\;\; \boldsymbol{\alpha} \leq \boldsymbol{u} \leq \boldsymbol{q} + \boldsymbol{\alpha} \right\}$$

Utilitarian SWF

$$W(\boldsymbol{u}) = \sum_j u_j$$

Proposition. An optimal utilitarian distribution, subject to a resource constraint, allocates **all utility** to individual 1:

$$u_1 = B/a_1, \quad u_j = 0 \text{ for } j = 2, \ldots, n$$

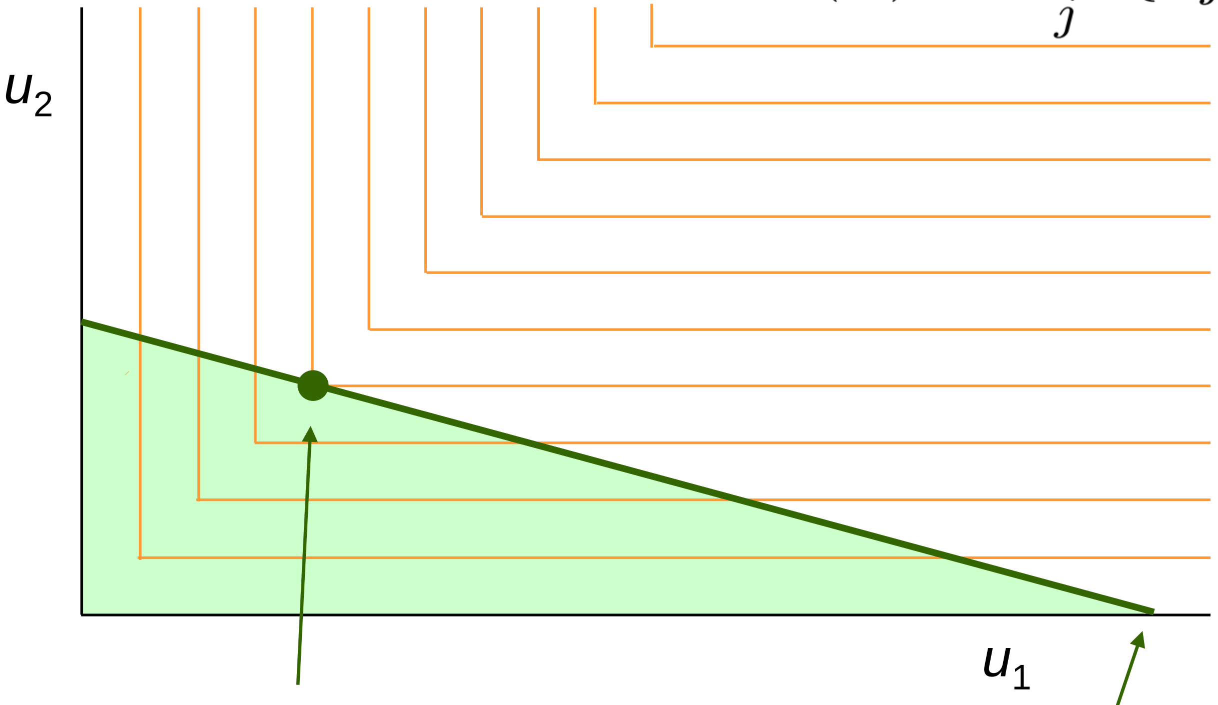


J. Bentham (1776)

Maximin SWF

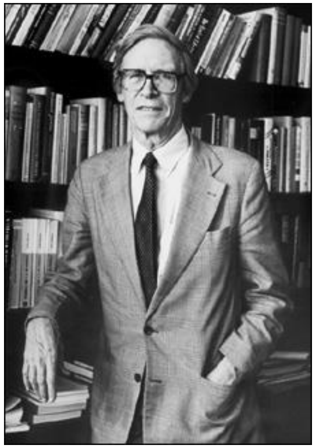
$$W(\boldsymbol{u}) = \min_j \{u_j\}$$

Optimal distribution subject to budget constraint



Utility equalized, Patient 2 gets most of the resources.

Substantial sacrifice of Patient 1



J. Rawls (1971)

Maximin SWF

$$W(\boldsymbol{u}) = \min_j \{u_j\}$$

Proposition. An optimal maximin distribution, subject to a resource constraint, distributes utility **equally**:

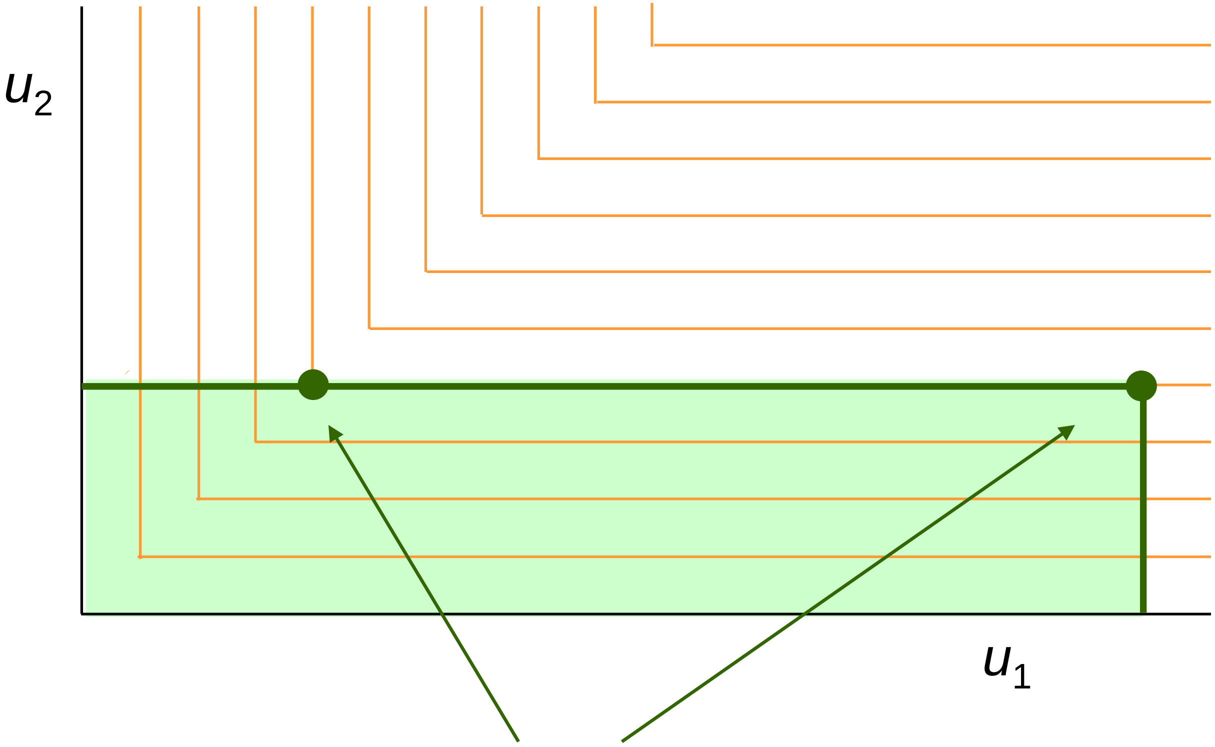
$$u_j = \frac{B}{\sum_i a_i} \text{ for } j = 1, \ldots, n$$

Note: Rawls intended this criterion to apply only to the design of **social institutions** and the distribution of ”**primary goods**.”

Maximin SWF

$$W(\boldsymbol{u}) = \min_j \{u_j\}$$

Optimal distribution subject to **resource bounds**



These solutions have same social welfare!

Maximin SWF

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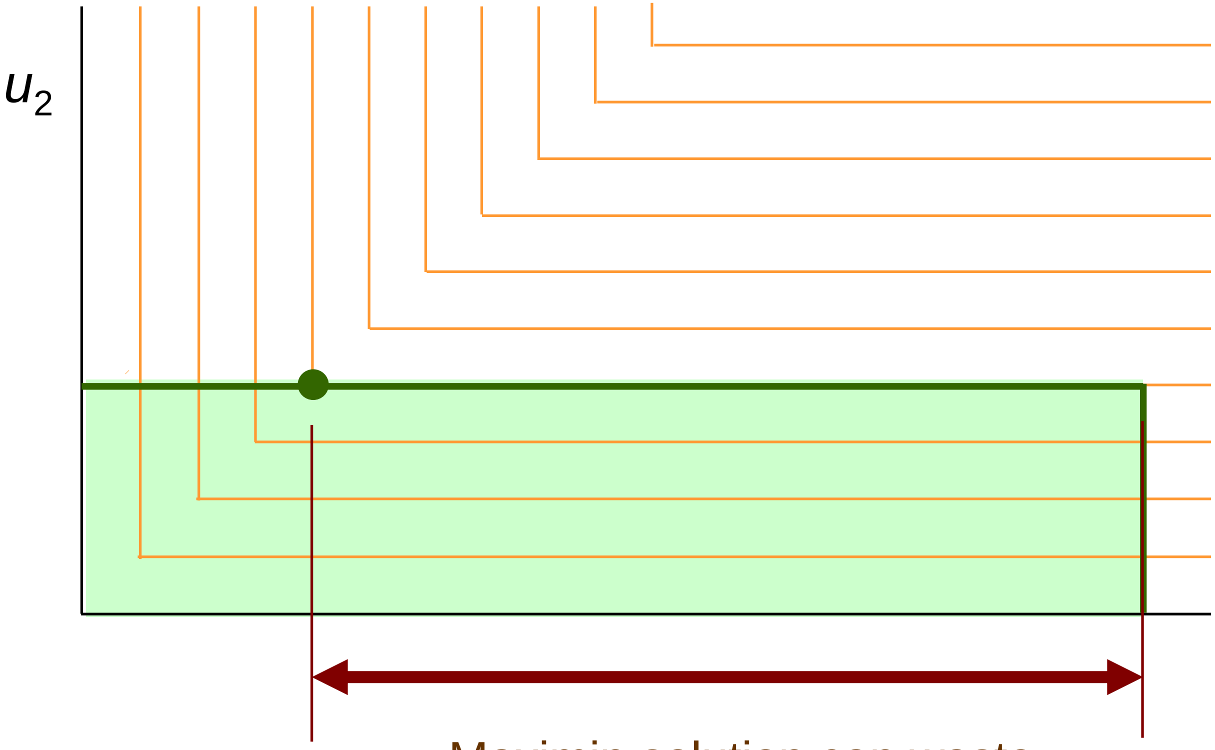
Optimal distribution subject to **resource bounds**



Maximin solution can waste most of the resources!

Maximin SWF

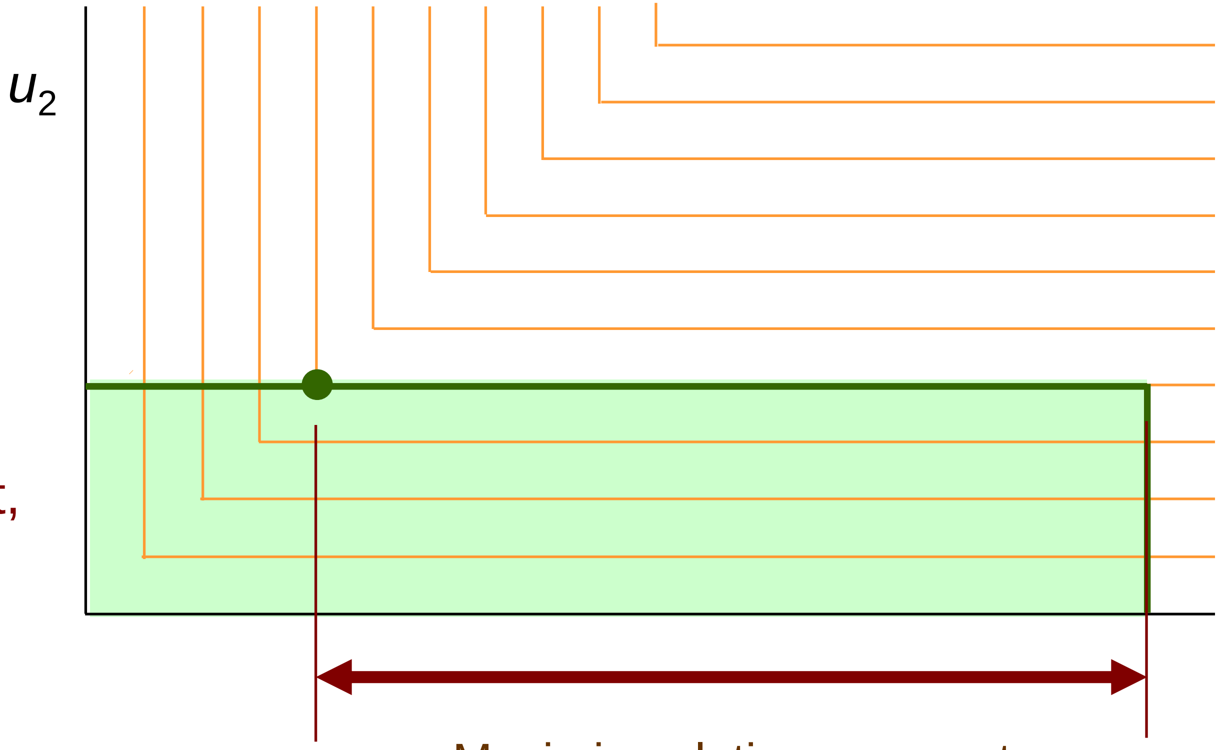
$$W(\boldsymbol{u}) = \min_j \{u_j\}$$

Optimal distribution subject to **resource bounds**

Remedy: use **leximax** solution

Maximize smallest, then 2nd smallest, etc.

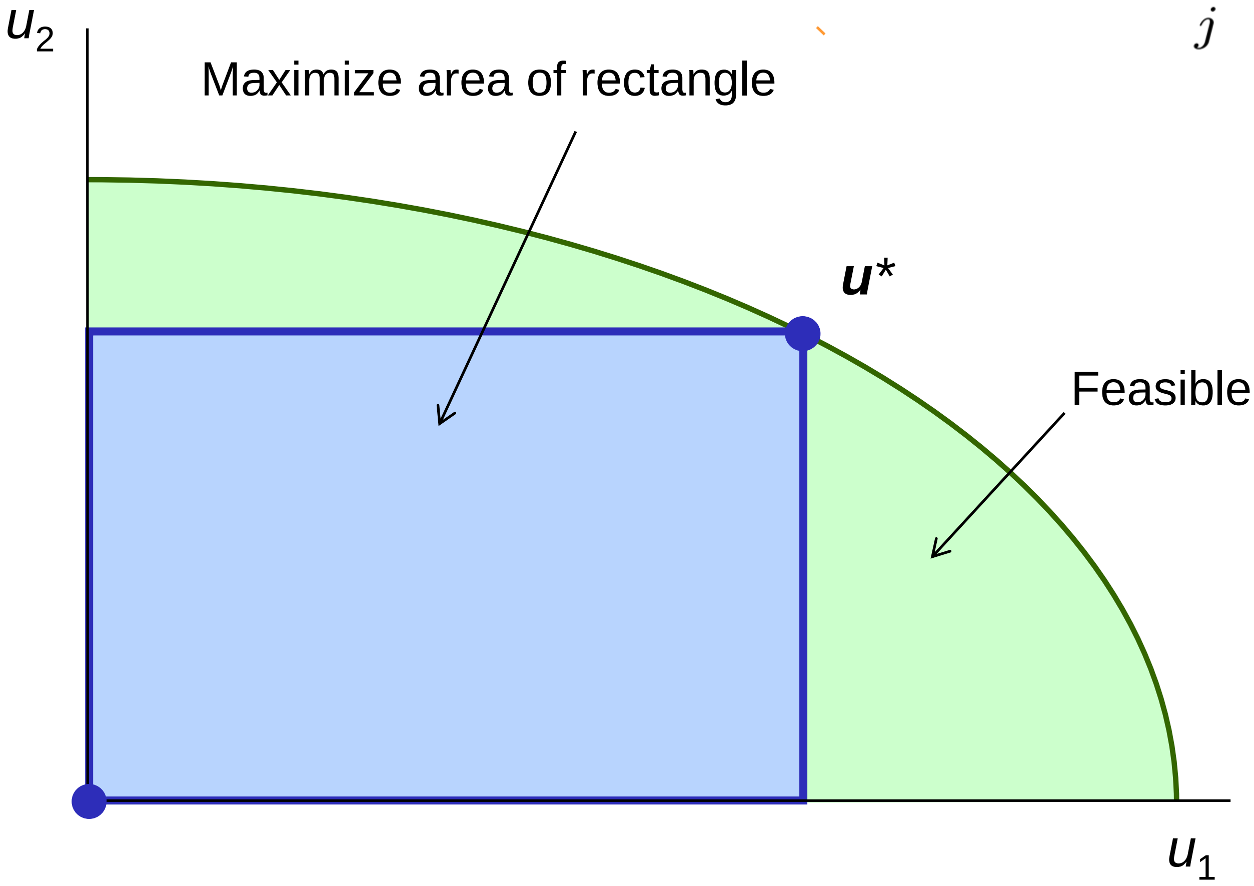
Extreme **sacrifice** can remain.

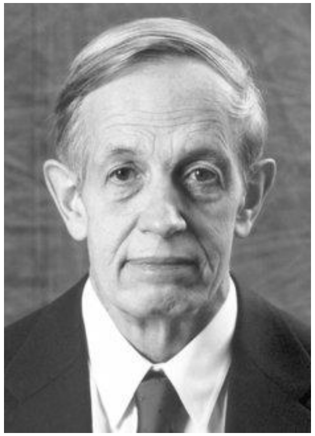


Maximin solution can waste most of the resources!

Nash Bargaining Solution
= Proportional fairness

$$W(\boldsymbol{u}) = \prod_j u_j$$



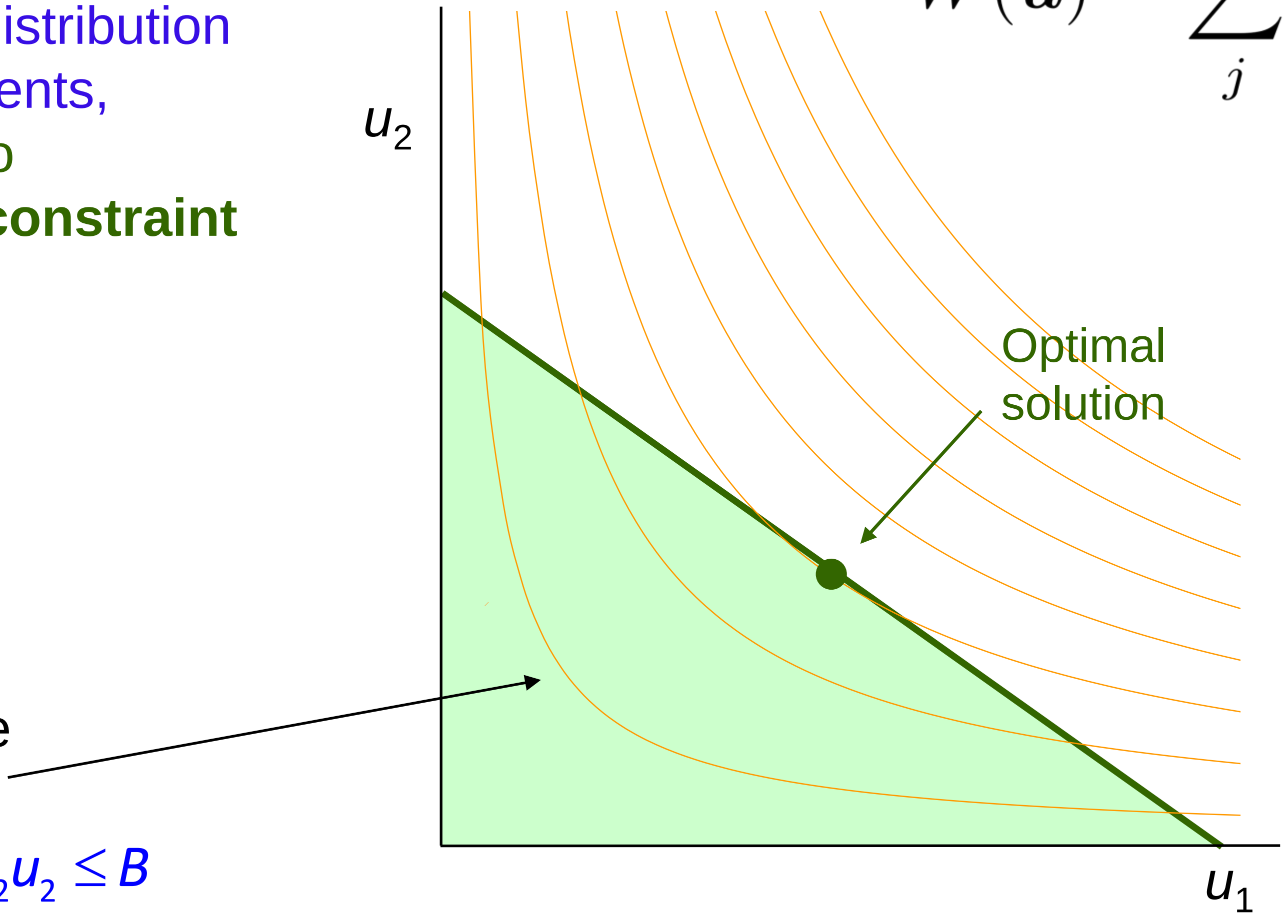


J. Nash
(1950)

Nash Bargaining Solution

Optimal social welfare distribution for 2 patients, subject to **budget constraint**

$$W(\boldsymbol{u}) = \sum_j log(u_j)$$



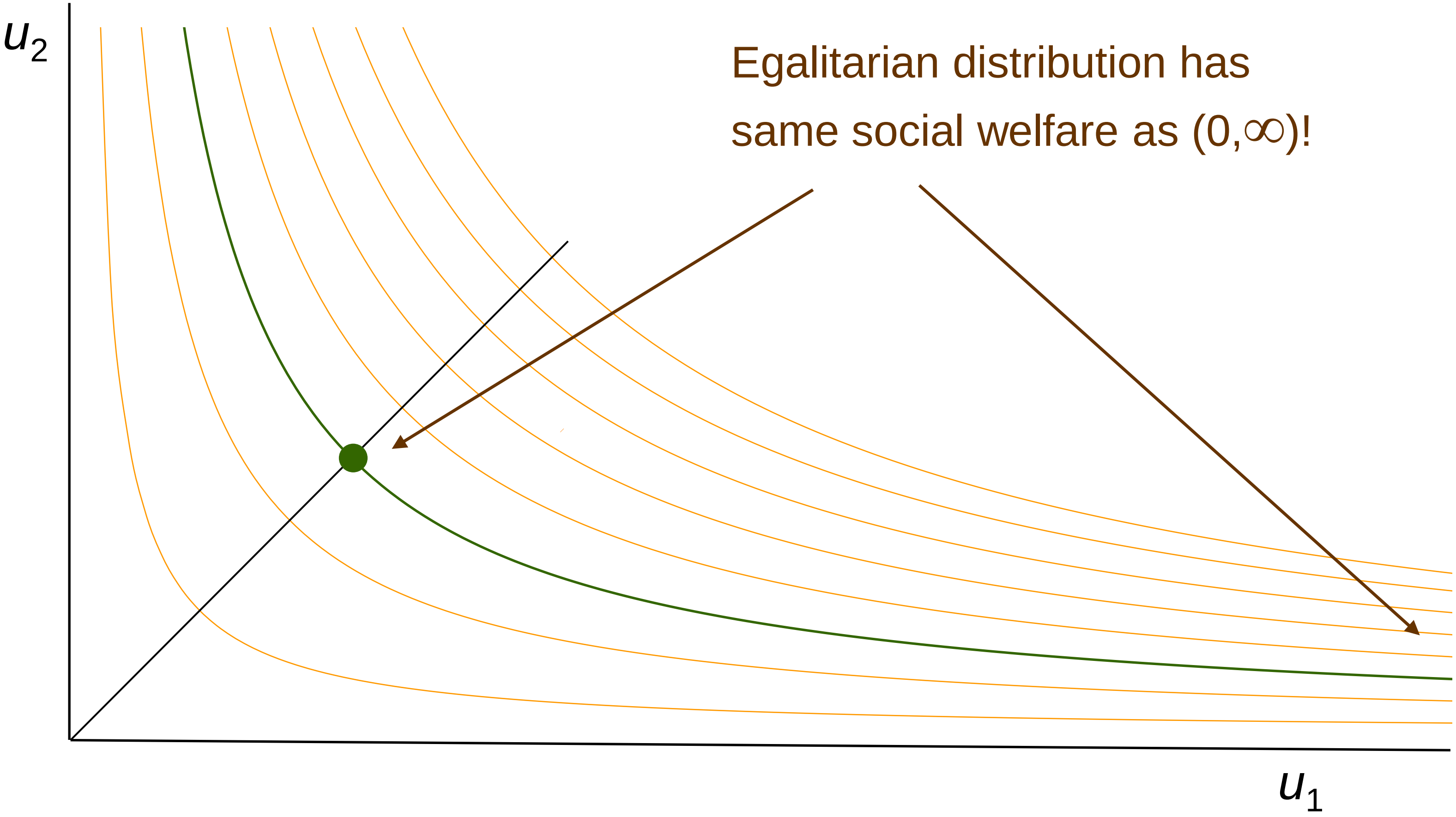
Optimal solution

Feasible region

$a_1u_1 + a_2u_2 \leq B$

Nash Bargaining Solution

$$W(\boldsymbol{u}) = \sum_j log(u_j)$$



Egalitarian distribution has same social welfare as (0,∞)!

Alpha Fairness

Generalizes proportional fairness, which corresponds to $\alpha = 1$

$$W(\boldsymbol{u}) = \begin{cases} \frac{1}{1-\alpha} \sum_j u_j^{1-\alpha}, & \text{for } \alpha \geq 0 \text{ and } \alpha \neq 1 \\ \sum_j \log(u_j), & \text{for } \alpha = 1 \end{cases}$$

Larger α corresponds to greater fairness.

$\alpha = 0$: utilitarian, $\alpha = \infty$: maximin

Alpha Fairness

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Larger α corresponds to greater fairness.

$\alpha = 0$: utilitarian, $\alpha = \infty$: maximin

Proposition. An optimal alpha fairness distribution, subject to a budget constraint, is

$$u_j = \frac{B}{a_j^{1/\alpha} \sum_i a_i^{1-1/\alpha}} \text{ for } j = 1, \ldots, n$$

Alpha Fairness

Proposition. When $\alpha \geq 1$, an egalitarian distribution has the same social welfare as one with **arbitrarily great inequality**.

Specifically, $(1, 1)$ has the same social welfare as $\left(\infty, 2^{1/(1-\alpha)}\right)$

Alpha Fairness

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Specifically, $(1, 1)$ has the same social welfare as $\left(\infty, 2^{1/(1-\alpha)}\right)$

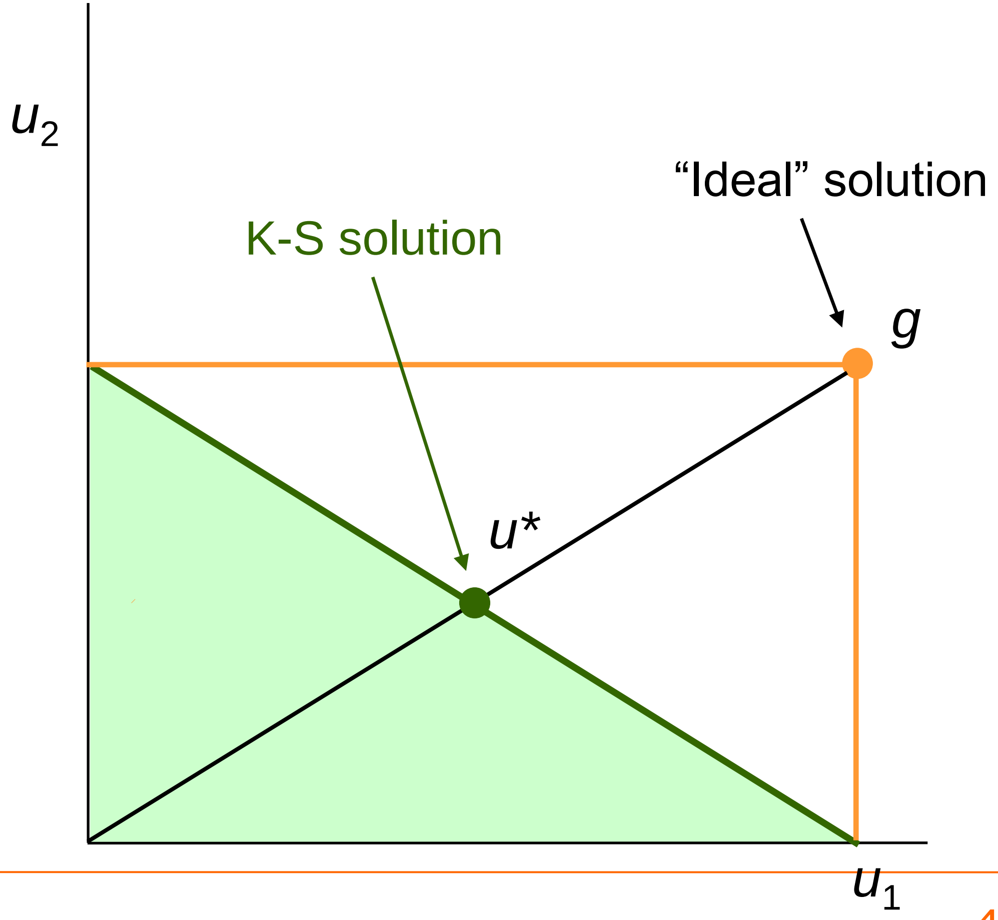
Another problem: How to choose α?

Kalai-Smorodinsky Bargaining

Patients receive an equal fraction of their possible utility gains.

Budget constraint.

$$\frac{u_1^*}{u_2^*} = \frac{g_1}{g_2}$$



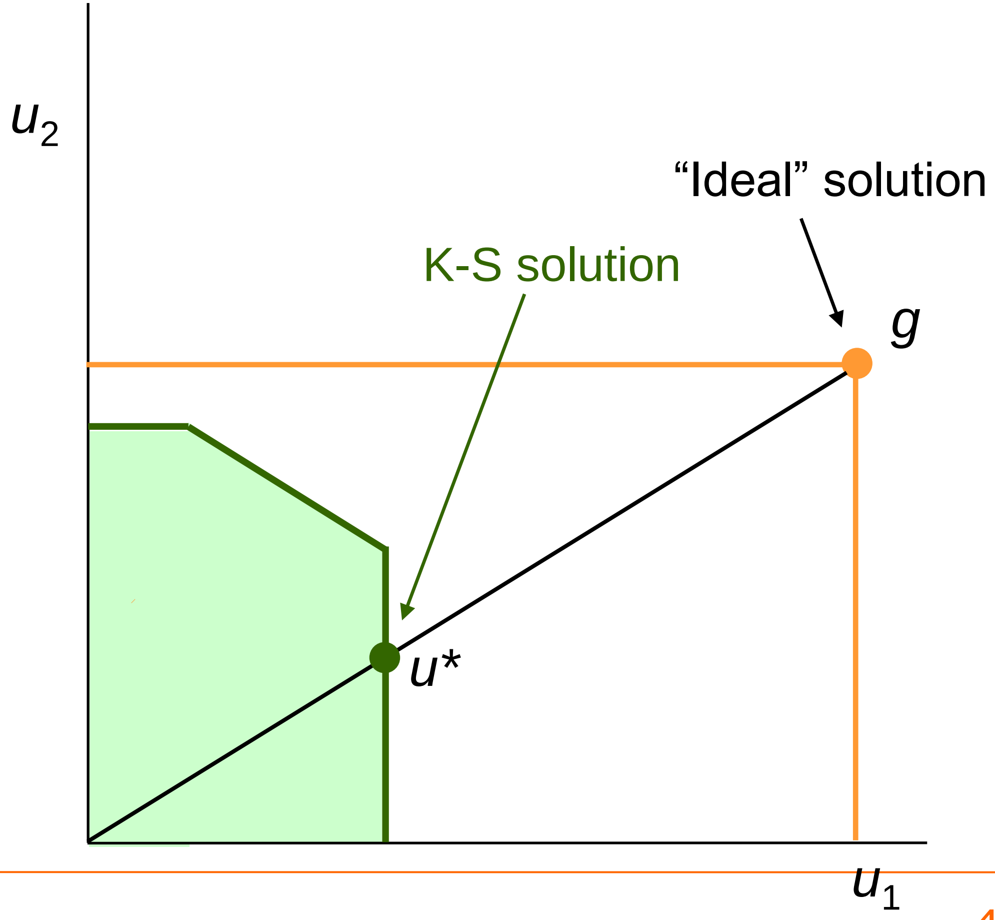
Kalai & Smorodinsky (1975)

Kalai-Smorodinsky Bargaining

Patients receive an equal fraction of their possible utility gains.

Budget constraint + bounds

Kalai & Smorodinsky (1975)



Kalai-Smorodinsky Bargaining

Proposition. The Kalai-Smorodinsky bargaining solution, subject to a budget constraint and bounds d_j, is

$$u_j = \frac{Bd_j}{\sum_i a_i d_i}, \ \text{all } j$$

Without bounds, the solution is

$$u_j = \frac{1}{n} \cdot \frac{B}{a_j}, \ \text{all } j$$

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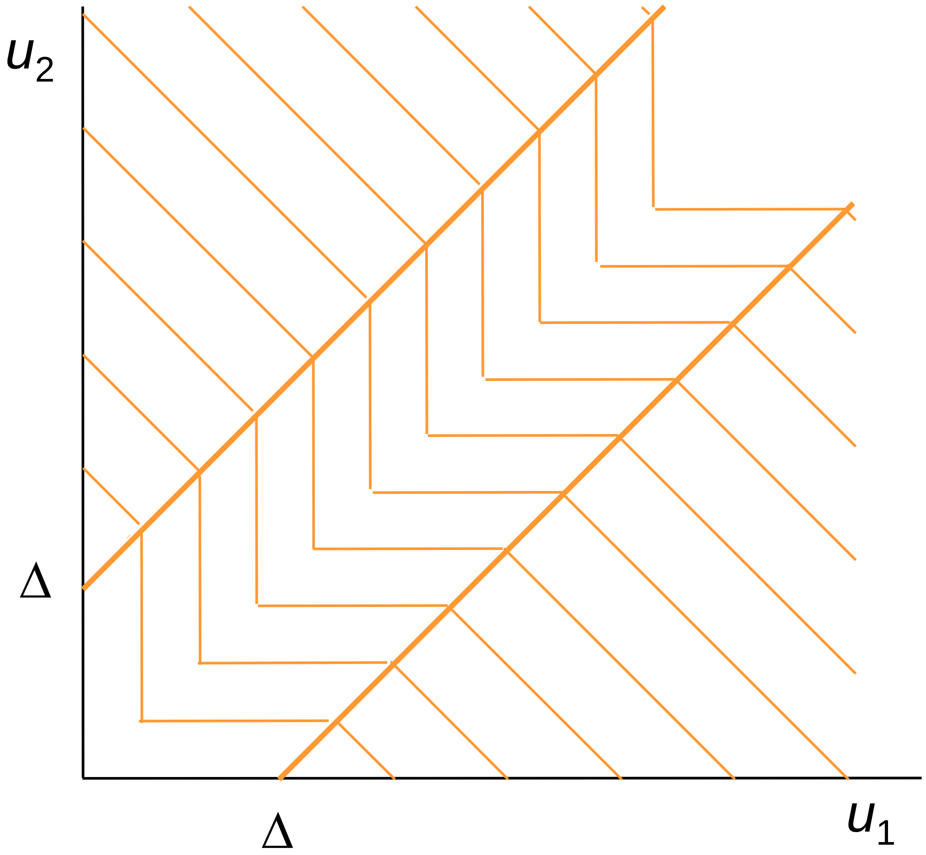
$$u_j = \frac{1}{n} \cdot \frac{B}{a_j}, \text{ all } j$$

Can be suitable for wage or price negotiation.

Questionable for medical applications. Transfers resources from cancer patients to sufferers of common cold to equalize their relative concession.

A Threshold SWF

Use maximin criterion until it results in excessive sacrifice by some patients



A. Williams & R. Cookson (2000)

A Threshold SWF

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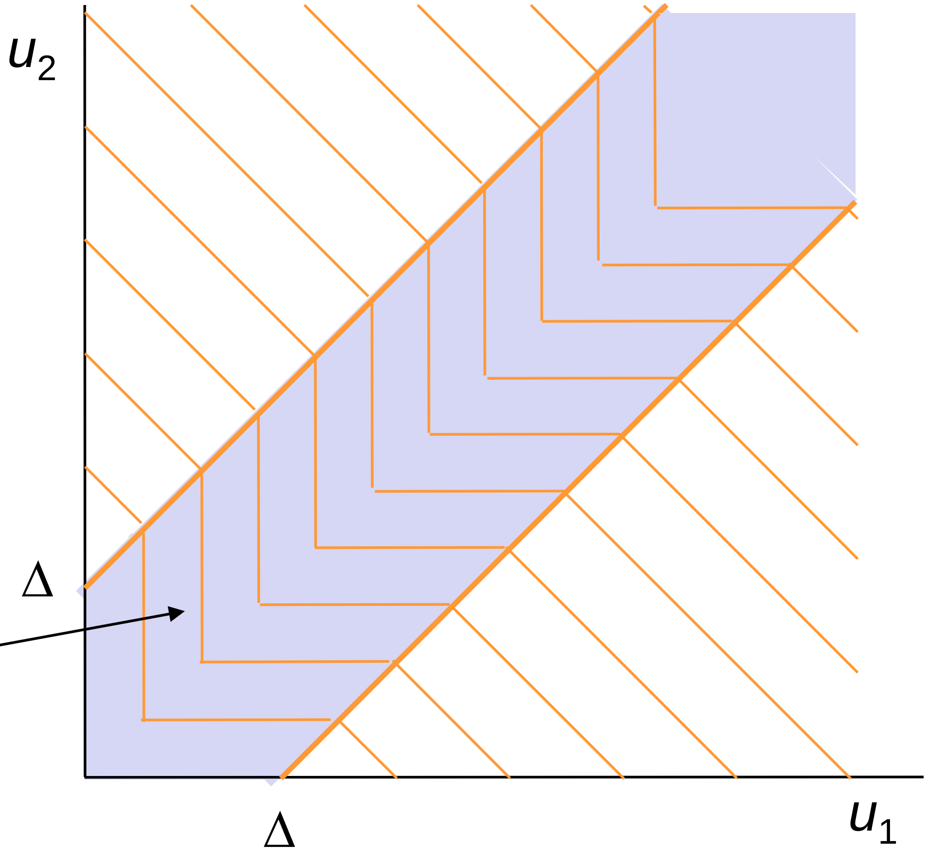


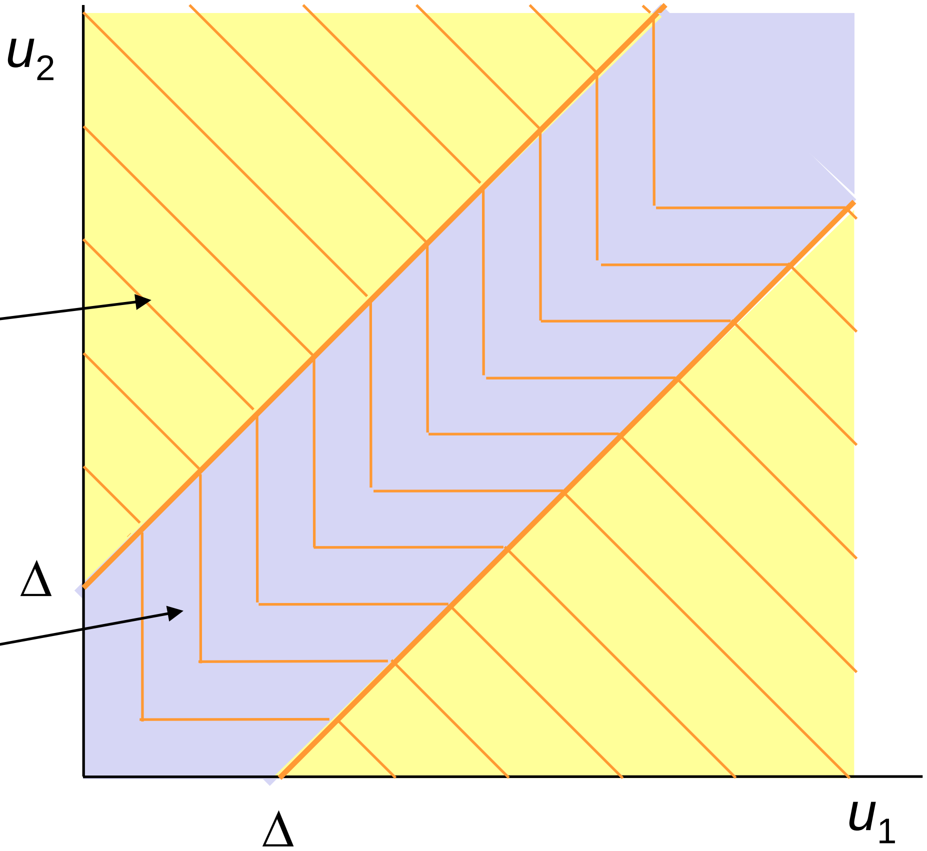
Maximin region

$$2\min\{u_1, u_2\} + \Delta$$

A. Williams & R. Cookson (2000)

A Threshold SWF

Use maximin criterion until it results in excessive sacrifice by some patients



Utilitarian region

$u_1 + u_2$

Maximin region

$2\min\{u_1, u_2\} + \Delta$

Ensures continuous contours

A Threshold SWF

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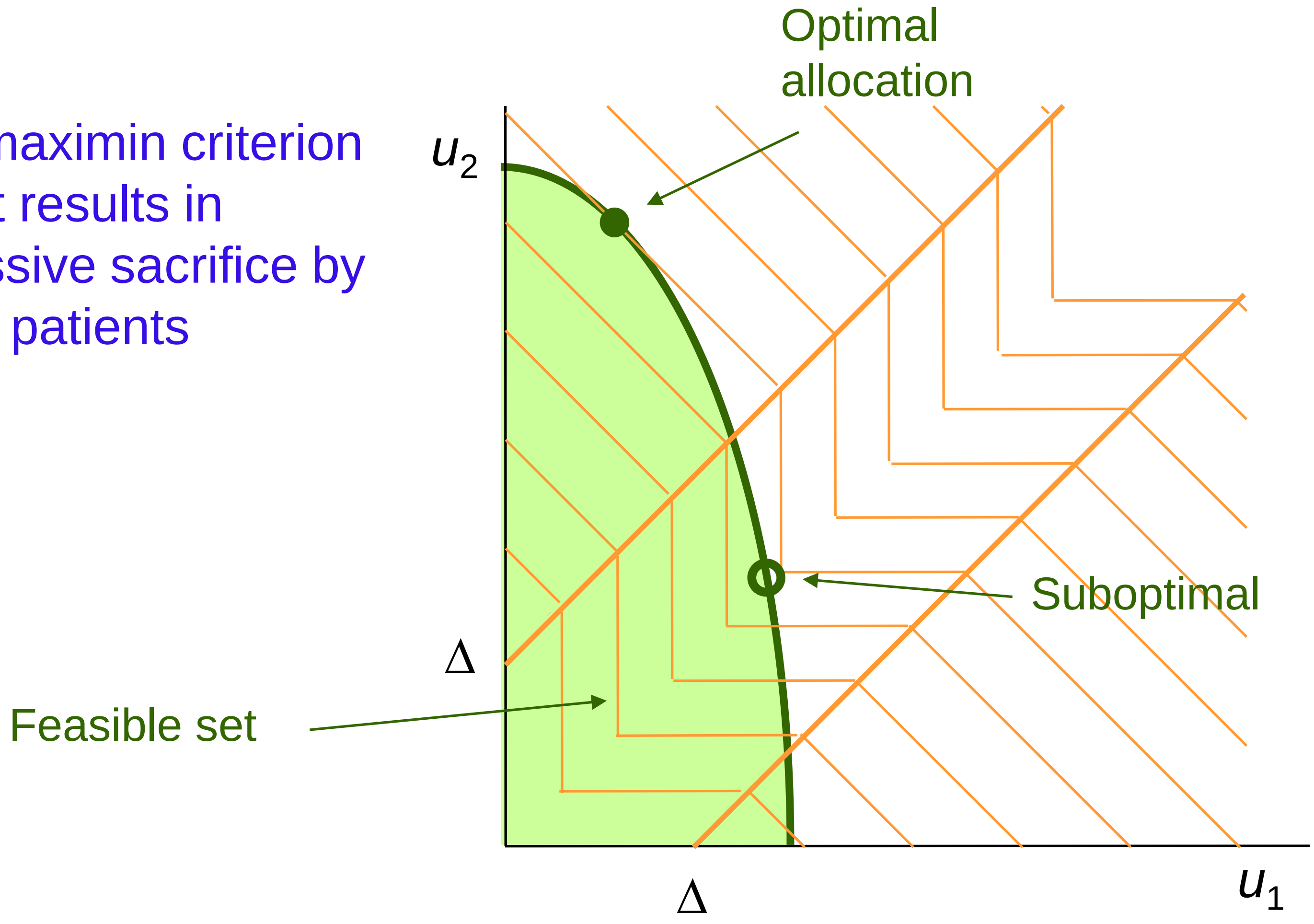


A. Williams & R. Cookson (2000)

A Threshold SWF

Generalize to *n* persons:

$$W(\boldsymbol{u}) = (n-1)\Delta + n u_{\min} + \sum_i \max\left\{0,\ u_i - u_{\min} - \Delta\right\}$$

JH & H. P. Williams (2012)

Disadvantaged individuals receive some priority.

Choose ***Δ*** so that those with utilities in **fair region** (within *Δ* of smallest, $u_{\min}$) **deserve priority**.

Δ **= 0:** utilitarian SWF (no fair region)

Δ **= ∞:** maximin SWF (all utilities in fair region)

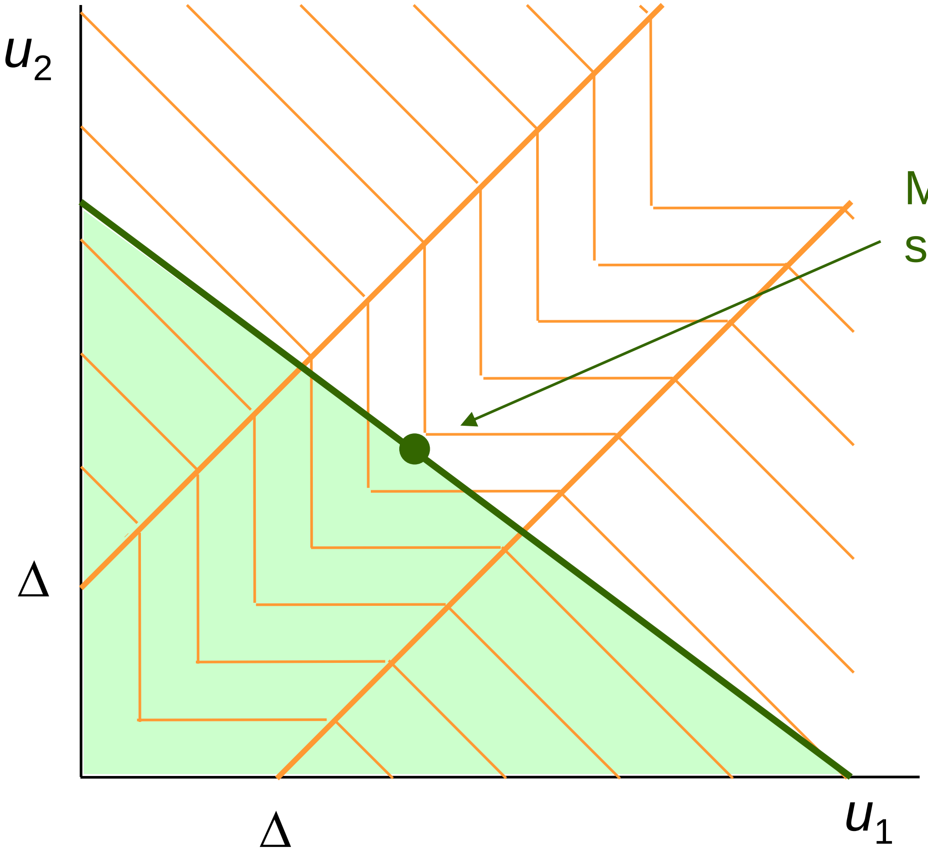
Utilities in fair region are equated with smallest utility, which receives weight equal to number of utilities in fair region.

A Threshold SWF

Maximize threshold SWF subject to **budget constraint**

Optimal solution is **maximin** or **utilitarian**, depending on Δ and cost coefficients a_j

Patients have **similar** treatment costs, or Δ is **large**.



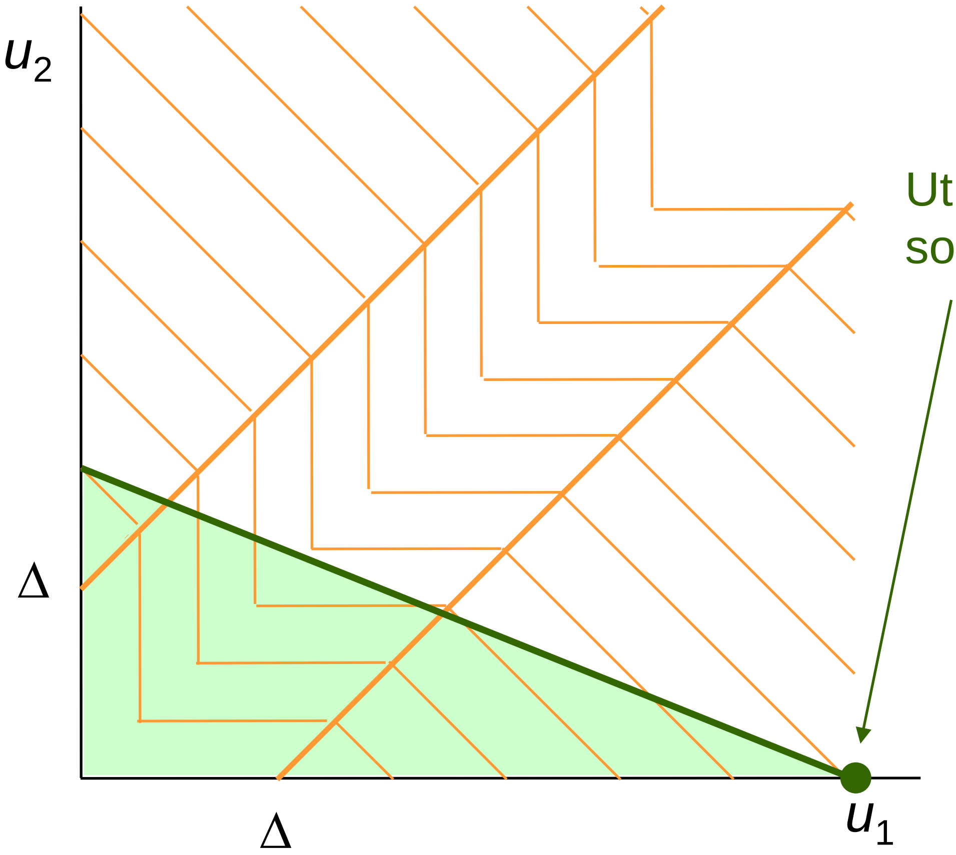
Maximin solution

A Threshold SWF

Maximize threshold SWF subject to **budget constraint**

Optimal solution is **maximin** or **utilitarian**, depending on Δ and cost coefficients a_j

Patients have **very different** treatment costs, or Δ is **small**.



Utilitarian solution

A Threshold SWF

Proposition. The threshold solution, subject to a budget constraint, is purely **maximin** if

$$\Delta \geq B\left(\frac{1}{a_1} - \frac{n}{\sum_i a_i}\right)$$

and purely **utilitarian** otherwise.

We again have **extreme solutions**, although we can adjust Δ to choose between utilitarian and maximin.

A Threshold SWF

Proposition. The threshold solution, subject to a budget constraint, is purely **maximin** if

$$\Delta \geq B\left(\frac{1}{a_1} - \frac{n}{\sum_i a_i}\right)$$

and purely **utilitarian** otherwise.

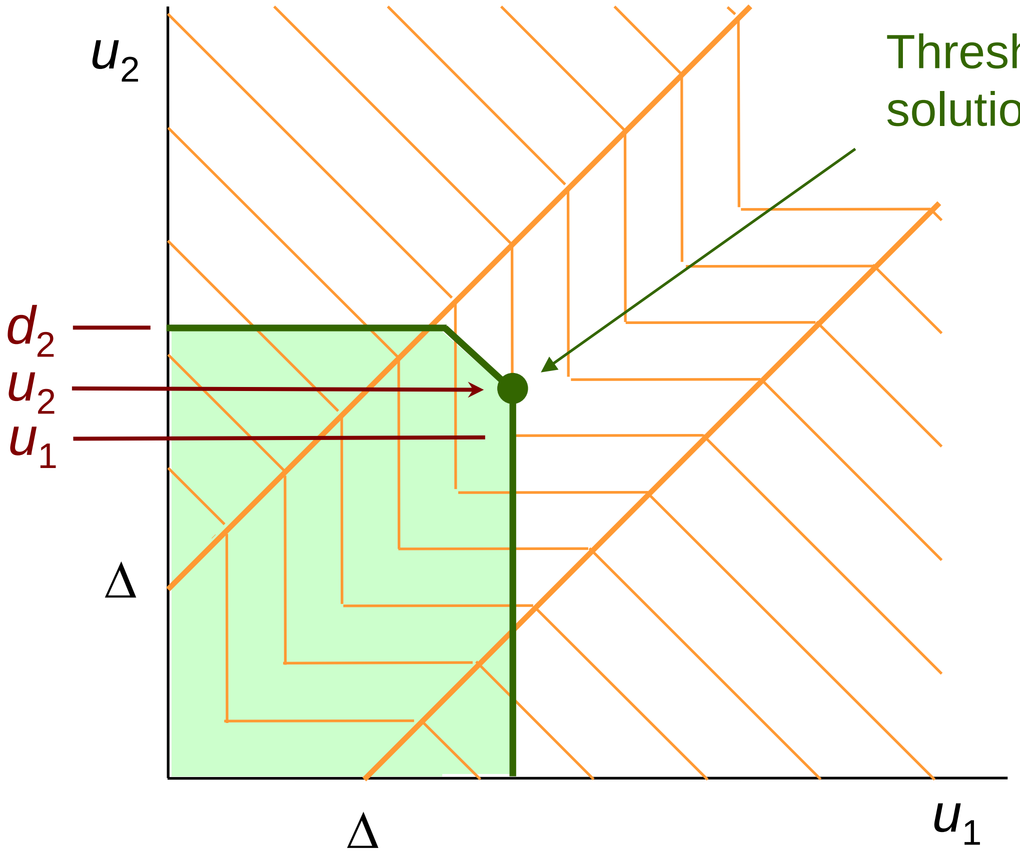
We again have **extreme solutions**, although we can adjust Δ to choose between utilitarian and maximin.

Solutions are more reasonable, and more interesting, when we add **utility bounds**...

A Threshold SWF

Maximize threshold SWF subject to **budget constraint and bounds**

One utility u_2 is **strictly between** the corresponding upper bound d_2 and the smallest utility.



A Threshold SWF

Proposition. In a threshold solution subject to a budget constraint and bounds, **at most one** utility u_j is **strictly between** its upper bound d_j and the smallest utility $\min_j \{u_j\}$

So in a threshold solution...

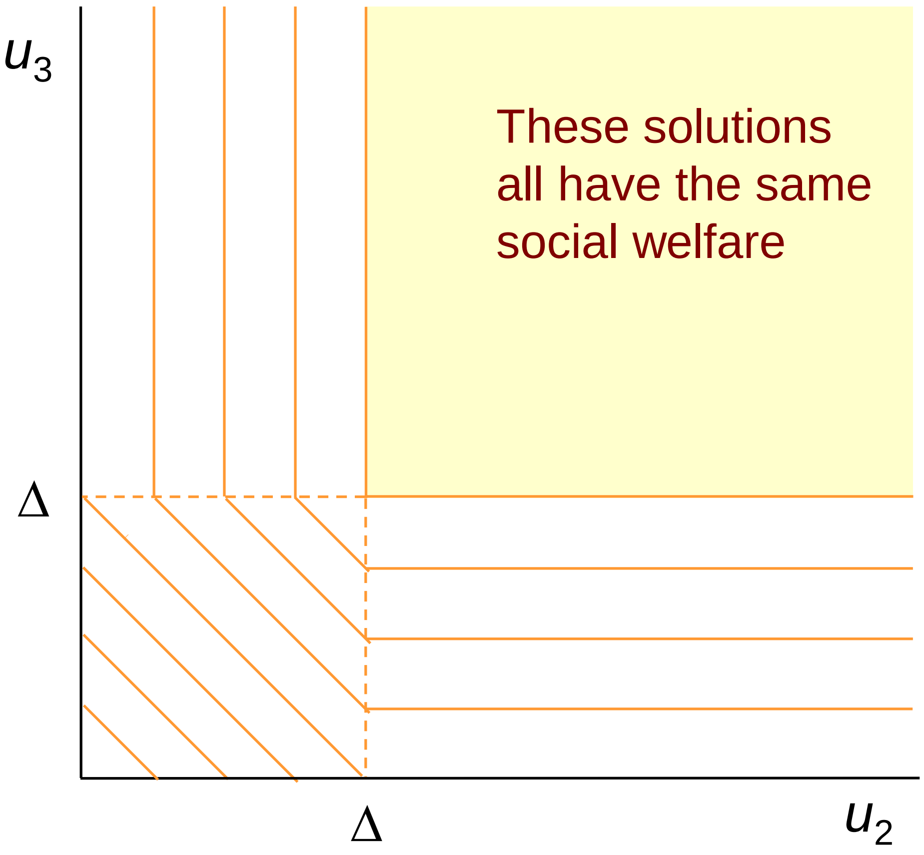
... patients (with one possible exception) are **either** as well off as they could be, **or** are one of the worst-off...

...keeping in mind that we are giving **substantial priority** to the worst-off (i.e., priority proportional to the number of utilities within Δ of the lowest).

A Threshold SWF

Contours
of $W(0,u_2,u_3)$
in a 3-patient
example

Problem:
Too many
solutions with
different equity
properties
have the same
social welfare.



...because we combine utility with **maxmin**

Threshold SWF with Leximax

Solution: Combine utilitarian and **leximax** criteria by maximizing a sequence of SWFs:

i-th smallest utility

$$W_k(\boldsymbol{u}) = \sum_{i=1}^{k-1}(n-i+1)\boxed{u_{\langle i\rangle}} + (n-k+1)\min\left\{u_{\langle 1\rangle}+\Delta, u_{\langle k\rangle}\right\} + \sum_{i=k}^{n}\max\left\{0, u_{\langle i\rangle}-u_{\langle 1\rangle}-\Delta\right\}, \quad k=2,\ldots,n$$

that determine smallest utility, 2nd smallest, etc., with decreasing priority.

This is more sensitive to equity for disadvantaged patients other than the very worst-off.

V. Chen & JH (2021)

Threshold SWF with leximax

Proposition: In a socially optimal solution subject to a **budget constraint**, solution may be **neither utilitarian nor maximin**.

Proposition: In a socially optimal solution subject to a budget constraint and **bounds**, **several** utilities may lie strictly between their upper bounds and the smallest utility.

We **may no longer have extreme solutions** when maximizing social welfare subject to simple generic constraints.

Healthcare Example

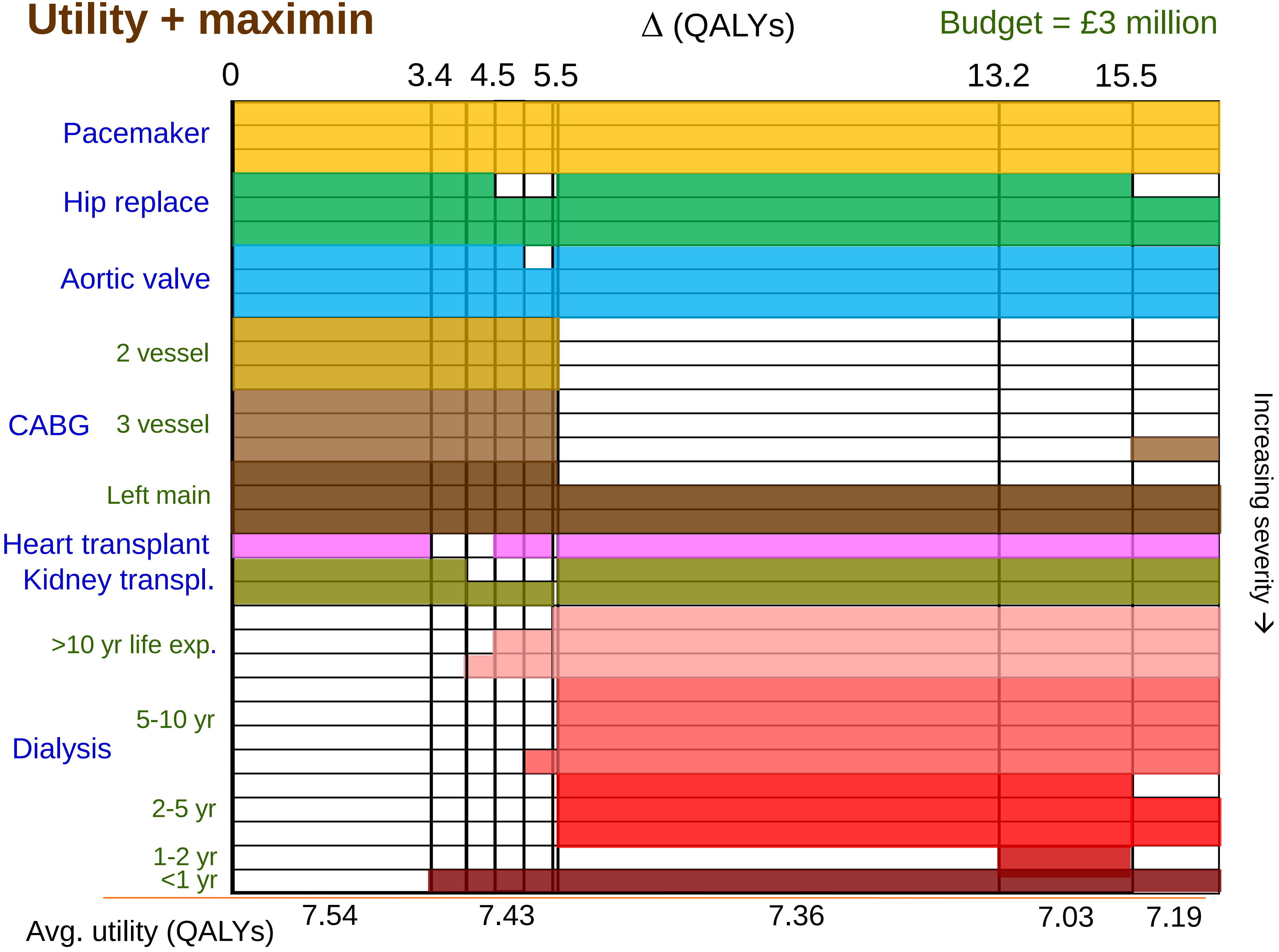
Returning to the healthcare example, we examine socially optimal solutions using **threshold SWFs** with

utility + maximin and **utility + leximax**

The solutions are **quite different**...

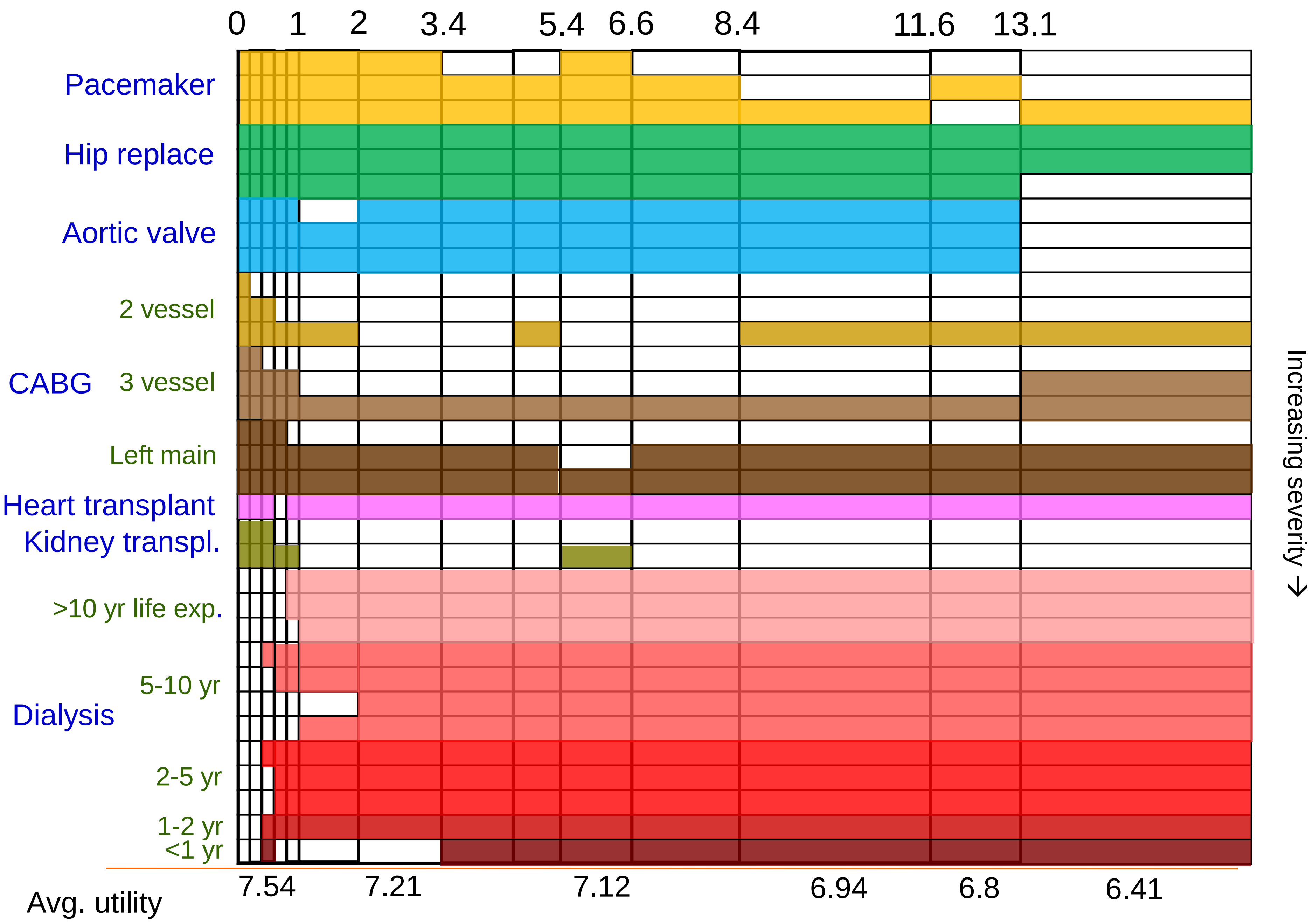
Utility + maximin

Budget = £3 million

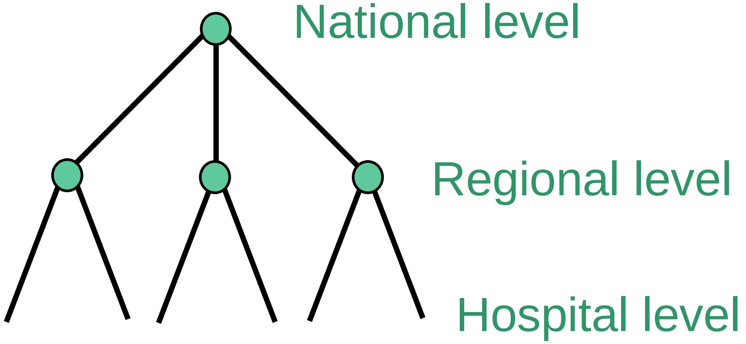


Utility + leximax

Budget = £3 million



Further Results



- Hierarchical distribution.
  - **More sophisticated SWFs** are less likely to be **regionally decomposable.**
  - That is, regions must consider national picture when allocating supplies received from national level, even if they use the same SWF as the national authority.

Survey Articles

V. Chen and J. N. Hooker, A guide to formulating equity and fairness in an optimization model, forthcoming.

O. Karsu and A. Morton, Inequality averse optimisation in operational research. *European Journal of Operational Research* (2015) 343-359.

W. Ogryczak et al., Fair optimization and networks: A survey. *Journal of Applied Mathematics* (2014) 1-25.

J. N. Hooker, Moral implications of rational choice theories, *Handbook of the Philosophical Foundations of Business Ethics* (2013) 1459-1476.

Further work

O. Karsu and A. Morton, Inequality averse optimisation in operations research, *EJOR* (2015) 343-359,

V. Chen and J. N. Hooker, Balancing fairness and efficiency in an optimization model, submitted (available on arXiv).