Equitable Allocation of Scarce Medical Resources

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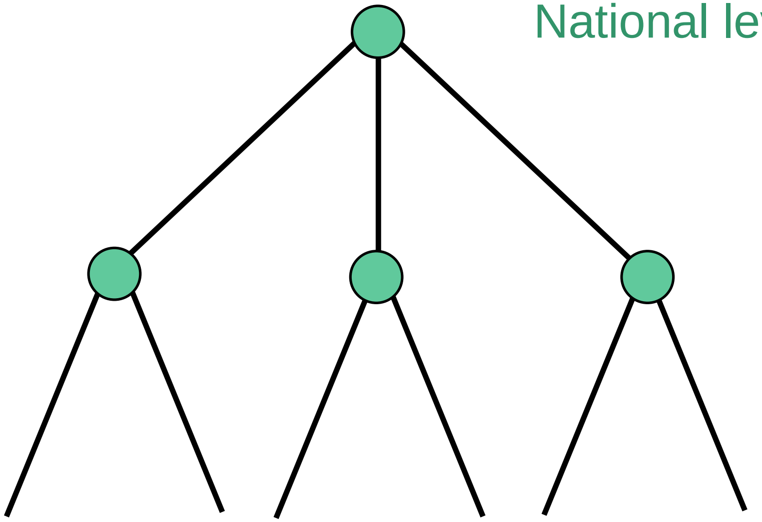
The Problem

- Allocation of scare medical resources.
    - Across regions, hospitals.
    - Consider **equity** as well as **efficiency**.
    - Use **robust optimization** to allow for demand uncertainty and take advantage of expert knowledge.
- Overall research goal.
    - **Multi-period** model on **general network**.
- This talk.
    - **Single-period** model on **hierarchical** network.
    - Focus on **balancing** equity and efficiency

The Problem



At each node, an equitable distribution problem based on available resources.

Simchi-Levi, Trichakis, Zhang (2019)

Robust Optimization

Budget constraints on resource distribution

$$\max_{\boxed{\boldsymbol{x}}} \left\{ f(\boldsymbol{x}) + \min_{\omega \in \Omega} \left\{ \mathbb{W}\big(\boldsymbol{u}(\boldsymbol{x}, \omega)\big) \right\} \;\middle|\; \boxed{\boldsymbol{x} \in X} \right\}$$

Resource allocation vector

Robust Optimization

Cost of allocation *x* (negative social welfare)

Budget constraints on resource distribution

$$\max_{\boxed{\boldsymbol{x}}} \left\{ \boxed{f(\boldsymbol{x})} + \min_{\omega \in \Omega} \left\{ \mathbb{W}\big(\boldsymbol{u}(\boldsymbol{x}, \omega)\big) \right\} \;\middle|\; \boxed{\boldsymbol{x} \in X} \right\}$$

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Resource allocation vector

Find worst-case social welfare over uncertainty set Ω

Robust Optimization

$$\max_{\boldsymbol{x}} \left\{ f(\boldsymbol{x}) + \min_{\omega \in \Omega} \{ \mathbb{W}(\boldsymbol{u}(\boldsymbol{x}, \omega)) \} \;\middle|\; \boldsymbol{x} \in X \right\}$$

Cost of allocation *x* (negative social welfare)

Social welfare function

Budget constraints on resource distribution

Resource allocation vector

Find worst-case social welfare over uncertainty set Ω

Utilities are functions of resource distribution and scenario (perhaps QALYs)

Robust Optimization

$$\max_{\boldsymbol{x}} \left\{ f(\boldsymbol{x}) + \min_{\omega\in\Omega} \left\{ \mathbb{W}\Big(\boldsymbol{u}(\boldsymbol{x},\omega)\Big) \right\} \;\middle|\; \boldsymbol{x} \in X \right\}$$

Possible constraints on *u*:

$$u_i \le M_i(\boldsymbol{x},\omega), \quad \text{all } i$$

Upper bounds on individual utilities, perhaps due to demand ceilings

$$\boldsymbol{c}^{\mathsf{T}}\boldsymbol{u} \le B(\boldsymbol{x},\omega)$$

Resource budget constraint on utilities imposed by allocation ***x.***

If utility is nonlinear function of resources, use several budget constraints as piecewise linear approximation

Robust Optimization

$$\max_{\boldsymbol{x}} \left\{ f(\boldsymbol{x}) + \min_{\omega \in \Omega} \left\{ \mathbb{W}\big(\boldsymbol{u}(\boldsymbol{x},\omega)\big) \right\} \;\middle|\; \boldsymbol{x} \in X \right\}$$

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$$\boldsymbol{c}^{\mathsf{T}}\boldsymbol{u} \le B(\boldsymbol{x},\omega)$$

Resource budget constraint on utilities imposed by allocation ***x.***

Possible problem:
Social welfare function is **too hard** to solve over many scenarios

If utility is **nonlinear** function of resources, use **several budget constraints** as piecewise linear approximation

Combining Equity and Efficiency

- Purely **efficient** solution may be viewed as **unfair**.
  - Unequal distribution to regions, hospitals.
  - Neglect of expensive-to-treat patients to boost average health outcome.

- Exclusive focus on **fairness** may be **inefficient**.
  - Insufficient distribution to regions, hospitals with greater need.
  - High expenditure on a few gravely ill patients at the expense of overall health outcome.

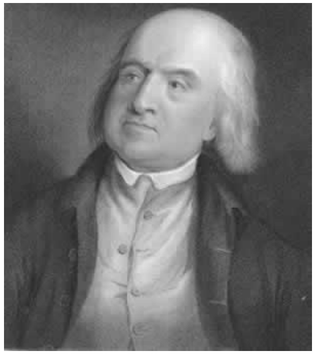
Combining Equity and Efficiency

- Two classical criteria for distributive justice:
- **Utilitarianism** (max total benefit)

$$\max_{\boldsymbol{x}} \left\{ \mathbb{W}\big(\boldsymbol{u}(\boldsymbol{x})\big) \;\middle|\; \boldsymbol{x} \in X \right\}$$

$$\mathbb{W}(\boldsymbol{u}) = \sum_i u_i$$

Bentham (1776)

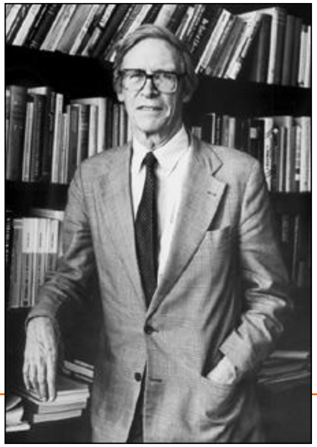


- **Rawlsian difference principle = maximin** (max welfare of worst off)

$$\max_{\boldsymbol{x}} \left\{ \mathbb{W}\big(\boldsymbol{u}(\boldsymbol{x})\big) \;\middle|\; \boldsymbol{x} \in X \right\}$$

$$\mathbb{W}(\boldsymbol{u}) = \max_i \{u_i\}$$

Rawls (1971)



Combining Equity and Efficiency

- Some proposals for combining equity and efficiency:
  - **Alpha-fairness**
  - **Proportional fairness**
  - **Kalai-Smorodinksy bargaining solution**
  - **Convex combination of utility and maximin**
  - **Product of utility and (1 – Gini coefficient)**
  - **H-W combination of utility & maximin (our choice for now)**

Combining Equity and Efficiency

Alpha fairness

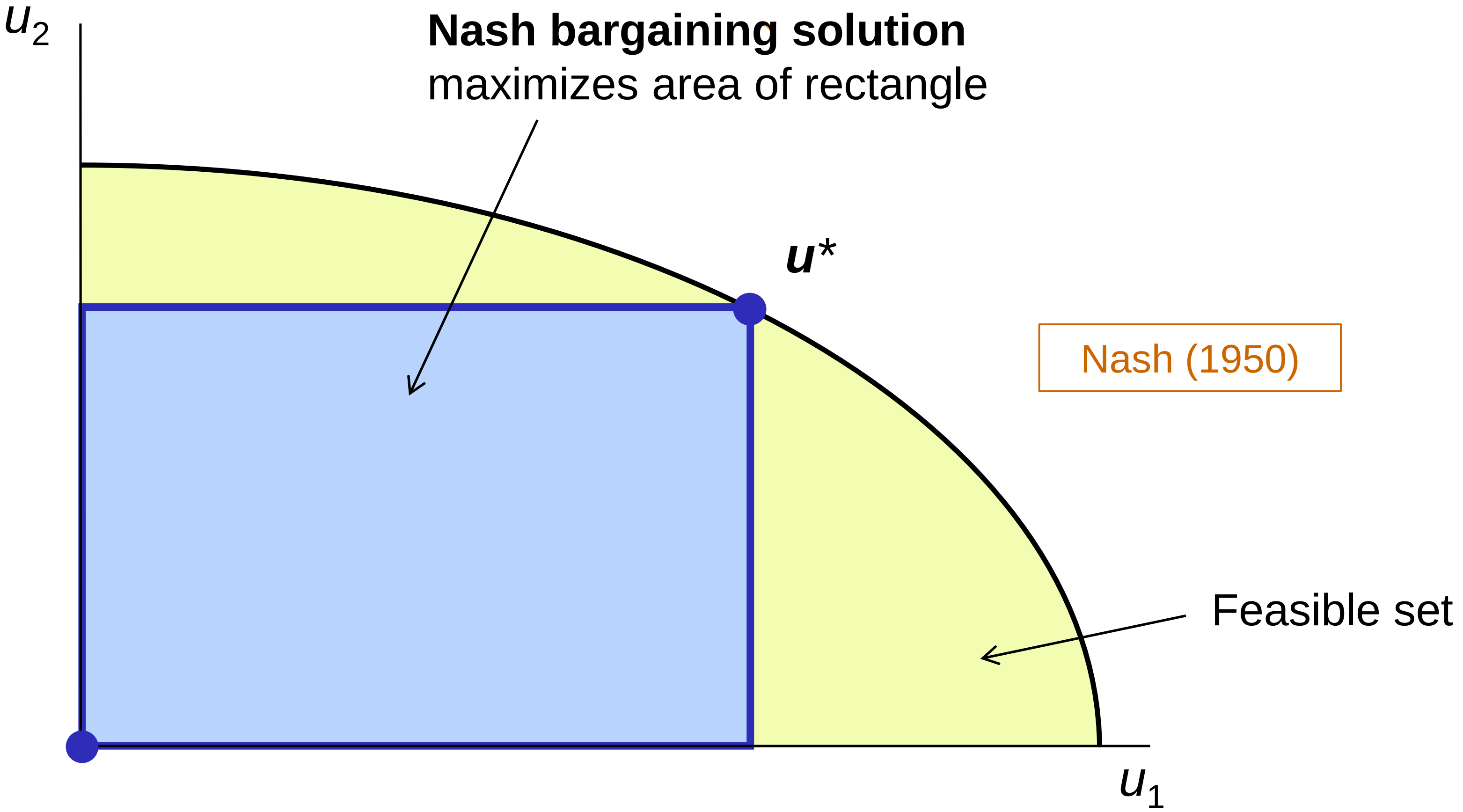
$$\mathbb{W}_\alpha(\boldsymbol{u}) = \begin{cases} \dfrac{1}{1-\alpha} \displaystyle\sum_i u_i^{1-\alpha} & \text{for } \alpha \ge 0,\ \alpha \ne 1 \\ \displaystyle\sum_i \log(u_i) & \text{for } \alpha = 1 \end{cases}$$

***α* = 0:** utilitarian ***α* = ∞:** maximin **larger *α*:** more emphasis on fairness

Problems: Nonlinear. How to choose *α*?

Special case: *α* = 1. **Proportional fairness** (Nash bargaining solution)

Combining Equity and Efficiency

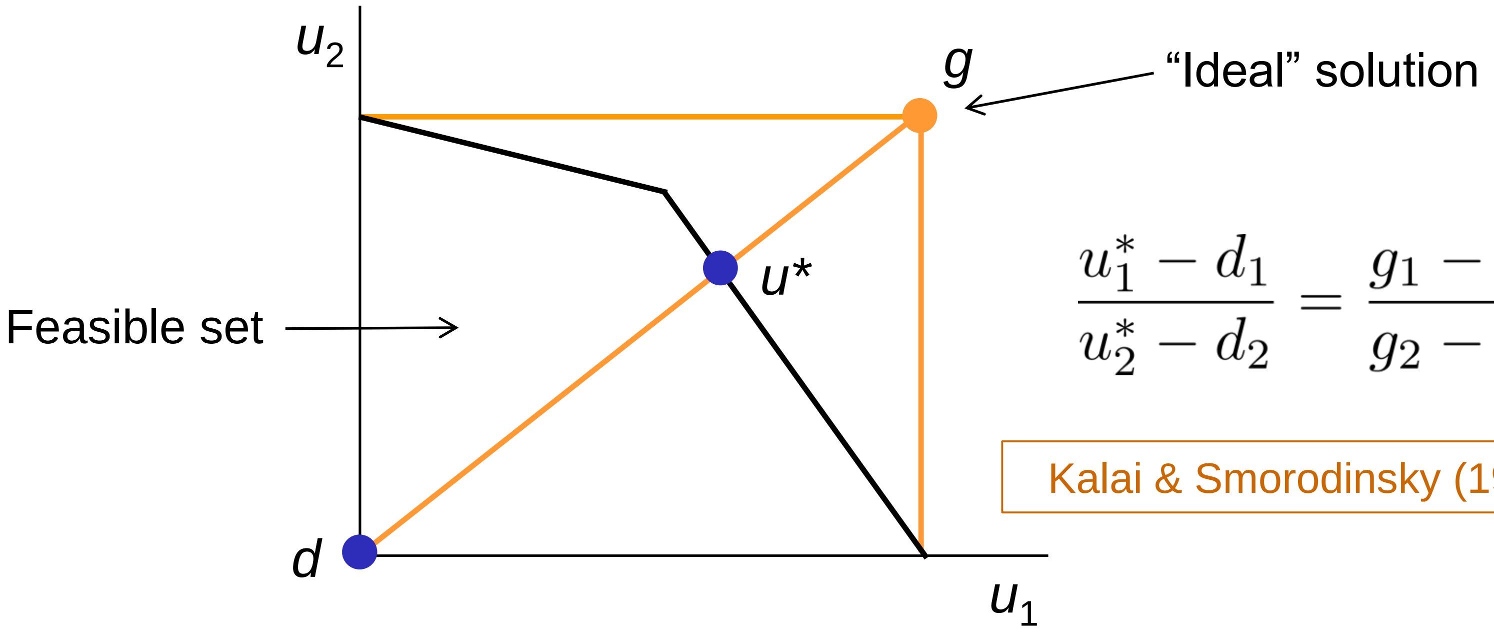


Problems: Nonlinear. Axiomatic and bargaining arguments rely on a strong assumption (lack of cardinal interpersonal comparability)

Combining Equity and Efficiency

Kalai-Smorodinsky bargaining solution

Players receive an equal fraction of their possible utility gains.
Justification: Rational bargaining arrives at optimal equilibrated relative concessions.



$$\frac{u_1^* - d_1}{u_2^* - d_2} = \frac{g_1 - d_1}{g_2 - d_2}$$

Kalai & Smorodinsky (1975)

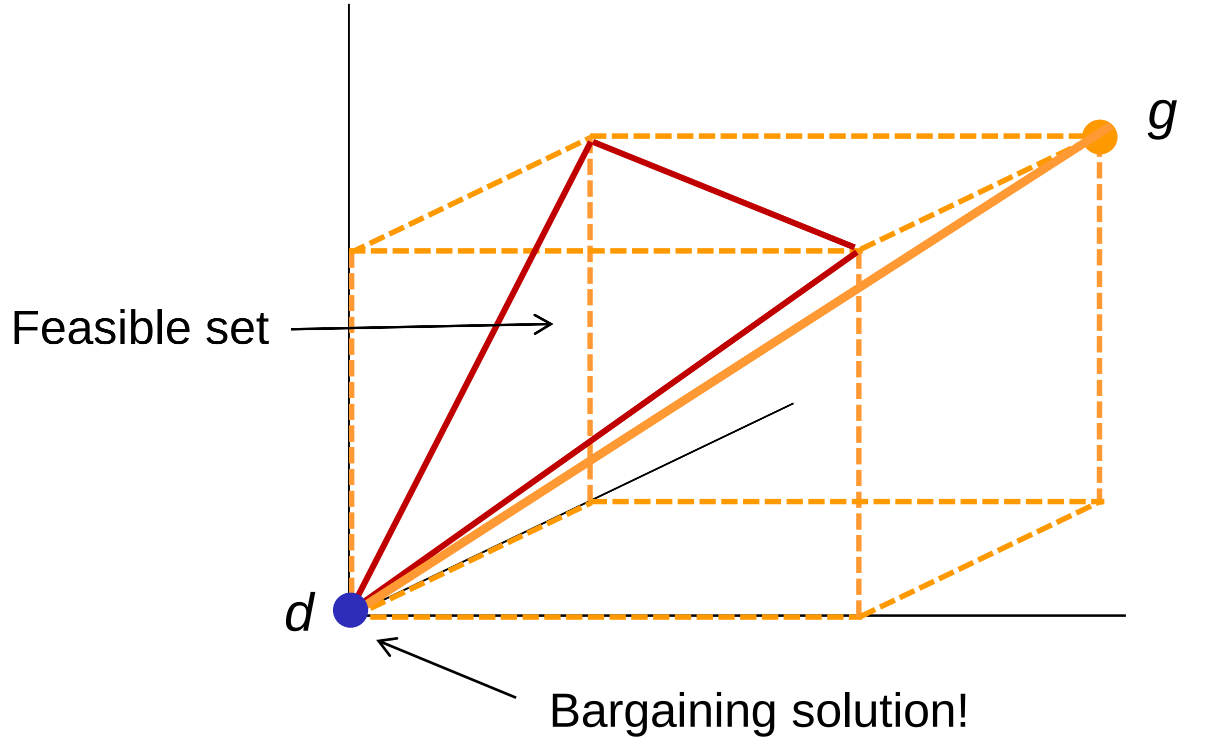
Combining Equity and Efficiency

Problems with Kalai-Smorodinsky solution:

Axiomatic treatment again assumes cardinal noncomparability.

No way to parameterize equity/efficiency trade-off.

Leads to counterintuitive result for ≥ 3 players:



Combining Equity and Efficiency

Convex combinination

$$\mathbb{W}(\boldsymbol{u}) = (1-\lambda)\sum_i u_i + \lambda \min_i\{u_i\}$$

Problem: No idea how to choose λ.

Combining Equity and Efficiency

Product of utility and equality measure

$$\mathbb{W}(\boldsymbol{u}) = \Big(\sum_i u_i\Big)\big(1 - G(\boldsymbol{u})\big)$$

where ***G***(***u***) is Gini coefficient $G(\boldsymbol{u}) = \dfrac{\sum_{i<j} |u_i - u_j|}{n \sum_i u_i}$

Eisenhandler & Tzur (2019)

This simplifies to expression that has a linear model:

$$\mathbb{W}(\boldsymbol{u}) = \sum_i u_i - \frac{1}{n}\sum_{i<j} |u_j - u_i|$$

Problems: Equality ≠ fairness.
This is just a convex combination of utility and an equality measure (negative mean absolute difference) with equal weights. **Why equal weights?**

H-W Model

Our choice (for now): H-W combination of utility and maximin

JH & Williams (2012)

Parameter *Δ* regulates equity-efficiency tradeoff, has **practical meaning**.

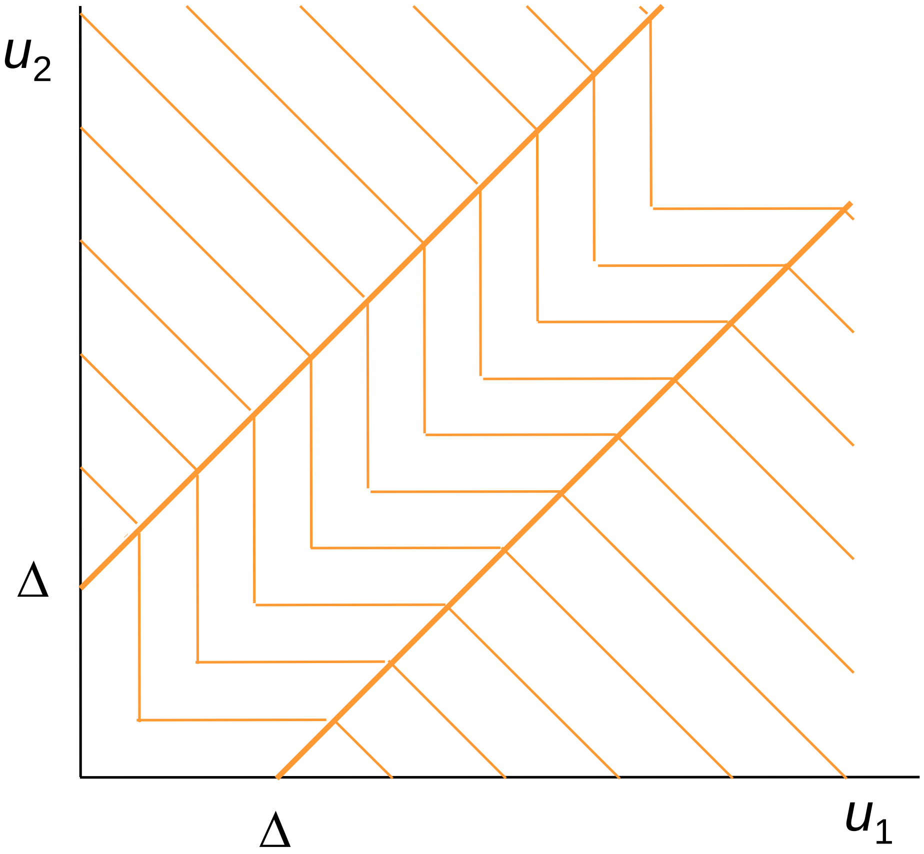
Social welfare function has practical **mixed integer (MIP) model**.

2-person model:

$$\mathbb{W}(u_1, u_2) = \begin{cases} u_1 + u_2 & \text{if } \left|u_1 - u_2\right| \geq \Delta \\ 2\min\{u_1, u_2\} + \Delta & \text{otherwise} \end{cases}$$

H-W Model

Contours of **social welfare function** for 2 persons.



H-W Model

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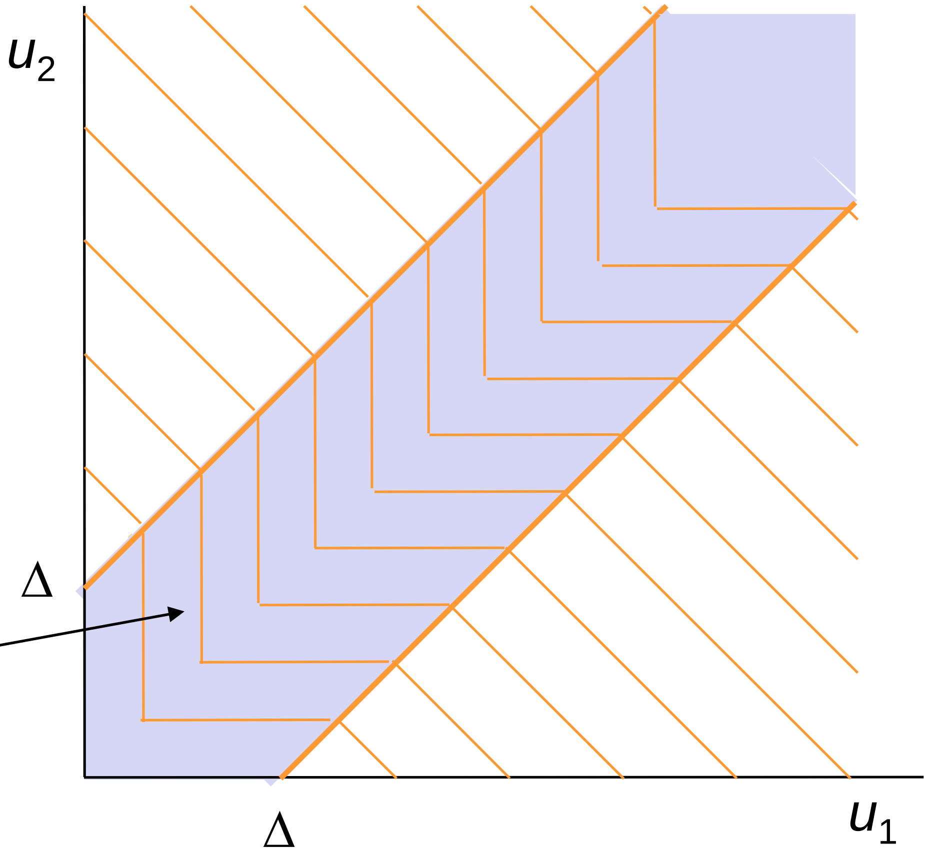


Maximin region

$2\min\{u_1, u_2\} + \Delta$

H-W Model

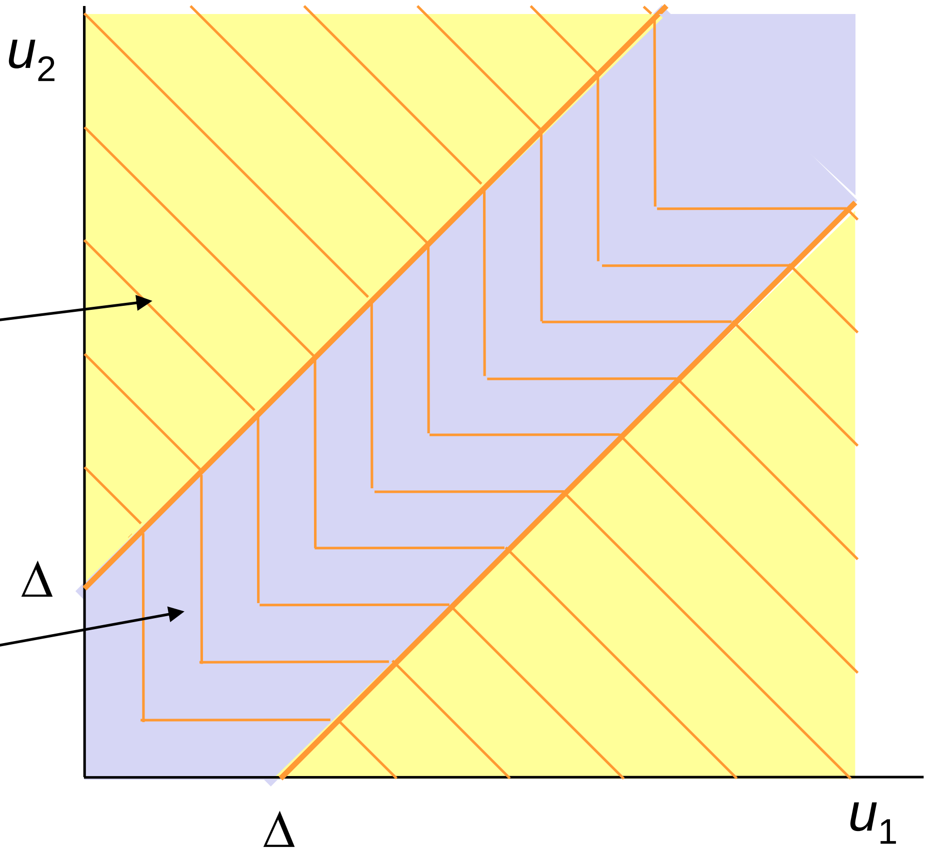
Contours of **social welfare function** for 2 persons.

Utilitarian region

$u_1 + u_2$

Maximin region

$2\min\{u_1, u_2\} + \Delta$



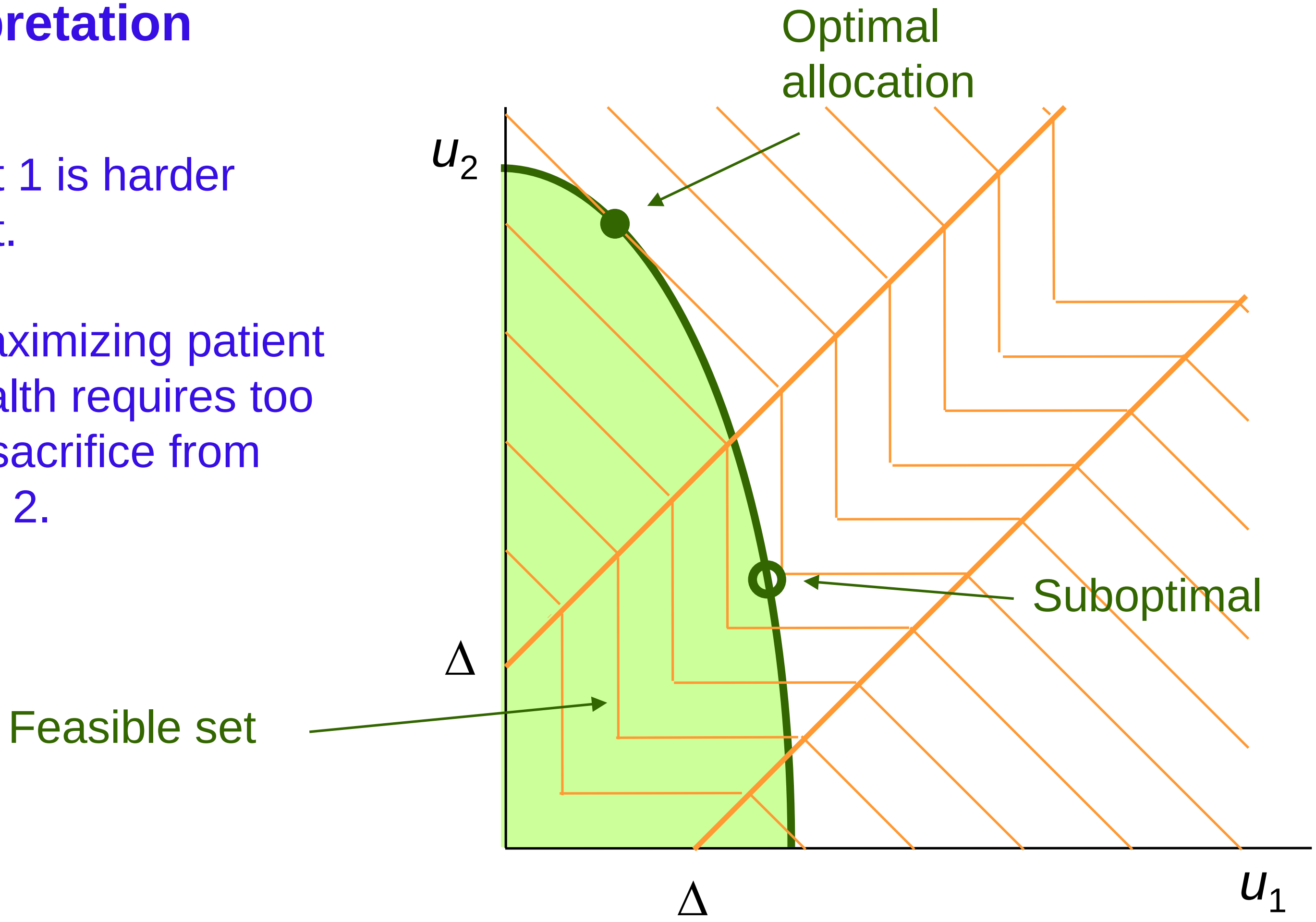
Ensures continuous contours

H-W Model

Healthcare interpretation

Patient 1 is harder to treat.

But maximizing patient 1's health requires too much sacrifice from patient 2.



H-W Model

***n*-person social welfare function**

$$\mathbb{W}(\boldsymbol{u}) = (n-1)\Delta + nu_{\min} + \sum_i \max\left\{0,\ u_i - u_{\min} - \Delta\right\}$$

Disadvantaged individuals receive some priority.

Choose ***Δ*** so that those with utilities in **fair region** (within *Δ* of smallest, $u_{\min}$) **deserve priority**.

***Δ* = 0:** utilitarian SWF (no fair region)

***Δ* = ∞:** maximin SWF (all utilities in fair region)

Utilities in fair region are equated with smallest utility, which receives weight equal to number of utilities in fair region.

H-W Model

MIP model of H-W social welfare function:

$$\max \ \sigma$$

$$\sigma \leq (n-1)\Delta + \sum_i v_i$$

Assumes $u_i \leq M$ for all i
to ensure MIP representability

$$v_i \leq u_i - \Delta\delta_i, \ \text{all } i$$

$$w \leq v_i \leq w + (M - \Delta)\delta_i, \ \text{all } i$$

$$u_i \geq 0, \ \delta_i \in \{0,1\}, \ , \ \text{all } i$$

Theorem. The model is **correct** (not easy to prove).

Theorem. The model is **sharp** (before resource constraints are added).

H-W Health Example

Measure utility in **QALY**s (quality-adjusted life years).

QALY and cost data based on Briggs & Gray, (2000) etc.

Decide whether to **fund** each disease/treatment pair.

Distinguish **severity levels** of each disease.

Treatment decisions are discrete, so funding is **all-or-nothing** for each category.

H-W Health Example

Add constraints to define feasible set…

$$\max \ \sigma$$

H-W model

$$u_i = q_i y_i + \alpha_i, \ \text{all } i$$

$$\sum_i n_i c_i y_i \le B$$

$$u_i \le q_i + \alpha_i, \ \text{all } i$$

$$y_i \in \{0, 1\}, \ \text{all } i$$

y_i indicates whether disease category *i* is funded

Budget constraint

Individual utility bounds

Model is slightly modified to accommodate patient groups of different sizes n_i

QALY & cost data

Part 1

Intervention	Cost per person c_i (£)	QALYs gained q_i	Cost per QALY (£)	QALYs without intervention α_i	Subgroup size n_i
Pacemaker for atrioventricular heart block					
Subgroup A	3500	3	1167	13	35
Subgroup B	3500	5	700	10	45
Subgroup C	3500	10	350	5	35
Hip replacement					
Subgroup A	3000	2	1500	3	45
Subgroup B	3000	4	750	4	45
Subgroup C	3000	8	375	5	45
Valve replacement for aortic stenosis					
Subgroup A	4500	3	1500	2.5	20
Subgroup B	4500	5	900	3	20
Subgroup C	4500	10	450	3.5	20
CABG[1] for left main disease					
Mild angina	3000	1.25	2400	4.75	50
Moderate angina	3000	2.25	1333	3.75	55
Severe angina	3000	2.75	1091	3.25	60
CABG for triple vessel disease					
Mild angina	3000	0.5	6000	5.5	50
Moderate angina	3000	1.25	2400	4.75	55
Severe angina	3000	2.25	1333	3.75	60
CABG for double vessel disease					
Mild angina	3000	0.25	12,000	5.75	60
Moderate angina	3000	0.75	4000	5.25	65
Severe angina	3000	1.25	2400	4.75	70

QALY & cost data

Part 2

Intervention	Cost per person c_i (£)	QALYs gained q_i	Cost per QALY (£)	QALYs without intervention α_i	Subgroup size n_i
	22,500	4.5	5000	1.1	2
Kidney transplant					
Subgroup A	15,000	4	3750	1	8
Subgroup B	15,000	6	2500	1	8
Kidney dialysis					
Less than 1 year survival					
Subgroup A	5000	0.1	50,000	0.3	8
1-2 years survival					
Subgroup B	12,000	0.4	30,000	0.6	6
2-5 years survival					
Subgroup C	20,000	1.2	16,667	0.5	4
Subgroup D	28,000	1.7	16,471	0.7	4
Subgroup E	36,000	2.3	15,652	0.8	4
5-10 years survival					
Subgroup F	46,000	3.3	13,939	0.6	3
Subgroup G	56,000	3.9	14,359	0.8	2
Subgroup H	66,000	4.7	14,043	0.9	2
Subgroup I	77,000	5.4	14,259	1.1	2
At least 10 years survival					
Subgroup J	88,000	6.5	13,538	0.9	2
Subgroup K	100,000	7.4	13,514	1.0	1
Subgroup L	111,000	8.2	13,537	1.2	1

H-W Results

Total budget £3 million

Δ range	Pace-maker	Hip repl.	Aortic valve	CABG L	CABG 3	CABG 2	Heart trans.	Kidney trans.	Kidney dialysis <1	Kidney dialysis 1-2	Kidney dialysis 2-5	Kidney dialysis 5-10	Kidney dialysis >10	Avg. QALYs.
0-3.3	111	111	111	111	111	111	1	11	0	0	000	0000	000	7.54
3.4-4.0	111	111	111	111	111	111	0	11	1	0	000	0000	000	7.54
4.0-4.4	111	111	111	111	111	111	0	01	1	0	000	0000	001	7.51
4.5-5.01	111	011	111	111	111	111	1	01	1	0	000	0000	011	7.43
5.02-5.55	111	011	011	111	111	111	1	01	1	0	000	0001	011	7.36
5.56-5.58	111	011	011	111	111	011	0	01	1	0	000	0001	111	7.36
5.59	111	011	011	110	111	111	0	01	1	0	000	0001	111	7.20
5.60-13.1	111	111	111	101	000	000	1	11	1	0	111	1111	111	7.06
13.2-14.2	111	011	111	011	000	000	1	11	1	1	111	1111	111	7.03
14.3-15.4	111	111	111	011	000	000	1	11	1	1	101	1111	111	7.13
15.5 up	111	011	111	011	001	000	1	11	1	0	011	1111	111	7.19

H-W Results

Utilitarian solution

Δ	Pace-	Hip	Aortic	CABG			Heart	Kidney	Kidney dialysis					Avg.
range	maker	repl.	valve	L	3	2	trans.	trans.	<1	1-2	2-5	5-10	>10	QALYs.
0-3.3	111	111	111	111	111	111	1	11	0	0	000	0000	000	7.54
3.4-4.0	111	111	111	111	111	111	0	11	1	0	000	0000	000	7.54
4.0-4.4	111	111	111	111	111	111	0	01	1	0	000	0000	001	7.51
4.5-5.01	111	011	111	111	111	111	1	01	1	0	000	0000	011	7.43
5.02-5.55	111	011	011	111	111	111	1	01	1	0	000	0001	011	7.36
5.56-5.58	111	011	011	111	111	011	0	01	1	0	000	0001	111	7.36
5.59	111	011	011	110	111	111	0	01	1	0	000	0001	111	7.20
5.60-13.1	111	111	111	101	000	000	1	11	1	0	111	1111	111	7.06
13.2-14.2	111	011	111	011	000	000	1	11	1	1	111	1111	111	7.03
14.3-15.4	111	111	111	011	000	000	1	11	1	1	101	1111	111	7.13
15.5 up	111	011	111	011	001	000	1	11	1	0	011	1111	111	7.19

H-W Results

Maximin solution

Δ range	Pace-maker	Hip repl.	Aortic valve	CABG L	CABG 3	CABG 2	Heart trans.	Kidney trans.	Kidney dialysis <1	1-2	2-5	5-10	>10	Avg. QALYs.
0-3.3	111	111	111	111	111	111	1	11	0	0	000	0000	000	7.54
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4.0-4.4	111	111	111	111	111	111	0	01	1	0	000	0000	001	7.51
4.5-5.01	111	011	111	111	111	111	1	01	1	0	000	0000	011	7.43
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13.2-14.2	111	011	111	011	000	000	1	11	1	1	111	1111	111	7.03
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15.5 up	111	011	111	011	001	000	1	11	1	0	011	1111	111	7.19

H-W Results

Most resources come from heart bypass surgery

As Δ increases, transfer resources to dialysis (very expensive, but these patients are the worst off without treatment)

Δ range	Pace-maker	Hip repl.	Aortic valve	CABG L	CABG 3	CABG 2	Heart trans.	Kidney trans.	Kidney dialysis <1	1-2	2-5	5-10	>10	Avg. QALYs.
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14.3-15.4	111	111	111	011	000	000	1	11	1	1	101	1111	111	7.13
15.5 up	111	011	111	011	001	000	1	11	1	0	011	1111	111	7.19

H-W Model

Problems

We need **rapid solution** for many scenarios. **But…**

- H-W model has **0-1 variables**.
- Model is **no longer sharp** when budget constraint, bounds are added.

H-W Model

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- Model is **no longer sharp** when budget constraint, bounds are added.

Current research goals

- Find **closed-form solution** for **single** budget constraint, **no bounds**.
- **Tighten** model that has **multiple** budget constraints to speed solution.
- Observe empirical behavior of **LP relaxation** with budget constraint and bounds (close to optimal?).

Single Budget Constraint

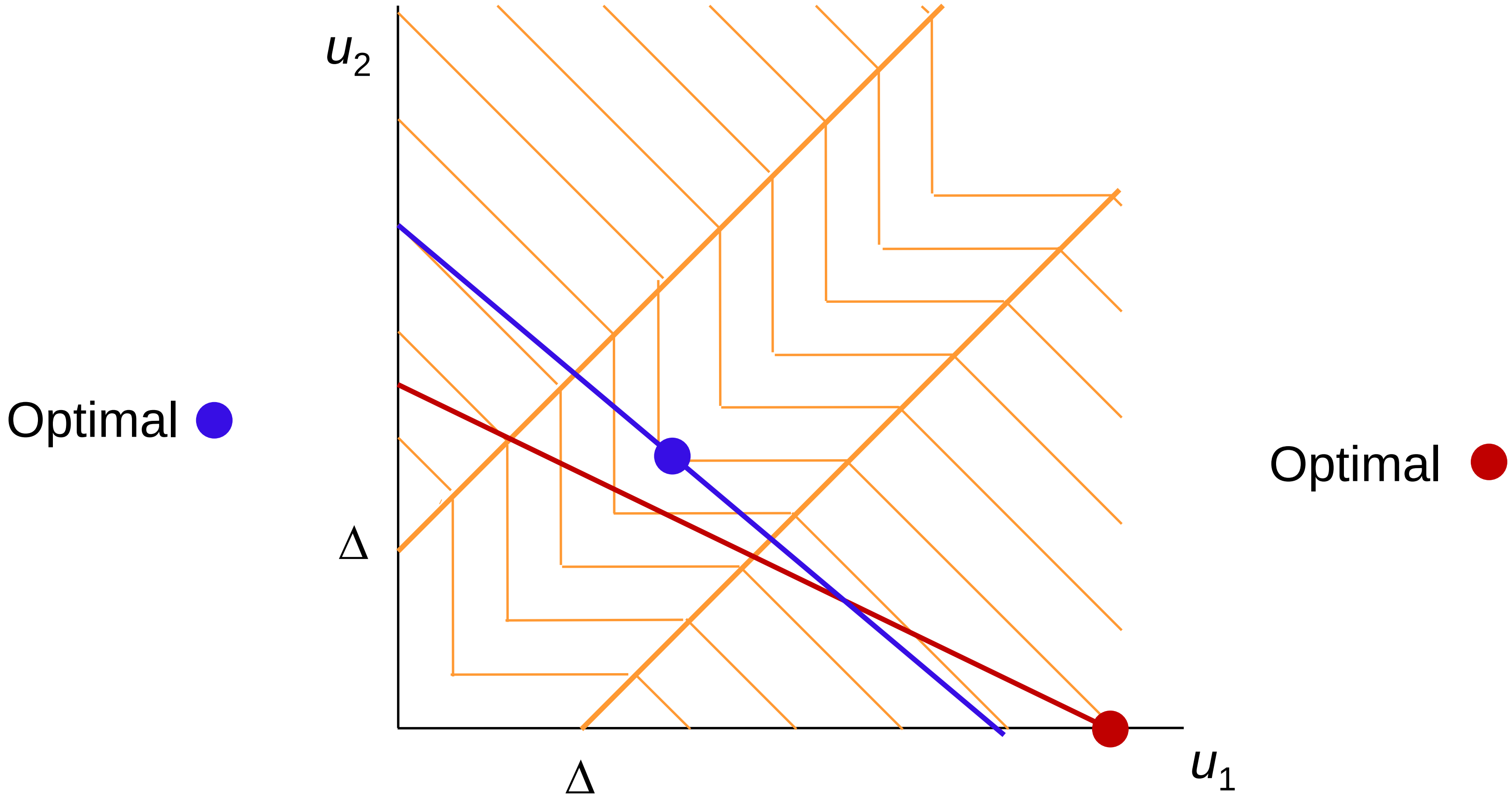
Suppose there is a single budget constraint $\sum_i c_i u_i \leq B$ and no bounds.

Theorem. The H-W solution is **purely utilitarian** or **purely maximin**. In particular, given $c_1 = \min_i\{c_i\}$, we have a utilitarian solution

$$u_i = \begin{cases} B/c_1 & \text{for } i = 1 \\ 0 & \text{for } i \geq 2 \end{cases} \quad \text{if } \Delta \leq \left(\frac{1}{c_1} - \frac{n}{\sum_i c_i}\right),$$

And maximin solution $u_i = \dfrac{B}{\sum_i c_i}$ for all i otherwise (all utilities equal).

Single Budget Constraint



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And maximin solution $u_i = \dfrac{B}{\sum_i c_i}$ for all i otherwise (all utilities equal).

Solution is **uninteresting** because cost is a **linear** function of utilities and there are **no individual bounds** on utilities.

More realistic: **multiple** linear budget constraints give piecewise linear approximation of **nonlinear** (concave) utility functions, or there are **individual bounds** on utilities (as in H-W problem).

Multiple Budget Constraints

Suppose there are multiple budget constraints of the form $\sum_i c_i u_i \leq B$.

Theorem. For each budget constraint, we have the valid inequalities

$$\sigma \leq (n-1)\Delta + \left(1 + (n-1)\frac{\Delta}{M_i}\right) + \sum_{j \neq i} \left(1 - \frac{\Delta}{\min\{M_i, M_j\}}\right) u_j, \quad \text{all } i$$

where $M_i = B/c_i$

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where $M_i = B/c_i$

These inequalities **tighten the MIP model** and therefore, potentially, quality of LP relaxation.

The inequalities remain valid, of course, when there are individual bounds on utilities.

Empirical Observations

LP relaxation of H-W model with single linear budget constraint and individual utility bounds **almost always yields optimal solution**.

We are exploring **special structure** of solutions of this and other problems.

Future work: Apply these results in robust optimization model.

References with Surveys

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