Optimization Models for Equity

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Modeling Equity

- There is a growing interest in incorporating **equity** considerations in mathematical programming models.
    - Not enough to minimize cost or maximize revenue.
    - Also concerned about **distribution** of resources/benefits.
    - Not obvious how to capture equity in the **objective function**.
    - Still less obvious how to **combine** it with an efficiency objective.

Modeling Equity

- Some applications…
    - Health care allocation.
    - Facility location (e.g., emergency services).
    - Taxation (revenue vs. progressivity).
    - Relief operations and disaster planning/response.
    - Telecommunications (lexmax, Nash bargaining solution)

Availability

- These slides are available on my website.
  - Google “John Hooker”
  - Originally presented at a workshop at London School of Economics, December 2010.
- Will give only a brief overview today.

Summary Outline

- Fairness and equality:
    - Utilitarianism
    - Piecewise Linear Modeling
    - Rawlsian Difference Principle
    - Axiomatics
    - Measures of Inequality
    - An Allocation Problem
- Bargaining and equality/efficiency combinations:
    - Nash Bargaining Solution
    - Raiffa-Kalai-Smorodinsky Bargaining
    - Disjunctive Modeling
    - Combining Equity and Efficiency
    - Health Care Example

Fairness and Equality Outline

- Utilitarianism
    - Utility and production functions
    - The optimization problem
    - Arguments for utilitarianism
- Piecewise Linear Modeling
    - LP model of concave maximization
    - MILP model of nonconcave maximization

Fairness and Equality Outline

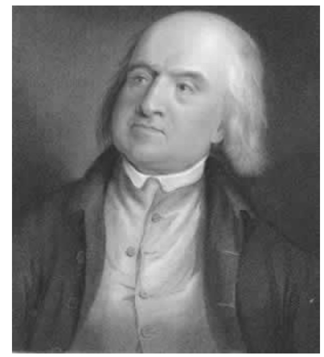
- Rawlsian Difference Principle
  - The social contract argument
  - The lexmax principle
  - The optimization problem
- Axiomatics
  - Interpersonal comparability
  - Axioms of rational choice
  - Social welfare functions

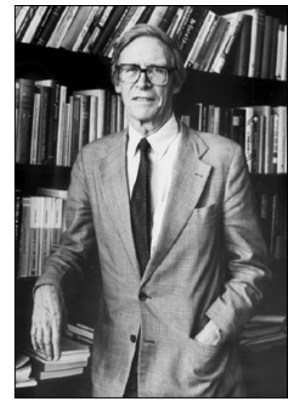
Fairness and Equality Outine

- Measures of Inequality
    - An example
    - Utrilitarian, maximin, and lexmax solution
    - Relative range, max, min
    - Relative mean deviation
    - Variance, coefficient of variation
    - McLoone index
    - Gini coefficient
    - Atkinson index
    - Hoover index
    - Theil index
- An Allocation Problem

Efficiency vs. Equity

- Two classical criteria for distributive justice:
  - **Utilitarianism (efficiency)**
  - **Difference principle** of **John Rawls (equity)**
- These have the must studied philosophical underpinnings.





Utilitarianism

- **Utilitarianism** seeks allocation of resources that maximizes total utility.
  - Let x_i = resources allocated to person *i*.
  - Let u_i = utility enjoyed by person *i*.
  - We have an optimization problem

$$\max \sum_i u_i$$

$$u_i = h_i(x_i), \quad \text{all } i$$

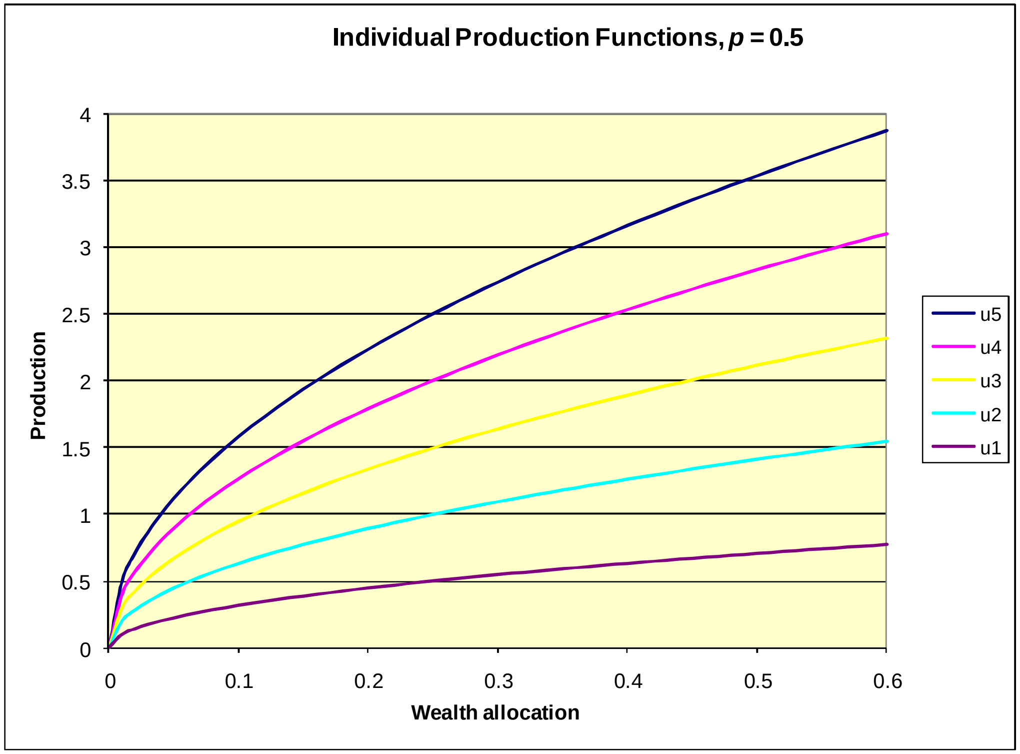
$$x \in S$$

Production functions

Set of feasible resource allocations

Utilitarianism

For example, $h_i(x_i) = a_i x_i^p$ with different a_is for 5 individuals



Utilitarianism

- The individual production function h_i has two components.
    - The **value** $v_i(x_i)$ created by the individual, as a result of receiving resources x_i.
    - The **utility** $u_i(v_i(x_i)) = h_i(x_i)$ of the value created (u_i is normally concave).
    - So a_i reflects the value function v_i (productivity), and p reflects the combined shape of both functions v_i and u_i.

Utilitarianism

Assume resource distribution is constrained only by a fixed budget.

We have the optimization problem

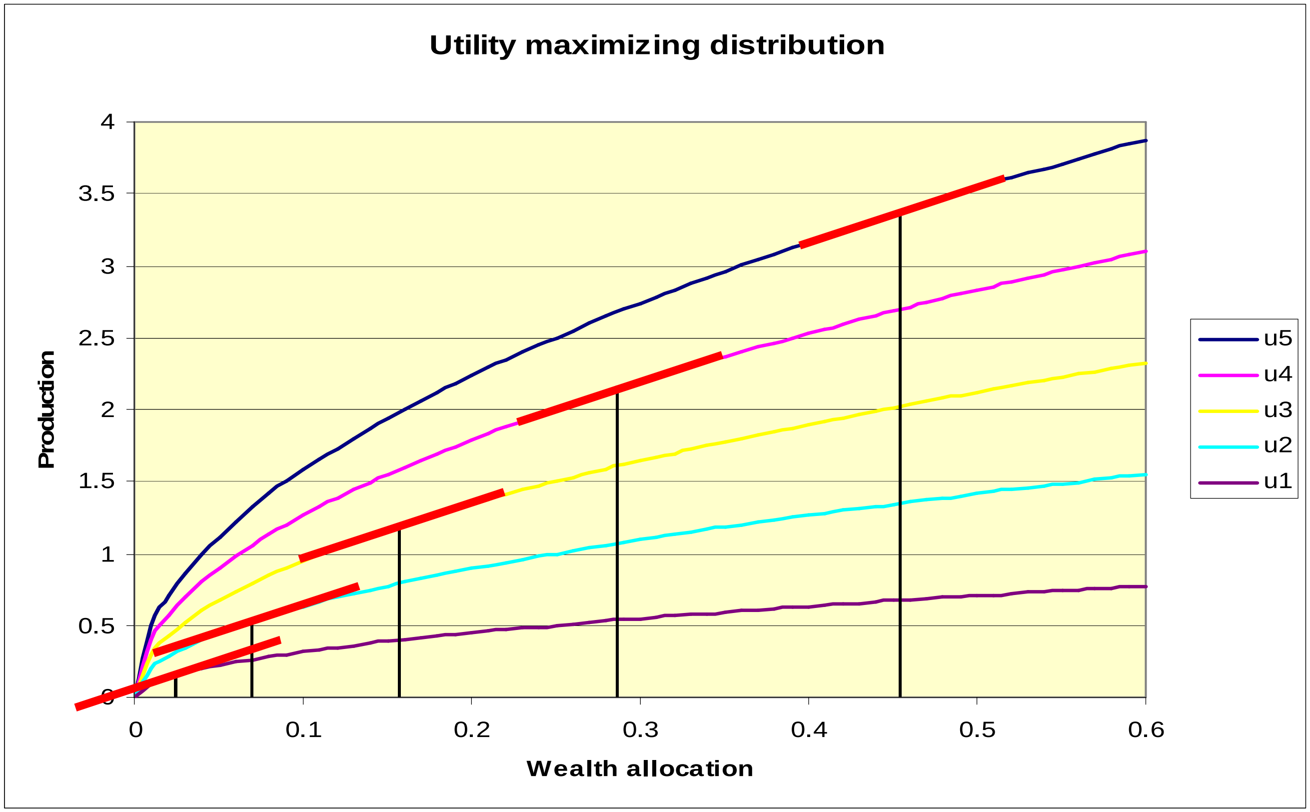
$$\begin{aligned}
&\max \sum_i u_i \\
&u_i = a_i x_i^p, \ \ \text{all } i \\
&\sum_i x_i = 1, \ \ x_i \geq 0, \ \text{all } i
\end{aligned}$$

This has a closed-form solution

$$x_i = a_i^{\frac{1}{1-p}} \left(\sum_{j=1}^{n} a_j^{\frac{1}{1-p}} \right)^{-1}$$

Utilitarianism

Optimal allocations equalize slope (i.e., equal marginal productivity).

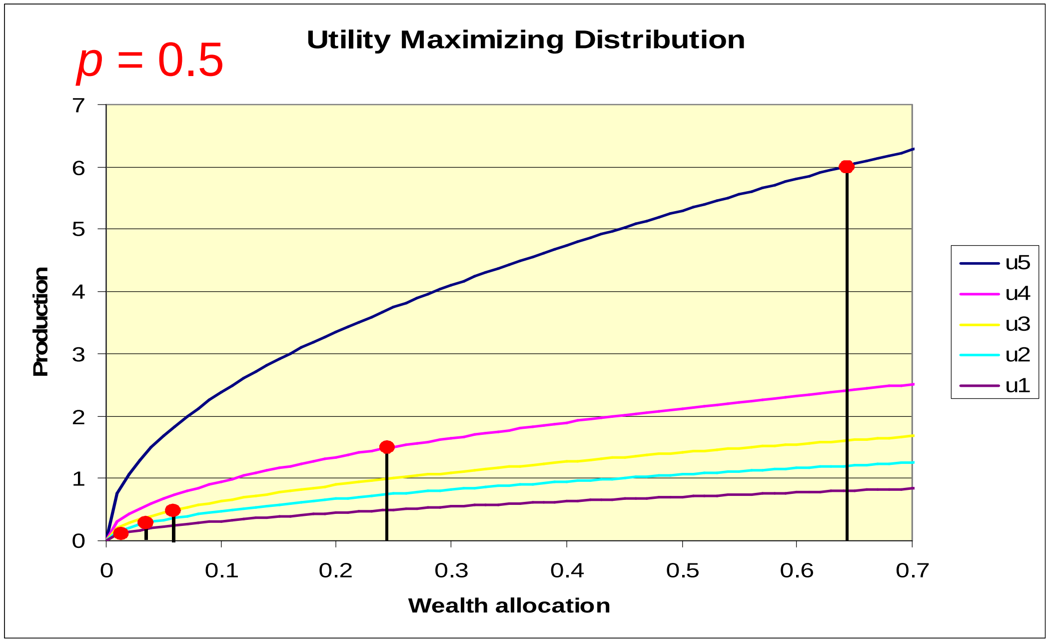


Utilitarianism

- **Arguments for utilitarianism**
    - Can define utility to suit context.
    - Utilitarian distributions incorporate some **egalitarian** factors:
    - With **concave** production functions, egalitarian distributions create more utility, *ceteris paribus*.
    - Inegalitarian distributions create **disutility**, due to social disharmony.

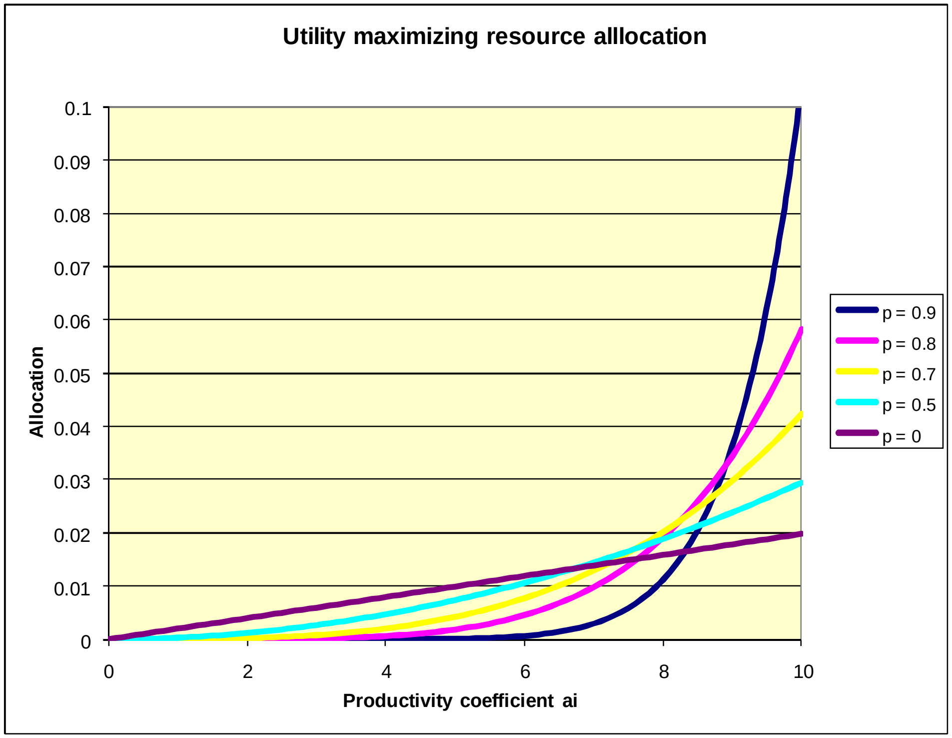
Utilitarianism

- Egalitarian distributions create more utility?
    - This effect is **limited**.
    - Utilitarian distributions can be very unequal. Productivity differences are magnified in the allocations.



Utilitarianism

- Egalitarian distributions create more utility?
  - In the example, the **most egalitarian** distribution ($p \rightarrow 0$) assigns resources in proportion to productivity.



Utilitarianism

- Unequal distributions create disutility?
  - Perhaps, but modeling this requires **nonseparable** utility functions

$$u_i = h_i(x_1, \cdots, x_n)$$

that may result in a problem that is hard to model and solve.

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  - More fundamentally, this defense of utilitarianism is based on **contingency**, **not principle**.
  - If we **evaluate** the fairness of utilitarian distribution, then there must be **another standard** of equitable distribution.
  - How do we model the standard we really have in mind?

Modeling Utility

- Ideally, production functions are **concave**, and feasible set is **convex**.
  - For example, $h_i(x_i) = a_i x_i^p$ for $0 < p < 1$ and linear constraints on x.
  - Then we solve the problem

$$\max \sum_i h_i(x_i)$$

$$Ax \leq b, \;\; x \geq 0$$

    by nonlinear programming.
  - Any local optimum is a global optimum.

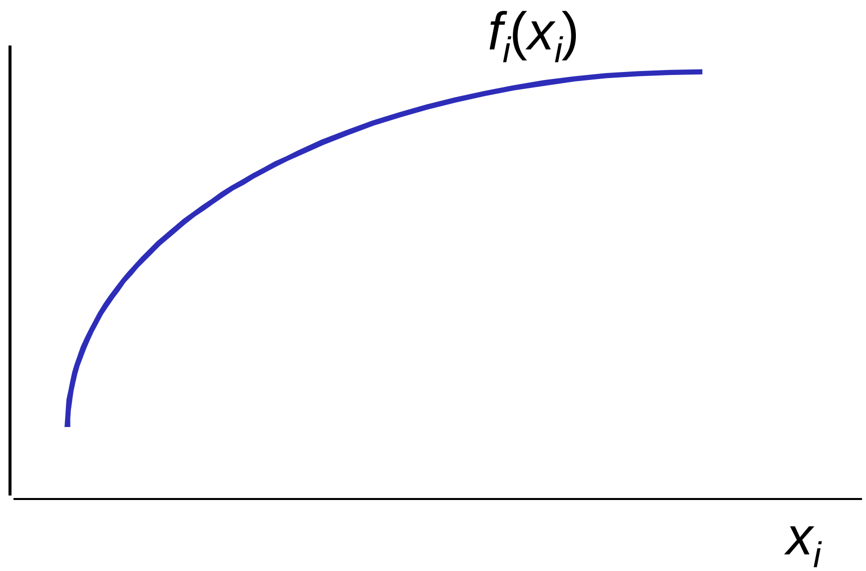
Piecewise Linear Modeling

- Piecewise linear modeling converts **nonlinear programming** to **LP** (linear programming) or **MILP** (mixed integer/linear programming).
  - A key technique.
  - Applies when functions are **separable**.
- Suppose we want to solve

$$\max \sum_i f_i(x_i)$$
$$Ax \le b,\;\; x \ge 0$$

Piecewise Linear Modeling

- If each f_i is **concave**, this reduces (approx.) to an **LP**.



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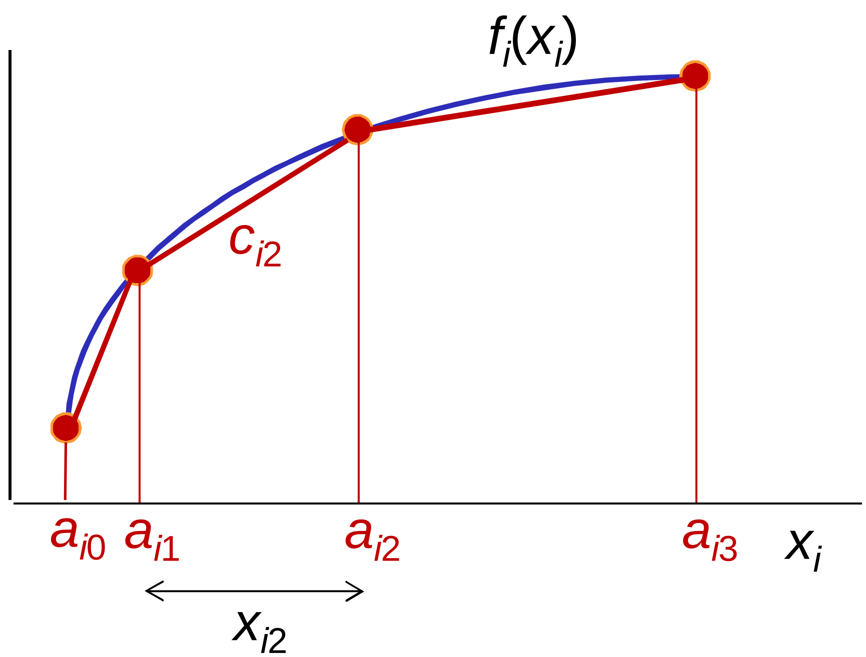

$$\max \sum_i v_i$$

$$v_i = f_i(a_0) + \sum_j \frac{\Delta f_{ij}}{\Delta a_{ij}} x_{ij}$$

$$x_i = \sum_j x_{ij}$$

$$Ax \le b, \quad x \ge 0$$

where

$$\Delta f_{ij} = f_i(a_{ij}) - f_i(a_{i,j-1})$$

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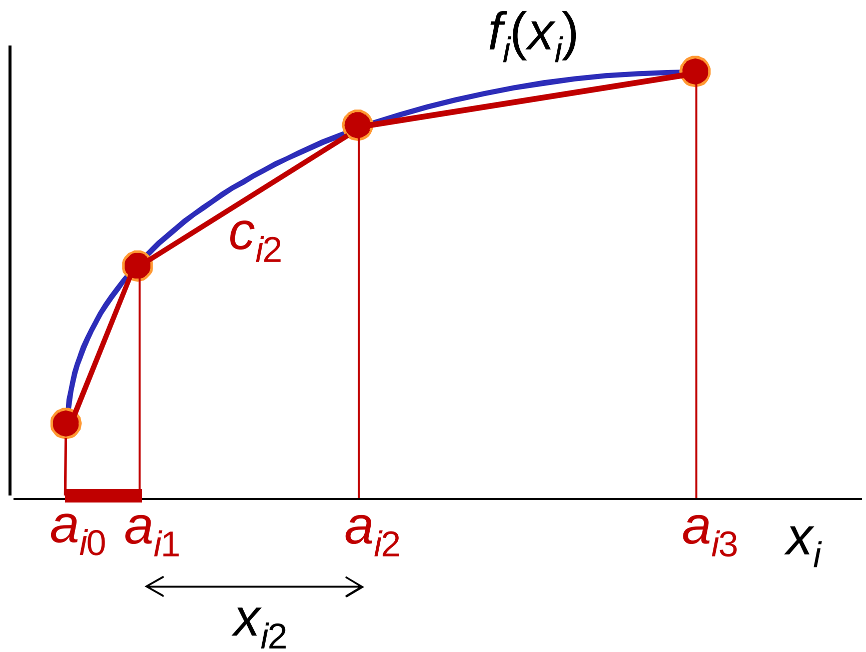

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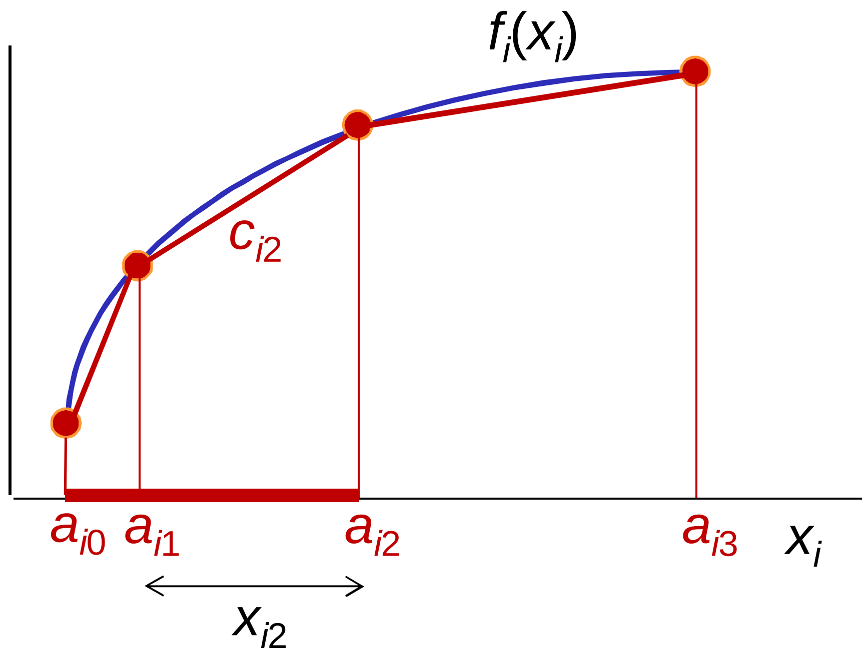

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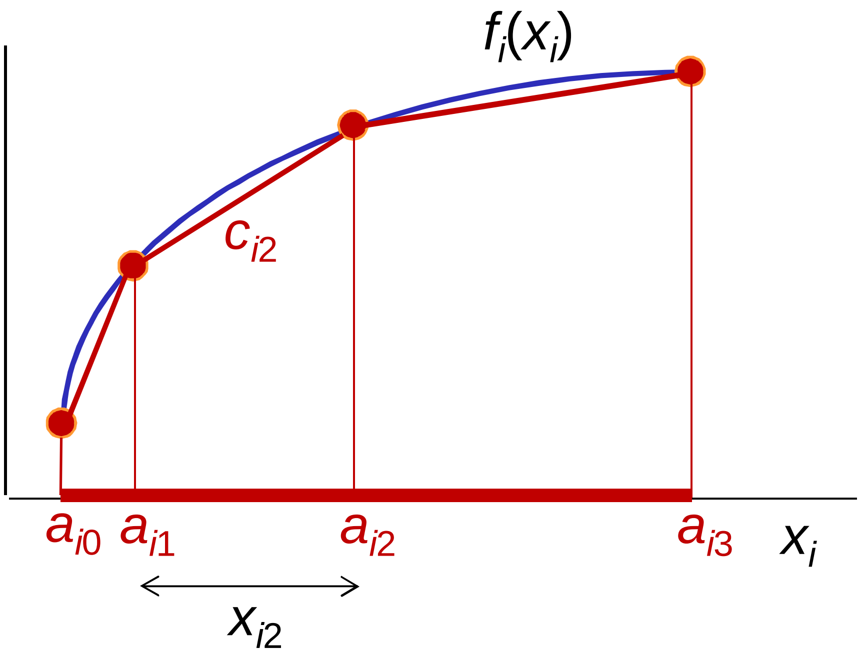

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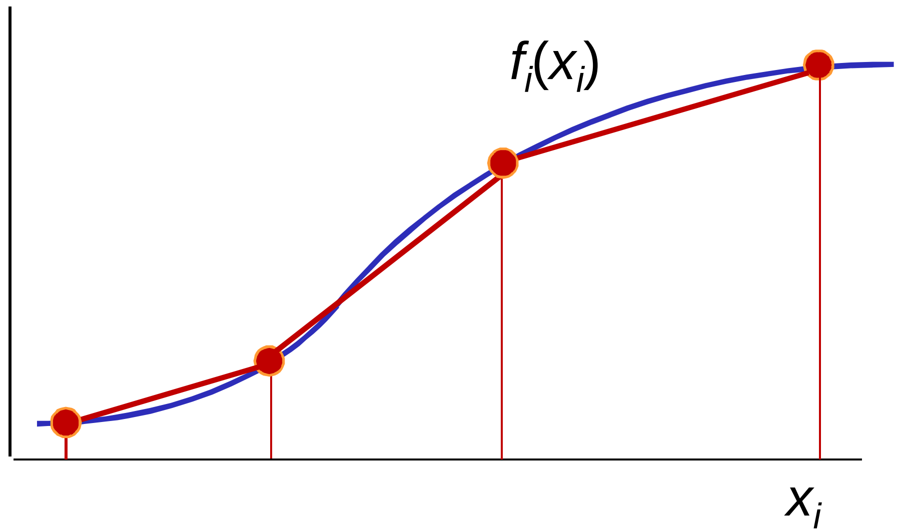
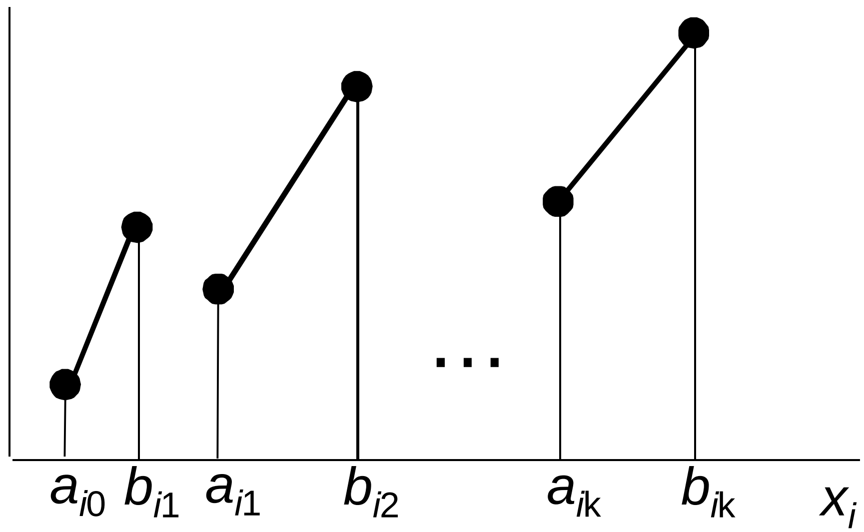
Piecewise Linear Modeling

- If f_i is **nonconcave**, we can use an **MILP** model of the piecewise linear approximation.



Piecewise Linear Modeling

- In general, a piecewise linear approximation v_i of f_i has the form



The function is continuous when $b_{ij} = a_{i,j+1}$

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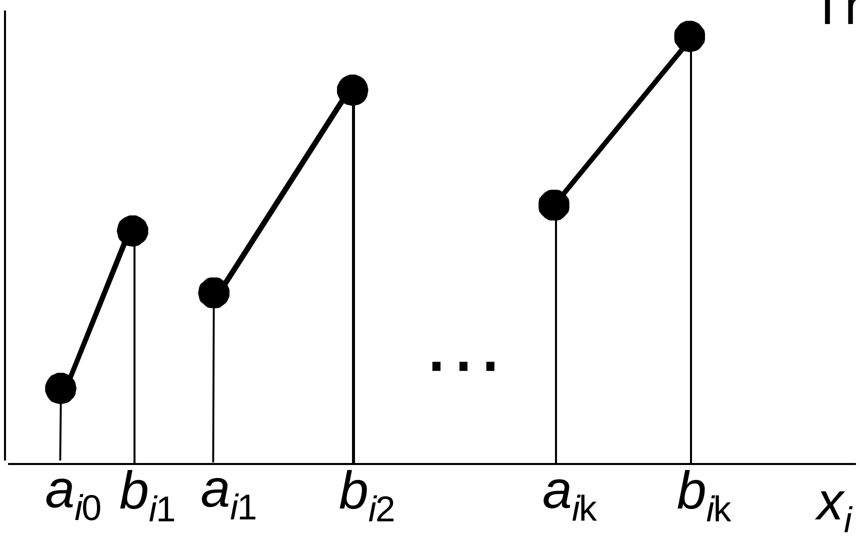

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The best MILP model is:

$$v_i = \sum_j \lambda_{ij} f_i(a_{ij}) + \mu_{ij} f_i(b_{ij})$$

$$x_i = \sum_j \lambda_{ij} a_{ij} + \mu_{ij} b_{ij}$$

$$\lambda_{ij} + \mu_{ij} = \delta_{ij}, \quad \text{all } j$$

$$\sum_j \delta_{ij} = 1$$

$$\lambda_{ij}, \mu_{ij} \geq 0, \quad \delta_{ij} \in \{0,1\}, \quad \text{all } j$$

Piecewise Linear Modeling

- When the piecewise linear function is continuous, **don't** use the "textbook" model

$$v_i = \sum_{j=1}^{k+1} \lambda_{ij} f_i(a_{ij})$$

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$$\lambda_{ij} \le \delta_{i,j-1} + \delta_{ij}, \ j = 2, \ldots, k$$

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and declare the λ_{ij} SOS2.

This sacrifices the tight relaxation of the next model…

Piecewise Linear Modeling

- The best model of a continuous piecewise v_i is the "incremental" formulation:

$$v_i = f_i(a_{i1}) + \sum_{j=2}^{k+1} \frac{\Delta f_{ij}}{\Delta a_{ij}} x_{ij}$$

$$x_i = a_{i1} + \sum_{j=1}^{k} x_{ij}$$

$$\Delta a_{ij}\delta_{ij} \le x_{ij} \le \Delta a_{ij}\delta_{i,j-1}, \quad j = 3,\dots,k$$

$$\Delta a_{i2}\delta_{ij} \le x_{i2} \le \Delta a_{i2}, \quad 0 \le x_{i,k+1} \le \Delta a_{i,k+1}\delta_{ik}$$

$$\delta_{ij} \in \{0,1\}, \; j = 2,\dots,k$$

Problems with Utilitarianism

- A utility maximizing distribution may be unjust.
  - Disabled or nonproductive people may be neglected.
  - Less talented people who work hard may receive meager wage.
  - Not all jobs can be equally productive. Those with less productive jobs may receive fewer resources.

Rawlsian Difference Principle

- Rawls' **Difference Principle** seeks to maximize the welfare of the worst off.
  - Also known as **maximin** principle.
  - Another formulation: inequality is permissible only to the extent that it is necessary to improve the welfare of those worst off.

$$\max \min_i \{u_i\}$$
$$u_i = h_i(x_i), \;\; \text{all } i$$
$$x \in S$$

Rawlsian Difference Principle

- The root idea is that when I make a decision for myself, I make a decision for **anyone** in similar circumstances.
    - It doesn't matter who I am.

- Social contract argument
    - I make decisions (formulate a social contract) in an **original position**, behind a **veil of ignorance** as to who I am.
    - I must find the decision acceptable **after** I learn who I am.
    - I cannot rationally assent to a policy that puts me on the bottom, unless I would have been even **worse off** under alternative policies.
    - So the policy must **maximize** the welfare of the **worst off**.

Rawlsian Difference Principle

- Applies only to **basic goods**.
  - Tings that people want, no matter what else they want.
  - Salaries, tax burden, medical benefits, etc.
  - For example, salary differentials may satisfy the principle if necessary to make the poorest better off.
- Applies to smallest **groups** for which outcome is predictable.
  - A lottery passes the test even though it doesn't maximize welfare of worst off – the loser is unpredictable.
  - …unless the lottery participants as a whole are worst off.

Rawlsian Difference Principle

- The difference rule implies a **lexmax** principle**.**
  - If applied recursively.

- **Lexmax (lexicographic maximum) principle**:
  - Maximize welfare of least advantaged class…
  - then next-to-least advantaged class…
  - and so forth.

Rawlsian Difference Principle

- There is apparently no practical math programming model for lexmax.

$$\begin{aligned}&\text{lexmax}\ \{u_1,\ldots,u_n\}\\&u_i = h_i(x_i),\ \ \text{all } i\\&x \in S\end{aligned}$$

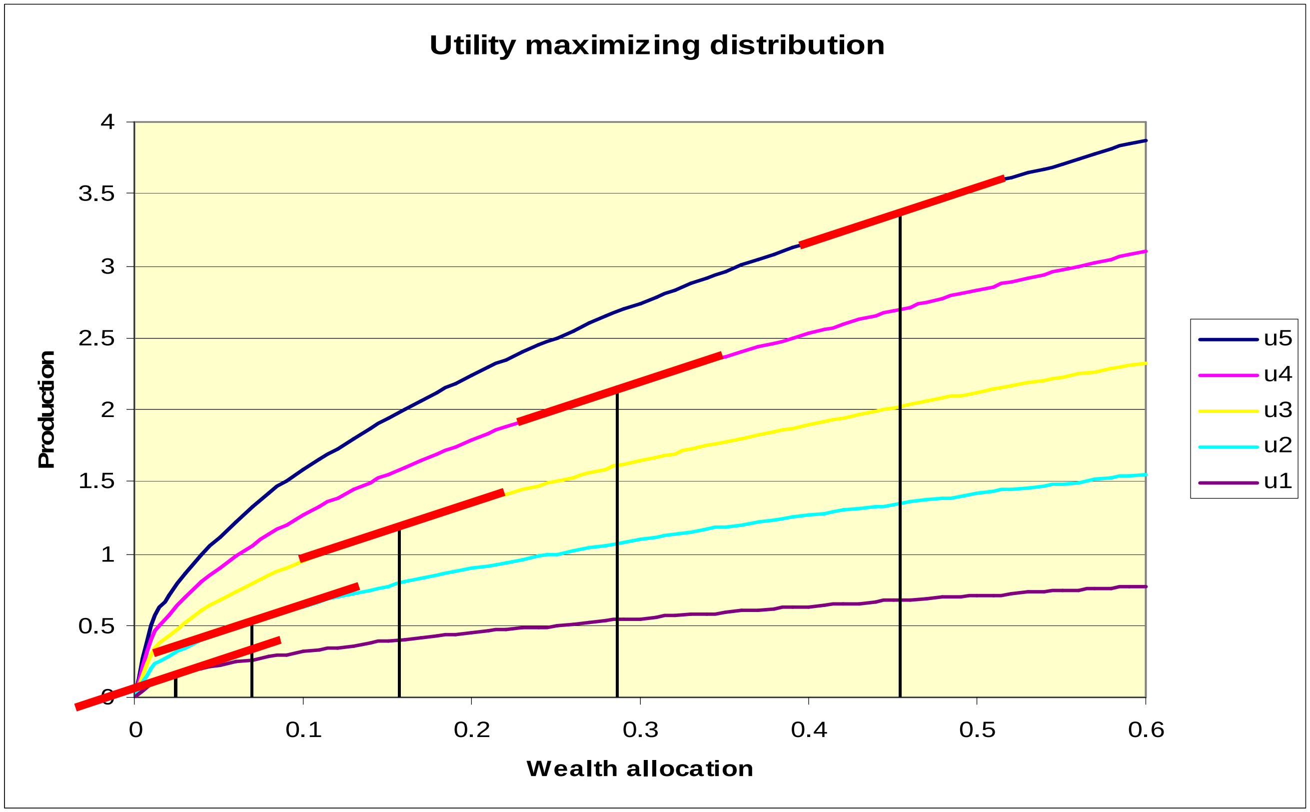
- We can solve the problem sequentially (pre-emptive goal programming).
    - Solve the maximin problem.
    - Fix the smallest u_i to its maximum value.
    - Solve the maximin problem over remaining u_is.
    - Continue to u_n.

Rawlsian Difference Principle

- The Difference and Lexmax Principles need not result in equality.
    - Consider the example presented earlier…

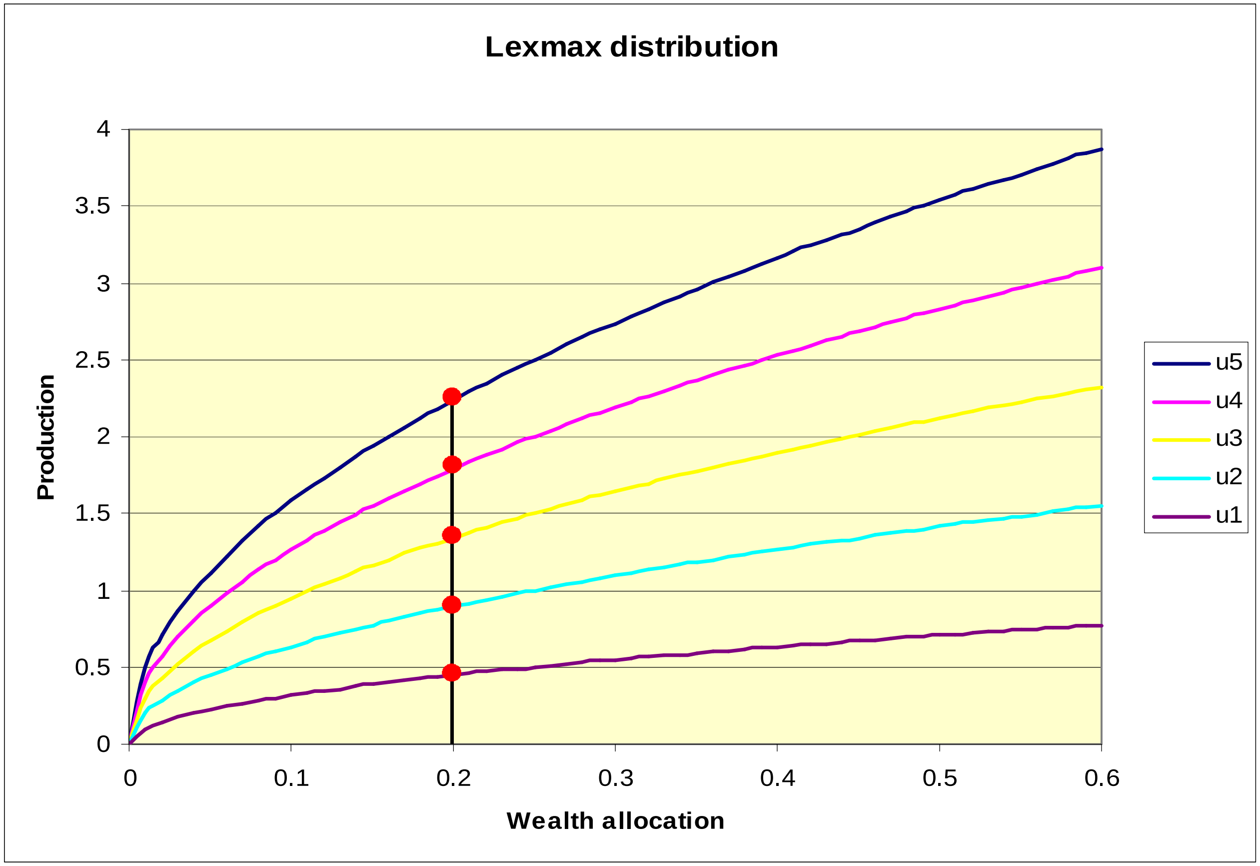
Rawlsian Difference Principle

Utilitarian distribution



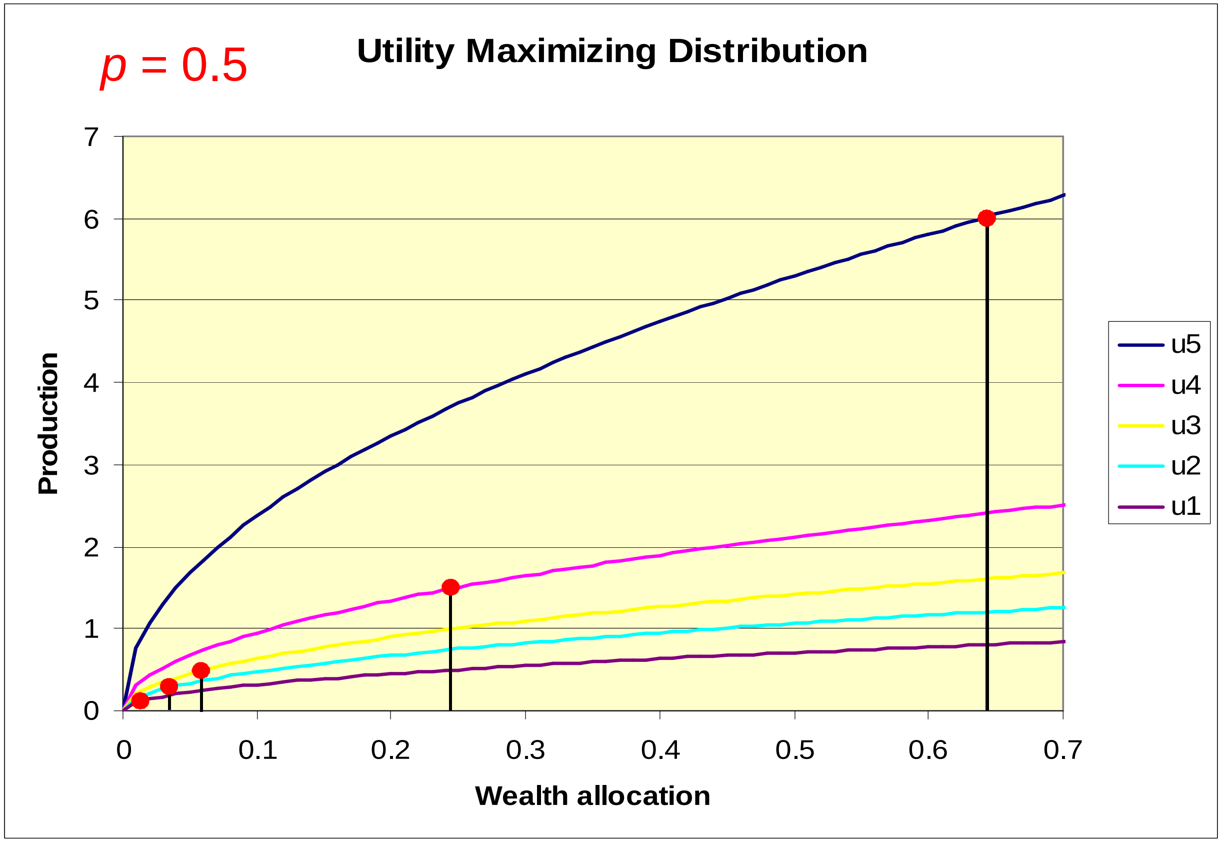
Rawlsian Difference Principle

Here, lexmax principle results in equality



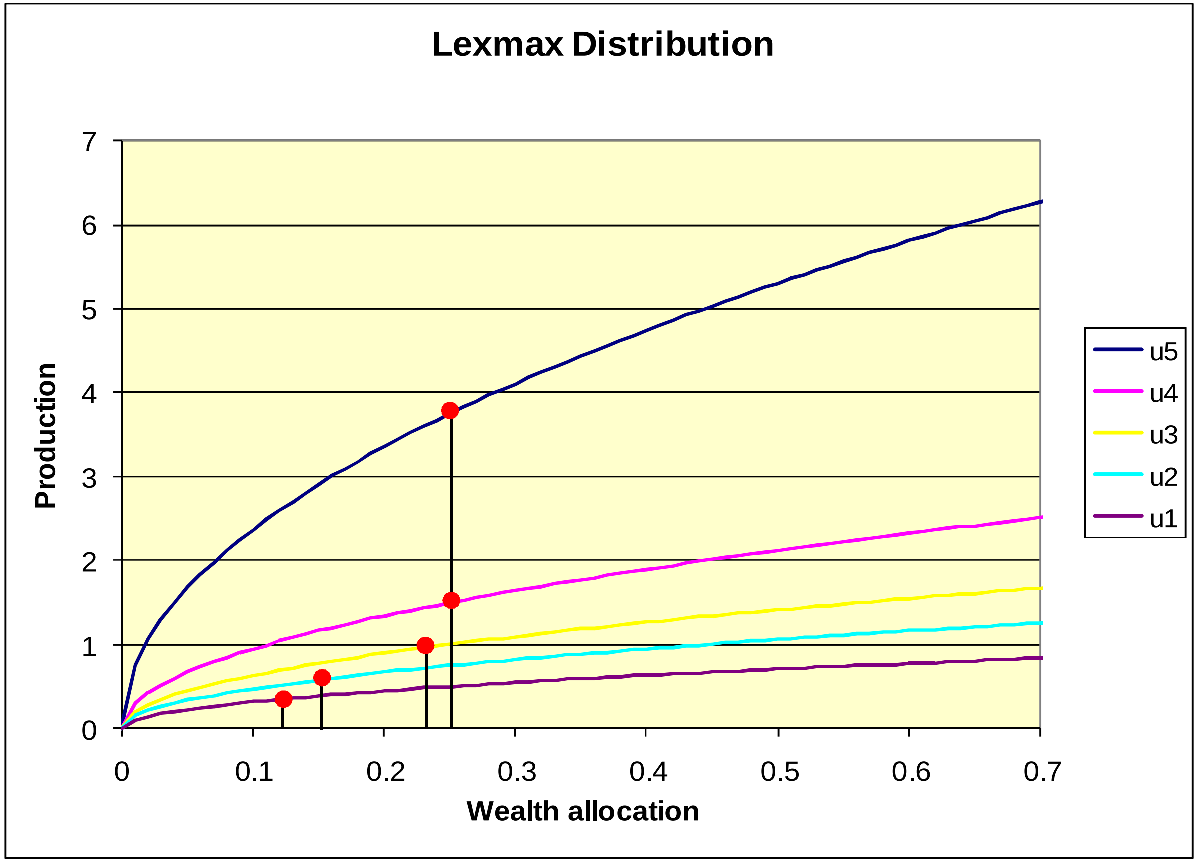
Utilitarianism

But consider this distribution…



Utilitarianism

Lexmax doesn’t result in equality



Axiomatics

- The economics literature derives social welfare functions from axioms of rational choice.
  - Some axioms are strong and hard to justify.
  - The social welfare function depends on degree of **interpersonal comparability** of utilities.
  - Arrow's impossibility theorem was the first result, but there are many others.

- **Social welfare function**
  - A function $f\,(u_1,\ldots,u_n)$ of individual utilities.
  - An optimization model can find a distribution of utility that maximizes social welfare.

Interpersonal Comparability

- Social Preferences
  - Let $u = (u_1,\ldots,u_n)$ be the vector of utilities allocated to individuals.
  - A social welfare function ranks distributions: u is preferable to u' if $f(u) > f(u')$.
- Invariance transformations.
  - These are transformations ϕ of utility vectors under which the ranking of distributions does not change.
  - Each $\phi = (\phi_1,\ldots,\phi_n)$, where ϕ_i is a transformation of individual utility u_i.

Interpersonal Comparability

- Ordinal noncomparability.
  - Any $\phi = (\phi_1,\ldots,\phi_n)$ with strictly increasing ϕ_is is an invariance transformation.
- Ordinal level comparability.
  - Any $\phi = (\phi_1,\ldots,\phi_n)$ with strictly increasing and identical ϕ_is is an invariance transformation.

Interpersonal Comparability

- Cardinal nonncomparability.
  - Any $\phi = (\phi_1,\ldots,\phi_n)$ with $\phi_i(u_i) = \alpha_i + \beta_i u_i$ and $\beta_i > 0$ is an invariance transformation.
- Cardinal unit comparability.
  - Any $\phi = (\phi_1,\ldots,\phi_n)$ with $\phi_i(u_i) = \alpha_i + \beta u_i$ and $\beta > 0$ is an invariance transformation.
- Cardinal ratio scale comparability
  - Any $\phi = (\phi_1,\ldots,\phi_n)$ with $\phi_i(u_i) = \beta u_i$ and $\beta > 0$ is an invariance transformation.

Axioms

- Anonymity
  - Social preferences are the same if indices of u_is are permuted.
- Strict pareto
  - If $u > u'$, then u is preferred to u'.
- Independence of irrelevant alternatives
  - The preference of u over u' depends only on u and u' and not on what other utility vectors are possible.
- Separability of unconcerned individuals
  - Individuals i for which $u_i = u_i'$ don't affect the ranking of u and u'.

Axiomatics

Theorem

Given **ordinal level comparability**, any social welfare function *f* that satisfies the axioms is lexicographically increasing or lexicographically decreasing. So we get a **lexmax** or **lexmin** objective.

Theorem

Given **cardinal unit comparability**, any social welfare function *f* that satisfies the axioms has the form $f(u) = \Sigma_i \, a_i u_i$ for
$a_i \geq 0$. Se we get a **utilitarian** objective.

Axiomatics

Theorem

Given **cardinal noncomparability**, any social welfare function f that satisfies the axioms (except anonimity and separability) has the form $f(u) = u_i$ for some fixed i. So individual i is a **dictator**.

Theorem

Given **cardinal ratio scale comparability**, any social welfare function f that satisfies the axioms has the form $f(u) = \Sigma_i u_i^p/p$. Se we get the production function used in the example.

Measures of Inequality

- Assume we wish to **minimize inequality**.
  - We will survey several measures of inequality.
  - They have different strengths and weaknesses.
  - Minimizing inequality may result in less total utility.

- **Pigou-Dalton** condition.
  - One criterion for evaluating an inequality measure.
  - If utility is transferred from one who is worse off to one who is better off, inequality should increase.

Measures of Inequality

- Measures of Inequality
  - An example
  - Utrilitarian, maximin, and lexmax solution
  - Relative range, max, min
  - Relative mean deviation
  - Variance, coefficient of variation
  - McLoone index
  - Gini coefficient
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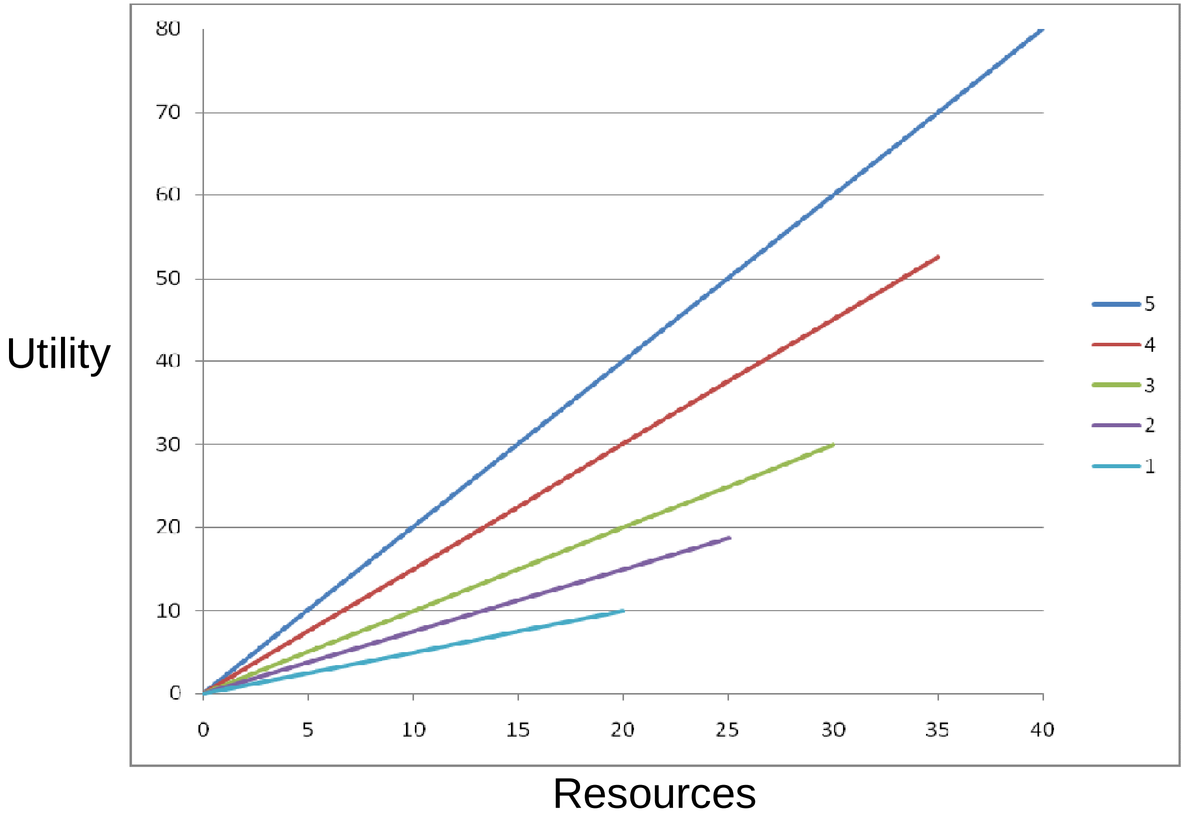
- An Allocation Problem

Example

Production functions for 5 individuals



Utilitarian

$$\max \sum_i u_i$$

LP model: $\max \sum_{i=1}^{5} u_i$

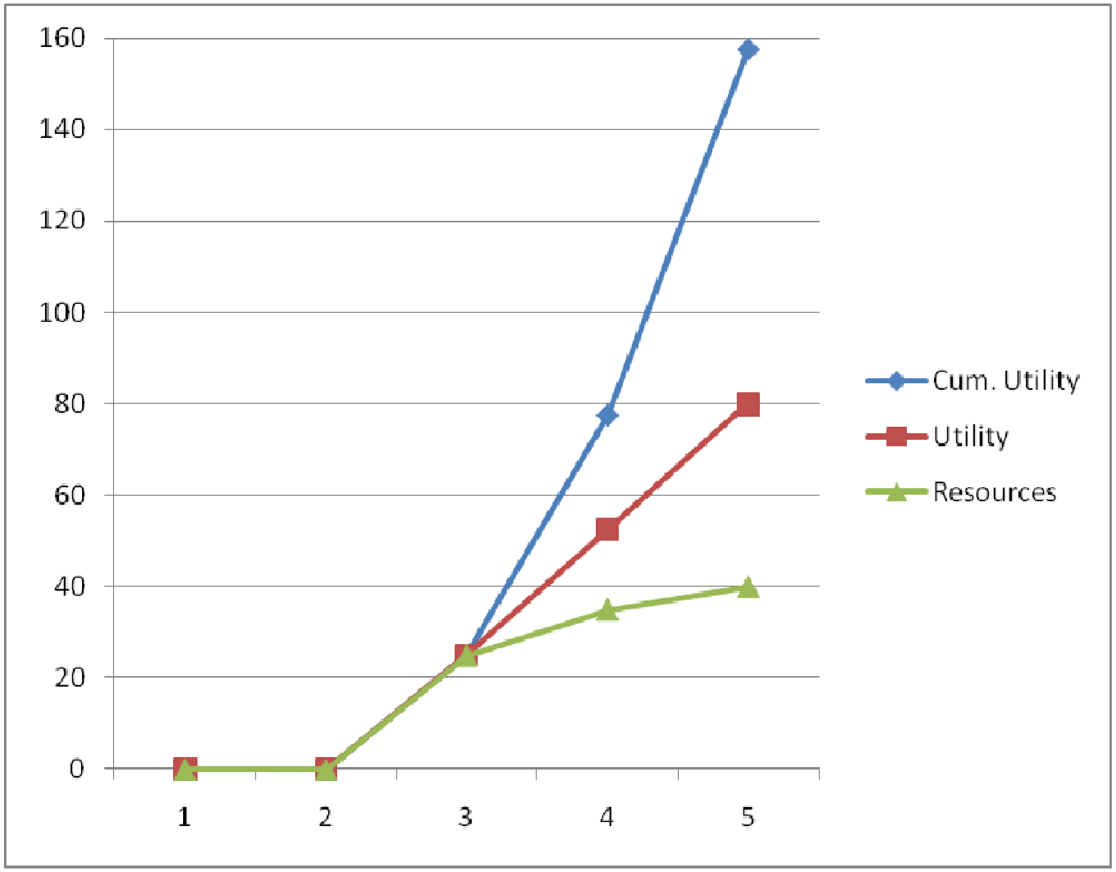
$$u_i = a_i x_i,\ 0 \le x_i \le b_i, \text{all } i, \quad \sum_i x_i = B$$

where $(a_1, \ldots a_5) = (0.5, 0.75, 1, 1.5, 2)$

$(b_1, \ldots, b_5) = (20, 25, 30, 35, 40)$

$B = 100$

Utilitarian



Rawlsian

$$\max \left\{ \min_i \{u_i\} \right\}$$

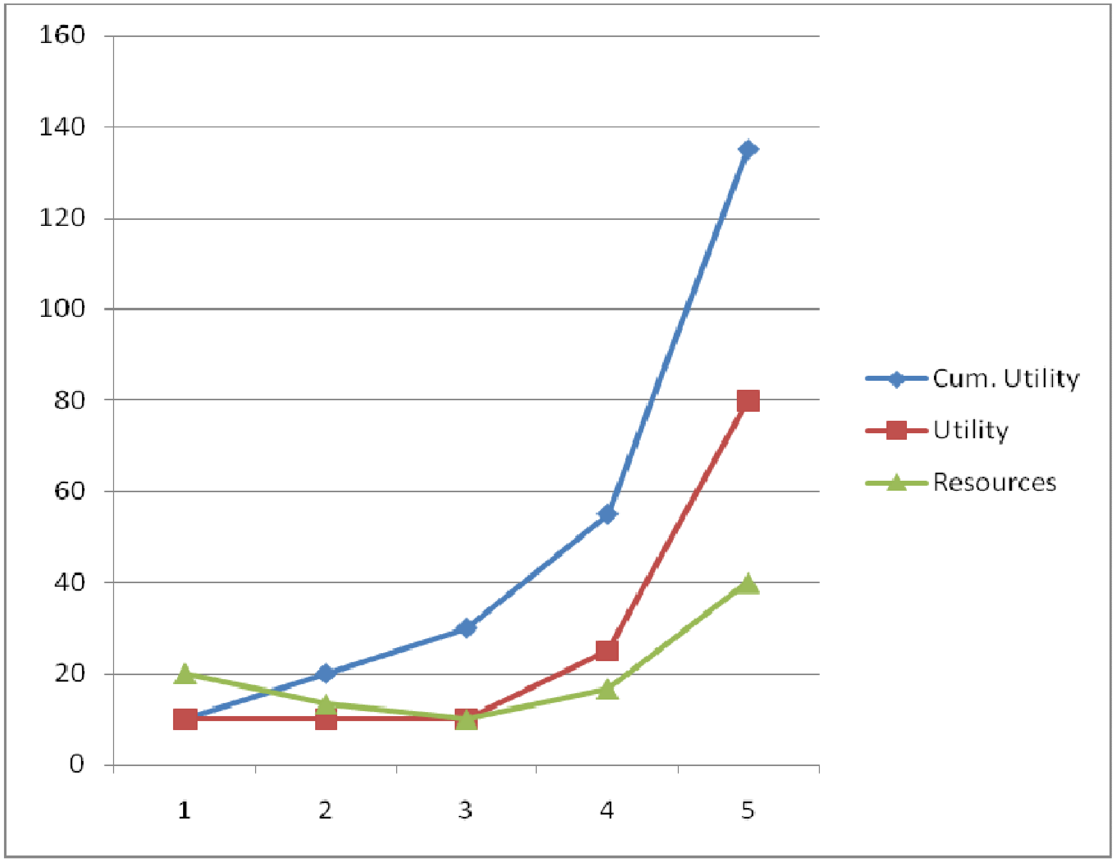
LP model: $\max u_{\min} + \in \sum_i u_i$

Ensures that solution is Pareto optimal

$$u_{\min} \le u_i, \quad \text{all } i$$

$$u_i = a_i x_i, \; 0 \le x_i \le b_i, \; \text{all } i, \quad \sum_i x_i = B$$

Rawlsian



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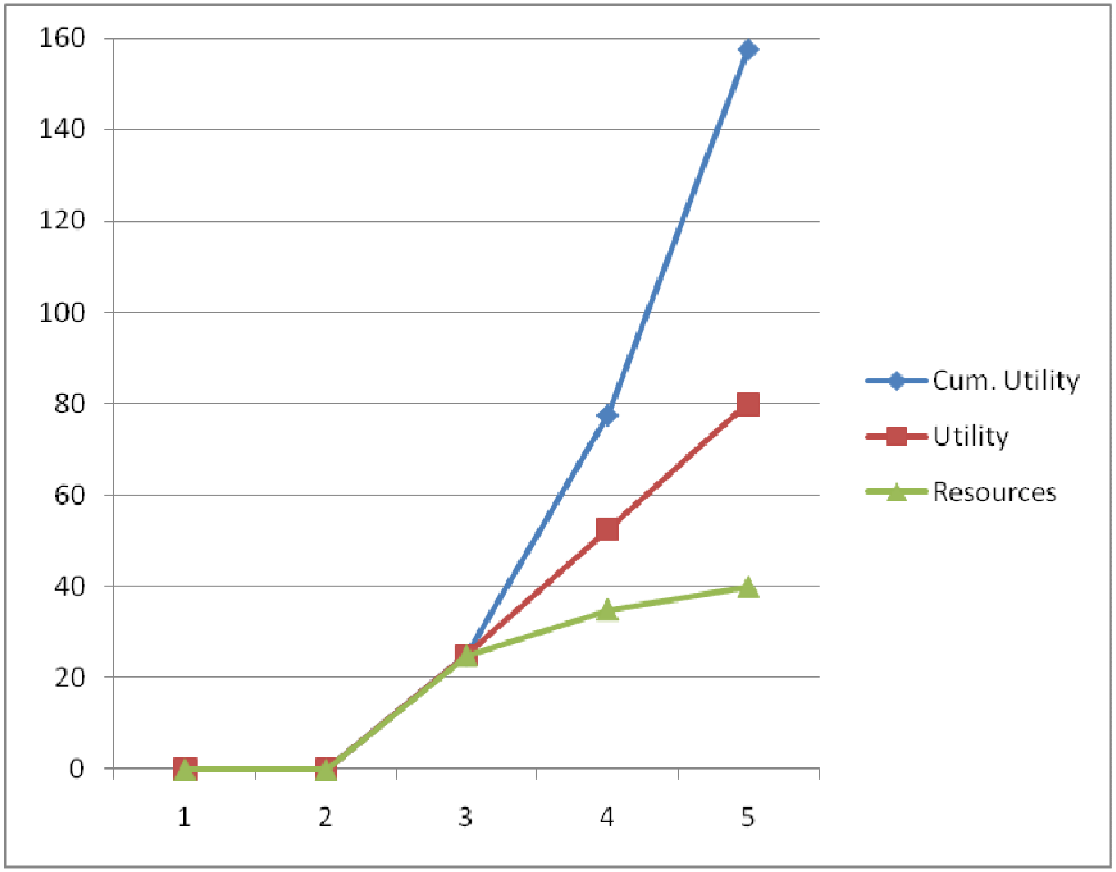
Cum. Utility
Utility
Resources

Utilitarian



Lexmax

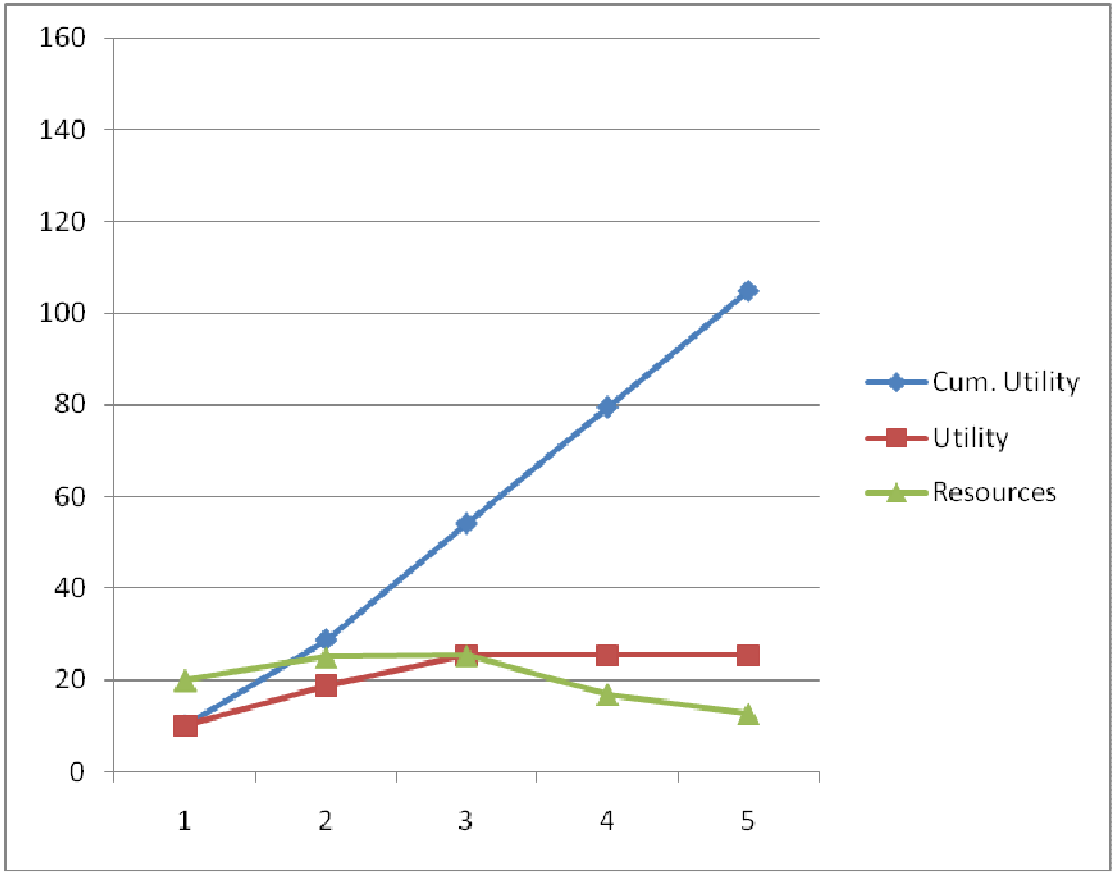
$$\text{lexmax}\left\{u_1,\ldots,u_n\right\}$$

Sequence of LP models, $k = 1, \ldots, n - 1$:

$$\begin{aligned}
&\max u_{\min} \\
&u_i = u_i^*, \ \text{ all } i < k \\
&u_{\min} \le u_i, \ \text{ all } i \ge k \\
&u_i = a_i x_i, \ 0 \le x_i \le b_i, \text{ all } i, \quad \sum_i x_i = B
\end{aligned}$$

Re-index for each k so that u_i for $i < k$ were fixed in previous iterations.

Lexmax



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Cum. Utility
Utility
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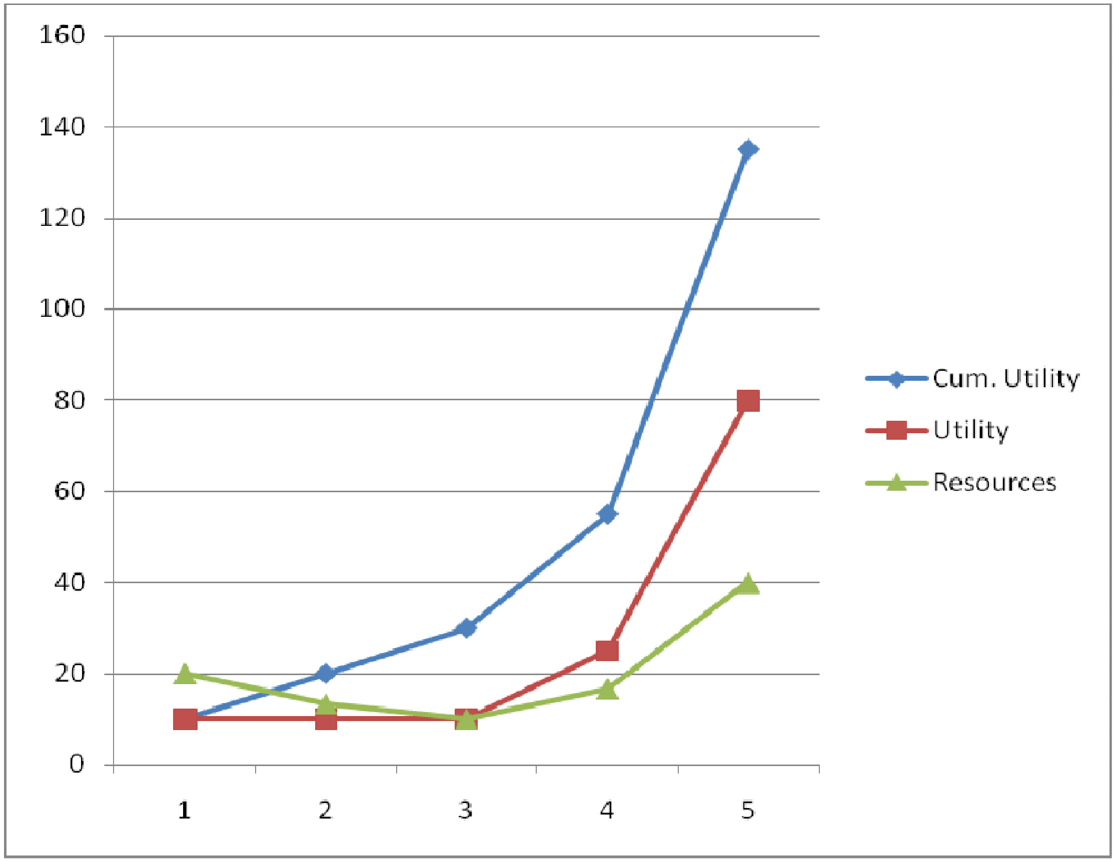
Rawlsian



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Cum. Utility
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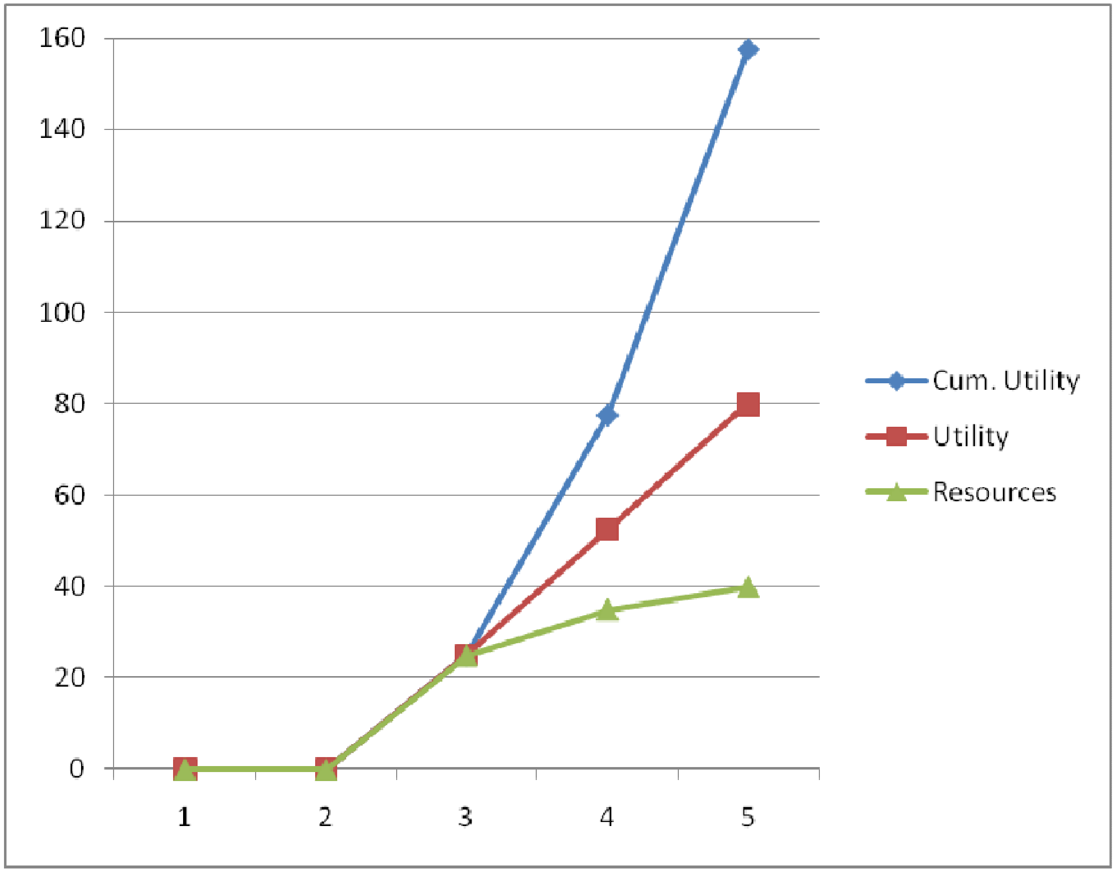
Utilitarian



Relative Range

$$\frac{u_{\max} - u_{\min}}{\bar{u}}$$

where $u_{\max} = \max_i \{u_i\}$ $\quad u_{\min} = \min_i \{u_i\}$ $\quad \bar{u} = (1/n)\sum_i u_i$

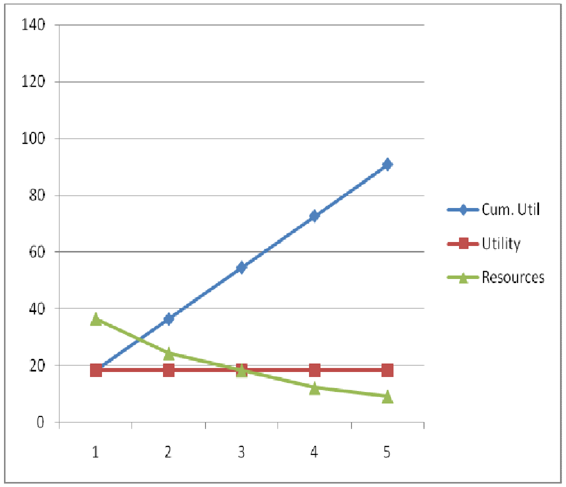
Rationale:

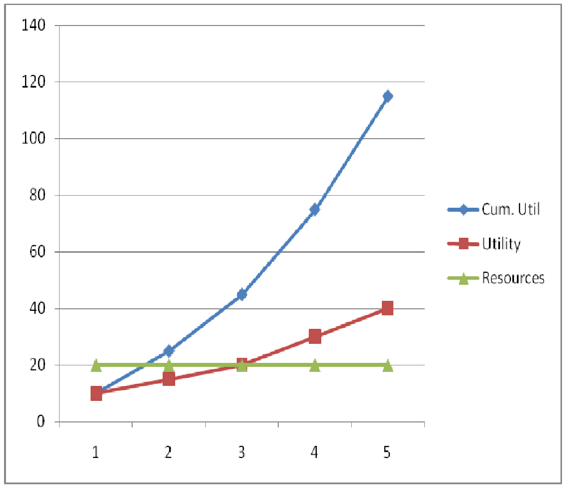
- Perceived inequality is relative to the best off.
- A distribution should be judged by the position of the worst-off.
- Therefore, minimize gap between top and bottom.

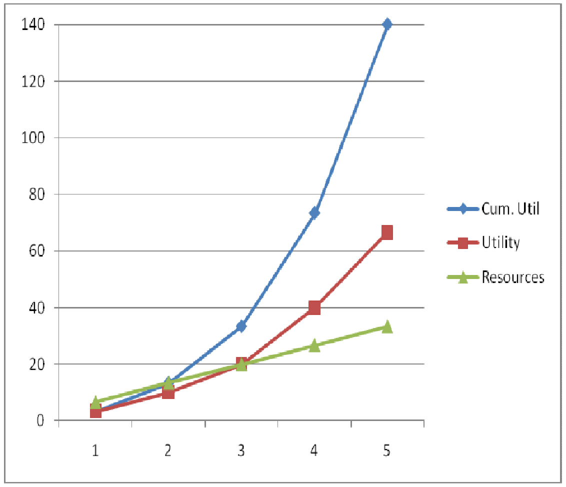
Problems:

- Ignores distribution between extremes.
- Violates Pigou-Dalton condition

Equality Measures: Comparison







Relative range: 0 1.30 2.26

Relative Range

$$\frac{u_{\max} - u_{\min}}{\bar{u}}$$

This is a **fractional linear programming** problem.

Use Charnes-Cooper transformation to an LP. In general,

$$\begin{aligned} &\min \frac{cx + c_0}{dx + d_0} \\ &Ax \geq b \\ &x \geq 0 \end{aligned} \qquad \text{becomes} \qquad \begin{aligned} &\min cx' + c_0 z \\ &Ax' \geq bz \\ &dx' + d_0 z = 1 \\ &x', z \geq 0 \end{aligned}$$

after change of variable $x = x'/z$ and fixing denominator to 1.

Relative Range

$$\frac{u_{\max} - u_{\min}}{\bar{u}}$$

Fractional LP model: $\min \dfrac{u_{\max} - u_{\min}}{(1/n)\sum_i u_i}$

$$u_{\max} \geq u_i,\ u_{\min} \leq u_i,\ \text{all } i$$

$$u_i = a_i x_i,\ 0 \leq x_i \leq b_i,\ \text{all } i,\quad \sum_i x_i = B$$

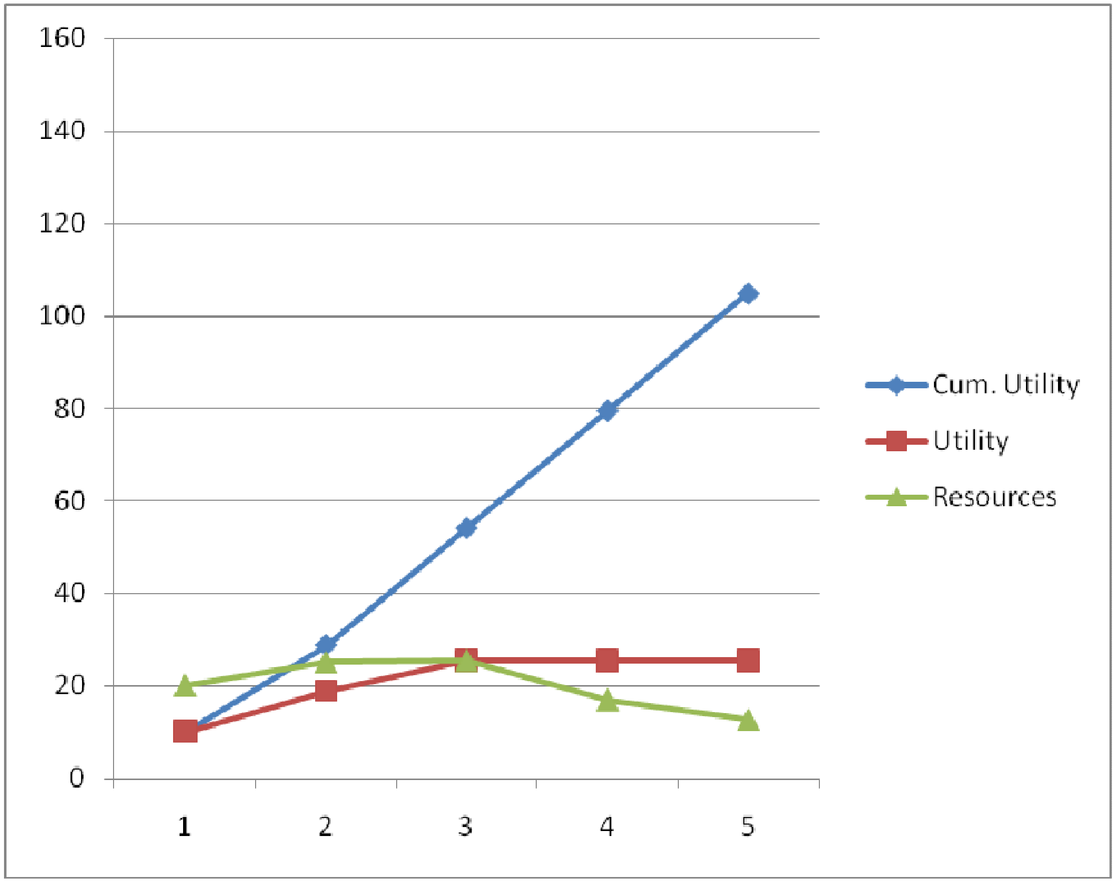
LP model: $\min u_{\max} - u_{\min}$

$$u_{\max} \geq u_i',\ u_{\min} \leq u_i',\ \text{all } i$$

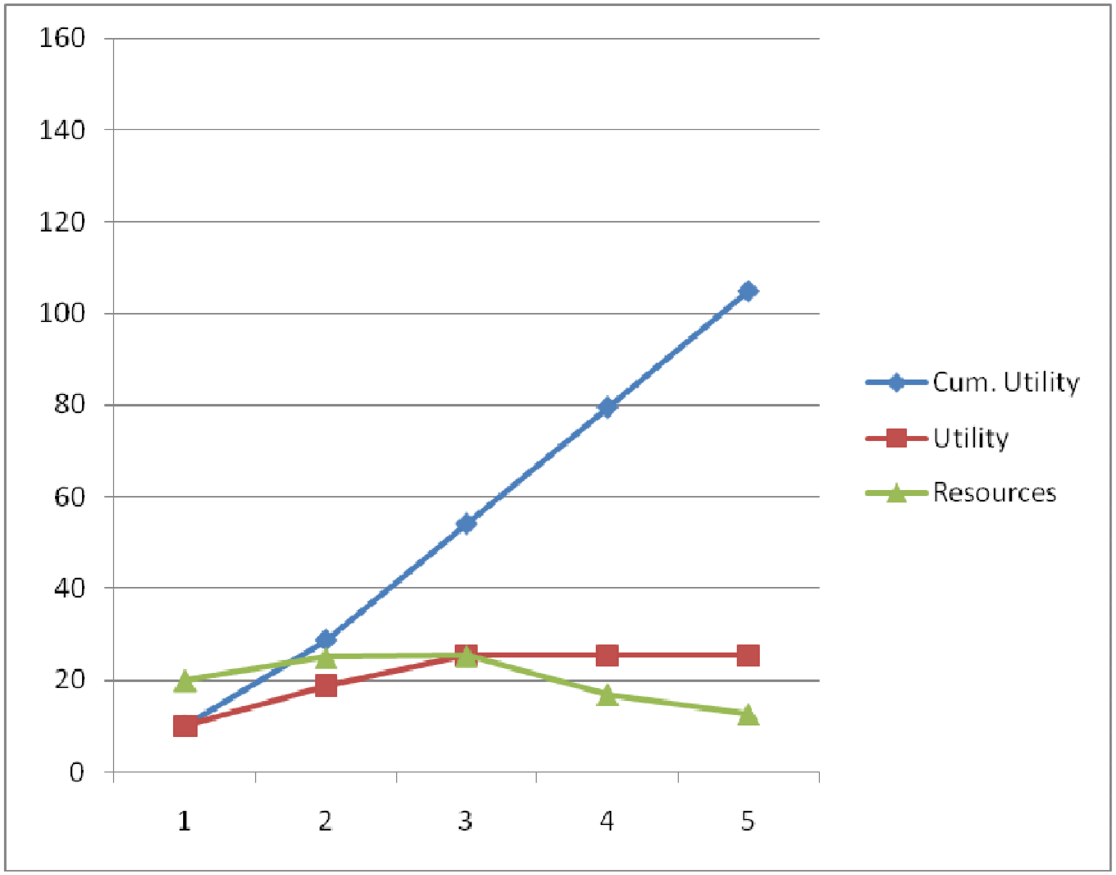
$$u_i' = a_i x_i',\ 0 \leq x_i' \leq b_i z,\ \text{all } i,\quad \sum_i x_i' = Bz$$

$$(1/n)\sum_i u_i' = 1$$

Relative Range



Lexmax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Cum. Utility
Utility
Resources

Relative Max

$$\frac{u_{\max}}{\bar{u}}$$

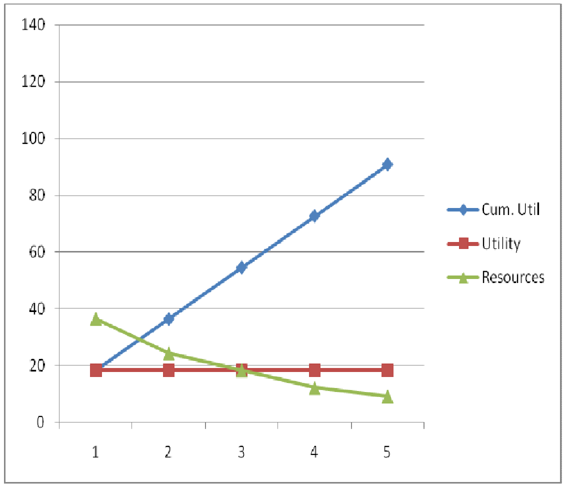
Rationale:

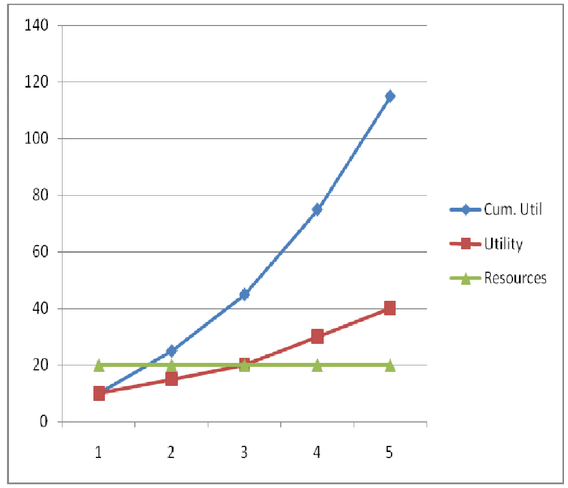
- Perceived inequality is relative to the best off.
- Possible application to salary levels (typical vs. CEO)

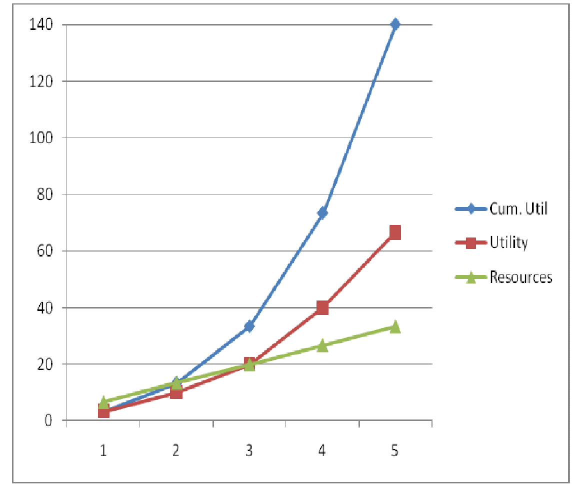
Problems:

- Ignores distribution below the top.
- Violates Pigou-Dalton condition

Equality Measures: Comparison







Relative range:	0	1.30	2.26
Relative max:	1	1.73	2.38

Relative Max

$$\frac{u_{\max}}{\bar{u}}$$

Fractional LP model:

$$\min \frac{u_{\max}}{(1/n)\sum_i u_i}$$

$$u_{\max} \geq u_i, \text{ all } i$$

$$u_i = a_i x_i,\ 0 \leq x_i \leq b_i,\ \text{ all } i,\quad \sum_i x_i = B$$

LP model:

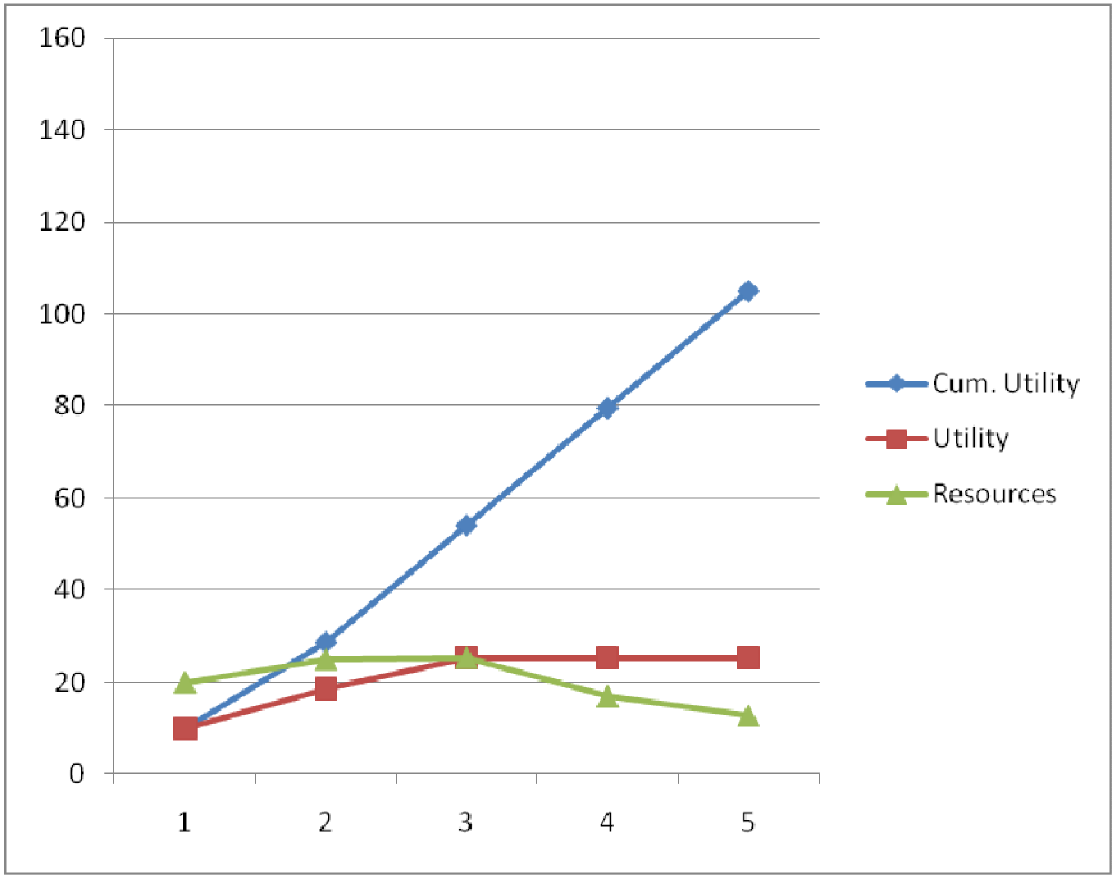
$$\min u_{\max}$$

$$u_{\max} \geq u_i' \text{ all } i$$

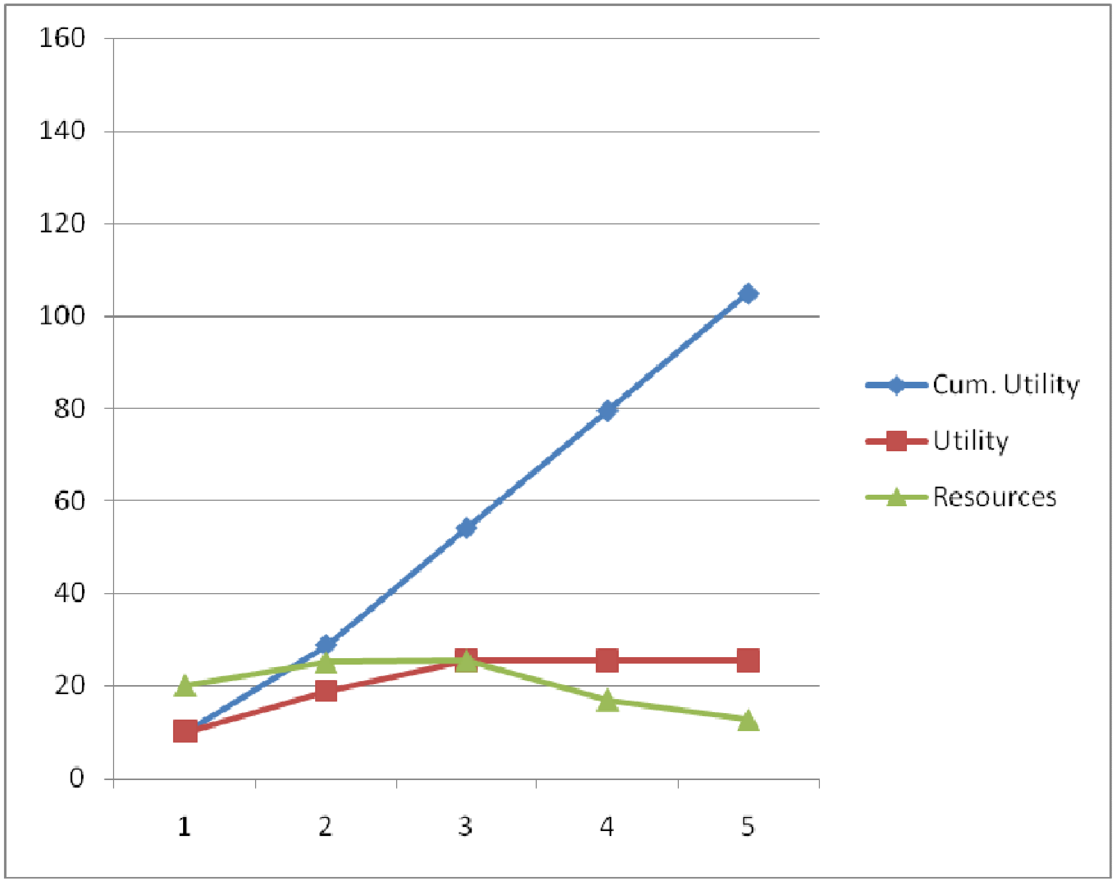
$$u_i' = a_i x_i',\ 0 \leq x_i' \leq b_i z,\ \text{ all } i,\quad \sum_i x_i' = Bz$$

$$(1/n)\sum_i u_i' = 1$$

Relative Max



Relative Range



Relative Min

$$\frac{u_{\min}}{\bar{u}}$$

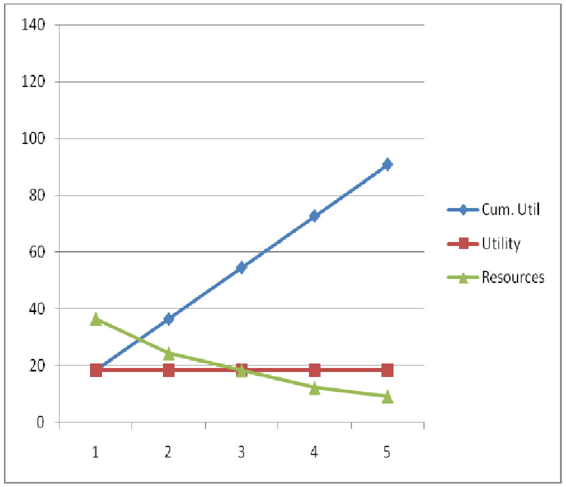
Rationale:

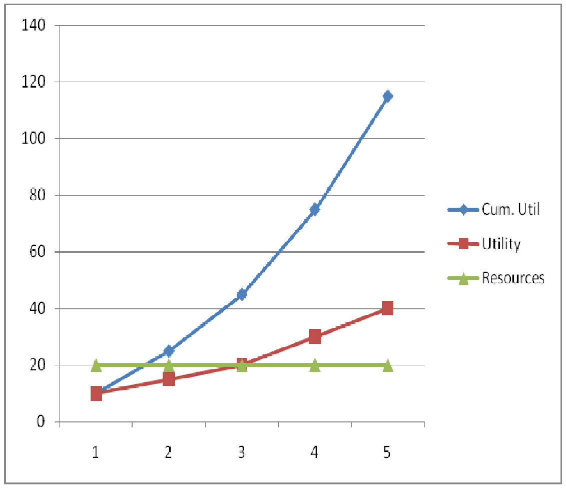
- Measures adherence to Rawlsian Difference Principle.
- …relativized to mean

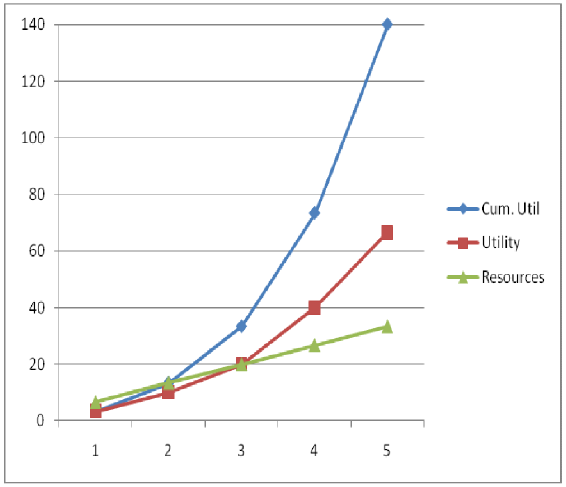
Problems:

- Ignores distribution above the bottom.
- Violates Pigou-Dalton condition

Equality Measures: Comparison







Relative range:	0	1.30	2.26
Relative max:	1	1.73	2.38
Relative min:	1	0.43	0.12

Relative Min

$$\frac{u_{\min}}{\bar{u}}$$

Fractional LP model:

$$\max \frac{u_{\min}}{(1/n)\sum_i u_i}$$

$$u_{\min} \le u_i, \text{ all } i$$

$$u_i = a_i x_i,\ 0 \le x_i \le b_i,\ \text{ all } i,\quad \sum_i x_i = B$$

LP model:

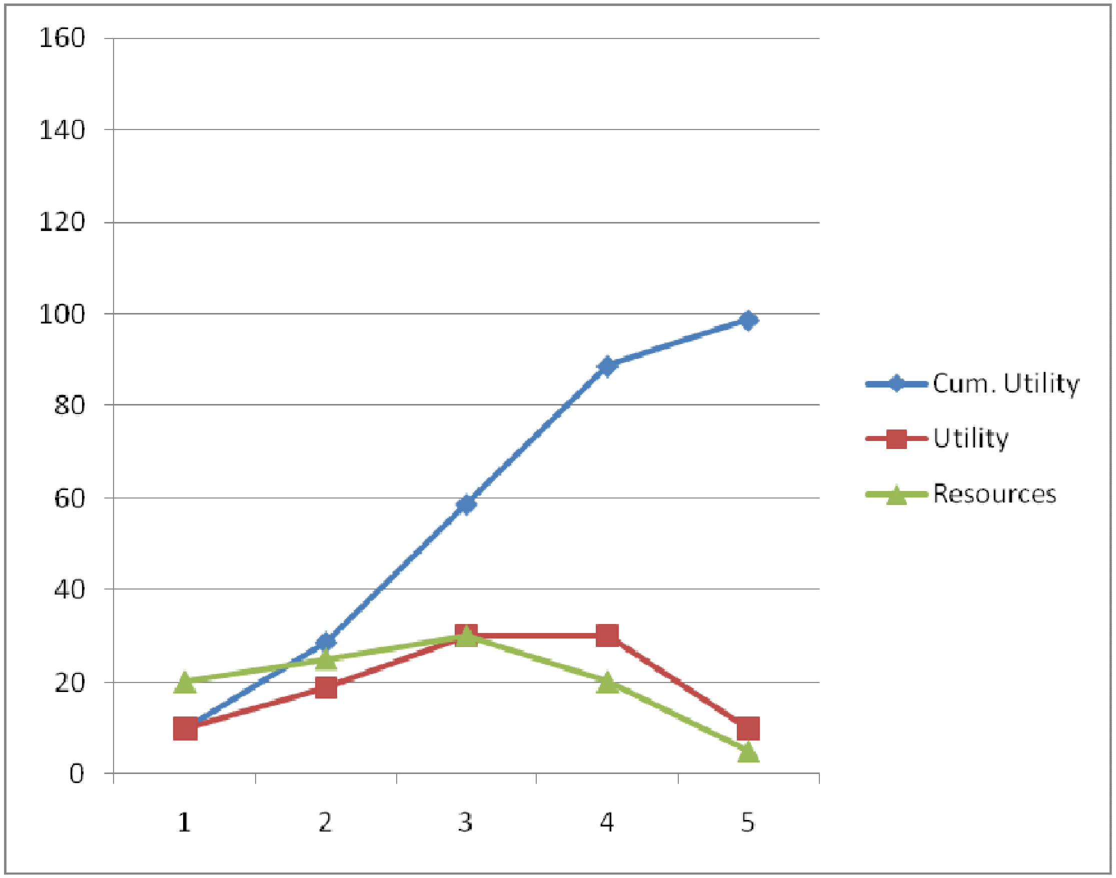
$$\max u_{\min}$$

$$u_{\min} \ge u_i' \text{ all } i$$

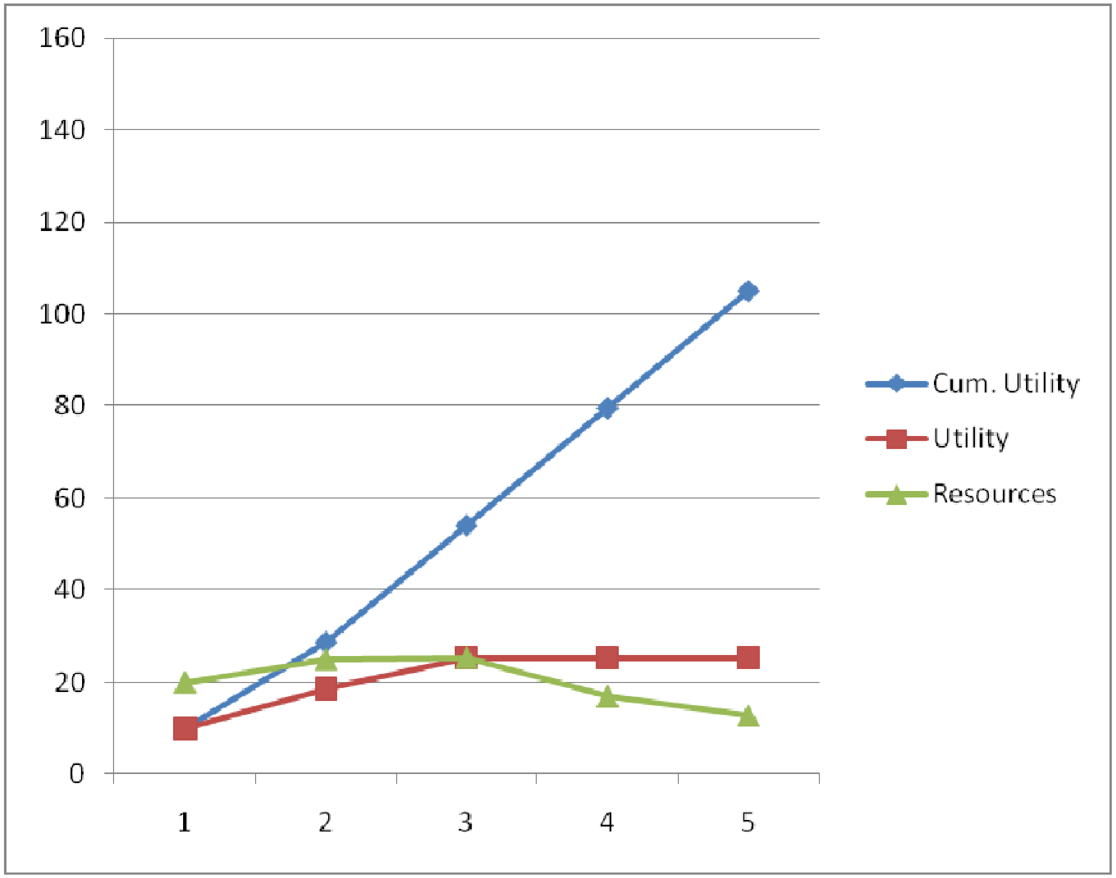
$$u_i' = a_i x_i',\ 0 \le x_i' \le b_i z,\ \text{ all } i,\quad \sum_i x_i' = Bz$$

$$(1/n)\sum_i u_i' = 1$$

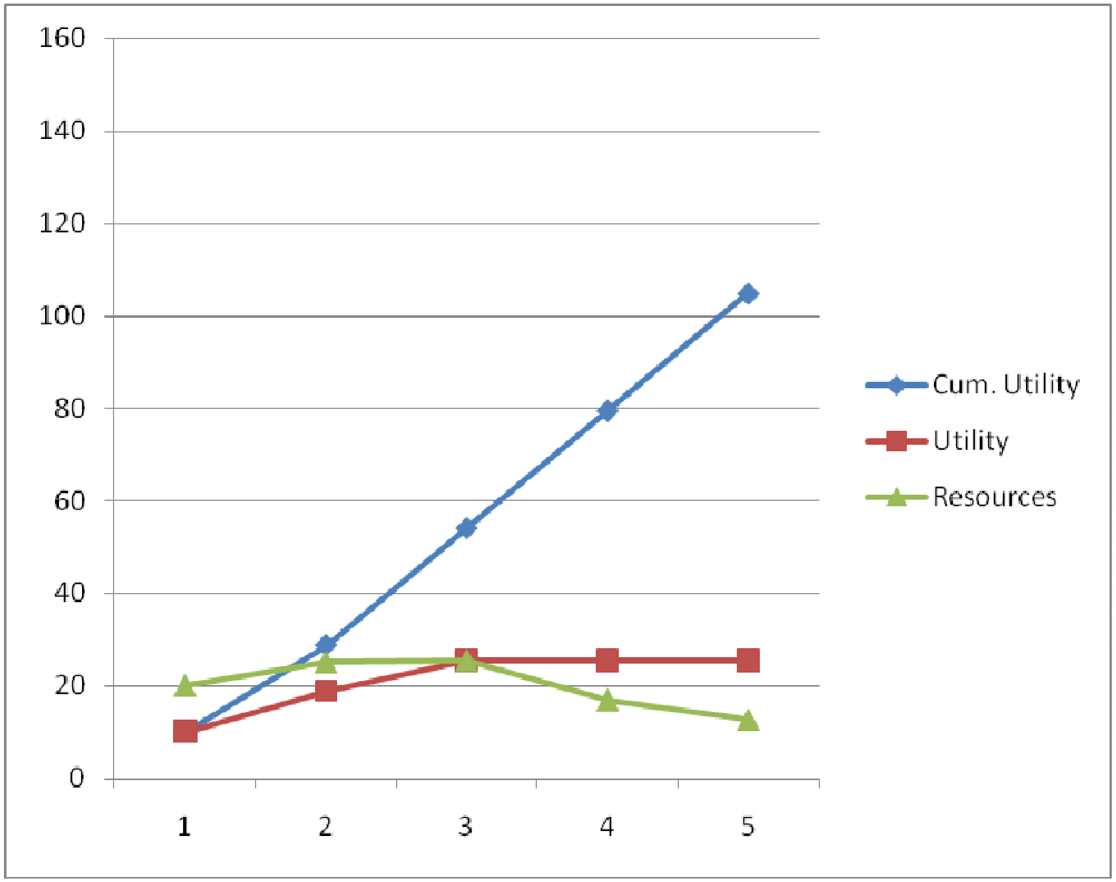
Relative Min



Relative Max



Relative Range



Relative Mean Deviation

$$\frac{\sum_i \left| u_i - \bar{u} \right|}{\bar{u}}$$

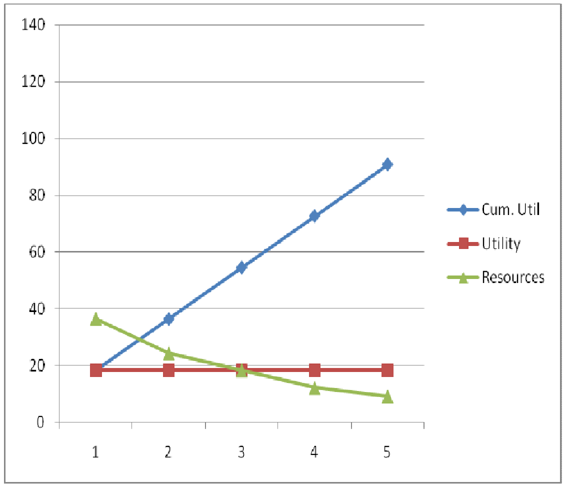
Rationale:

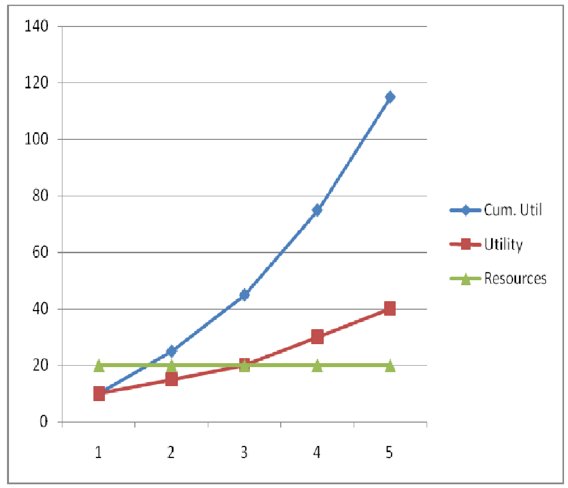
- Perceived inequality is relative to average.
- Entire distribution should be measured.

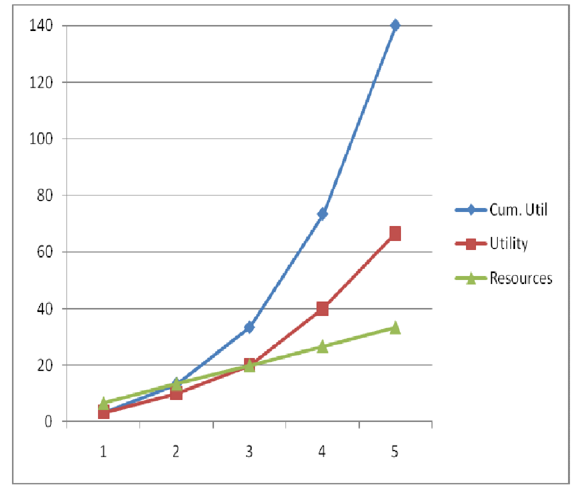
Problems:

- Violates Pigou-Dalton condition
- Insensitive to transfers on the same side of the mean.
- Insensitive to placement of transfers from one side of the mean to the other.

Equality Measures: Comparison







Relative range:	0	1.30	2.26
Rel. mean dev.:	0	0.42	0.72

Relative Mean Deviation

$$\frac{\sum_i |u_i - \bar{u}|}{\bar{u}}$$

Fractional LP model:

$$\max \frac{\sum_i (u_i^+ + u_i^-)}{\bar{u}}$$

$$u_i^+ \geq u_i - \bar{u},\ u_i^- \geq \bar{u} - u_i,\ \text{all } i$$

$$\bar{u} = (1/n)\sum_i u_i$$

$$u_i = a_i x_i,\ 0 \leq x_i \leq b_i,\ \text{all } i,\quad \sum_i x_i = B$$

LP model:

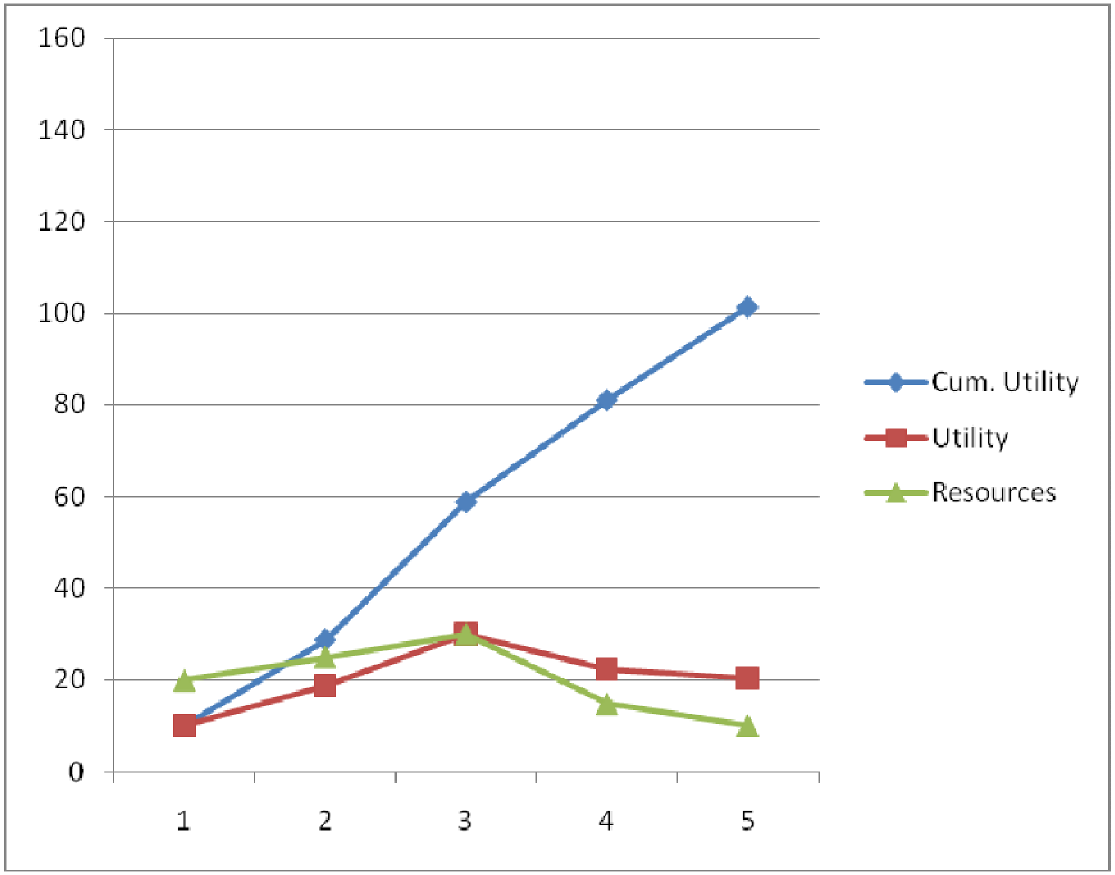
$$\max \sum_i (u_i^+ + u_i^-)$$

$$u_i^+ \geq u_i' - 1,\ u_i^- \leq u_i' - 1,\ \text{all } i$$

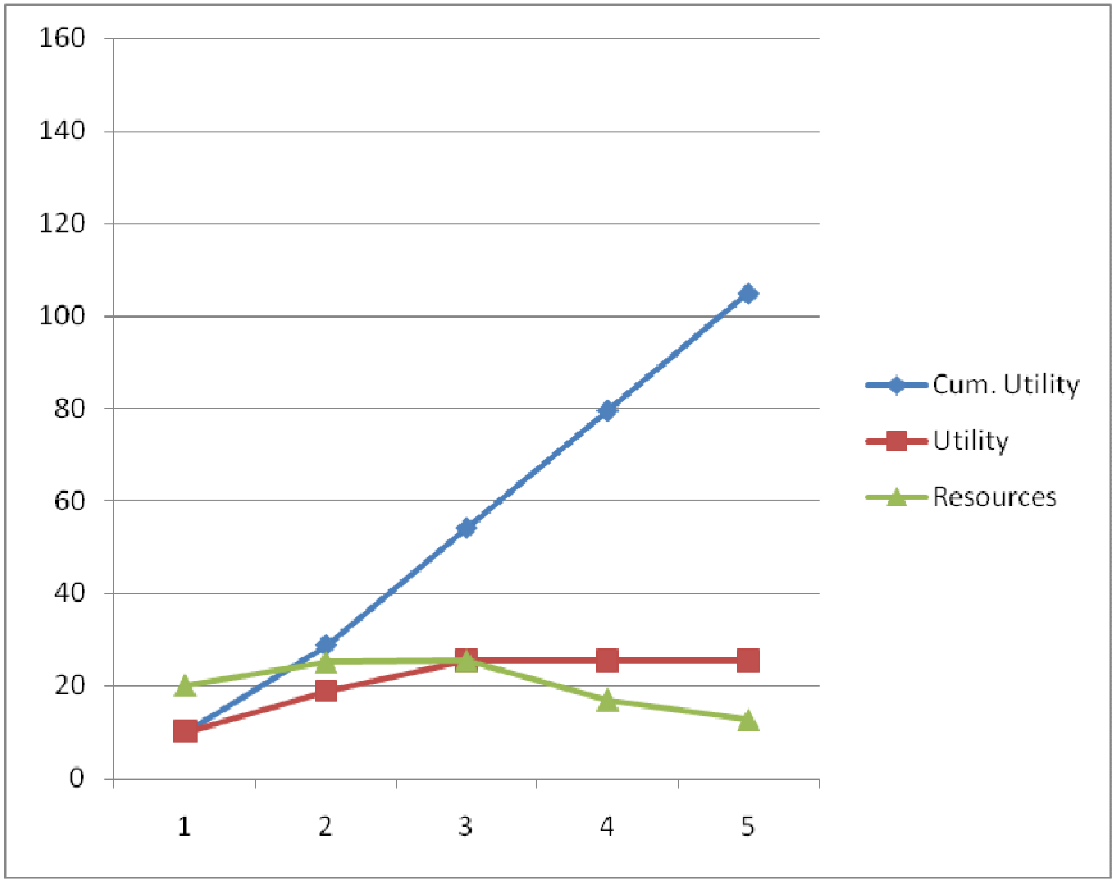
$$(1/n)\sum_i u_i' = 1$$

$$u_i' = a_i x_i',\ 0 \leq x_i' \leq b_i z,\ \text{all } i,\quad \sum_i x_i' = Bz$$

Relative Mean Deviation



Relative Range



Variance

$$(1/n)\sum_i (u_i - \bar{u})^2$$

Rationale:

- Weight each utility by its distance from the mean.
- Satisfies Pigou-Dalton condition.
- Sensitive to transfers on one side of the mean.
- Sensitive to placement of transfers from one side of the mean to the other.

Problems:

- Weighting is arbitrary?
- Variance depends on scaling of utility.

Variance

$$(1/n)\sum_i (u_i - \bar{u})^2$$

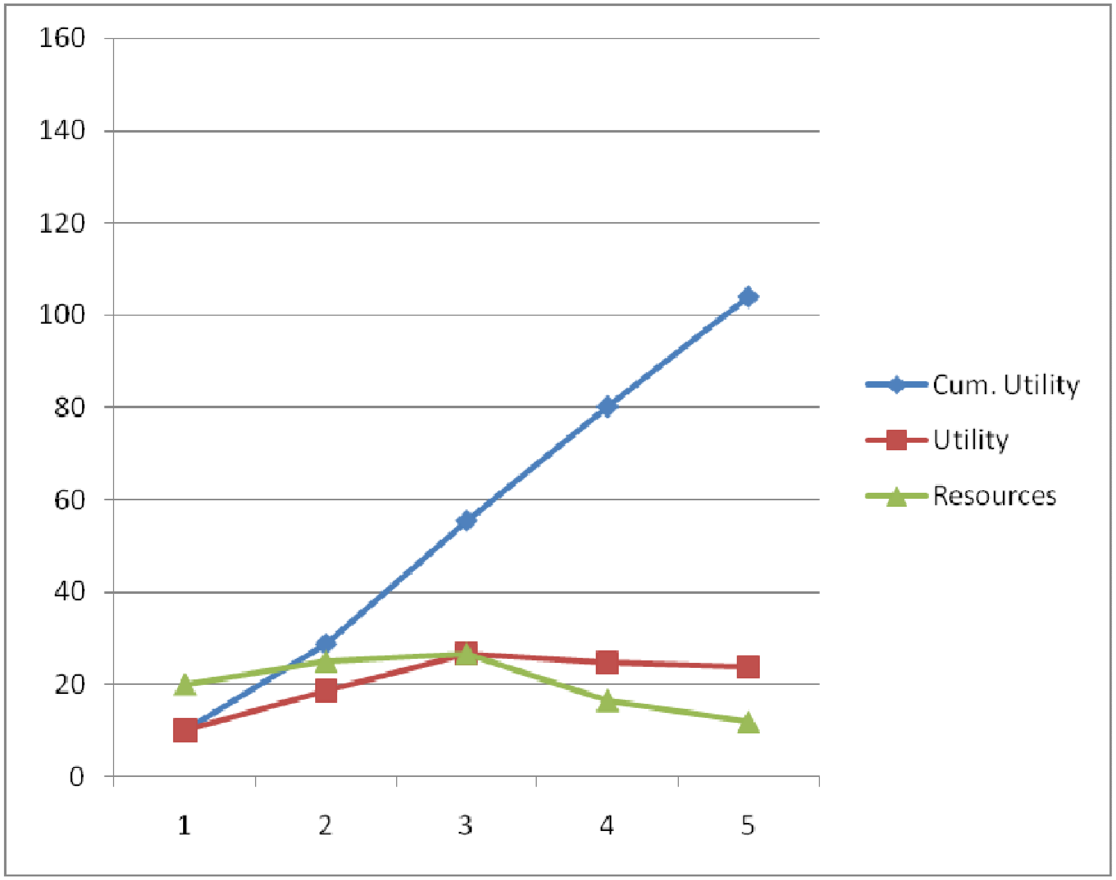
Convex nonlinear model:

$$\min\ (1/n)\sum_i (u_i - \bar{u})^2$$

$$\bar{u} = (1/n)\sum_i u_i$$

$$u_i = a_i x_i,\ \ 0 \le x_i \le b_i,\ \text{all } i, \quad \sum_i x_i = B$$

Variance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Cum. Utility
Utility
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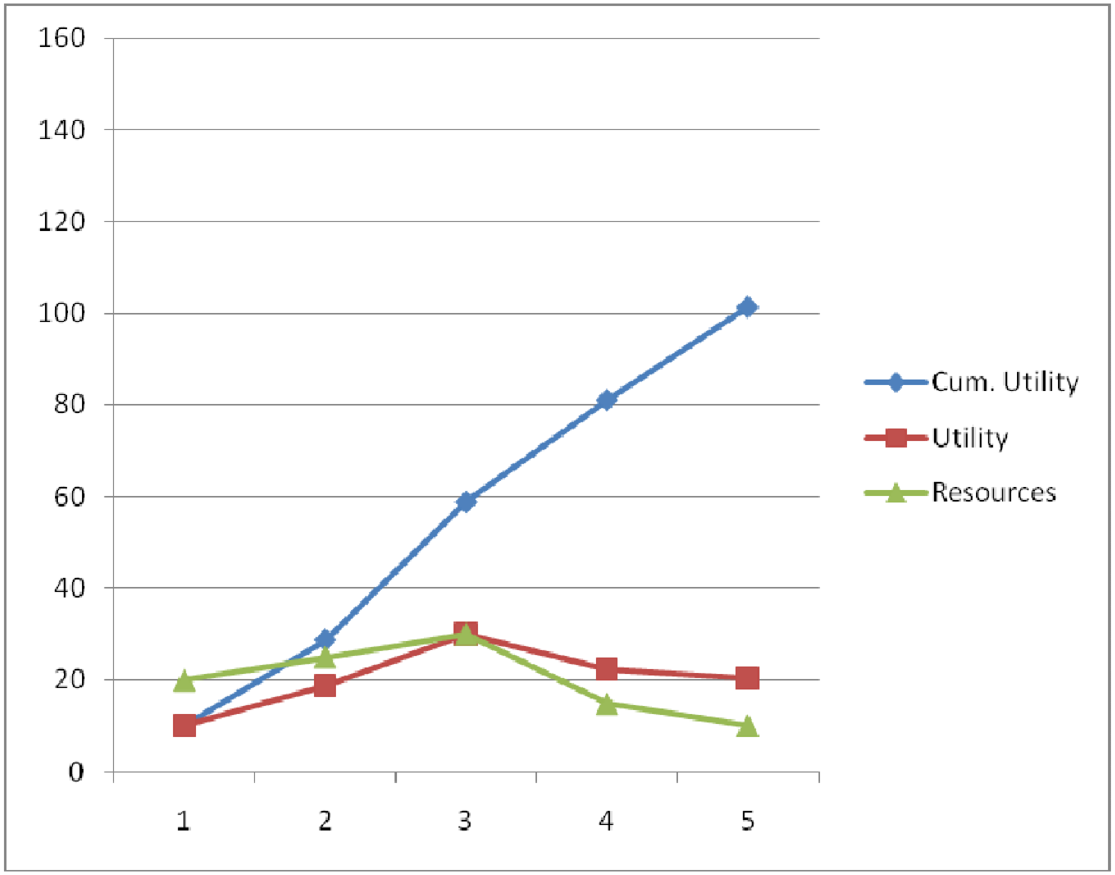
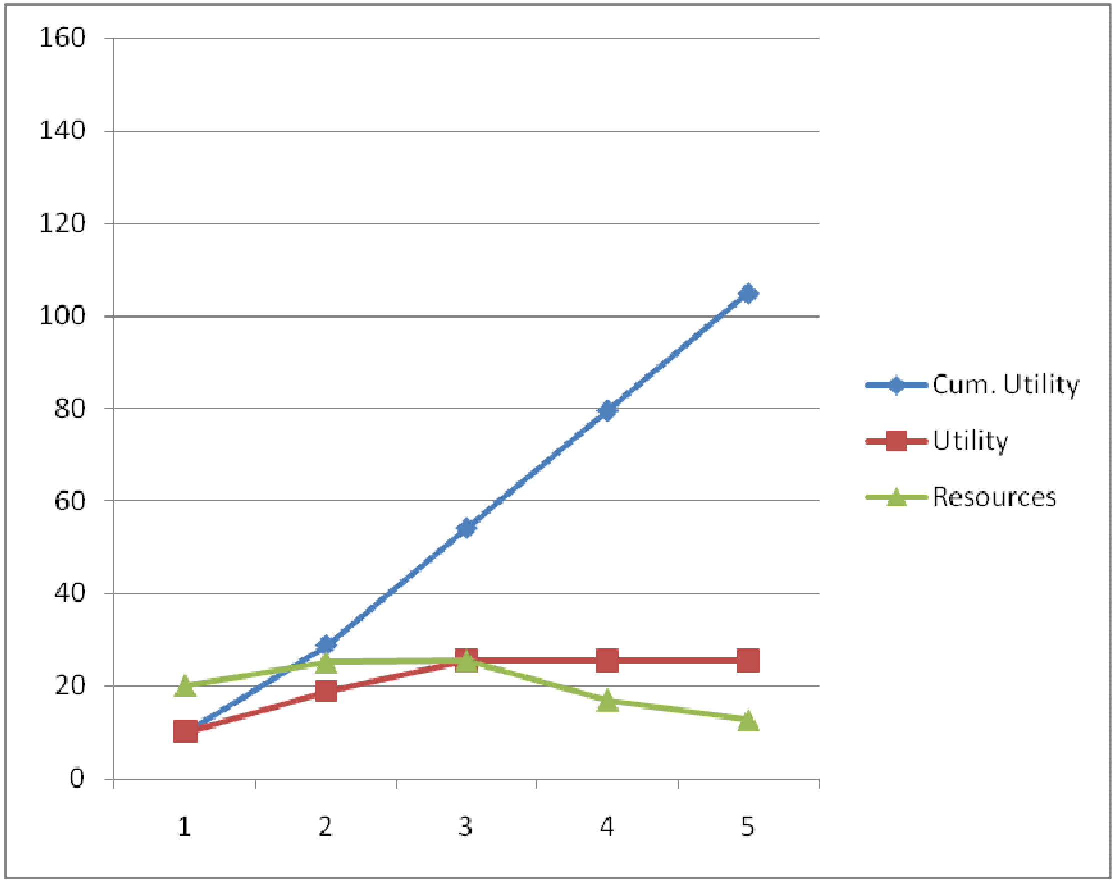
Relative Mean Deviation



Relative Range



Coefficient of Variation

$$\frac{\left((1/n)\sum_i (u_i - \bar{u})^2\right)^{1/2}}{\bar{u}}$$

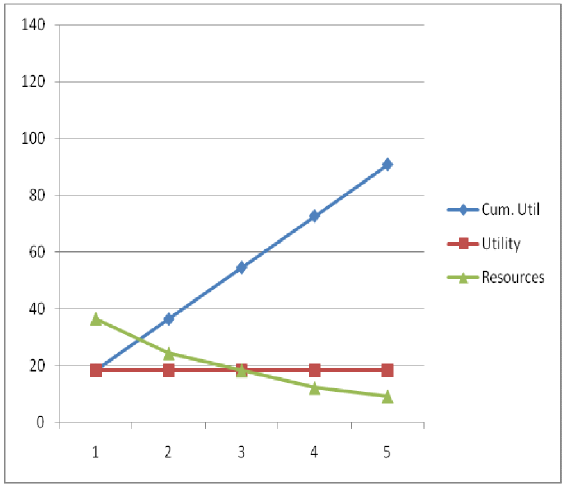
Rationale:

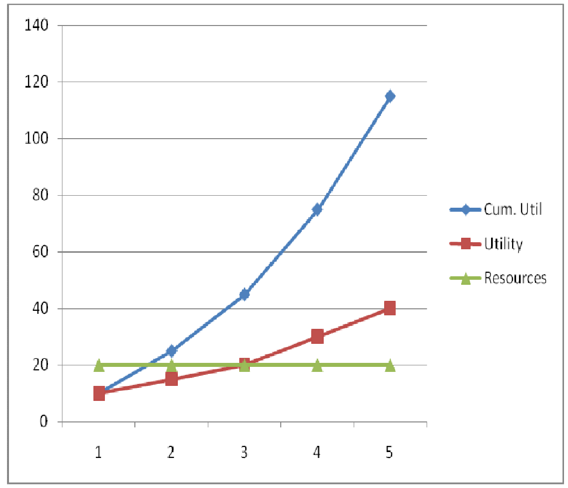
- Similar to variance.
- Invariant with respect to scaling of utilities.

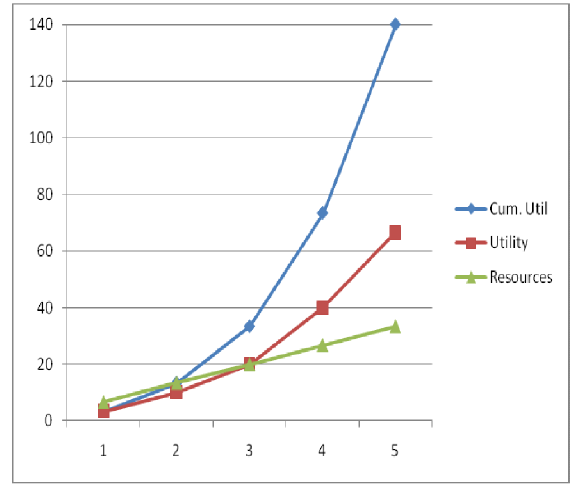
Problems:

- When minimizing inequality, there is an incentive to reduce average utility.
- Should be minimized only for fixed total utility.

Equality Measures: Comparison







Relative range:	0	1.30	2.26
Rel. mean dev.:	0	0.42	0.72
Coeff. of variation:	0	0.46	0.81

Coefficient of Variation

$$\frac{\left((1/n)\sum_i (u_i - \bar{u})^2\right)^{1/2}}{\bar{u}}$$

Again use change of variable $u = u'/z$ and fix denominator to 1.

$$\begin{aligned} \min\ & \frac{\left((1/n)\sum_i (u_i - \bar{u})^2\right)^{1/2}}{\bar{u}} \\ & Au \ge b \\ & u \ge 0 \end{aligned}$$

becomes

$$\begin{aligned} \min\ & \left((1/n)\sum_i (u'_i - 1)^2\right)^{1/2} \\ & Au' \ge bz \\ & (1/n)\sum_i u'_i = 1 \\ & u' \ge 0 \end{aligned}$$

Can drop exponent to make problem convex

Coefficient of Variation

$$\frac{\left((1/n)\sum_i (u_i - \bar{u})^2 \right)^{1/2}}{\bar{u}}$$

Fractional nonlinear model:

$$\max \frac{\left((1/n)\sum_i (u_i - \bar{u})^2 \right)^{1/2}}{\bar{u}}$$

$$\bar{u} = (1/n)\sum_i u_i$$

$$u_i = a_i x_i,\ 0 \le x_i \le b_i,\ \text{all } i, \quad \sum_i x_i = B$$

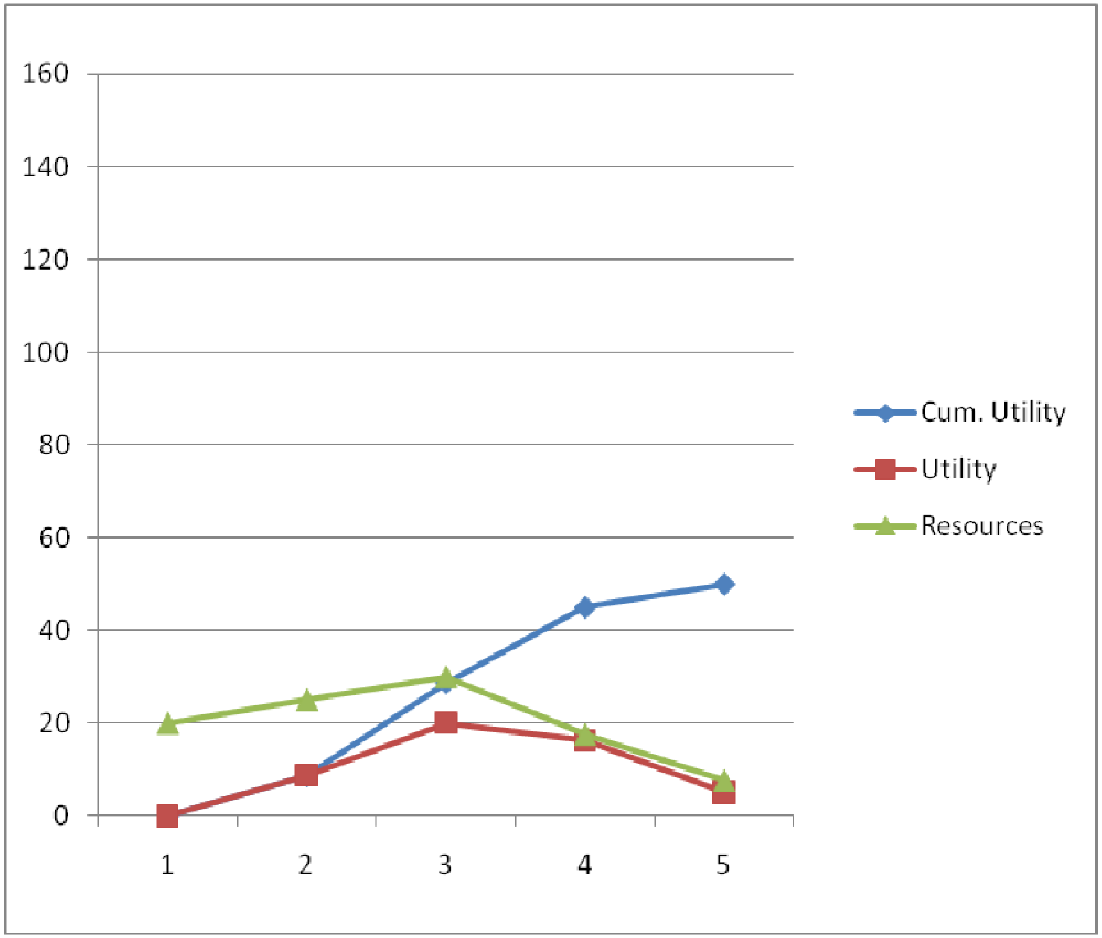
Convex nonlinear model:

$$\min\, (1/n)\sum_i (u_i' - 1)^2$$

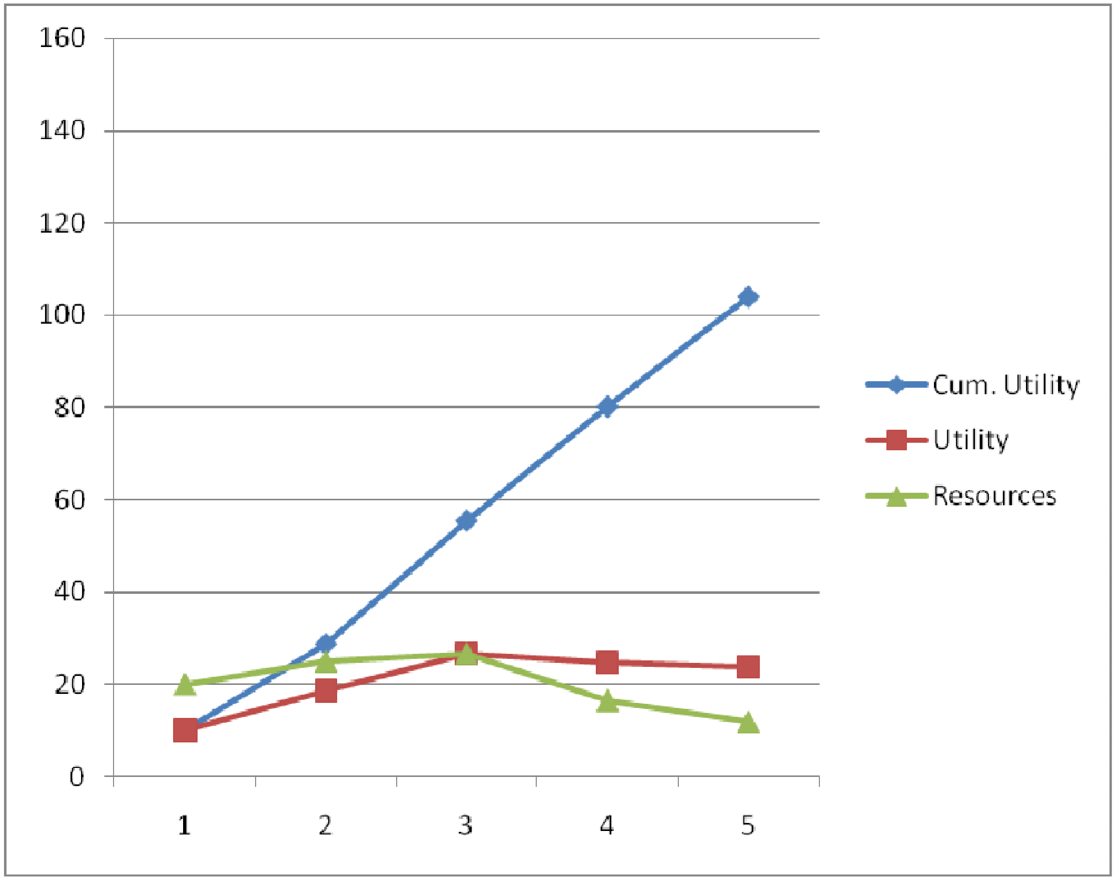
$$(1/n)\sum_i u_i' = 1$$

$$u_i' = a_i x_i',\ 0 \le x_i' \le b_i z,\ \text{all } i, \quad \sum_i x_i' = Bz$$

Coefficient of Variation



Variance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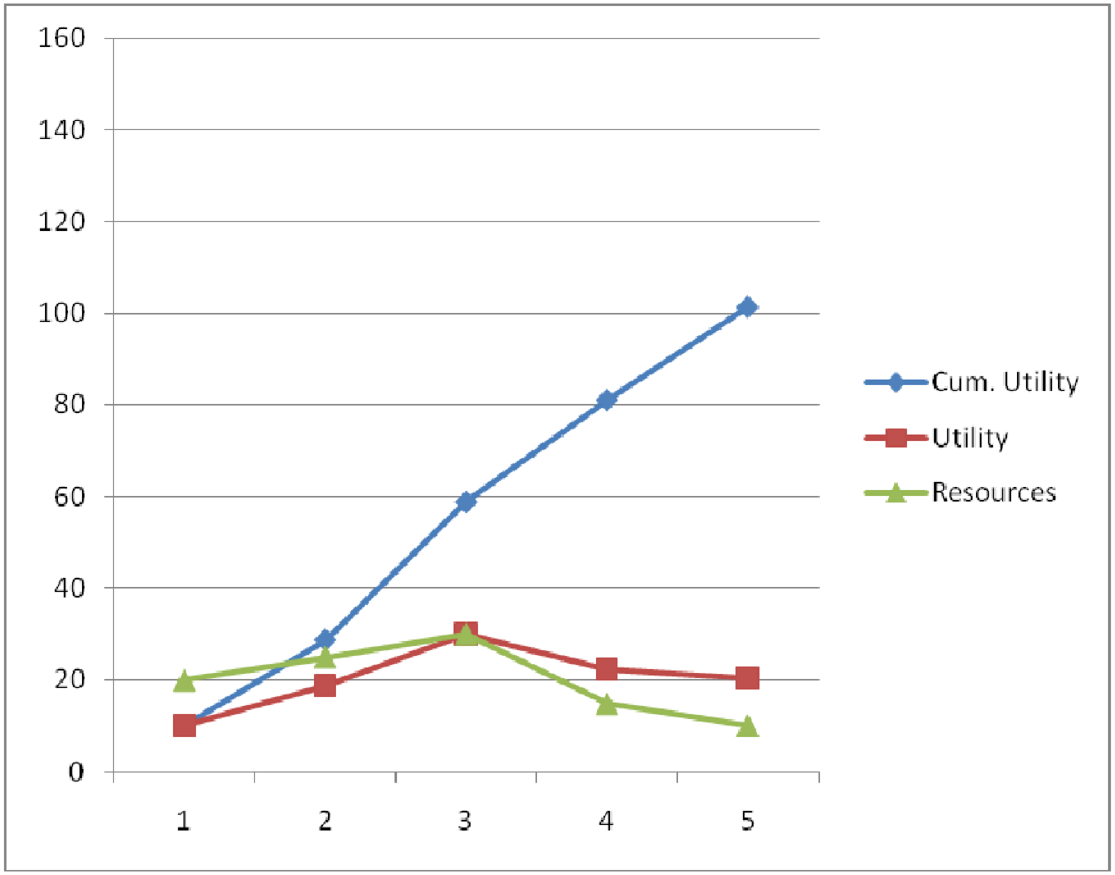
Cum. Utility
Utility
Resources

Relative Mean Deviation



McLoone Index

$$\frac{(1/2)\sum_{i:u_i<m} u_i}{\bar{u}}$$

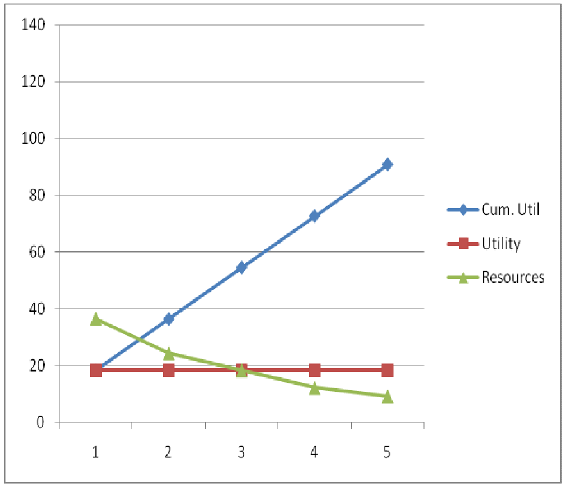
Rationale:

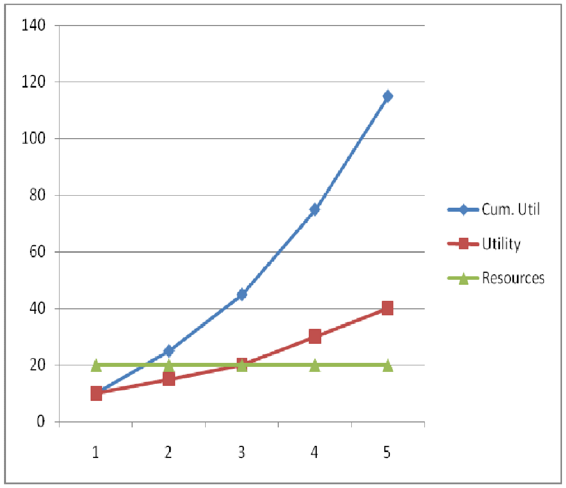
- Ratio of average utility below median to overall average.
- No one wants to be "below average."
- Pushes average up while pushing inequality down.

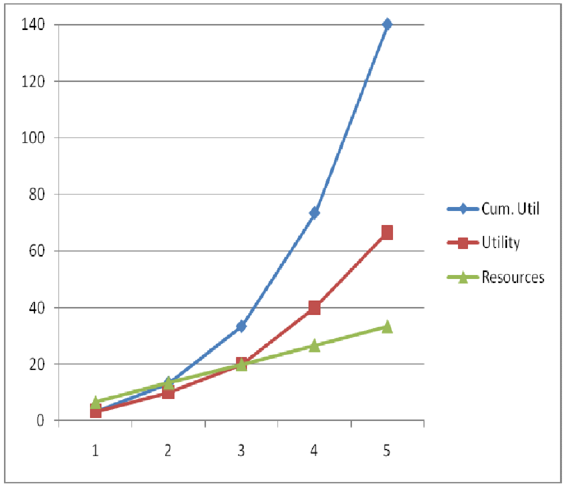
Problems:

- Violates Pigou-Dalton condition.
- Insensitive to upper half.

Equality Measures: Comparison







Relative range:	0	1.30	2.26
Rel. mean dev.:	0	0.42	0.72
Coeff. of variation:	0	0.46	0.81
McLoone:	1	0.54	0.23

McLoone Index

$$\frac{(1/2)\sum_{i:u_i<m} u_i}{\bar{u}}$$

Fractional MILP model:

$$\max \frac{\sum_i v_i}{\sum_i u_i}$$

Defines median m → $m - My_i \le u_i \le m + M(1-y_i)$, all i

Defines $v_i = u_i$ if u_i is below median → $v_i \le u_i, v_i \le My_i$, all i

Half of utilities are below median → $\sum_i y_i < n/2$

$u_i = a_i x_i$, $0 \le x_i \le b_i$, all i, $\sum_i x_i = B$

Selects utilities below median → $y_i \in \{0,1\}$, all i

McLoone Index

$$\frac{(1/2)\sum\limits_{i:u_i<m} u_i}{\bar{u}}$$

MILP model:

$$\max \sum_i v_i'$$

$$m' - My_i \le u_i' \le m' + M(1-y_i), \text{ all } i$$

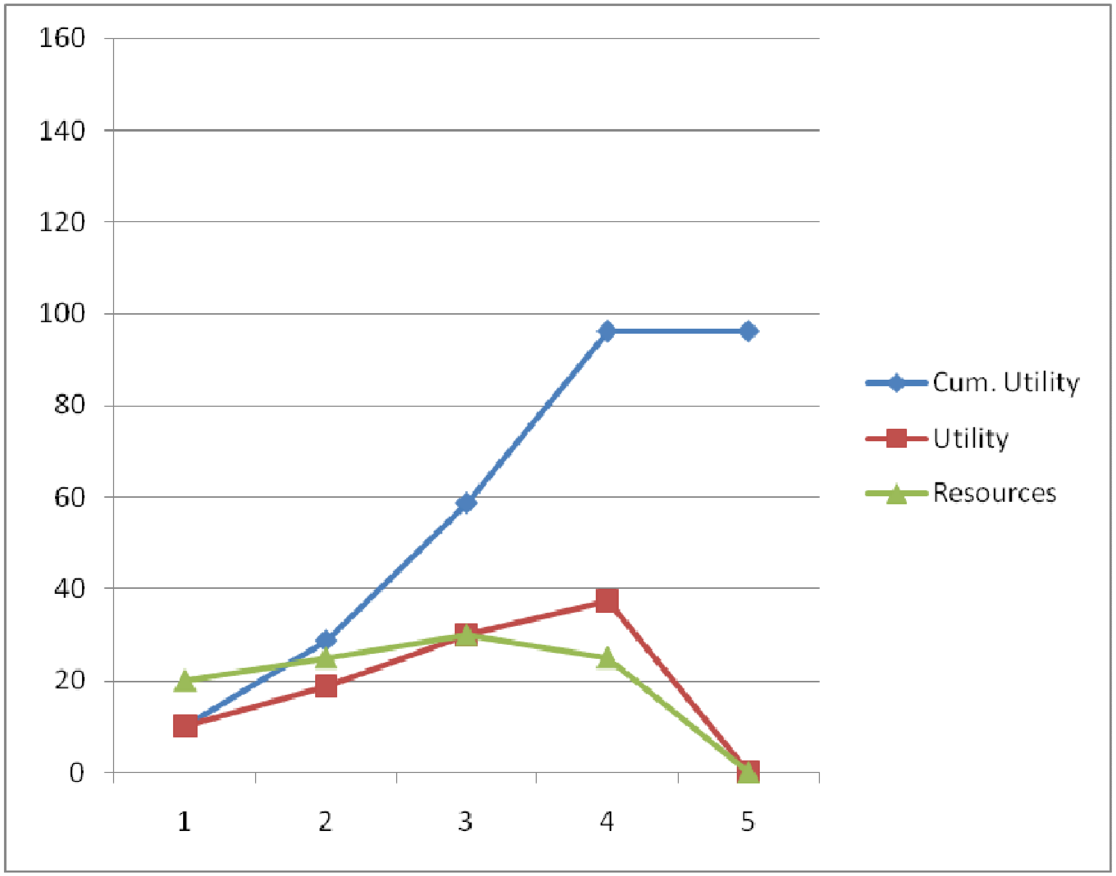
$$v_i' \le u_i', v_i' \le My_i, \text{ all } i$$

$$\sum_i y_i < n/2$$

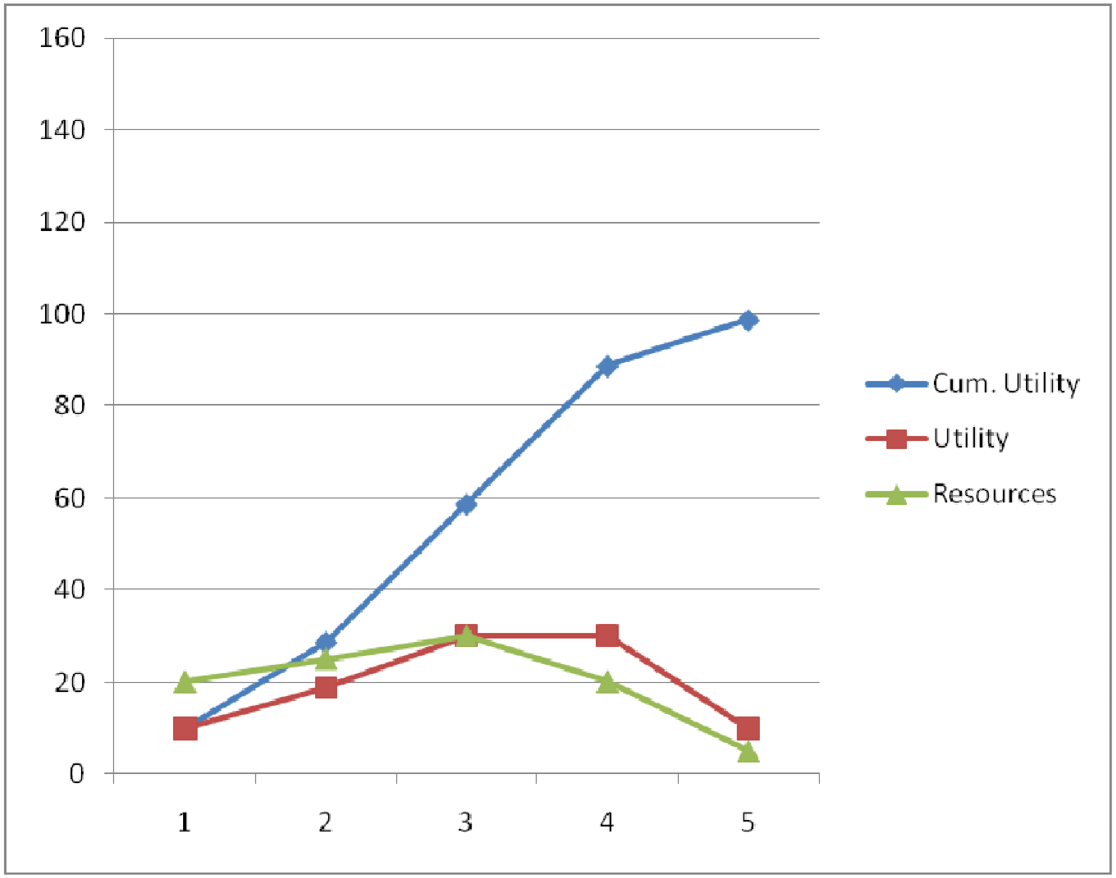
$$u_i' = a_i x_i', \; 0 \le x_i' \le b_i z, \text{ all } i, \quad \sum_i x_i' = Bz$$

$$y_i \in \{0,1\}, \text{ all } i$$

McLoone Index



Relative Min



Gini Coefficient

$$\frac{(1/n^2)\sum_{i,j}\left|u_i - u_j\right|}{2\bar{u}}$$

Rationale:

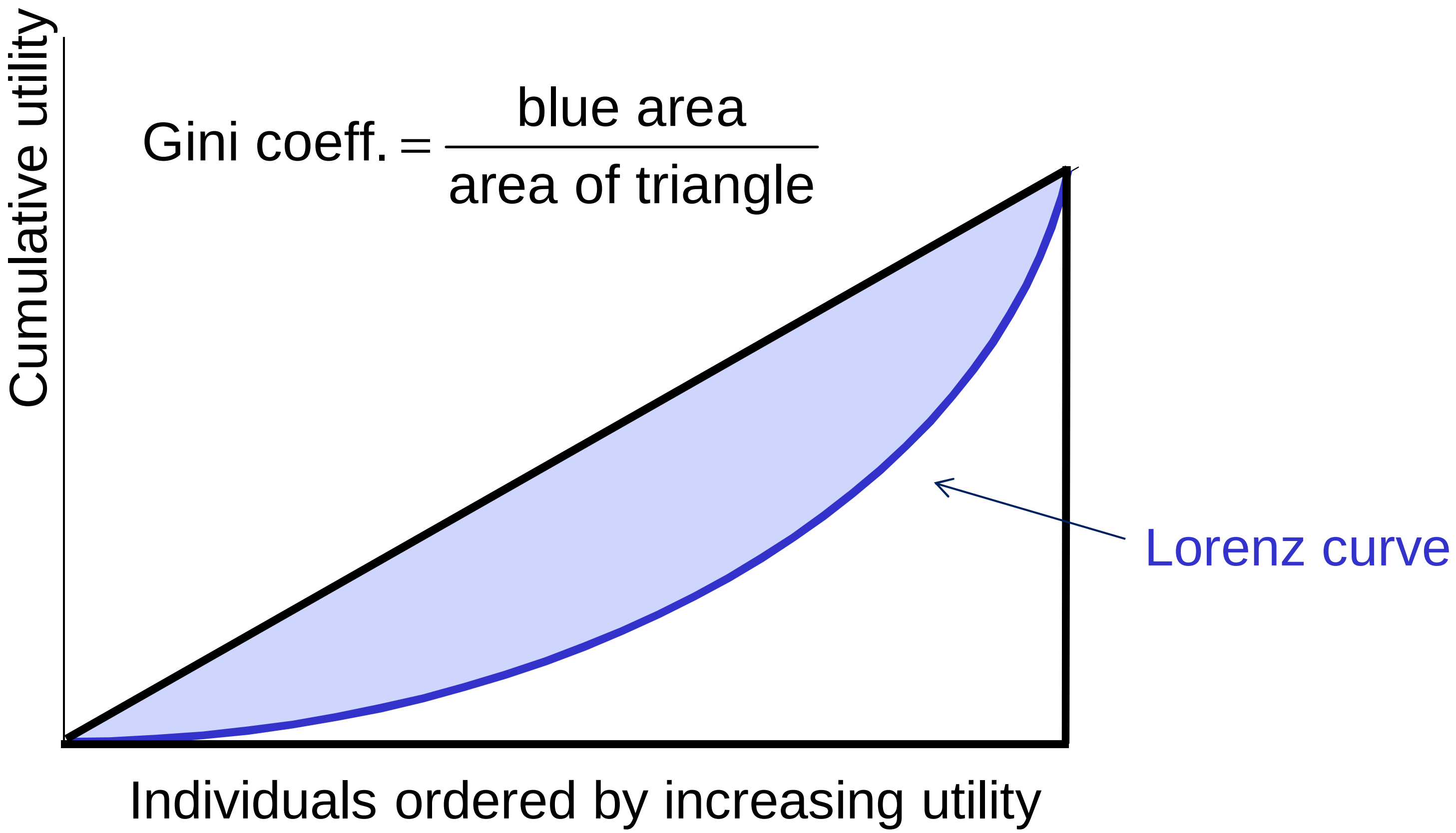
- Relative mean difference between all pairs.
- Takes all differences into account.
- Related to area above cumulative distribution (Lorenz curve).
- Satisfies Pigou-Dalton condition.

Problems:

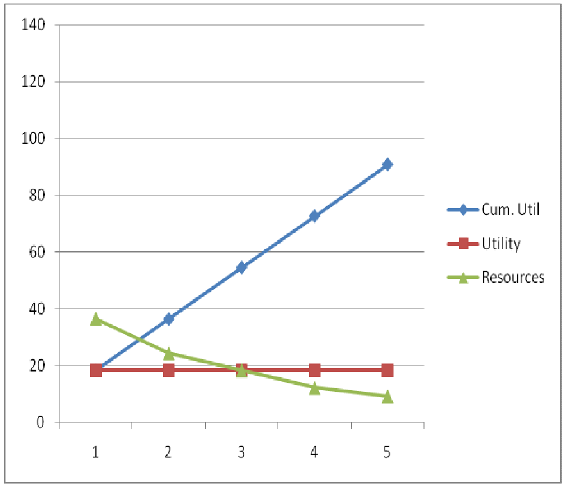
- Insensitive to shape of Lorenz curve, for a given area.

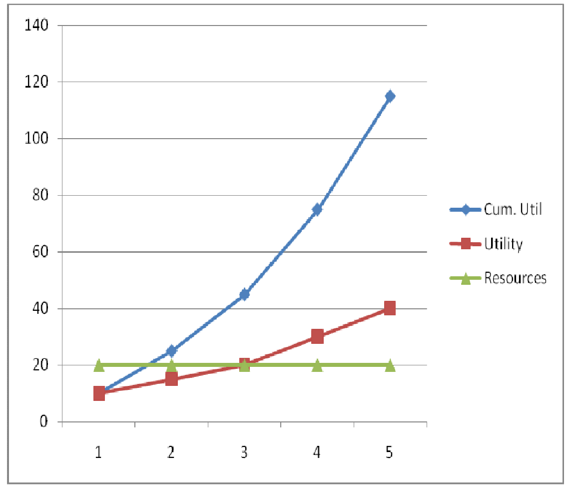
Gini Coefficient

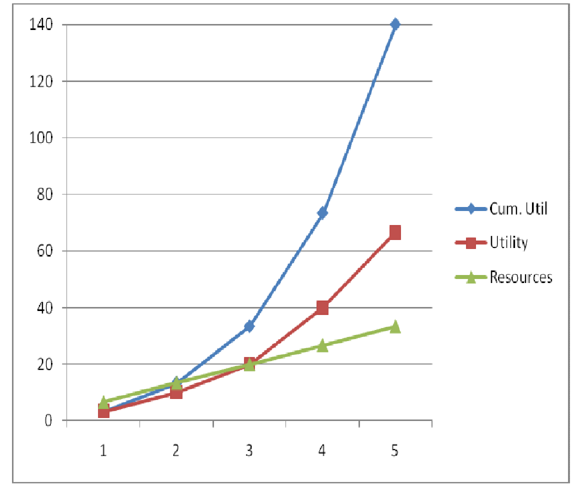
$$\frac{(1/n^2)\sum_{i,j}\left|u_i - u_j\right|}{2\bar{u}}$$



Equality Measures: Comparison







Relative range:	0	1.30	2.26
Rel. mean dev.:	0	0.42	0.72
Coeff. of variation:	0	0.46	0.81
McLoone:	1	0.54	0.23
Gini:	0	0.26	0.45

Gini Coefficient

$$\frac{(1/n^2)\sum_{i,j}\left|u_i - u_j\right|}{2\bar{u}}$$

Fractional LP model:

$$\max \frac{(1/2n^2)\sum_{ij}(u_{ij}^+ + u_{ij}^-)}{\bar{u}}$$

$$u_{ij}^+ \geq u_i - u_j,\ u_{ij}^- \geq u_j - u_i,\ \text{all } i, j$$

$$\bar{u} = (1/n)\sum_i u_i$$

$$u_i = a_i x_i,\ 0 \leq x_i \leq b_i,\ \text{all } i,\quad \sum_i x_i = B$$

LP model:

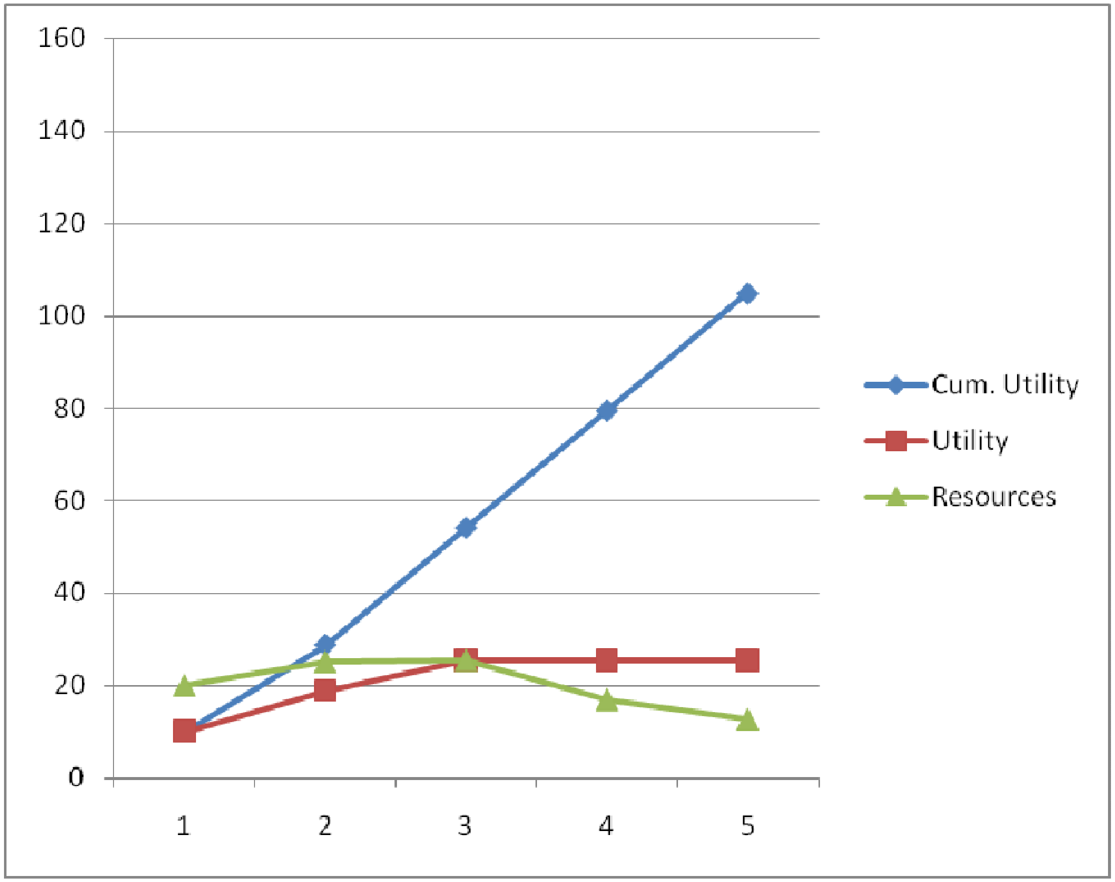
$$\max\ (1/2n^2)\sum_{ij}(u_{ij}^+ + u_{ij}^-)$$

$$u_{ij}^+ \geq u_i' - u_j',\ u_{ij}^- \geq u_j' - u_i',\ \text{all } i, j$$

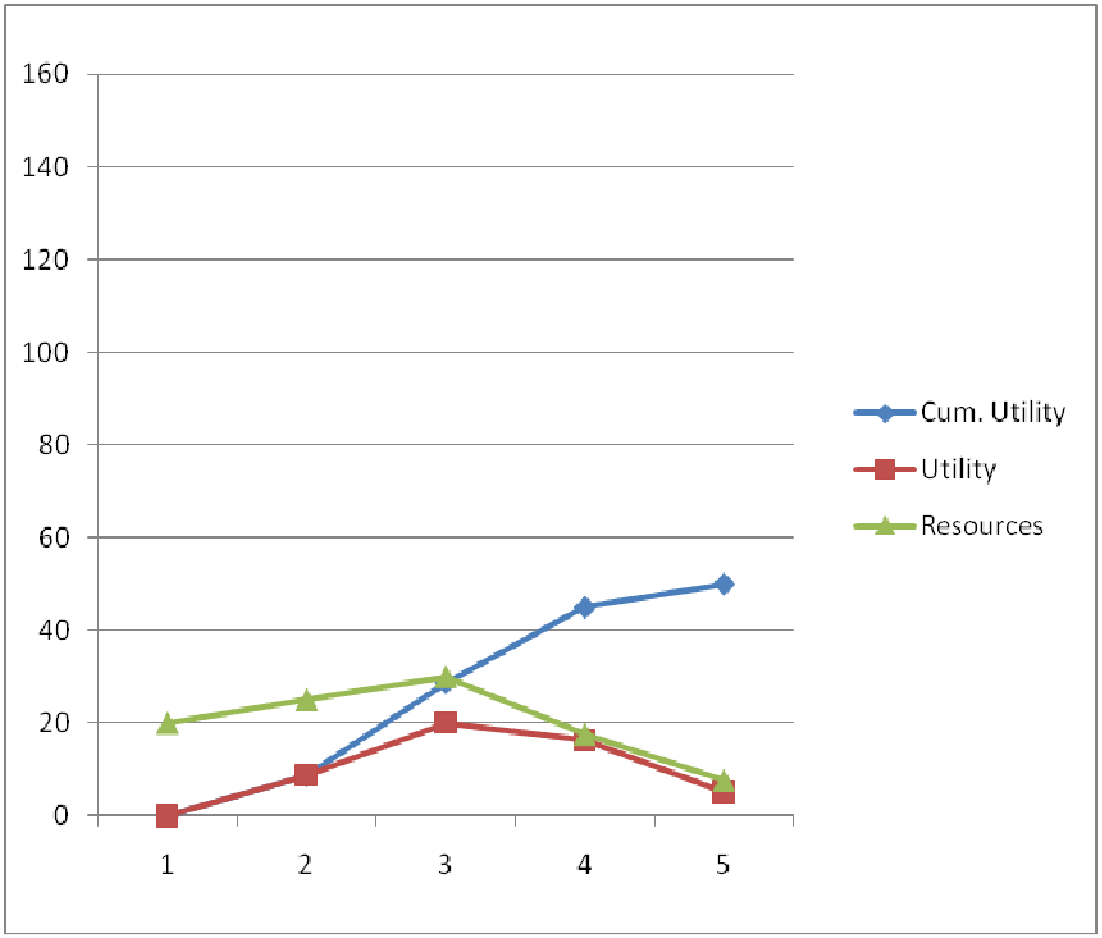
$$(1/n)\sum_i u_i' = 1$$

$$u_i' = a_i x_i',\ 0 \leq x_i' \leq b_i z,\ \text{all } i,\quad \sum_i x_i' = Bz$$

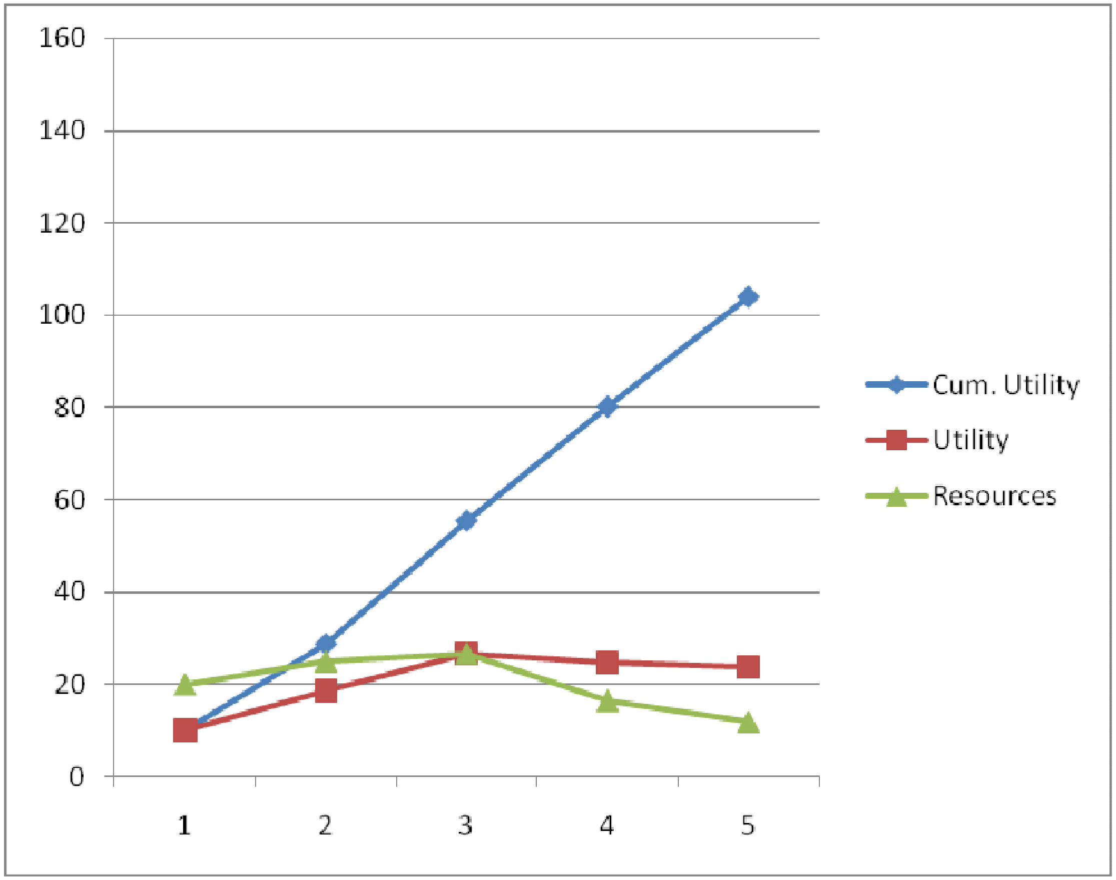
Gini Coefficient



Coefficient of Variation



Variance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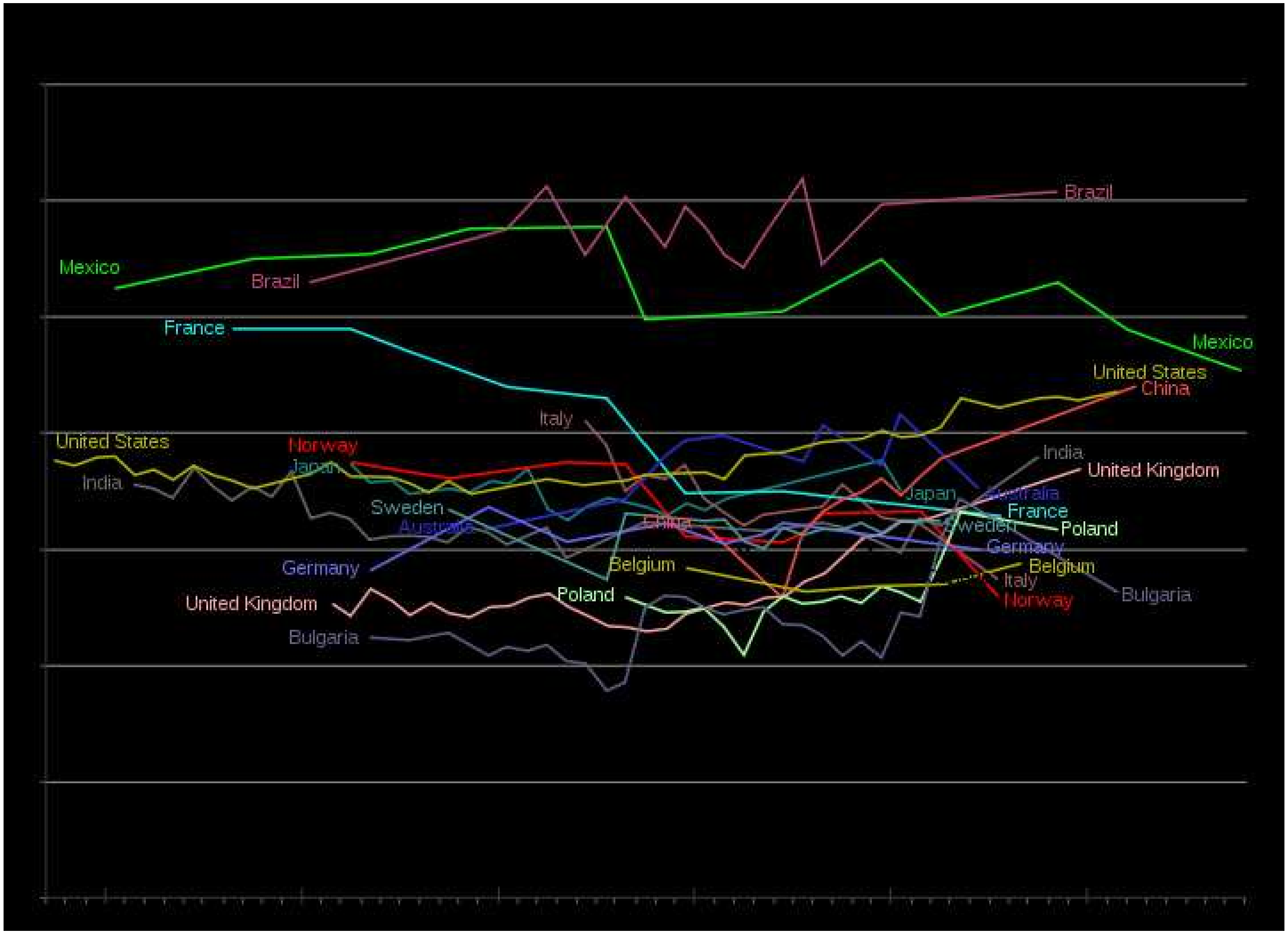
Cum. Utility
Utility
Resources

Historical Gini Coefficient, 1945-2010



Atkinson Index

$$1-\left(\boxed{(1/n)\sum_i \left(\frac{x_i}{\overline{x}}\right)^p}\right)^{1/p}$$

Rationale:

- Best seen as measuring inequality of **resources** x_i.
- Assumes allotment y of resources results in utility y^p
- This is average utility per individual.

Atkinson Index

$$1 - \boxed{\left((1/n) \sum_i \left(\frac{x_i}{\bar{x}} \right)^p \right)^{1/p}}$$

Rationale:

- Best seen as measuring inequality of **resources** x_i.
- Assumes allotment y of resources results in utility y^p
- This is average utility per individual.
- This is equal resource allotment to each individual that results in same total utility.

Atkinson Index

$$\boxed{1-\left((1/n)\sum_i\left(\frac{x_i}{\bar{x}}\right)^p\right)^{1/p}}$$

Rationale:

- Best seen as measuring inequality of **resources** x_i.
- Assumes allotment y of resources results in utility y^p
- This is average utility per individual.
- This is equal resource allotment to each individual that results in same total utility.
- This is additional resources per individual necessary to sustain inequality.

Atkinson Index

$$1 - \left((1/n) \sum_i \left(\frac{x_i}{\overline{x}} \right)^p \right)^{1/p}$$

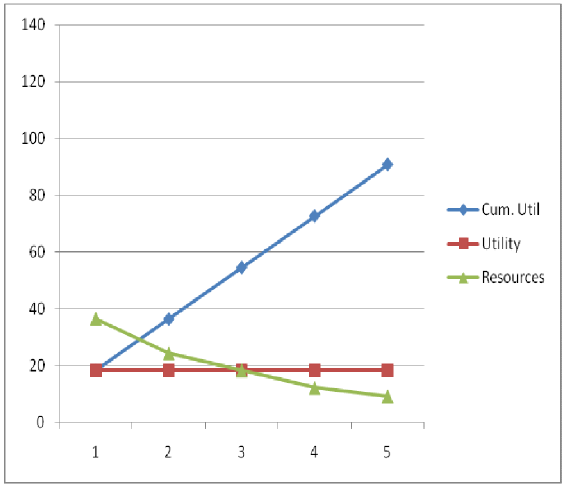
Rationale:

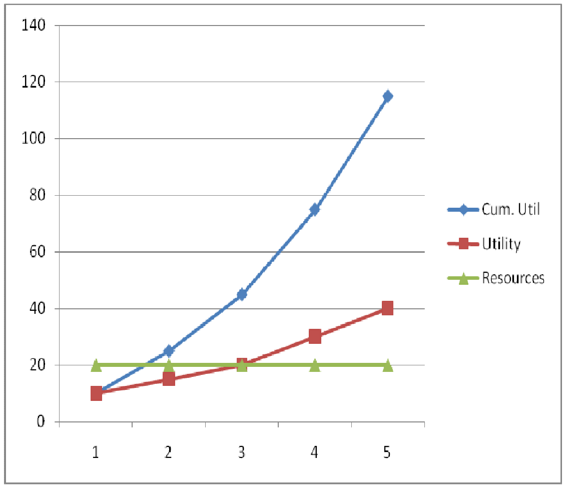
- p indicates "importance" of equality.
- Similar to L_p norm
- $p = 1$ means inequality has no importance
- $p = 0$ is Rawlsian (measures utility of worst-off individual).

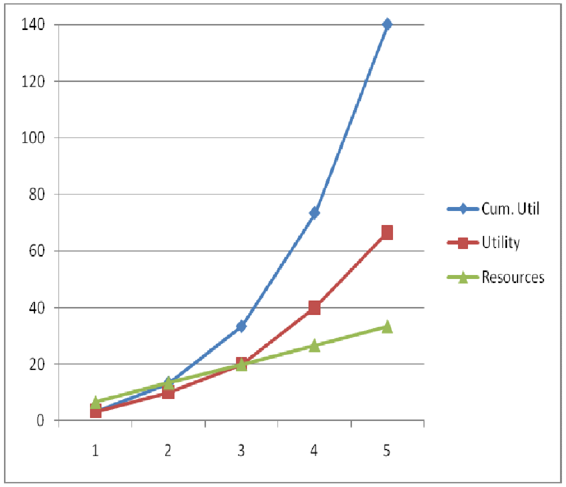
Problems:

- Measures utility, not equality.
- Doesn't evaluate distribution of utility, only of resources.
- p describes utility curve, not importance of equality.

Equality Measures: Comparison







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Rel. mean dev.:	0	0.42	0.72
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McLoone:	1	0.54	0.23
Gini:	0	0.26	0.45
Atkinson	0.06	0	0.06

Atkinson Index

$$1-\left((1/n)\sum_i\left(\frac{x_i}{\bar{x}}\right)^p\right)^{1/p}$$

To minimize index, solve fractional problem

$$\max \sum_i\left(\frac{x_i}{\bar{x}}\right)^p = \frac{\sum_i x_i^p}{\bar{x}^p}$$

$$Ax \ge b,\ \ x \ge 0$$

After change of variable $x_i = x_i'/z$, this becomes

$$\max \sum_i x_i'^p$$

$$(1/n)\sum_i x_i' = 1$$

$$Ax' \ge bz,\ \ x' \ge 0$$

Atkinson Index

$$1 - \left((1/n) \sum_i \left(\frac{x_i}{\bar{x}} \right)^p \right)^{1/p}$$

Fractional nonlinear model:

$$\max \frac{\sum_i x_i^p}{\bar{x}^p}$$

$$\bar{x} = (1/n) \sum_i x_i$$

$$\sum_i x_i = B, \; x \ge 0$$

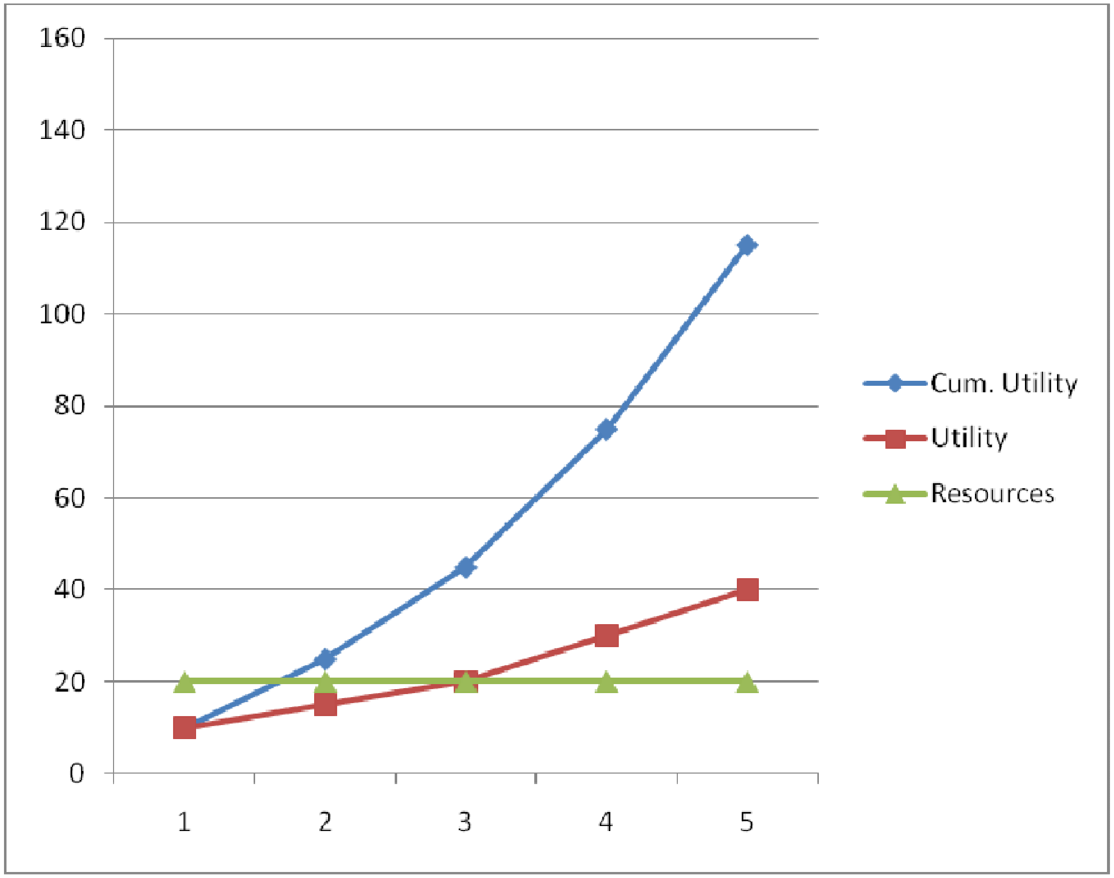
Concave nonlinear model:

$$\max \sum_i x_i'^p$$

$$(1/n) \sum_i x_i' = 1$$

$$\sum_i x_i' = Bz, \; x' \ge 0$$

Atkinson index



Hoover Index

$$(1/2)\frac{\sum_i |u_i - \bar{u}|}{\sum_i u_i}$$

Rationale:

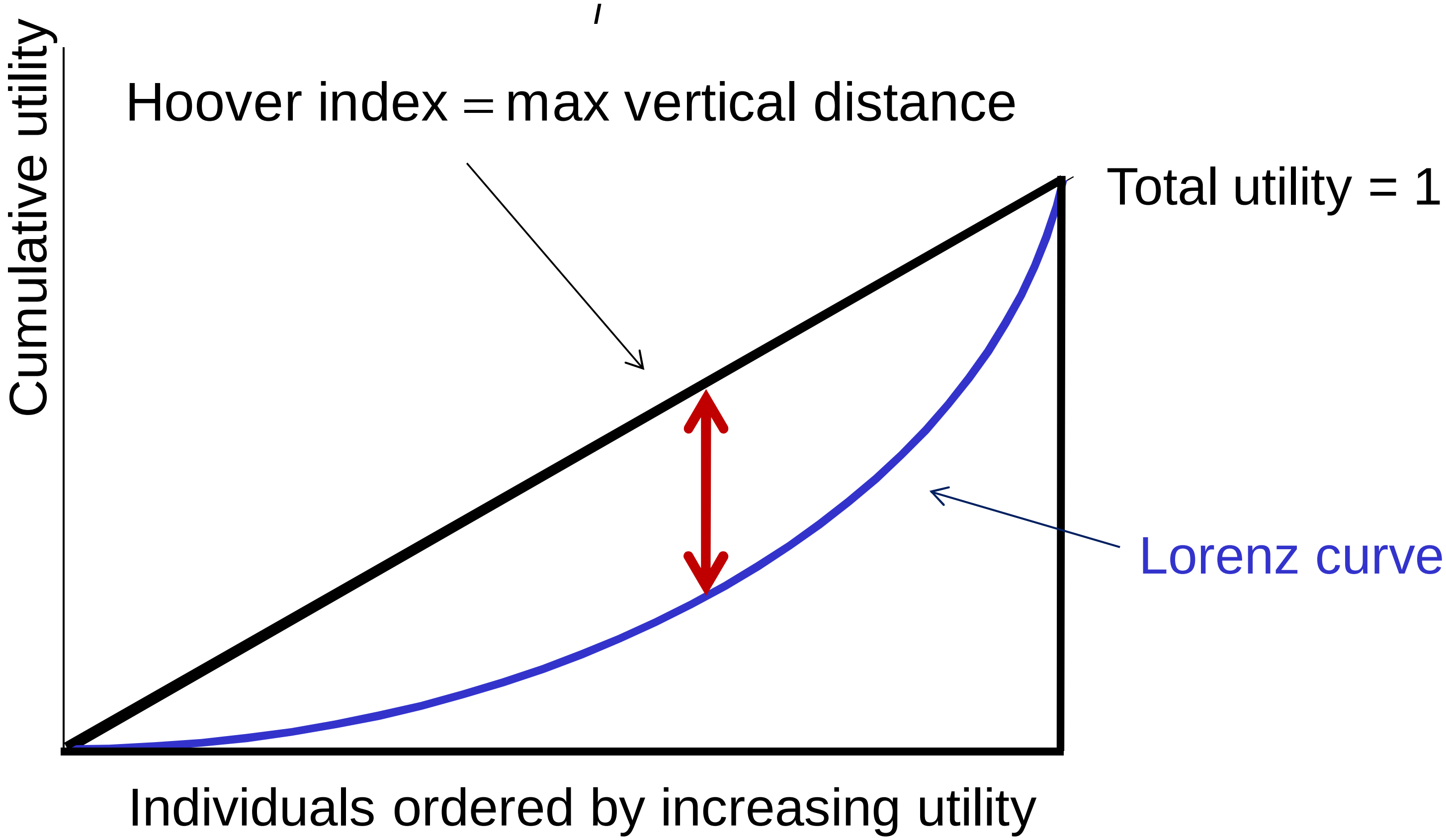
- Fraction of total utility that must be redistributed to achieve total equality.
- Proportional to maximum vertical distance between Lorenz curve and 45^o line.
- Originated in regional studies, population distribution, etc. (1930s).
- Easy to calculate.

Problems:

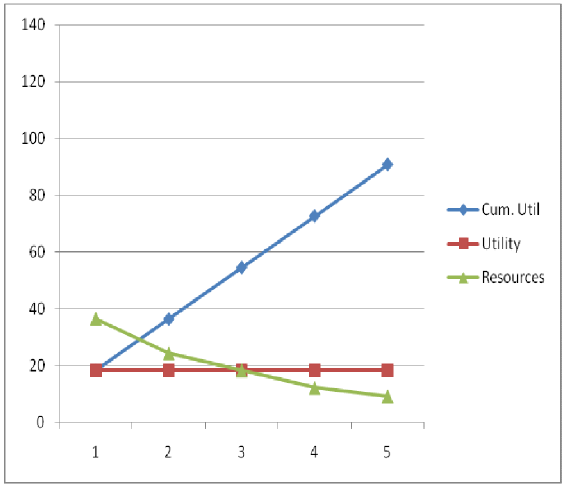
- Less informative than Gini coefficient?

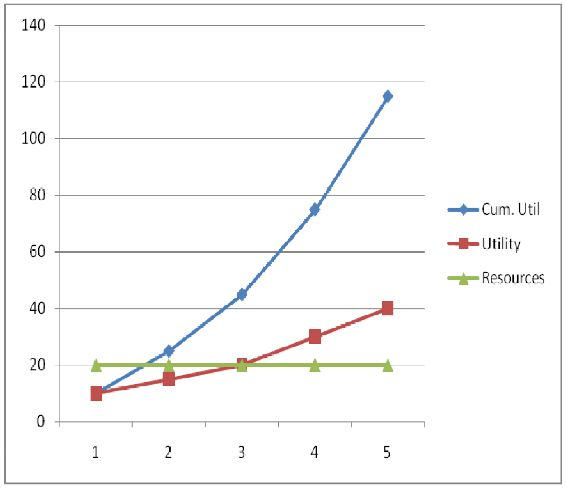
Hoover Index

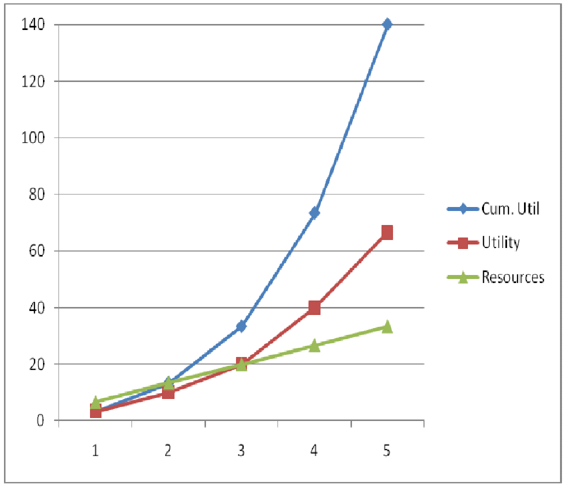
$$(1/2)\frac{\sum_i |u_i - \bar{u}|}{\sum_i u_i}$$



Equality Measures: Comparison







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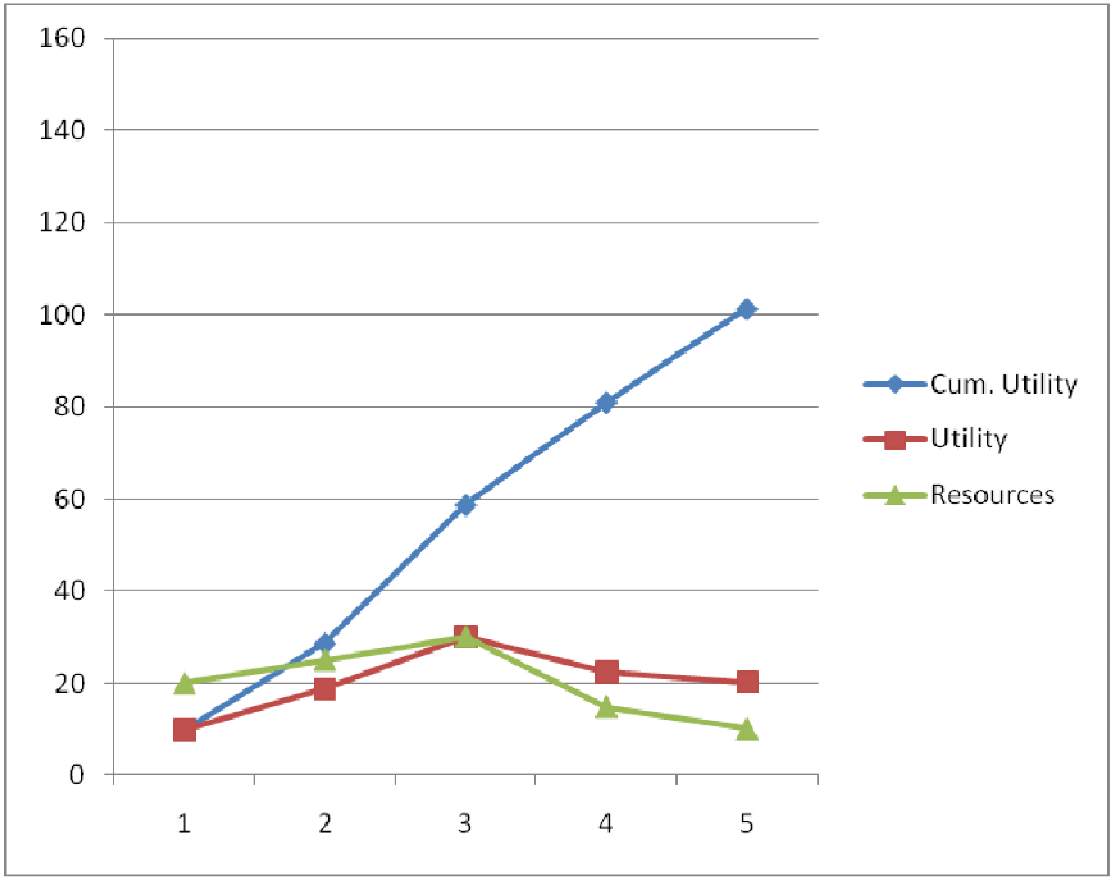
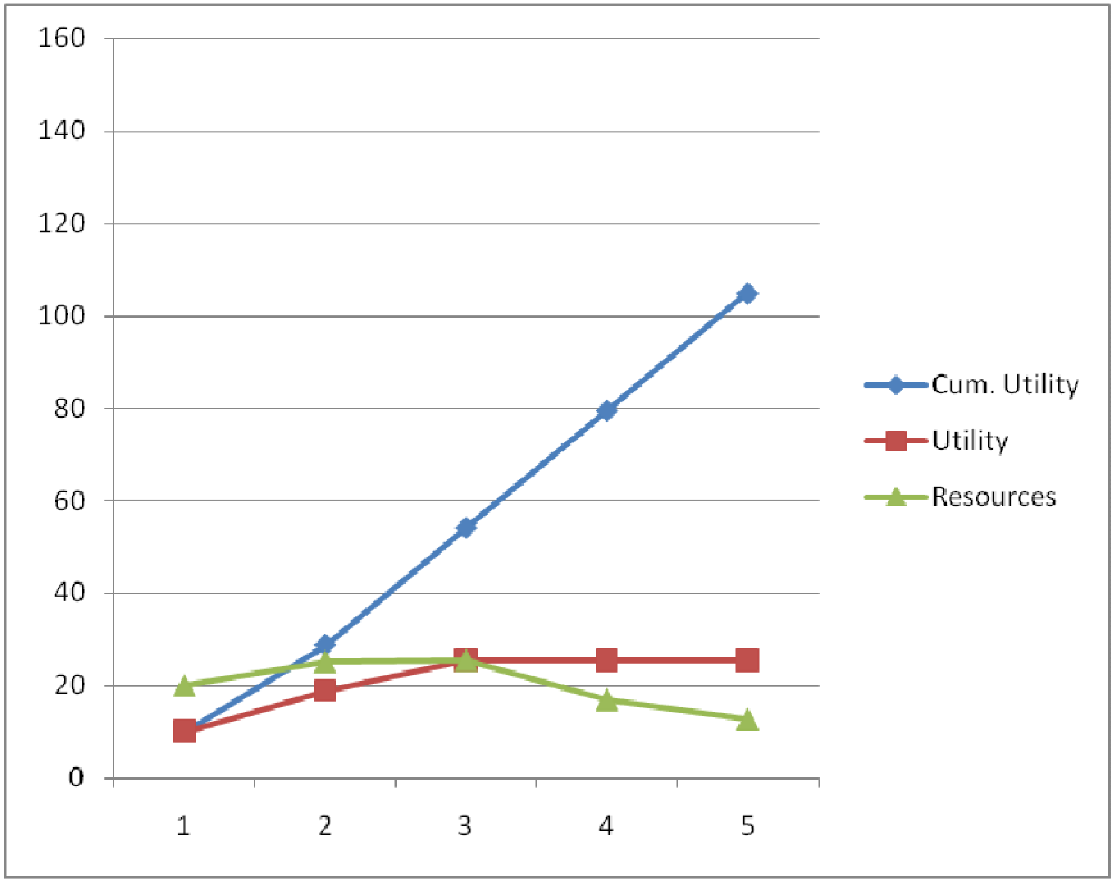
Atkinson:	0.06	0	0.06
Hoover:	0	0.15	0.28

Hoover Index



Gini Coefficient



Theil Index

$$(1/n)\sum_i \left(\frac{u_i}{\bar{u}} \ln \frac{u_i}{\bar{u}} \right)$$

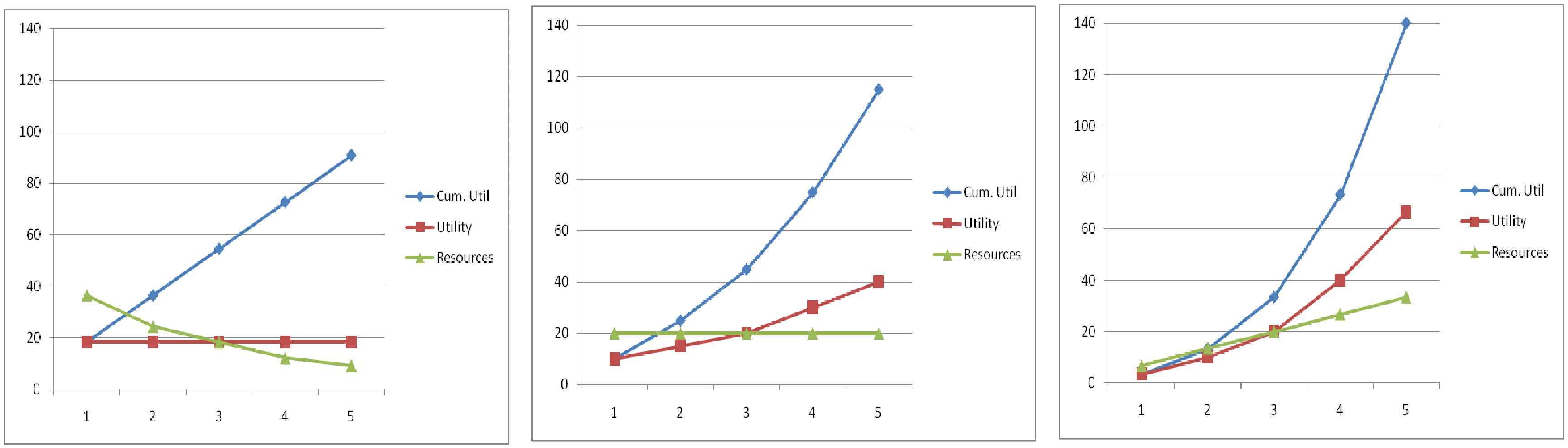
Rationale:

- One of a family of entropy measures of inequality.
- Index is zero for complete equality (maximum entropy)
- Measures nonrandomness of distribution.
- Described as stochastic version of Hoover index.

Problems:

- Motivation unclear.
- A. Sen doesn't like it.

Equality Measures: Comparison



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Gini:	0	0.26	0.45
Atkinson:	0.06	0	0.06
Hoover:	0	0.15	0.28
Theil:	0	0.27	0.86

Theil Index

$$(1/n)\sum_i \left(\frac{u_i}{\bar{u}} \ln \frac{u_i}{\bar{u}} \right)$$

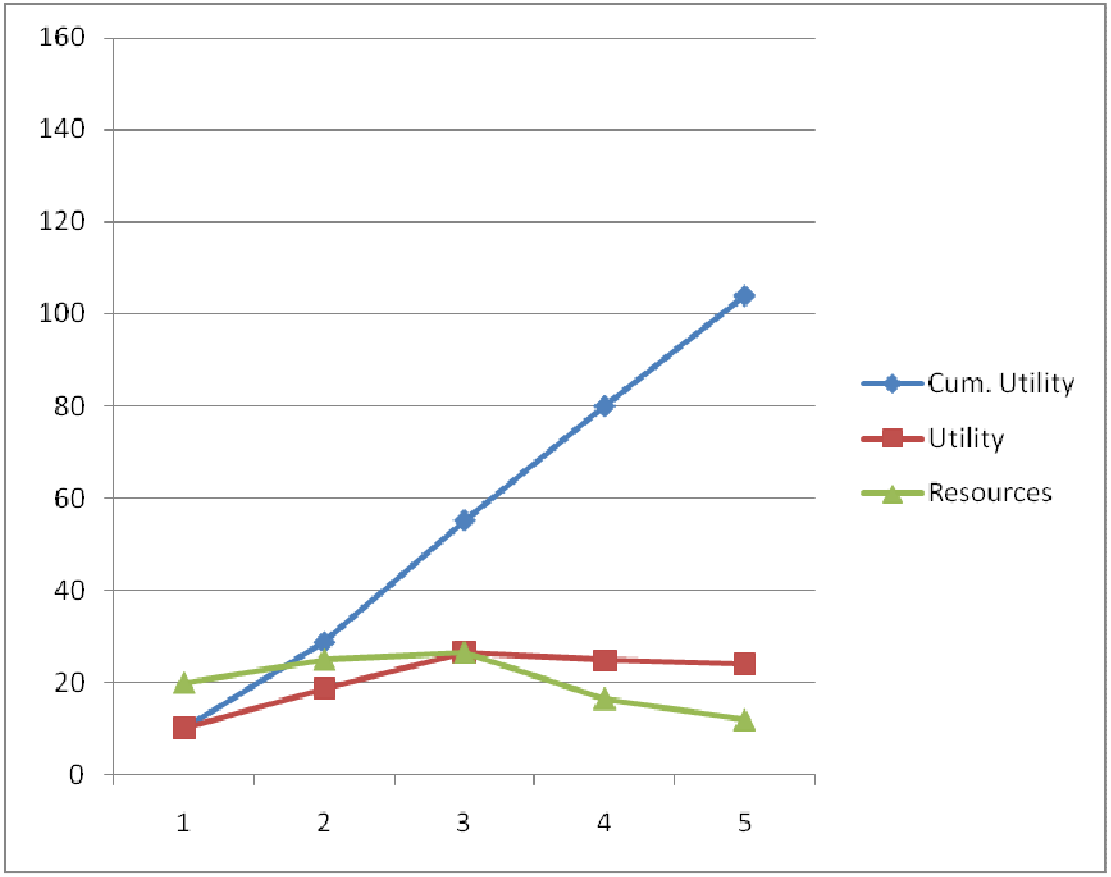
Nasty nonconvex model:

$$\min\,(1/n)\sum_i \left(\frac{u_i}{\bar{u}} \ln \frac{u_i}{\bar{u}} \right)$$

$$\bar{u} = (1/n)\sum_i u_i$$

$$u_i = a_i x_i,\ 0 \le x_i \le b_i,\ \text{all } i,\quad \sum_i x_i = B$$

Theil Index



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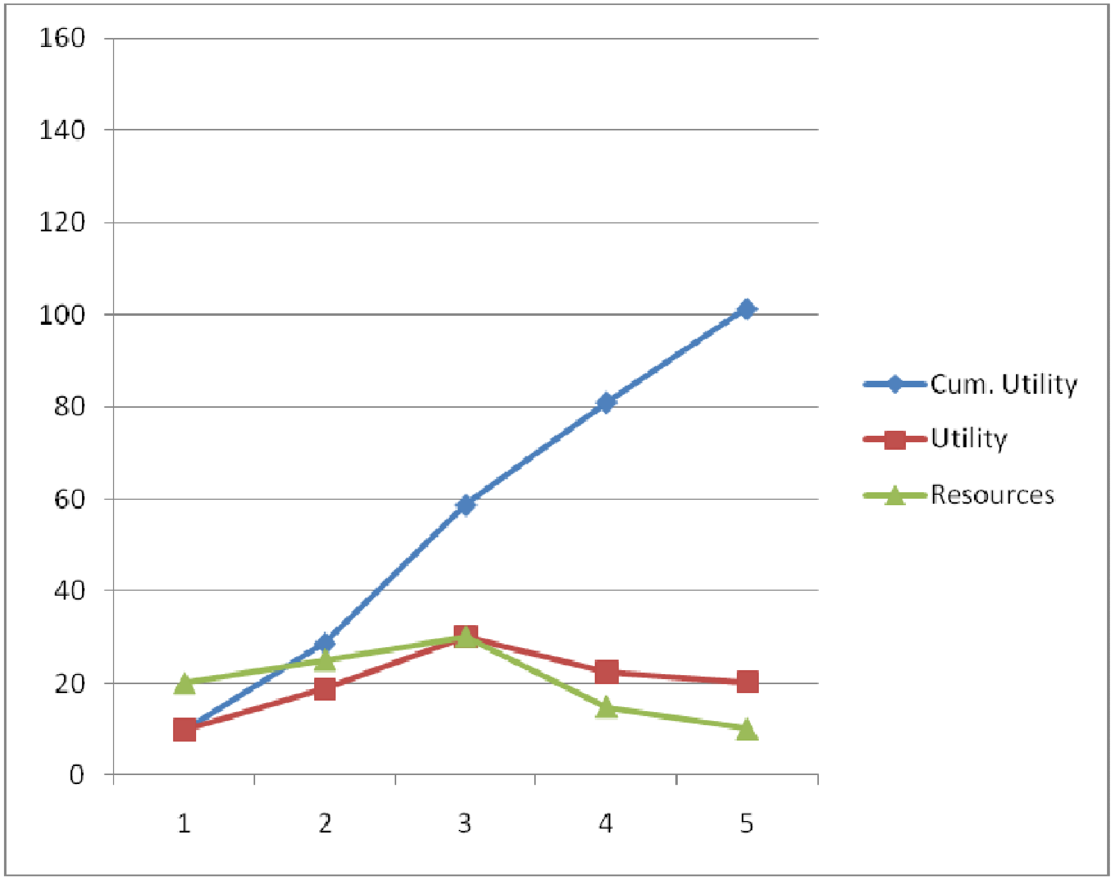
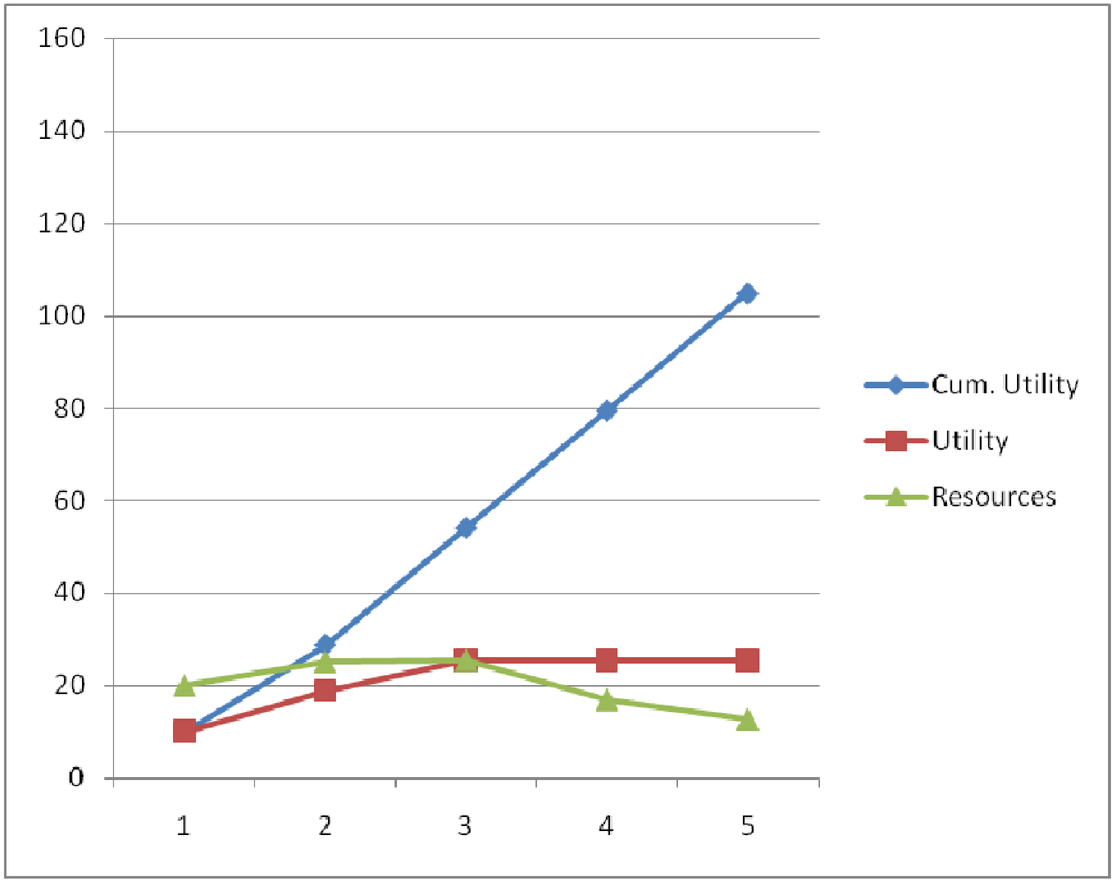
Cum. Utility
Utility
Resources

Hoover Index



Gini Coefficient



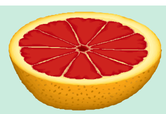
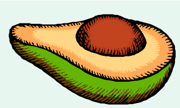
Outline

- Bargaining:
  - Nash Bargaining Solution
  - Raiffa-Kalai-Smorodinsky Bargaining
  - Disjunctive Modeling
- Combining Equity and Efficiency
  - Health Care Example

An Allocation Problem

- From Yaari and Bar-Hillel, 1983.
- **12 grapefruit** and **12 avocados** are to be divided between **Jones** and **Smith**.
- How to divide justly?

Utility provided by one fruit of each kind

	Jones	Smith
	100	50
	0	50

An Allocation Problem

The optimization problem:

Social welfare function

$$\max \; f(u_1, u_2)$$

$$u_1 = 100x_{11}, \;\; u_2 = 50x_{12} + 50x_{22}$$

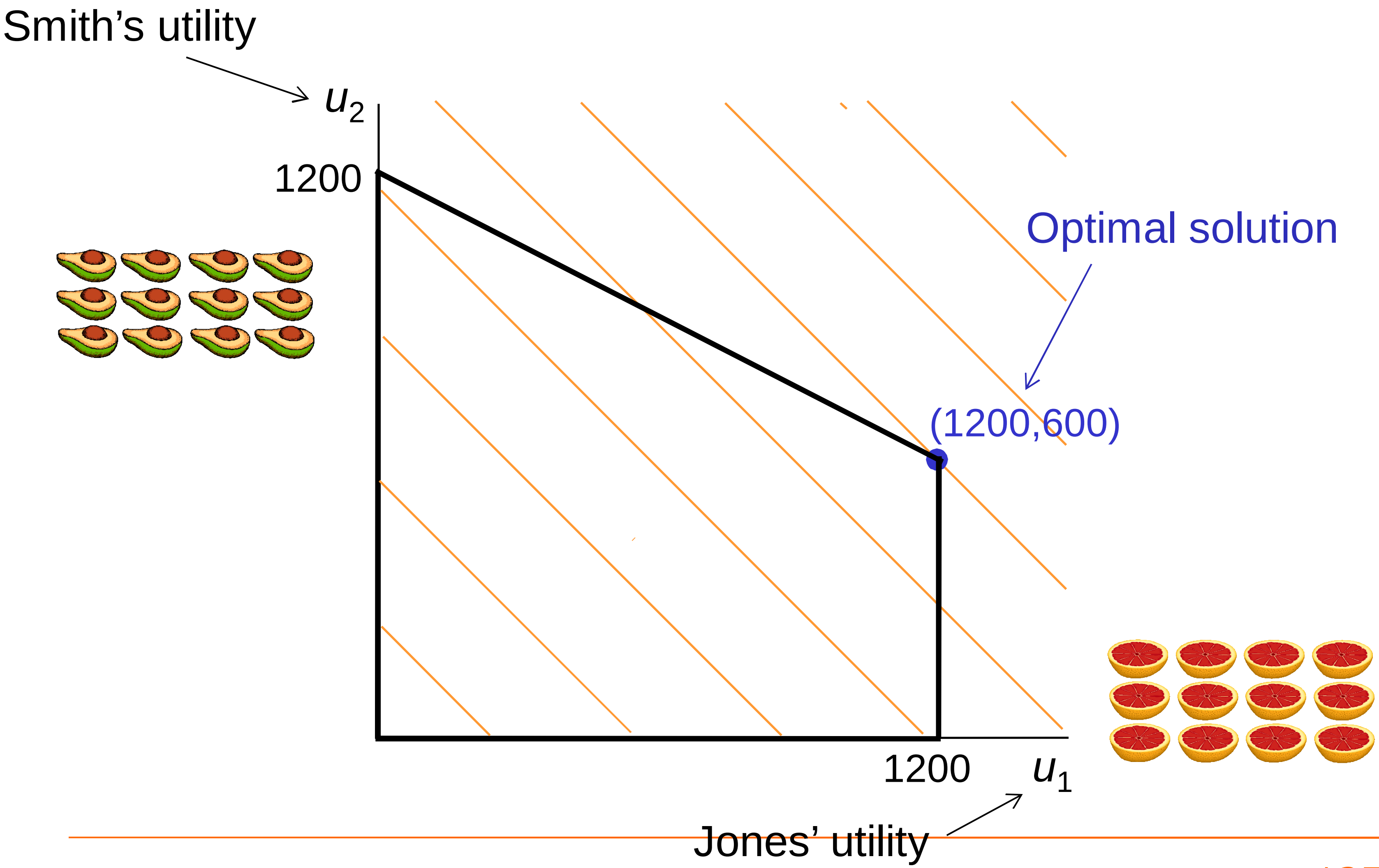
$$x_{i1} + x_{i2} = 12, \;\; i = 1,2$$

$$x_{ij} \geq 0, \;\; \text{all } i, j$$

where u_i = utility for person i (Jones, Smith)
x_{ij} = allocation of fruit i (grapefruit, avocados) to person j

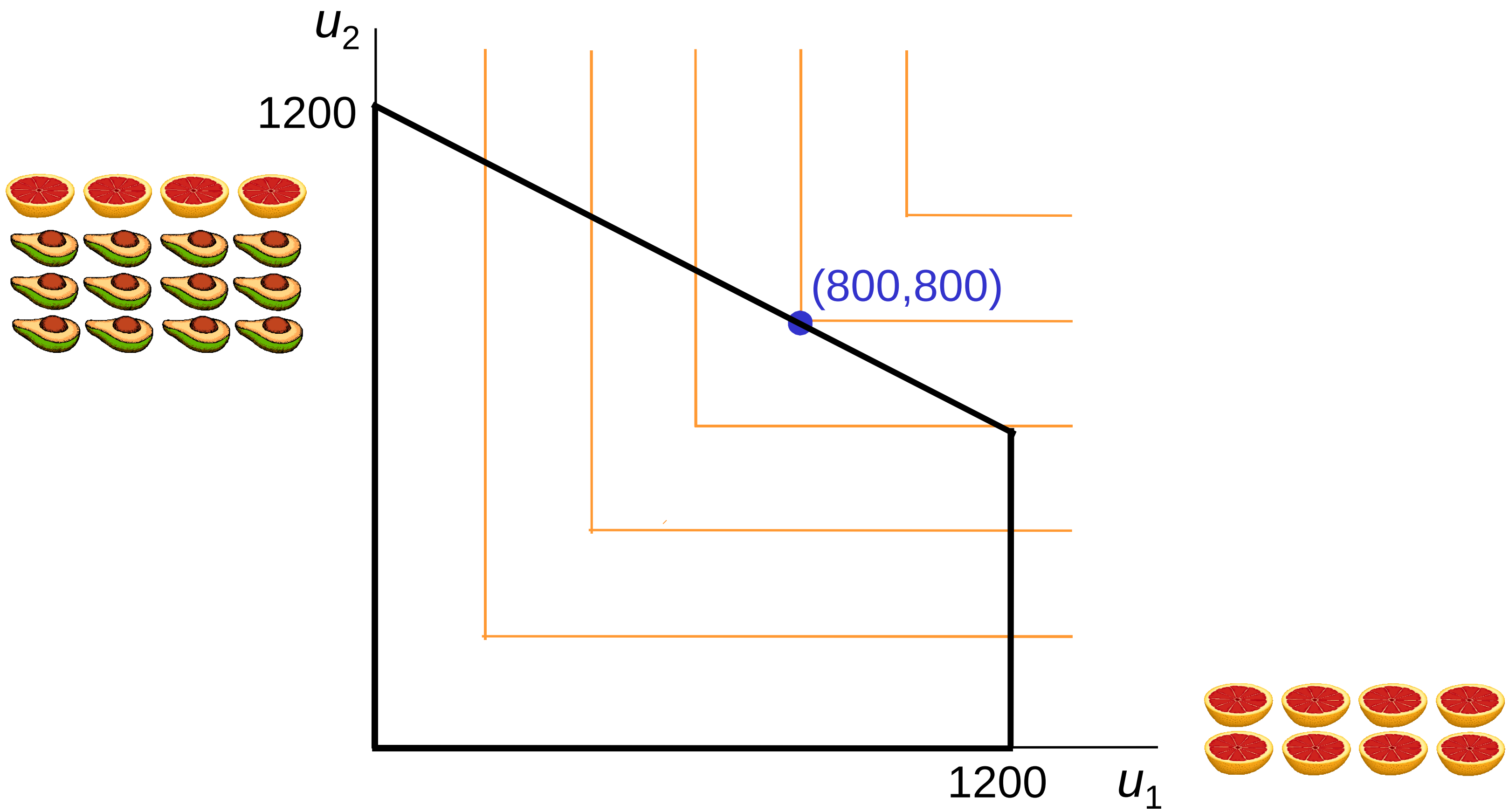
Utilitarian Solution

$$f(u_1,u_2) = u_1 + u_2$$



Rawlsian (maximin) solution

$$f(u_1,u_2)=\min\{u_1,u_2\}$$



Bargaining Solutions

- A **bargaining solution** is an equilibrium allocation in the sense that none of the parties wish to bargain further.
  - Because all parties are “satisfied” in some sense, the outcome may be viewed as “fair.”
  - Bargaining models have a **default** outcome, which is the result of a failure to reach agreement.
  - The default outcome can be seen as a **starting point**.

Bargaining Solutions

- Several proposals for the default outcome (starting point):
  - **Zero** for everyone. Useful when only the resources being allocated are relevant to fairness of allocation.
  - **Equal split**. Resources (not necessarily utilities) are divided equally. May be regarded as a "fair" **starting point**.
  - **Strongly pareto set**. Each party receives resources that can benefit no one else. Parties can always agree on this.

Nash Bargaining Solution

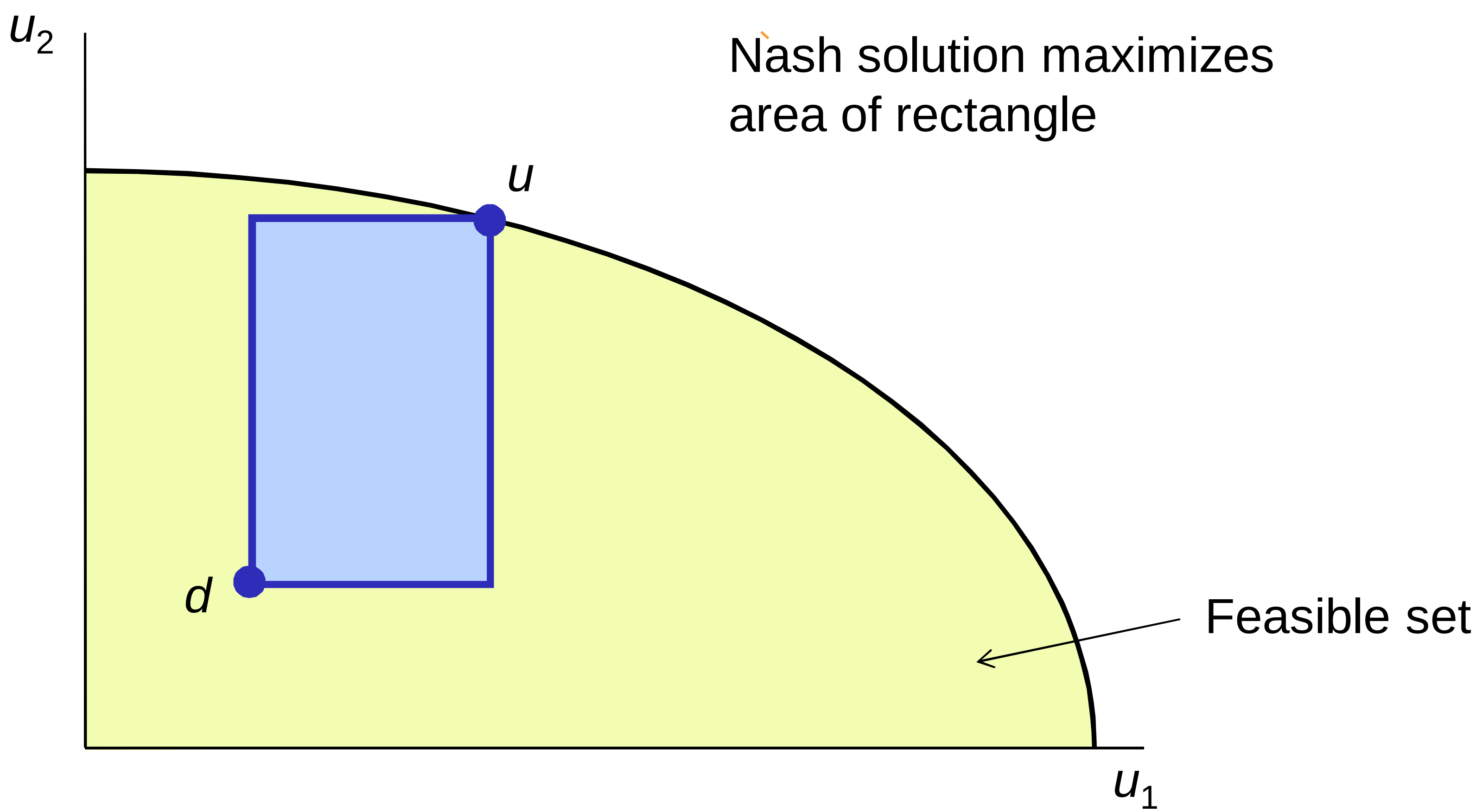
- The **Nash bargaining solution** maximizes the social welfare function

$$f(u) = \prod_i (u_i - d_i)$$

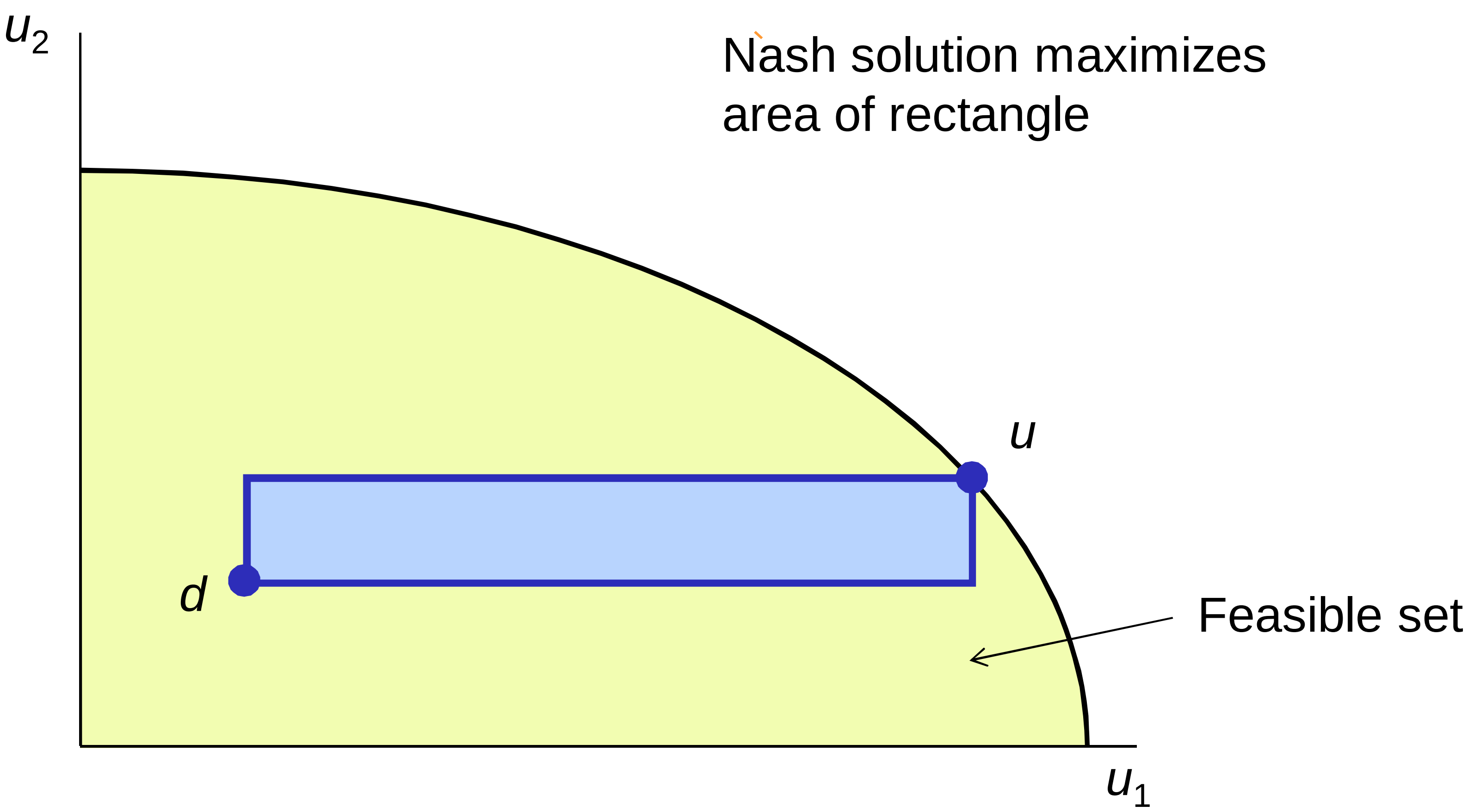
where d is the default outcome.

  - **Not** the same as **Nash equilibrium**.
  - It maximizes the **product of the gains** achieved by the bargainers, relative to the fallback position.
  - Assume feasible set is **convex**, so that Nash solution is unique (due to strict concavity of f).

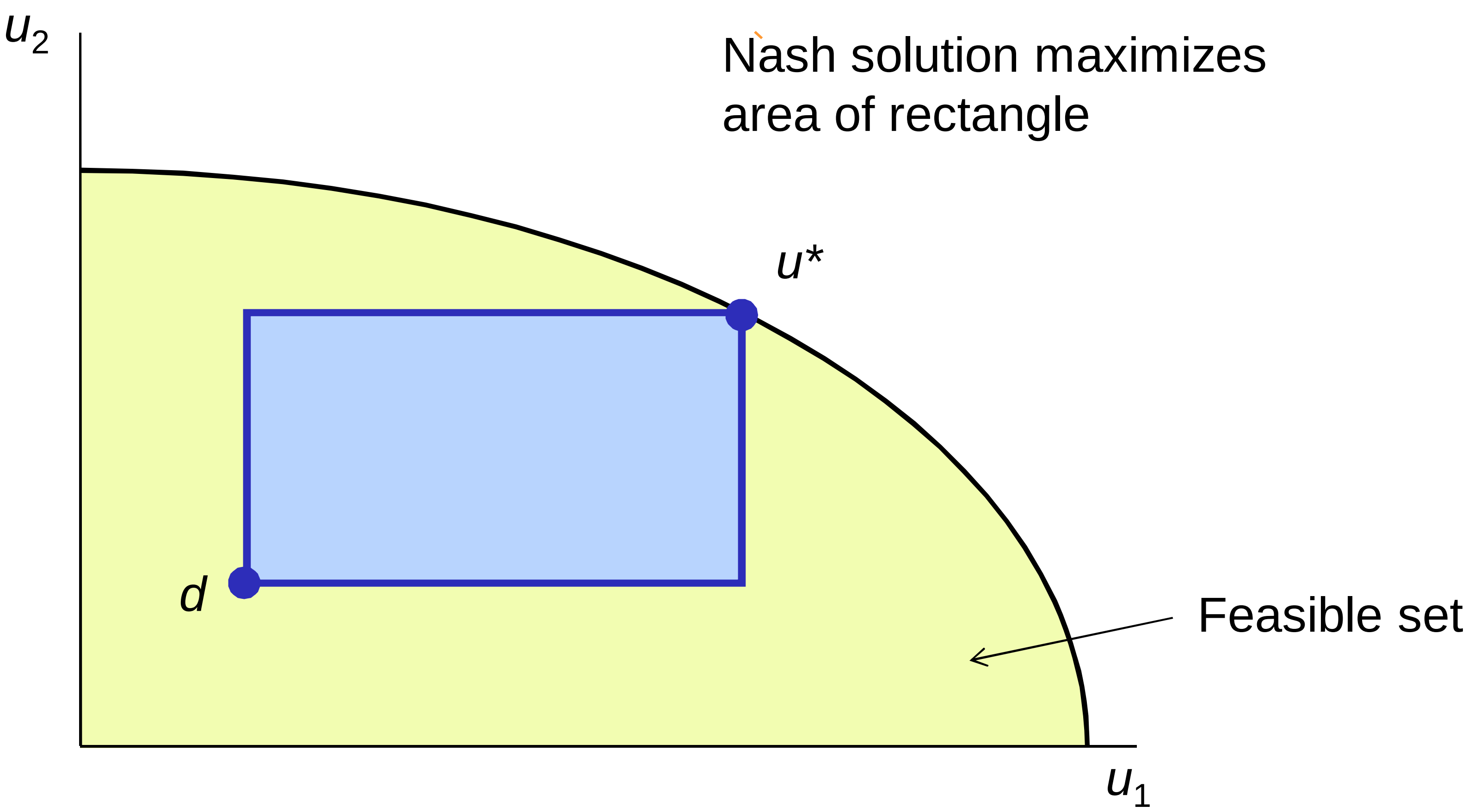
Nash Bargaining Solution



Nash Bargaining Solution



Nash Bargaining Solution



Nash Bargaining Solution

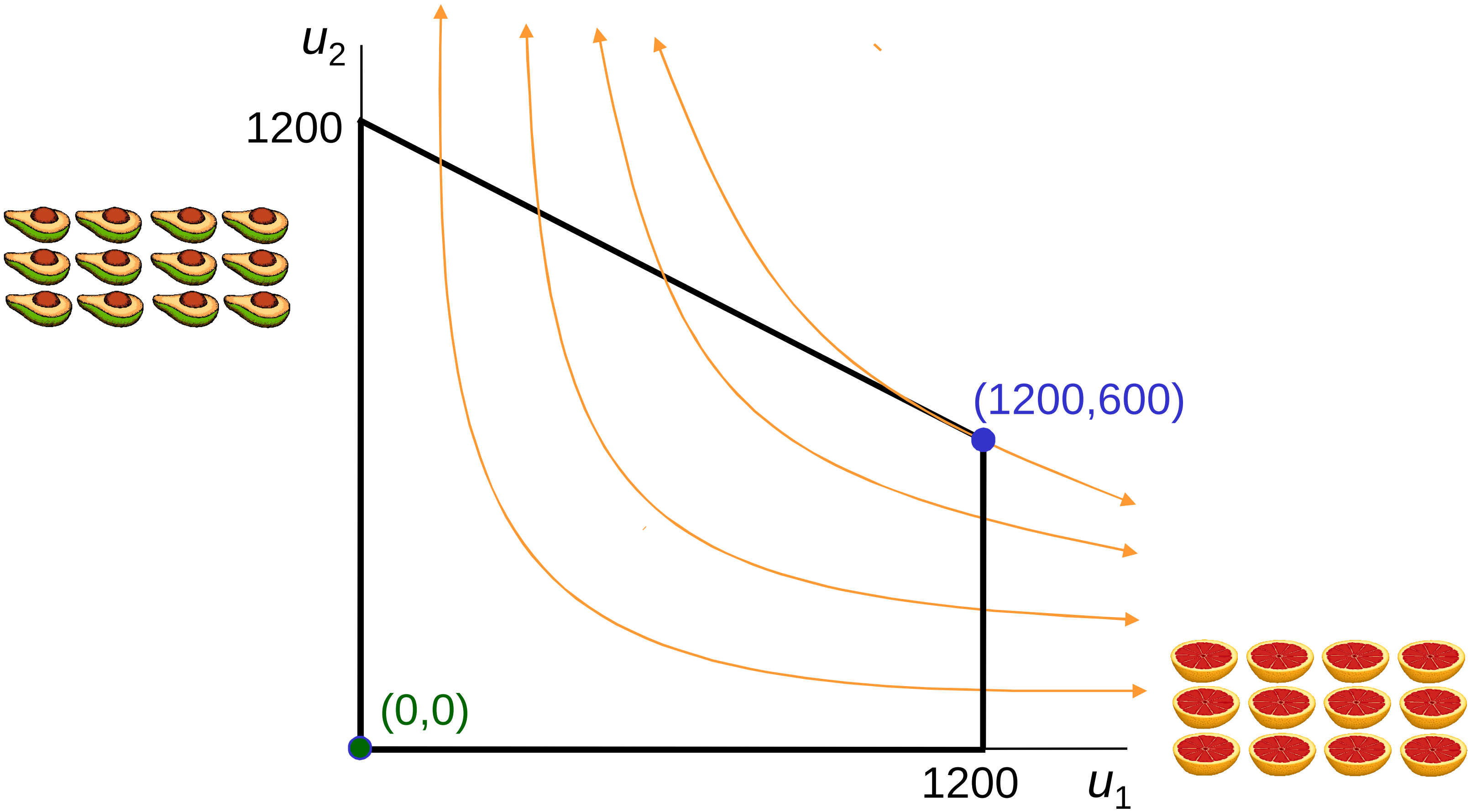
- The **optimization problem** has a concave objective function if we maximize log $f(u)$.

$$\max \log \prod_i (u_i - d_i) = \sum_i \log(u_i - d_i)$$
$$u \in S$$

- Problem is relatively easy if feasible set S is convex.

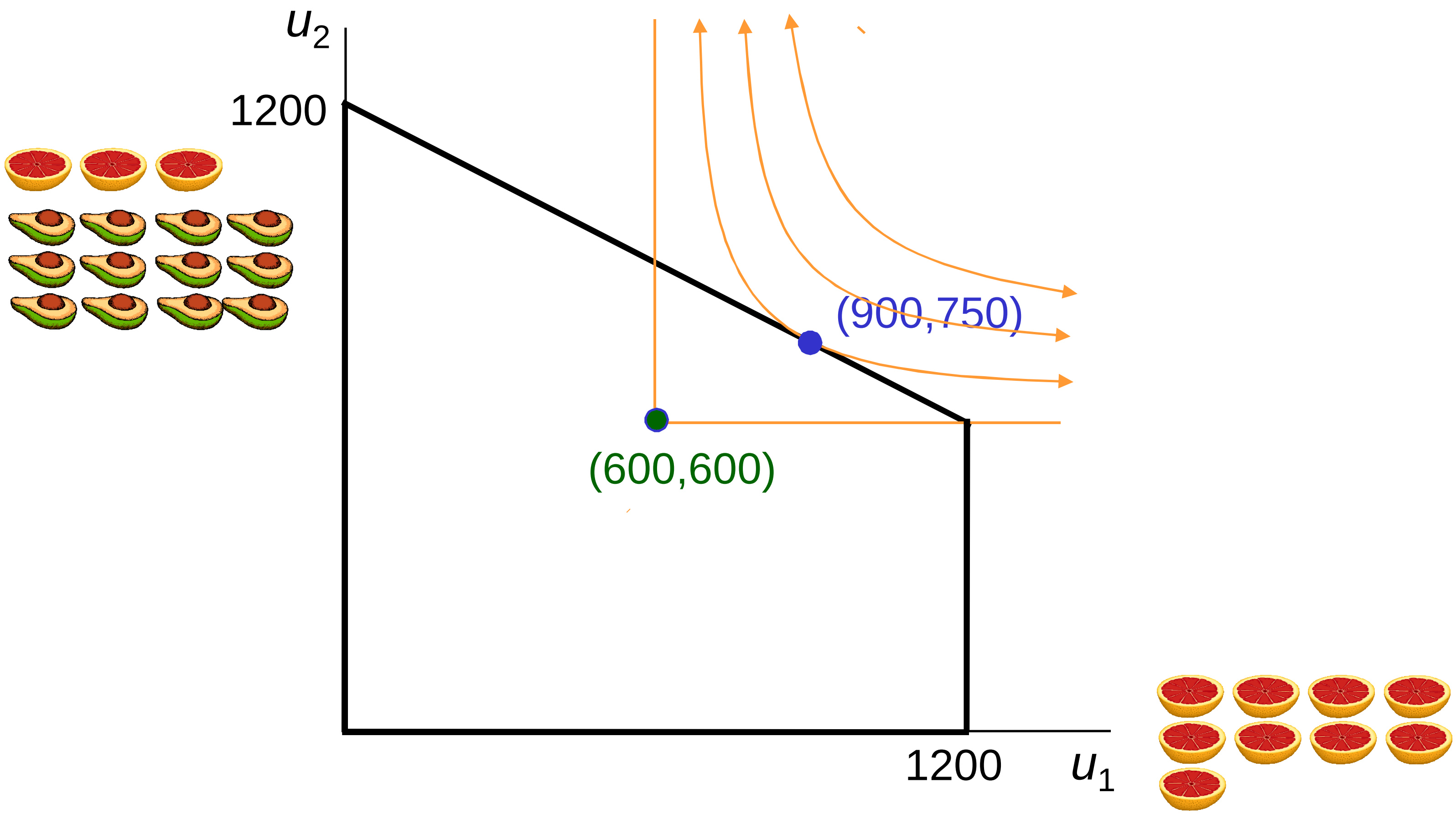
Nash Bargaining Solution

From Zero



Nash Bargaining Solution

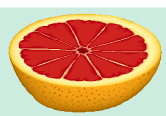
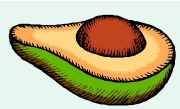
From Equality



Nash Bargaining Solution

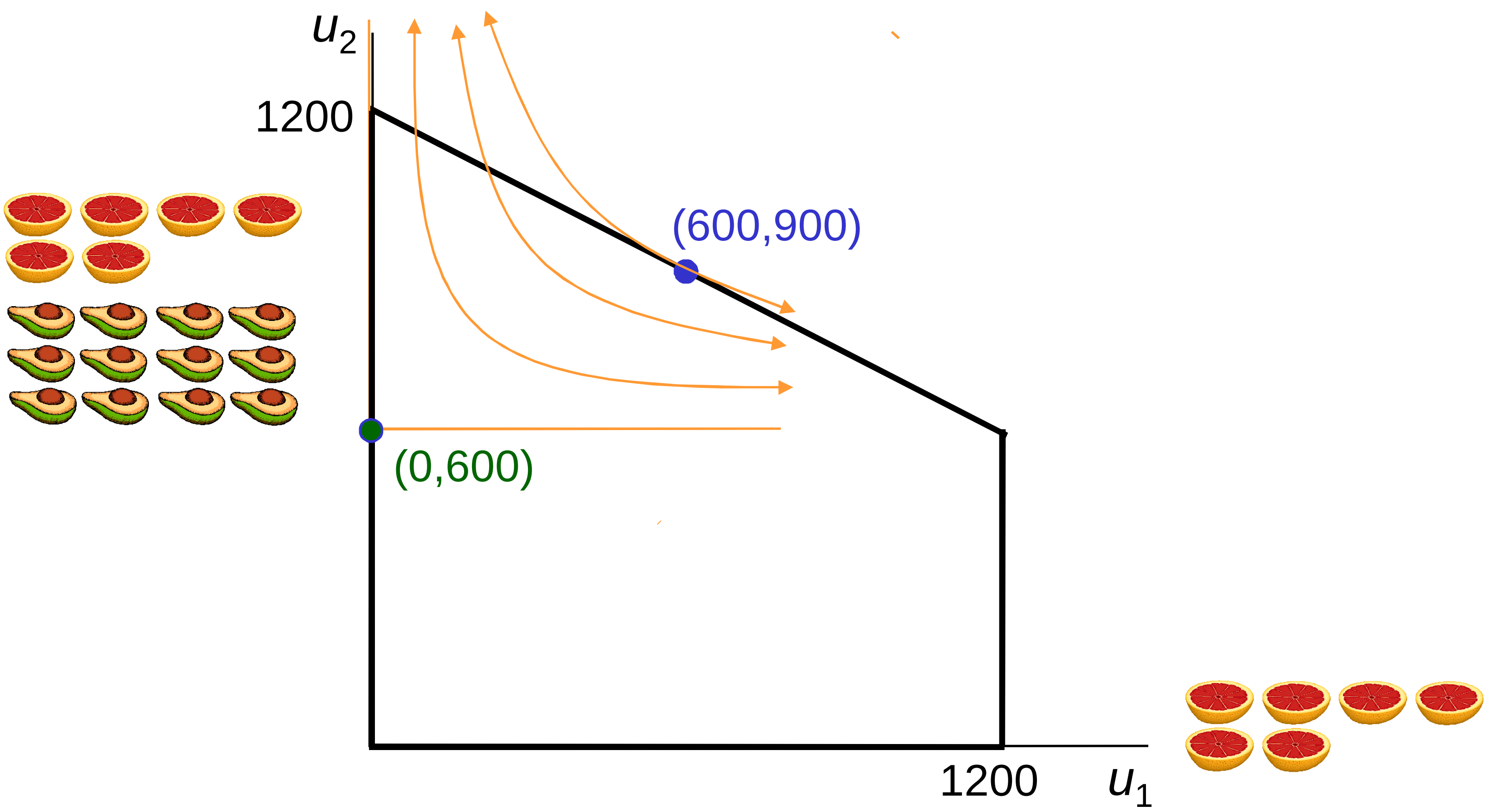
- **Strongly pareto set** gives Smith all 12 avocados.
  - Nothing for Jones.
  - Results in utility $(u_1,u_2) = (0,600)$

Utility provided by one fruit of each kind

	Jones	Smith
	100	50
	0	50

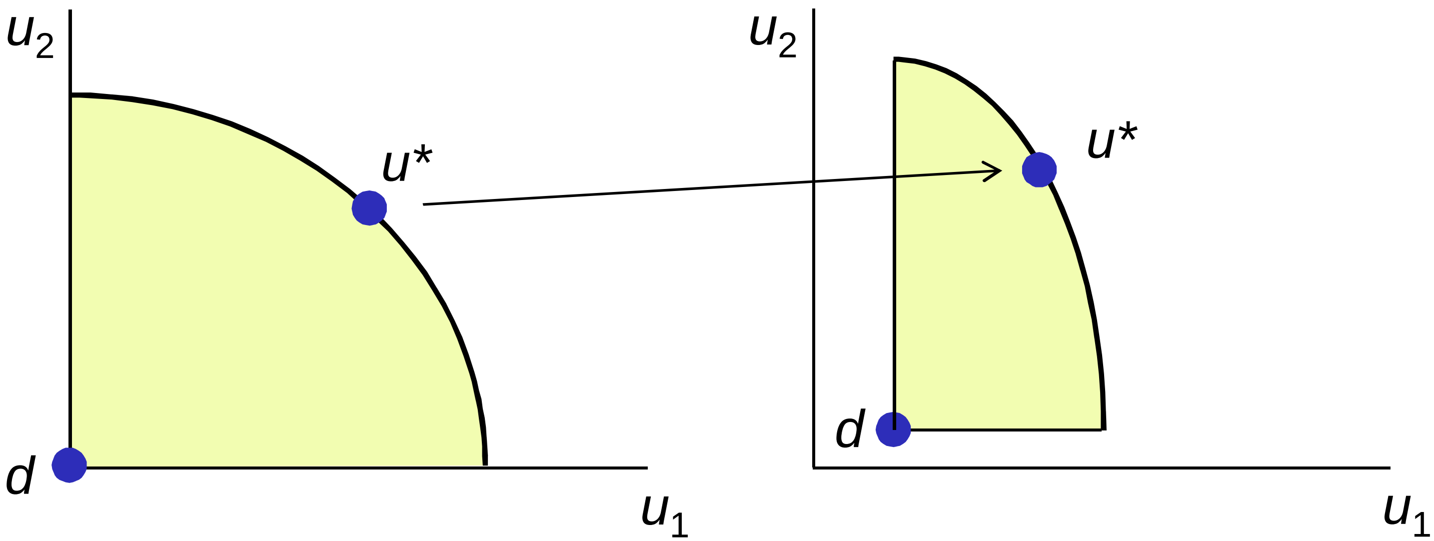
Nash Bargaining Solution

From Strongly Pareto Set



Axiomatic Justification

- **Axiom 1**. Invariance under translation and rescaling.
  - If we map $u_i \rightarrow a_i u_i + b_i$, $d_i \rightarrow a_i d_i + b_i$, then bargaining solution $u_i^* \rightarrow a_i u_i^* + b_i$.



This is **cardinal noncomparability**.

Axiomatic Justification

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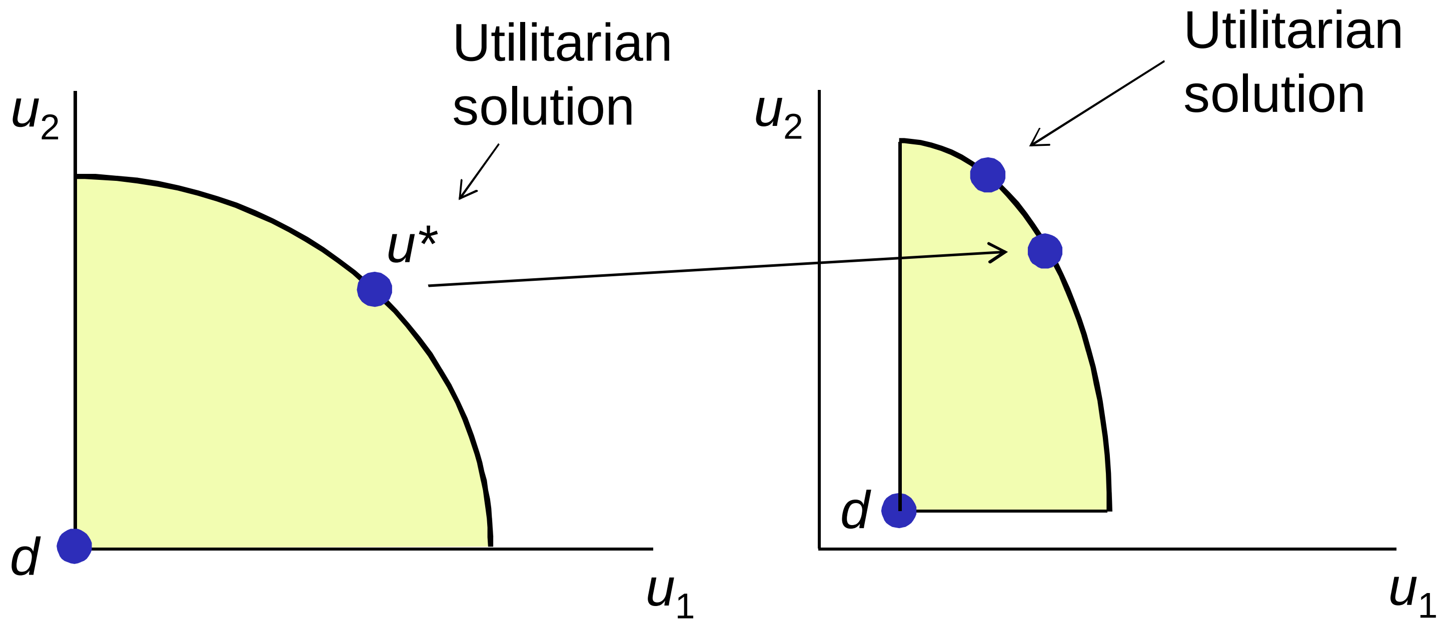
  - If we map $u_i \rightarrow a_i u_i + b_i$, $d_i \rightarrow a_i d_i + b_i$, then bargaining solution $u_i^* \rightarrow a_i u_i^* + b_i$.

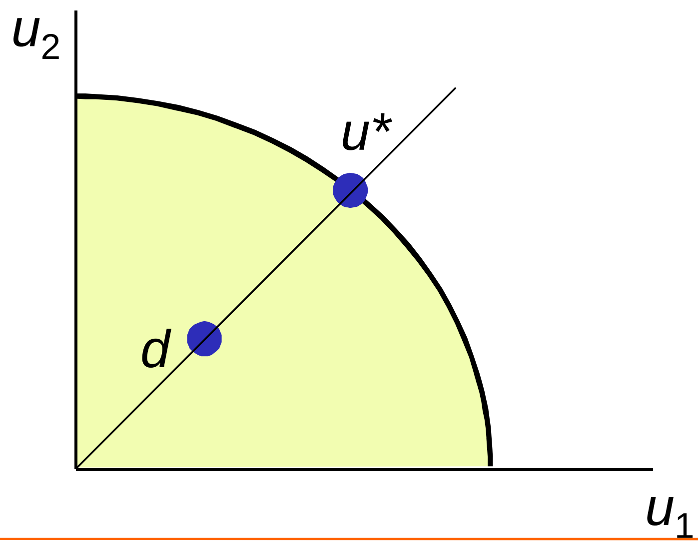


- **Strong assumption** – failed, e.g., by utilitarian welfare function

Axiomatic Justification

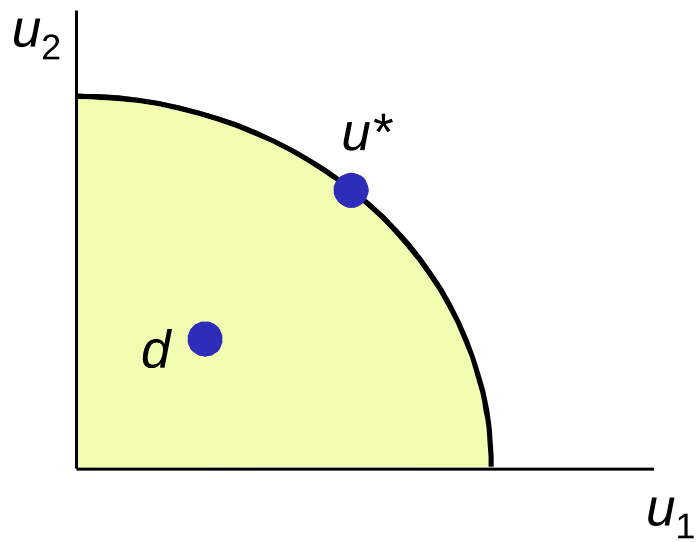
- **Axiom 2**. Pareto optimality.
  - Bargaining solution is pareto optimal.

- **Axiom 3.** Symmetry.
  - If all d_is are equal and feasible set is symmetric, then all u_i*s are equal in bargaining solution.



Axiomatic Justification

- **Axiom 4.** Independence of irrelevant alternatives.
  - Not the same as Arrow's axiom.
  - If u^* is a solution with respect to d…



Axiomatic Justification

- **Axiom 4.** Independence of irrelevant alternatives.
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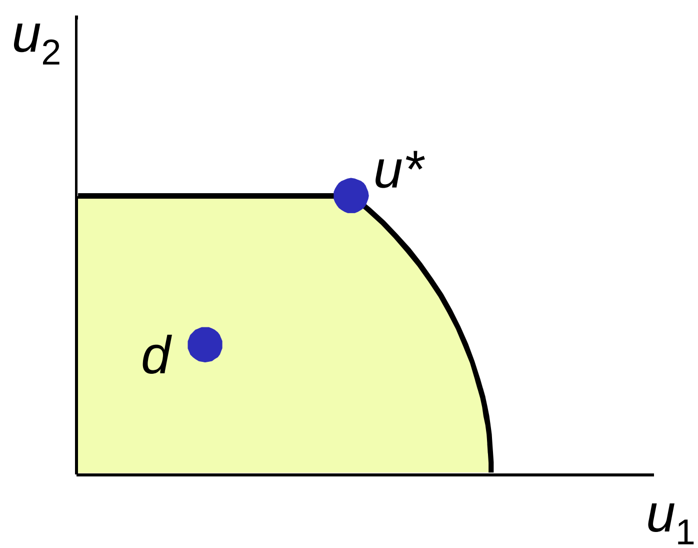
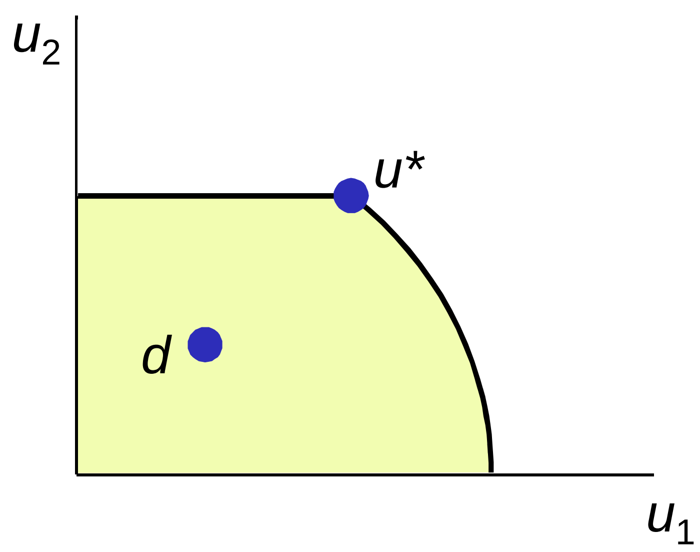
  - If u^* is a solution with respect to d, then it is a solution in a smaller feasible set that contains u^* and d.



Axiomatic Justification

- **Axiom 4.** Independence of irrelevant alternatives.
  - Not the same as Arrow's axiom.
  - If u^* is a solution with respect to d, then it is a solution in a smaller feasible set that contains u^* and d.
  - This basically says that the solution behaves like an **optimum**.



Axiomatic Justification

Theorem. Exactly one solution satisfies Axioms 1-4, namely the Nash bargaining solution.

Proof (2 dimensions).

First show that the Nash solution satisfies the axioms.

Axiom 1. Invariance under transformation. If

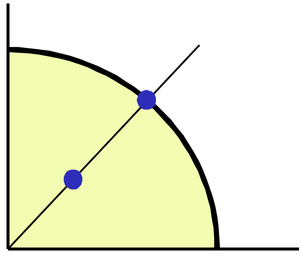
$$\prod_i (u_i^* - d_1) \geq \prod_i (u_i - d_1)$$

then

$$\prod_i \big((a_i u_i^* + b_i) - (a_i d_i + b_i)\big) \geq \prod_i \big((a_i u_i + b_i) - (a_i d_i + b_i)\big)$$

Axiomatic Justification

Axiom 2. Pareto optimality. Clear because social welfare function is strictly monotone increasing.



Axiom 3. Symmetry. Obvious.

Axiom 4. Independence of irrelevant alternatives. Follows from the fact that u^* is an optimum.

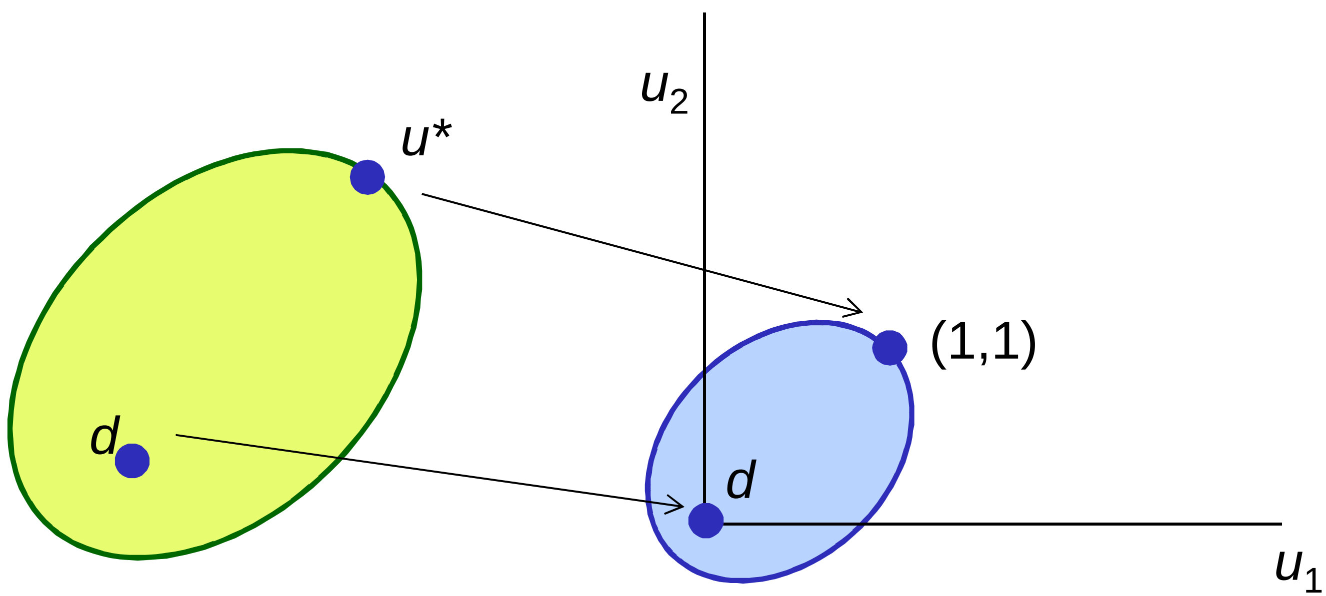
Now show that **only** the Nash solution satisfies the axioms…

Axiomatic Justification

Let u^* be the Nash solution for a given problem. Then it satisfies the axioms with respect to d. Select a transformation that sends

$$(u_1,u_2) \rightarrow (1,1), \quad (d_1,d_2) \rightarrow (0,0)$$

The transformed problem has Nash solution (1,1), by Axiom 1:



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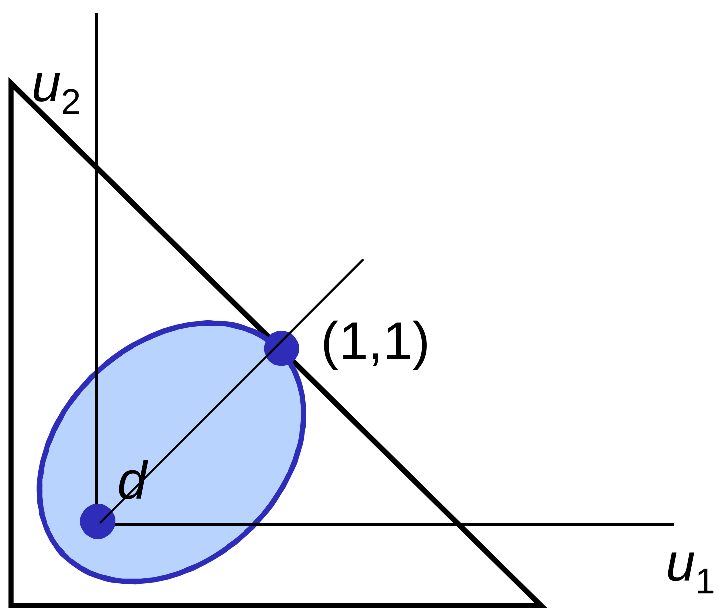
By Axioms 2 & 3, (1,1) is the **only** bargaining solution in the triangle:



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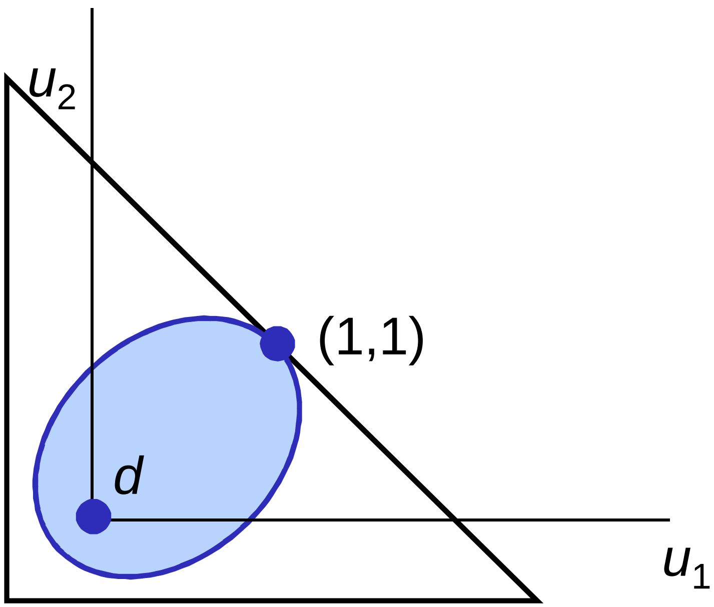

So by Axiom 4, (1,1) is the only bargaining solution in blue set.

Axiomatic Justification

Let u^* be the Nash solution for a given problem. Then it satisfies the axioms with respect to d. Select a transformation that sends

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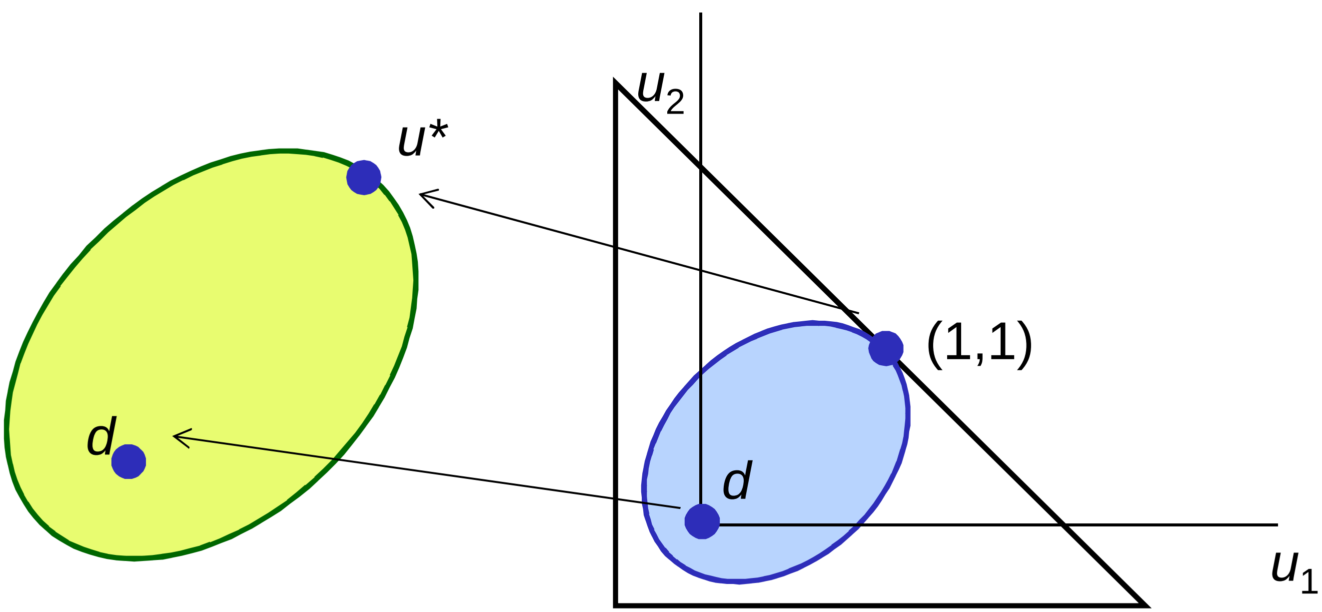
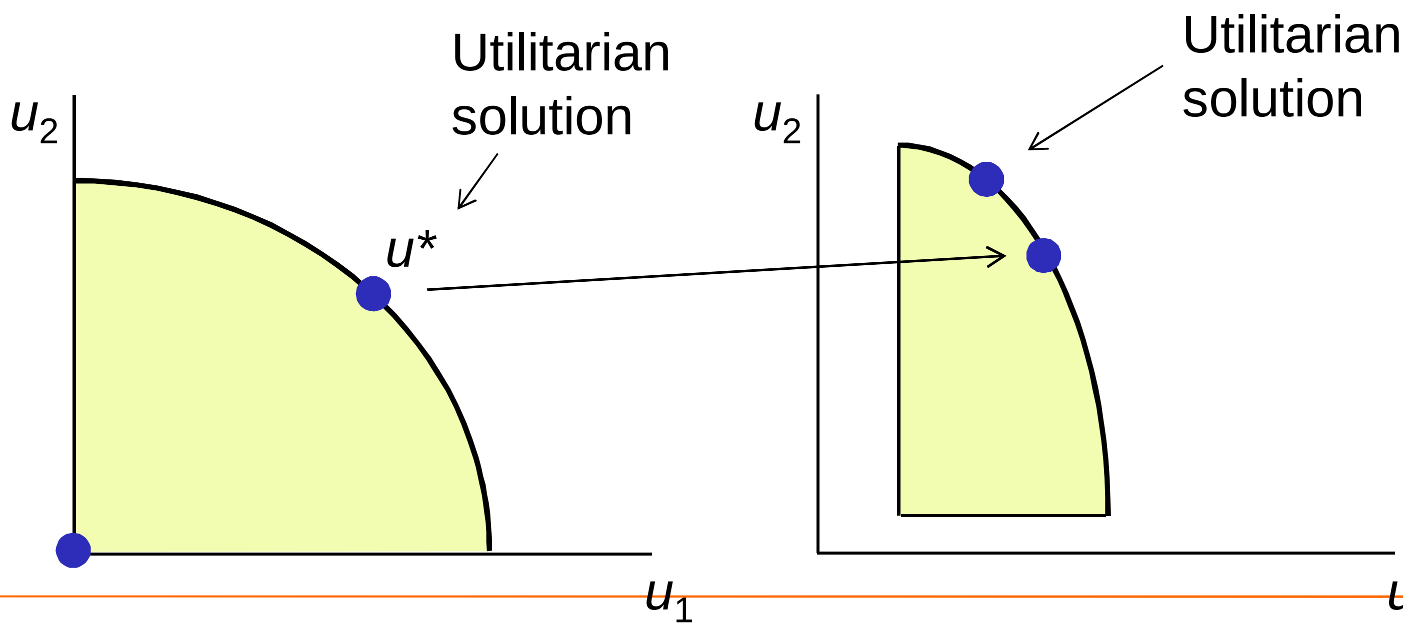


So by Axiom 4, (1,1) is the only bargaining solution in blue set.

By Axiom 1, u^* is the only bargaining solution in the original problem.

Axiomatic Justification

- **Problems** with axiomatic justification.
  - **Axiom 1** (invariance under transformation) is very strong.
  - Axiom 1 denies **interpersonal comparability**.
  - So how can it reflect moral concerns?



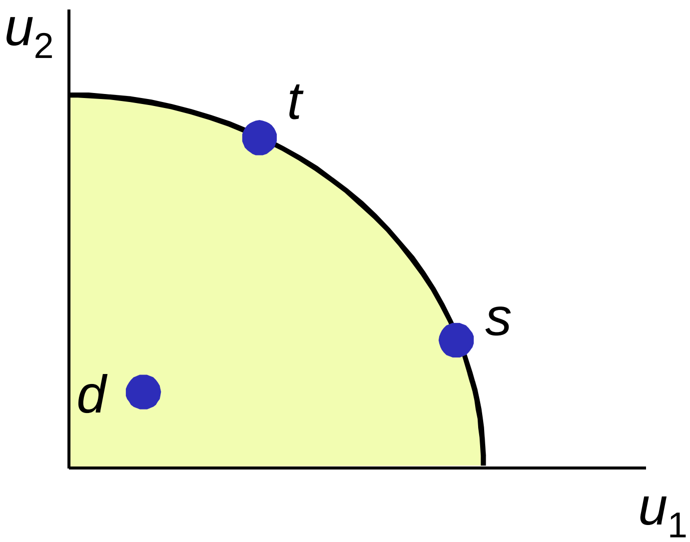
Axiomatic Justification

- **Problems** with axiomatic justification.
  - **Axiom 1** (invariance under transformation) is very strong.
  - Axiom 1 denies **interpersonal comparability**.
  - So how can it reflect moral concerns?

- Most attention has been focused on **Axiom 4** (independence of irrelevant alternatives).
  - Will address this later.

Bargaining Justification

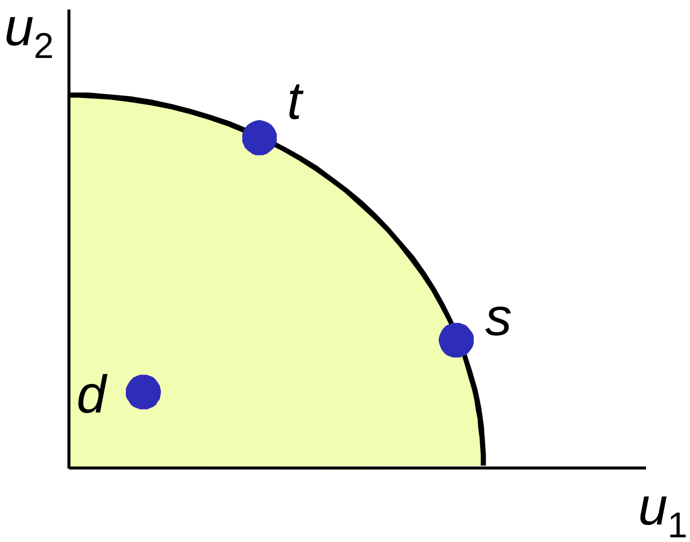
Players 1 and 2 make offers *s*, *t*.



Bargaining Justification

Players 1 and 2 make offers *s*, *t*.

Let $p = P(\text{player 2 will reject } s)$, as estimated by player 1.



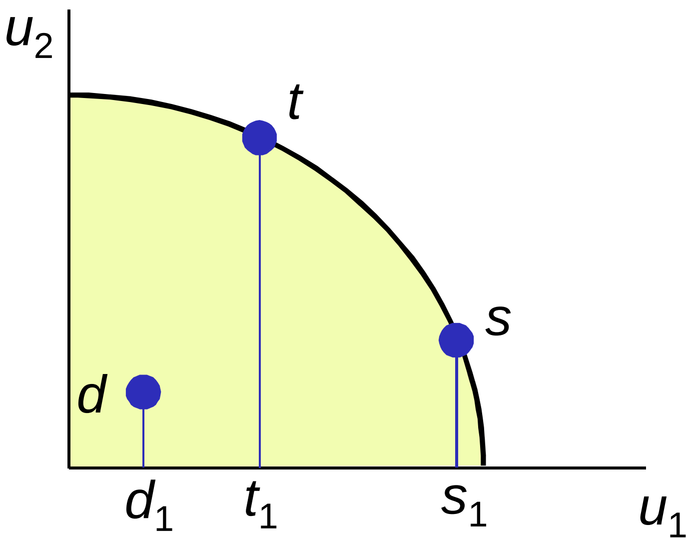
Bargaining Justification

Players 1 and 2 make offers *s*, *t*.

Let $p = P$(player 2 will reject *s*), as estimated by player 1.

Then player 1 will stick with *s*, rather than make a counteroffer, if

$$(1-p)s_1 + pd_1 \geq t_1$$



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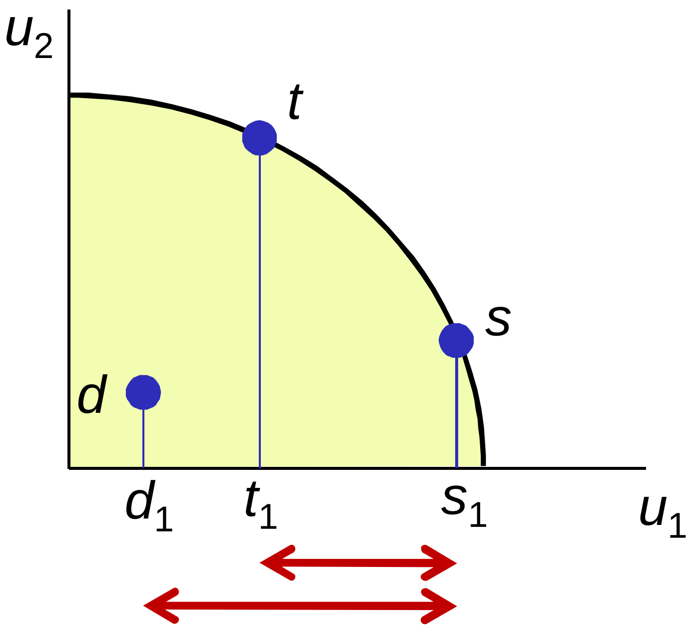


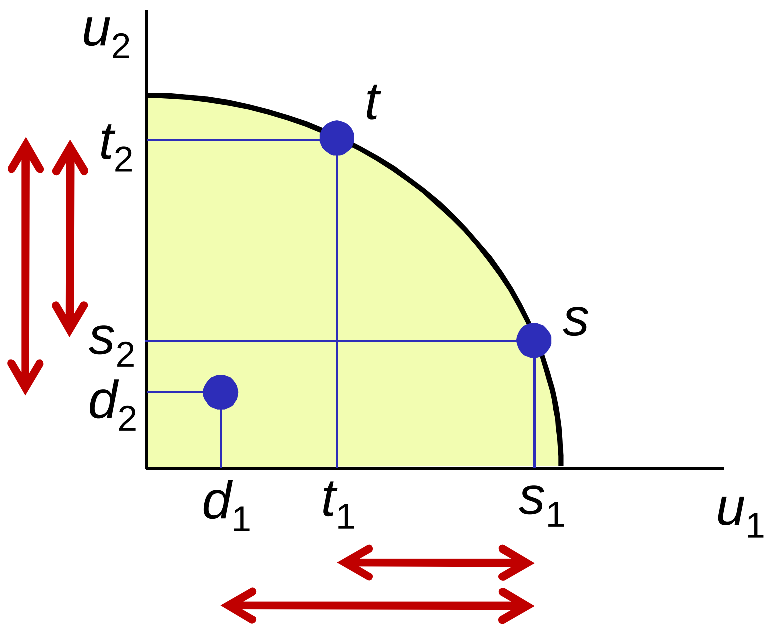
So player 1 will stick with *s* if

$$p \leq \frac{s_1 - t_1}{s_1 - d_1} = r_1$$

Bargaining Justification

It is rational for player 1 to make a counteroffer s', rather than player 2, if

$$r_1 = \frac{s_1 - t_1}{s_1 - d_1} \leq \frac{t_2 - s_2}{t_2 - d_2} = r_2$$



So player 1 will stick with s if

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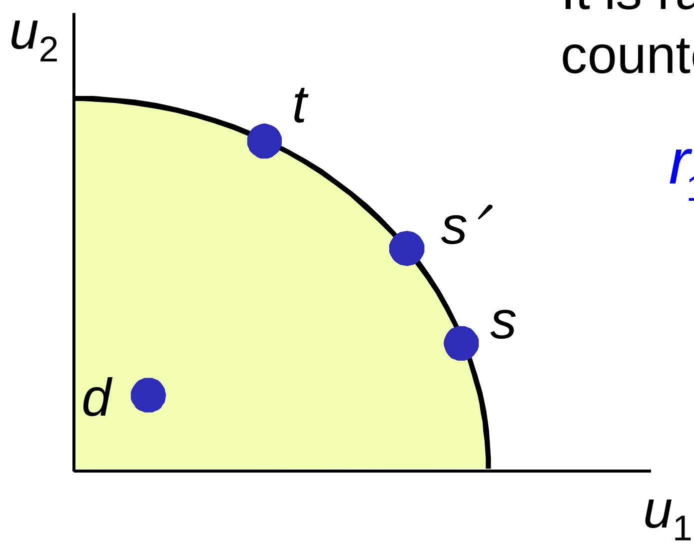
Bargaining Justification

It is rational for player 1 to make a counteroffer *s*′, rather than player 2, if

$$r_1 = \frac{s_1 - t_1}{s_1 - d_1} \leq \frac{t_2 - s_2}{t_2 - d_2} = r_2$$

It is rational for player 2 to make the next counteroffer if

$$r_1' = \frac{s_1' - t_1}{s_1' - d_1} \geq \frac{t_2 - s_2'}{t_2 - d_2} = r_2'$$



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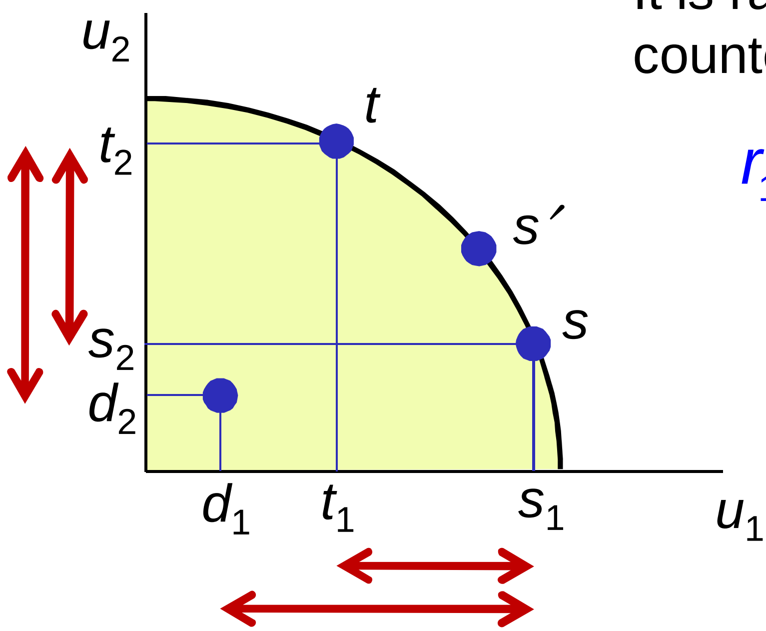

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But $$\frac{s_1 - t_1}{s_1 - d_1} \leq \frac{t_2 - s_2}{t_2 - d_2}$$

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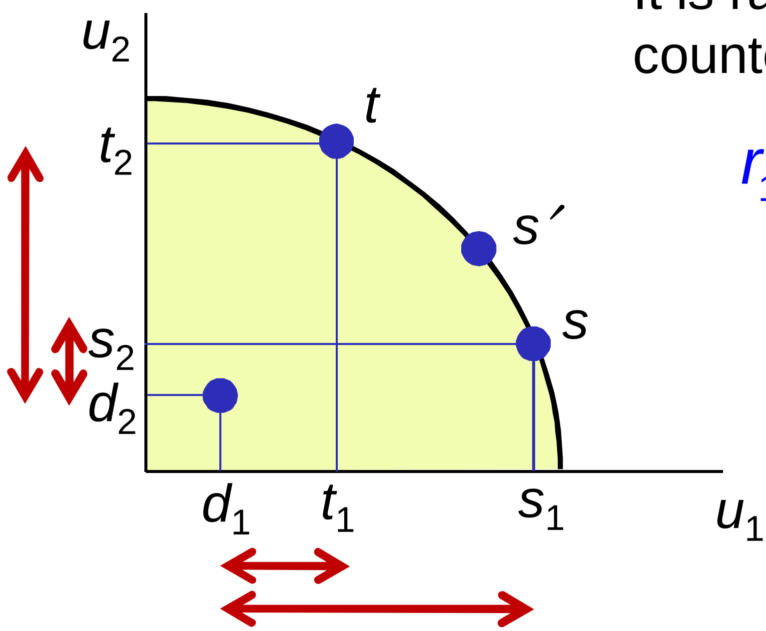

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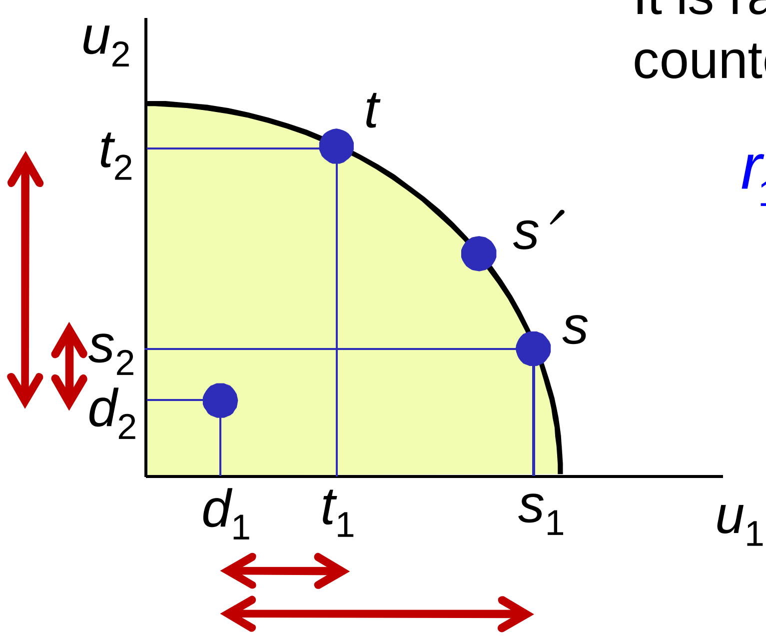
$$\frac{s_1 - t_1}{s_1 - d_1} \le \frac{t_2 - s_2}{t_2 - d_2}$$

$$\Longleftrightarrow \frac{t_1 - d_1}{s_1 - d_1} \ge \frac{s_2 - d_2}{t_2 - d_2}$$

Bargaining Justification

So we have $$(s_1 - d_1)(s_2 - d_2) \le (t_1 - d_1)(t_2 - d_2)$$



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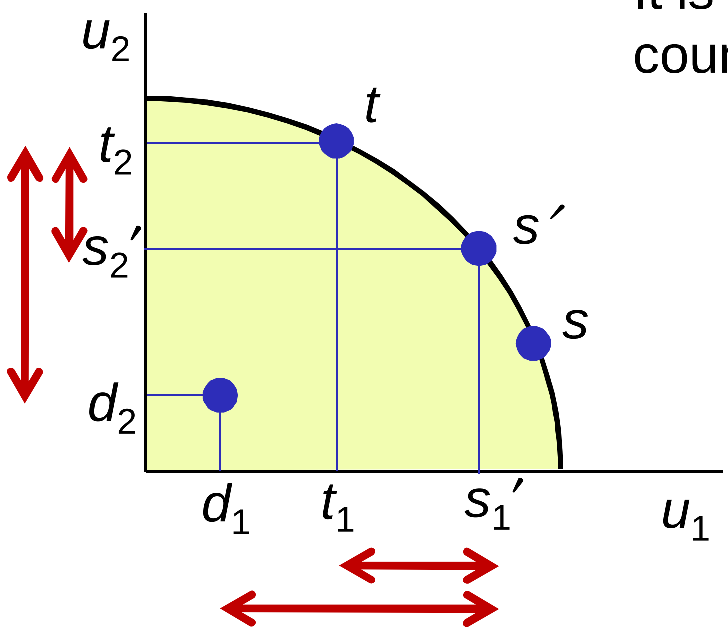

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Similarly $$\frac{s_1' - t_1}{s_1' - d_1} \geq \frac{t_2 - s_2'}{t_2 - d_2}$$

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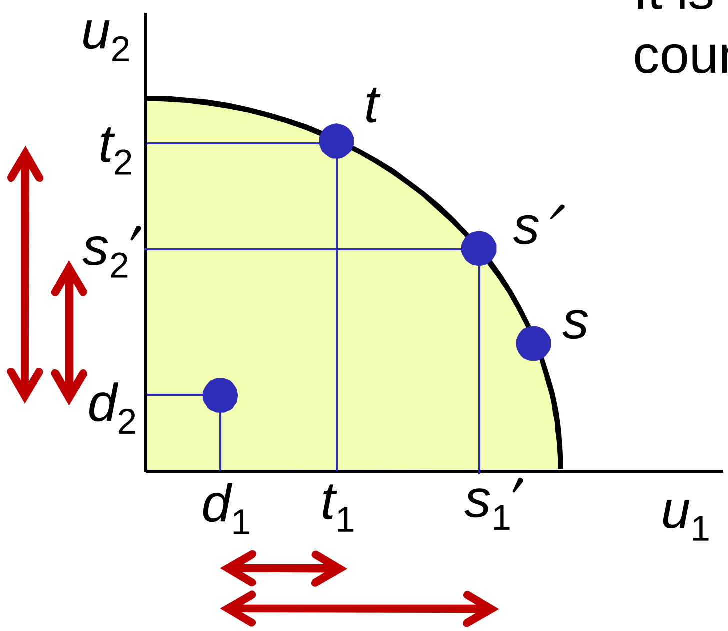
So we have $$(s_1 - d_1)(s_2 - d_2) \le (t_1 - d_1)(t_2 - d_2)$$



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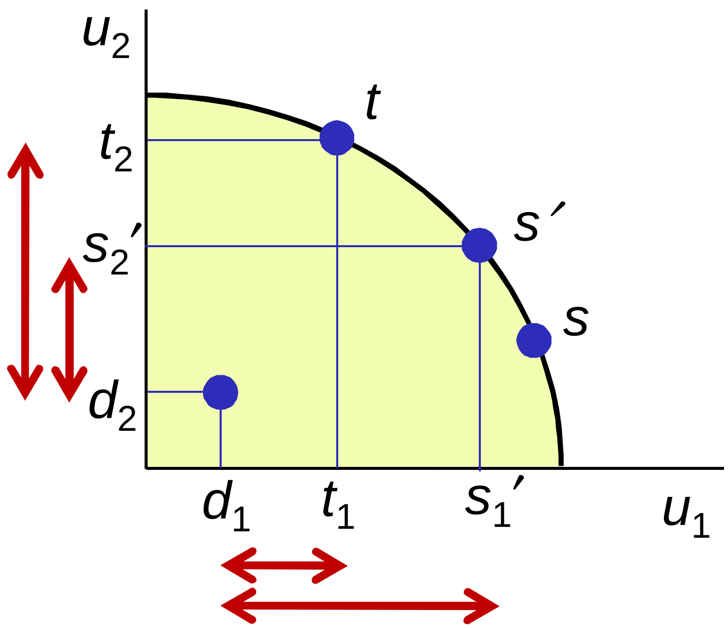
Similarly $$\frac{s_1' - t_1}{s_1' - d_1} \ge \frac{t_2 - s_2'}{t_2 - d_2}$$

$$\Longleftrightarrow \frac{t_1 - d_1}{s_1' - d_1} \le \frac{s_2' - d_2}{t_2 - d_2}$$

Bargaining Justification

So we have $(s_1 - d_1)(s_2 - d_2) \le (t_1 - d_1)(t_2 - d_2)$

and we have $(t_1 - d_1)(t_2 - d_2) \le (s_1' - d_1)(s_2' - d_2)$



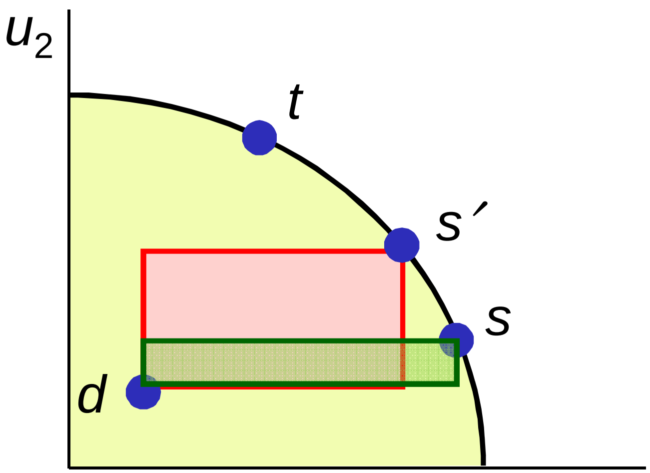
Similarly
$$\frac{s_1' - t_1}{s_1' - d_1} \ge \frac{t_2 - s_2'}{t_2 - d_2}$$

$$\Longleftrightarrow \frac{t_1 - d_1}{s_1' - d_1} \le \frac{s_2' - d_2}{t_2 - d_2}$$

Bargaining Justification

So we have $$(s_1 - d_1)(s_2 - d_2) \leq (t_1 - d_1)(t_2 - d_2)$$

and we have $$(t_1 - d_1)(t_2 - d_2) \leq (s'_1 - d_1)(s'_2 - d_2)$$

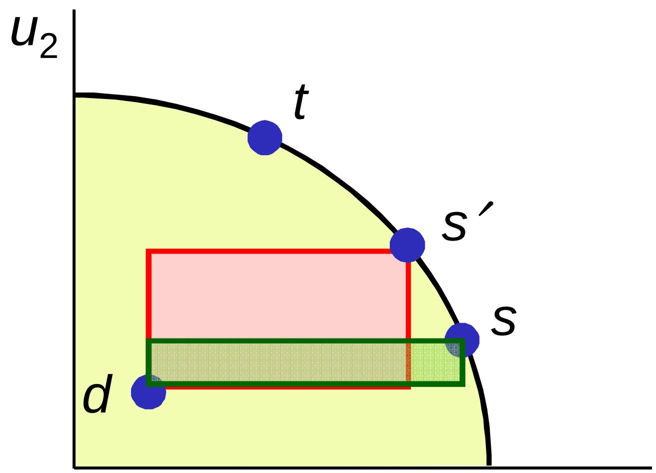


This implies an improvement in the Nash social welfare function

Bargaining Justification

So we have $(s_1 - d_1)(s_2 - d_2) \leq (t_1 - d_1)(t_2 - d_2)$

and we have $(t_1 - d_1)(t_2 - d_2) \leq (s_1' - d_1)(s_2' - d_2)$

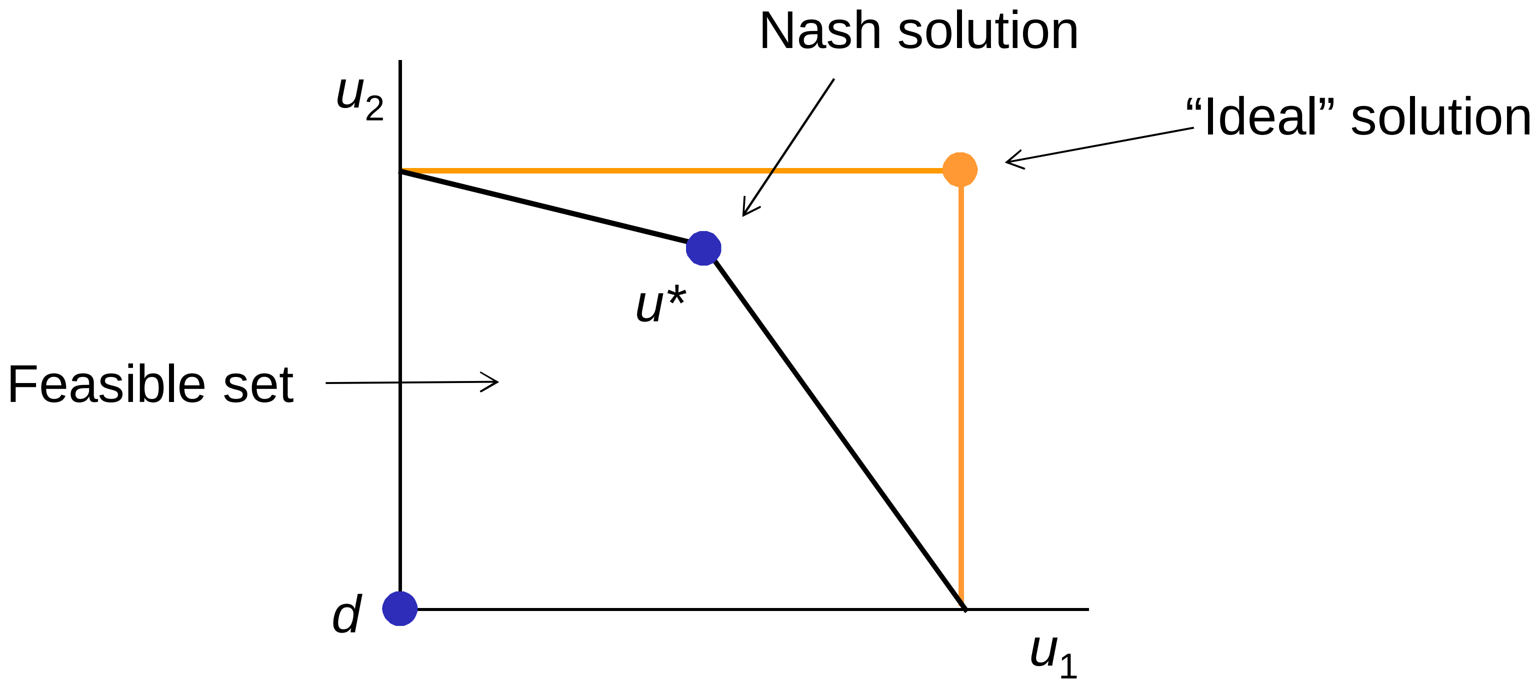


This implies an improvement in the Nash social welfare function.

Given a minimum distance between offers, continued bargaining converges to Nash solution.

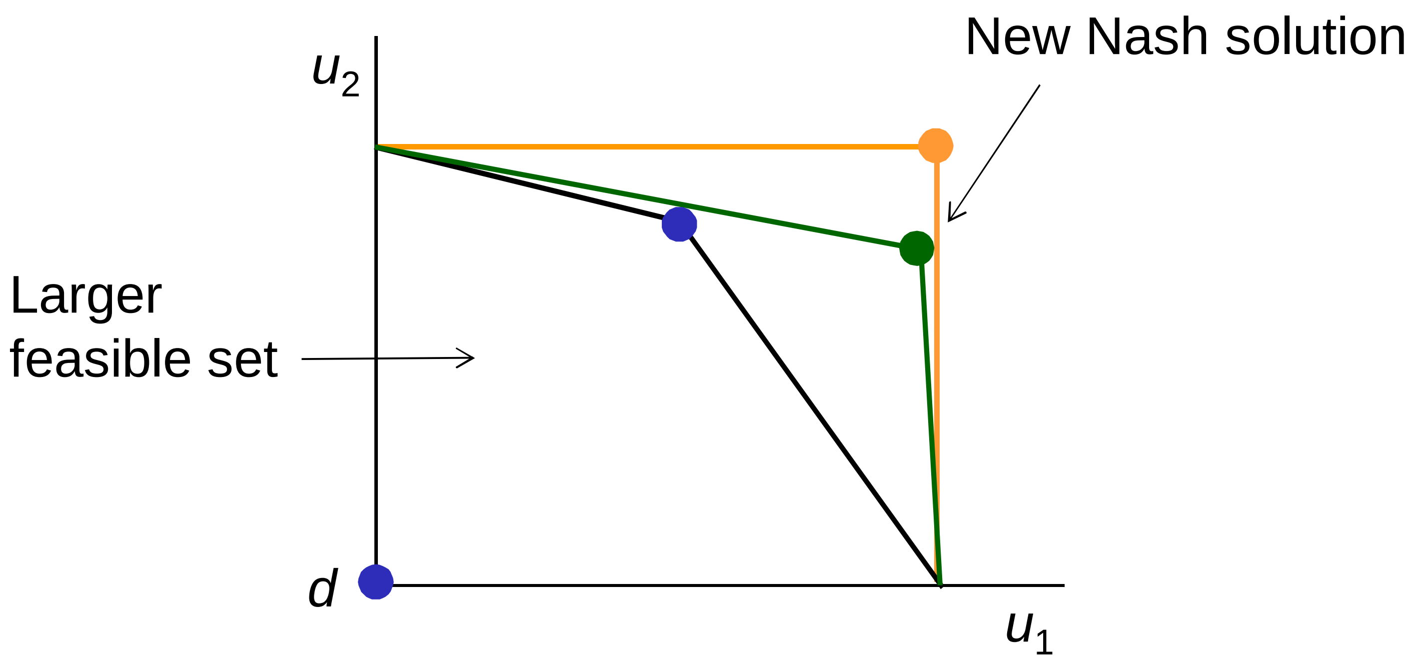
Raiffa-Kalai-Smorodinsky Bargaining Solution

- This approach begins with a critique of the Nash bargaining solution.



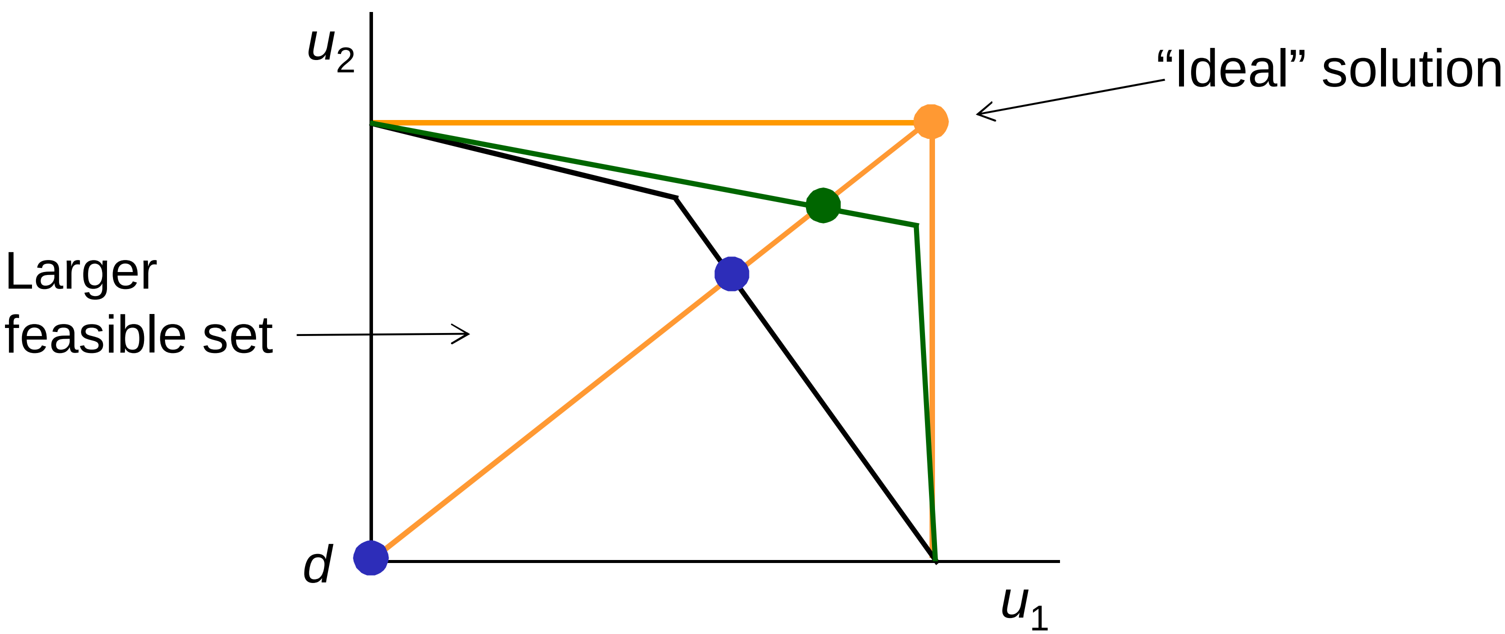
Raiffa-Kalai-Smorodinsky Bargaining Solution

- This approach begins with a critique of the Nash bargaining solution.
  - The new Nash solution is worse for player 2 even though the feasible set is larger.



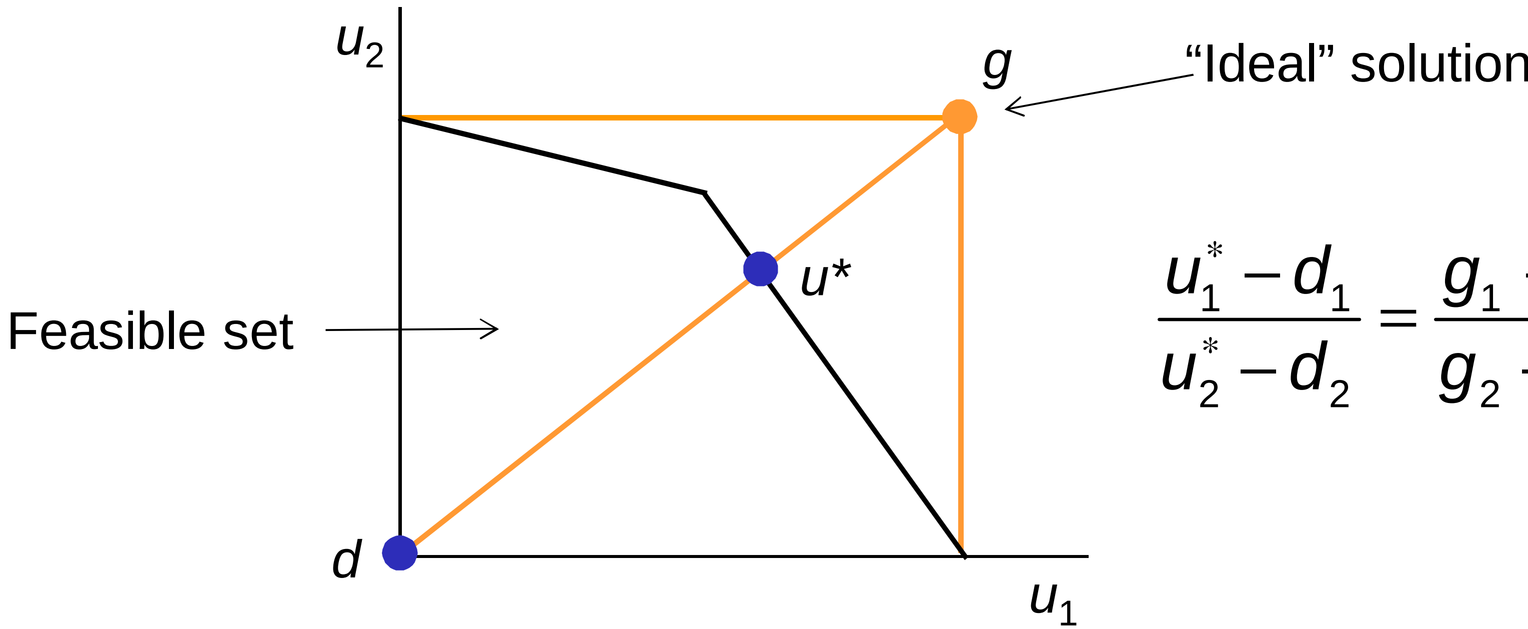
Raiffa-Kalai-Smorodinsky Bargaining Solution

- **Proposal**: Bargaining solution is pareto optimal point on line from *d* to ideal solution.



Raiffa-Kalai-Smorodinsky Bargaining Solution

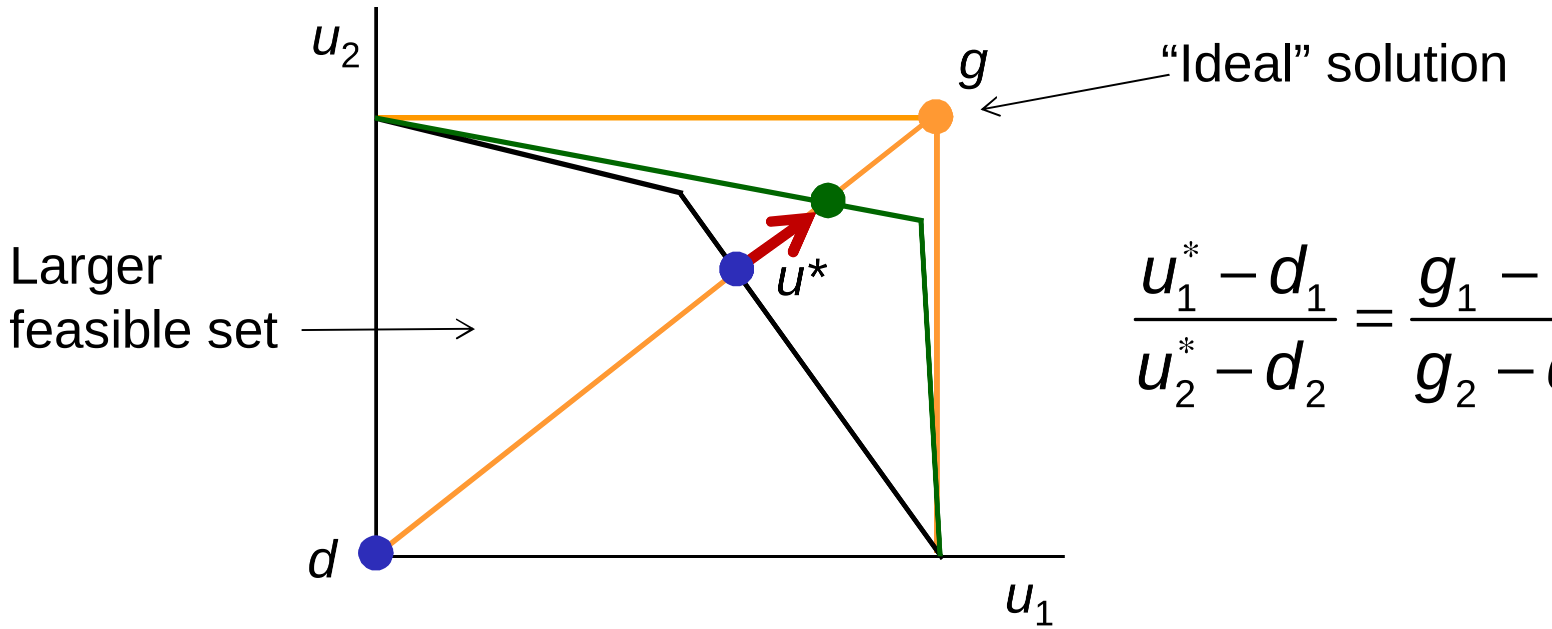
- **Proposal**: Bargaining solution is pareto optimal point on line from *d* to ideal solution.
  - The players receive an equal fraction of their possible utility gains.



$$\frac{u_1^* - d_1}{u_2^* - d_2} = \frac{g_1 - d_1}{g_2 - d_2}$$

Raiffa-Kalai-Smorodinsky Bargaining Solution

- **Proposal**: Bargaining solution is pareto optimal point on line from *d* to ideal solution.
  - Replace Axiom 4 with **Axiom 4′ (Monotonicity)**: A larger feasible set with same ideal solution results in a bargaining solution that is better (or no worse) for all players.



$$\frac{u_1^* - d_1}{u_2^* - d_2} = \frac{g_1 - d_1}{g_2 - d_2}$$

Raiffa-Kalai-Smorodinsky Bargaining Solution

- **Optimization model**.
  - Not an optimization problem over original feasible set (we gave up Axiom 4).
  - But it is an optimization problem (pareto optimality) over the line segment from *d* to ideal solution.

$$
\begin{aligned}
&\max \sum_i u_i \\
&(g_1 - d_1)(u_i - d_i) = (g_i - d_i)(u_1 - d_1), \quad \text{all } i \\
&u \in S
\end{aligned}
$$

$$\frac{u_1^* - d_1}{u_2^* - d_2} = \frac{g_1 - d_1}{g_2 - d_2}$$

Raiffa-Kalai-Smorodinsky Bargaining Solution

- **Optimization model**.
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constants

$$\max \sum_i u_i$$

$$\boxed{(g_1 - d_1)}(u_i - d_i) = \boxed{(g_i - d_i)}(u_1 - d_1), \quad \text{all } i$$

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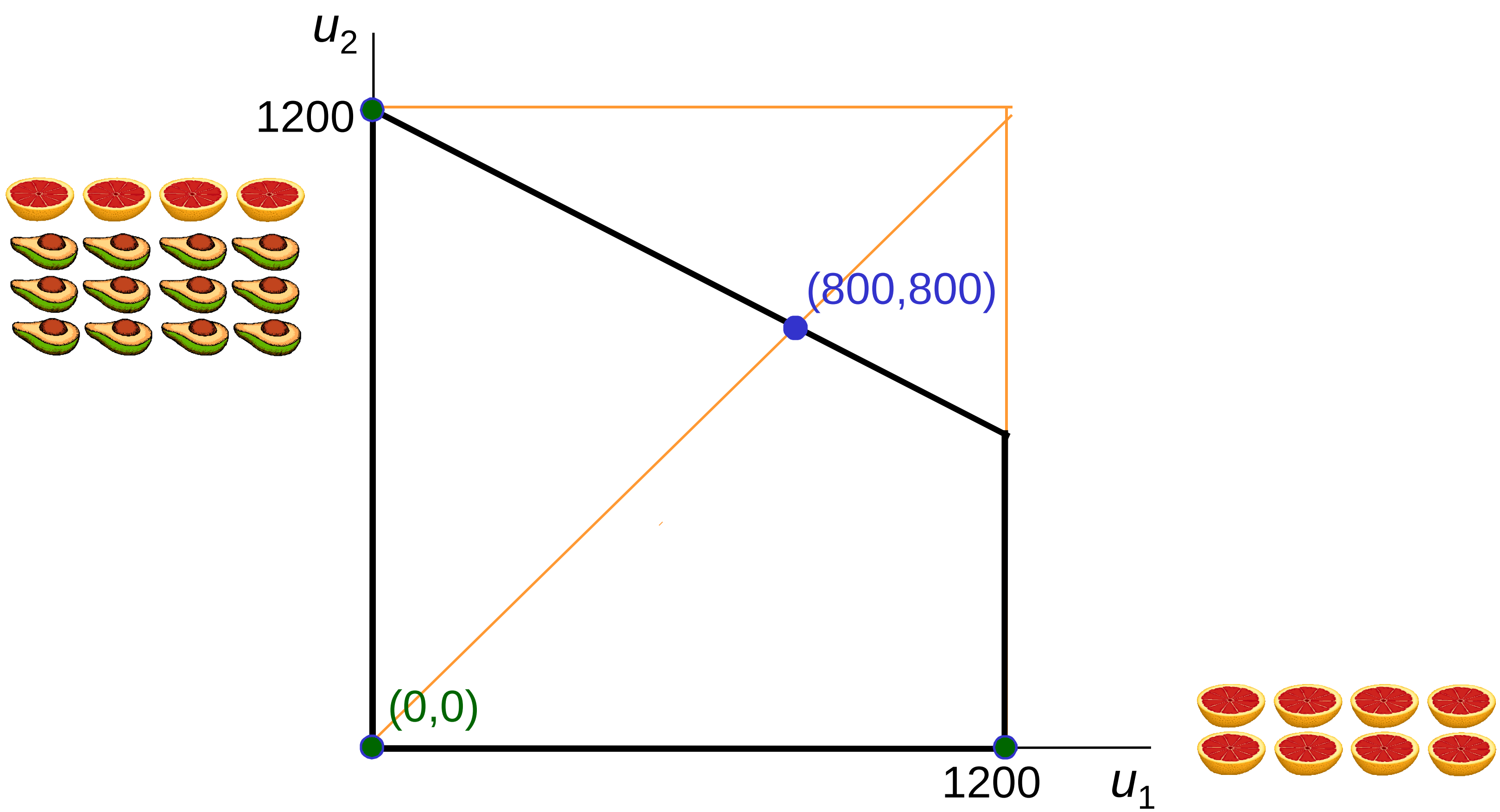
Linear constraint

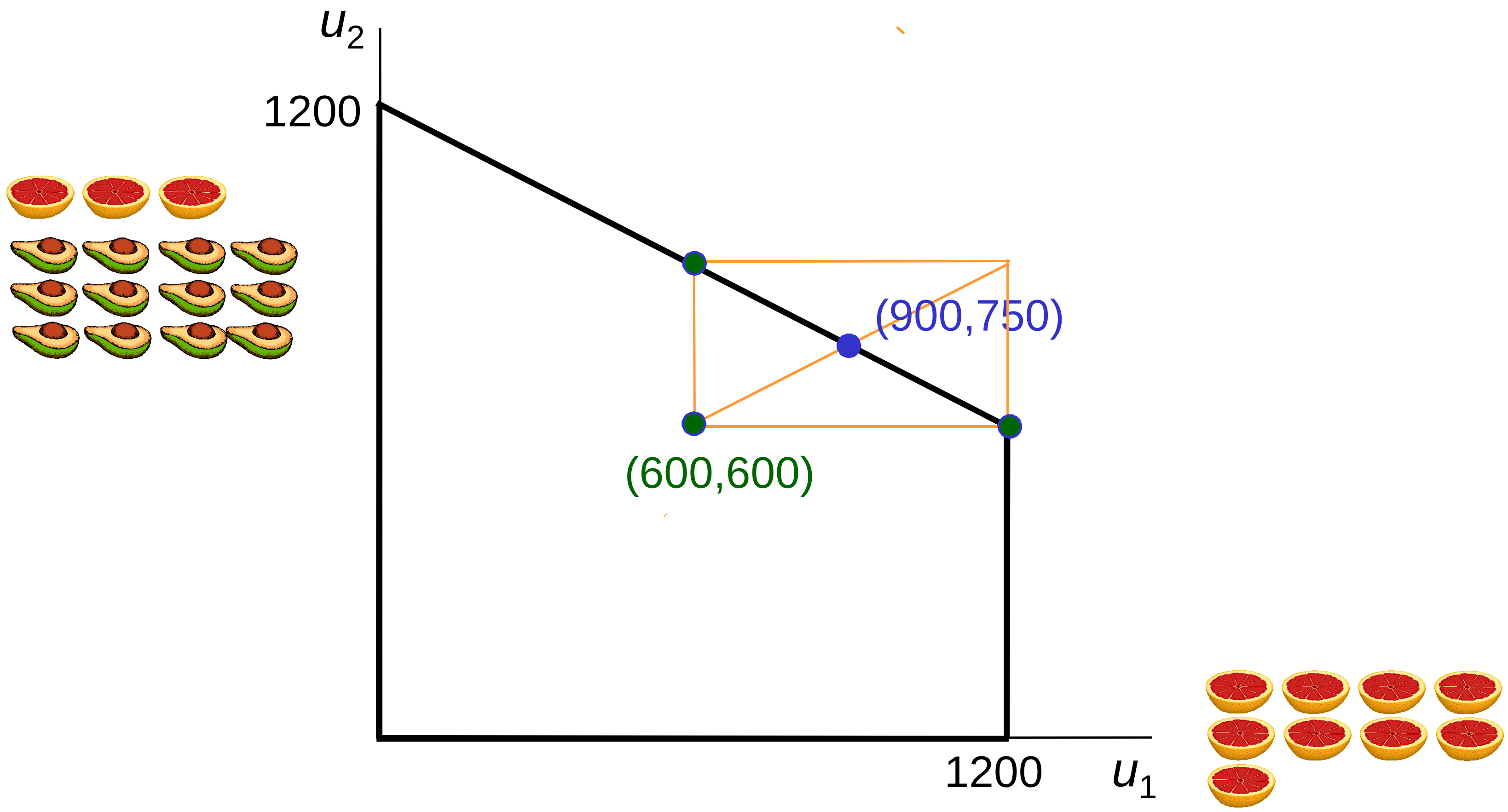
Raiffa-Kalai-Smorodinsky Bargaining Solution

From Zero



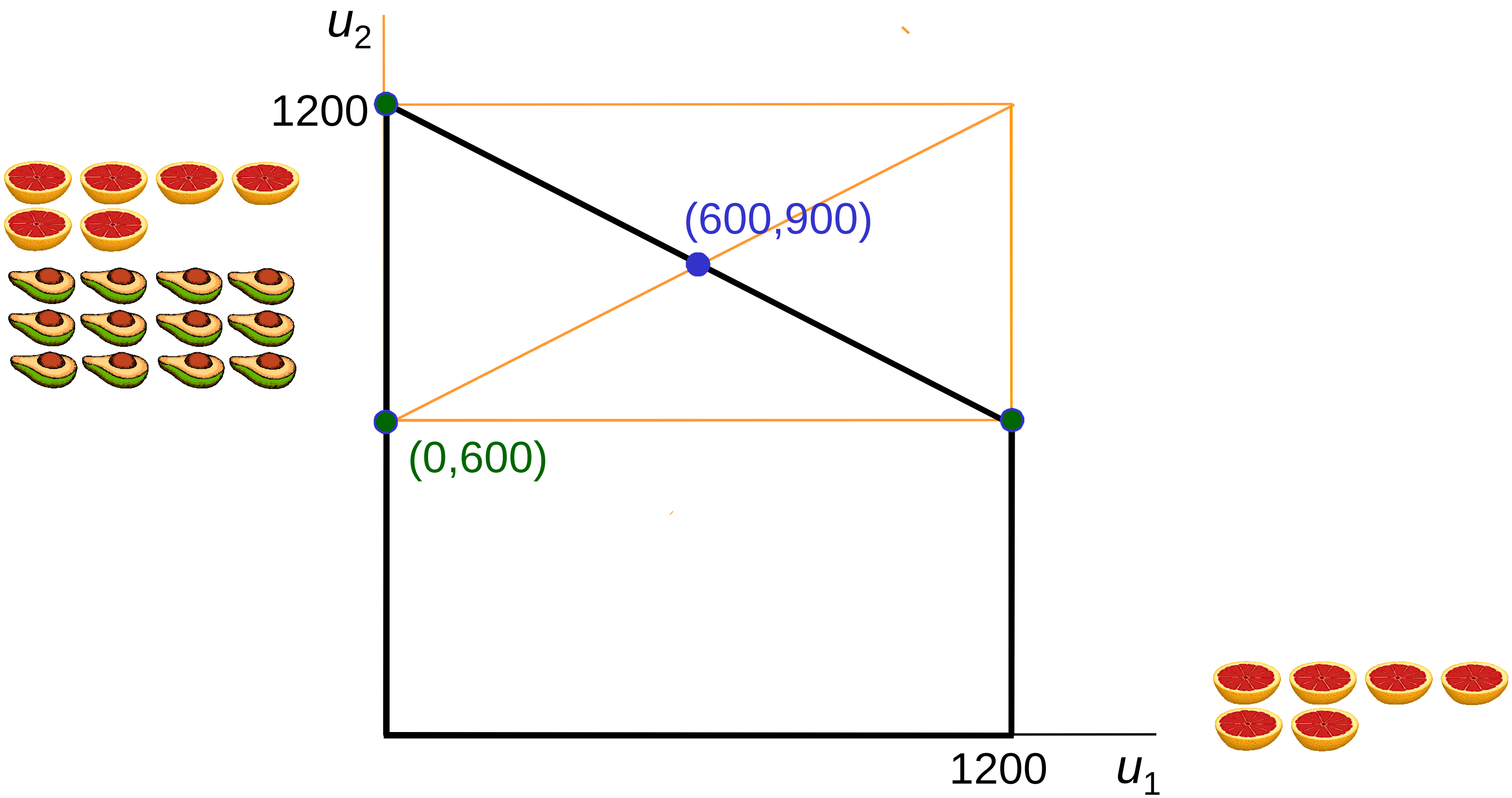
Raiffa-Kalai-Smorodinsky Bargaining Solution

From Equality



Raiffa-Kalai-Smorodinsky Bargaining Solution

From Strong Pareto Set



Axiomatic Justification

- **Axiom 1**. Invariance under transformation.
- **Axiom 2.** Pareto optimality.
- **Axiom 3.** Symmetry.
- **Axiom 4′.** Monotonicity.

Axiomatic Justification

Theorem. Exactly one solution satisfies Axioms 1-4′, namely the RKS bargaining solution.

Proof (2 dimensions).

Easy to show that RKS solution satisfies the axioms.

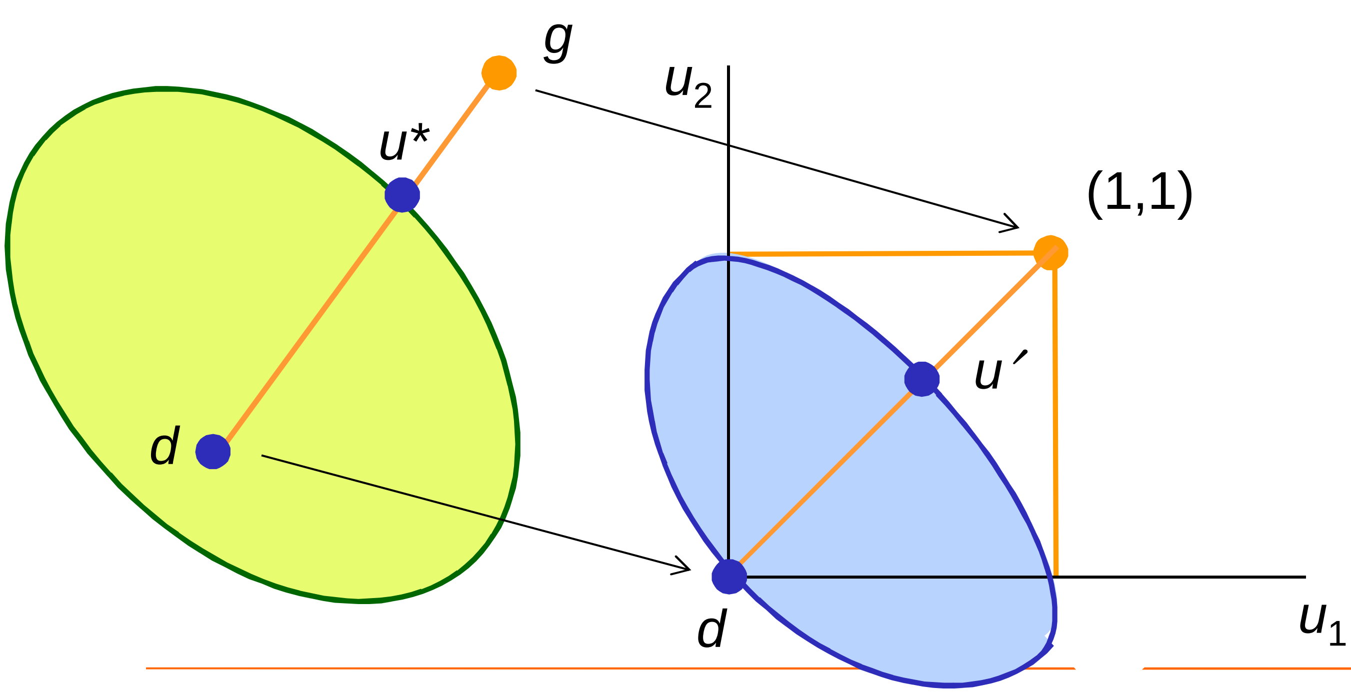
Now show that **only** the RKS solution satisfies the axioms.

Axiomatic Justification

Let u^* be the RKS solution for a given problem. Then it satisfies the axioms with respect to d. Select a transformation that sends

$$(g_1,g_2) \rightarrow (1,1), \quad (d_1,d_2) \rightarrow (0,0)$$

The transformed problem has RKS solution u', by Axiom 1:



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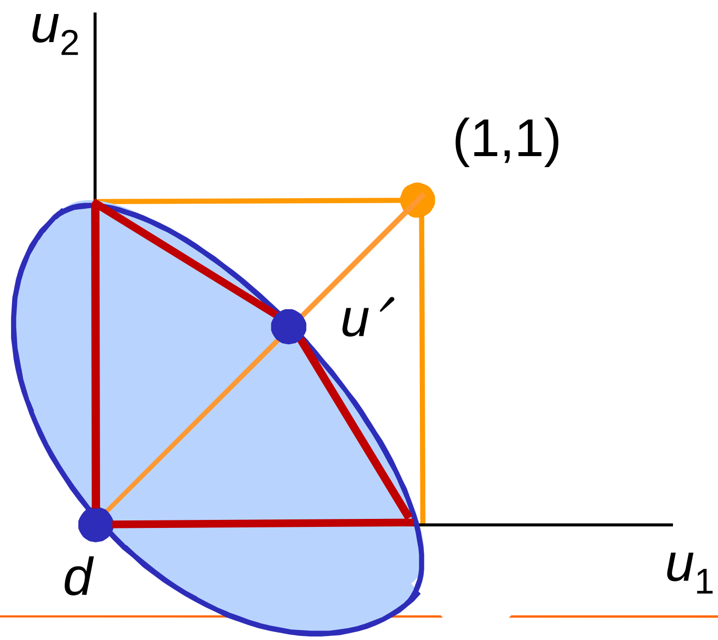
By Axioms 2 & 3, u' is the **only** bargaining solution in the red polygon:



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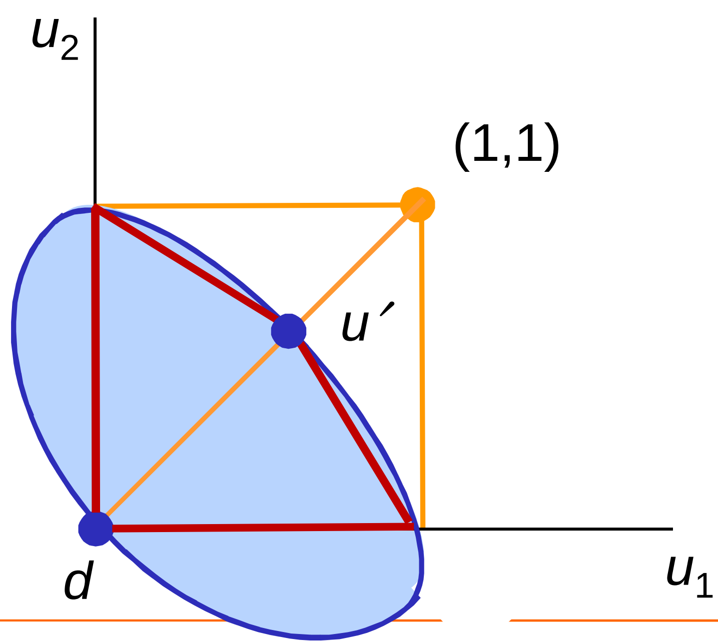

The red polygon lies inside blue set. So by Axiom 4′, its bargaining solution is no better than bargaining solution on blue set. So u' is the only bargaining solution on blue set.

Axiomatic Justification

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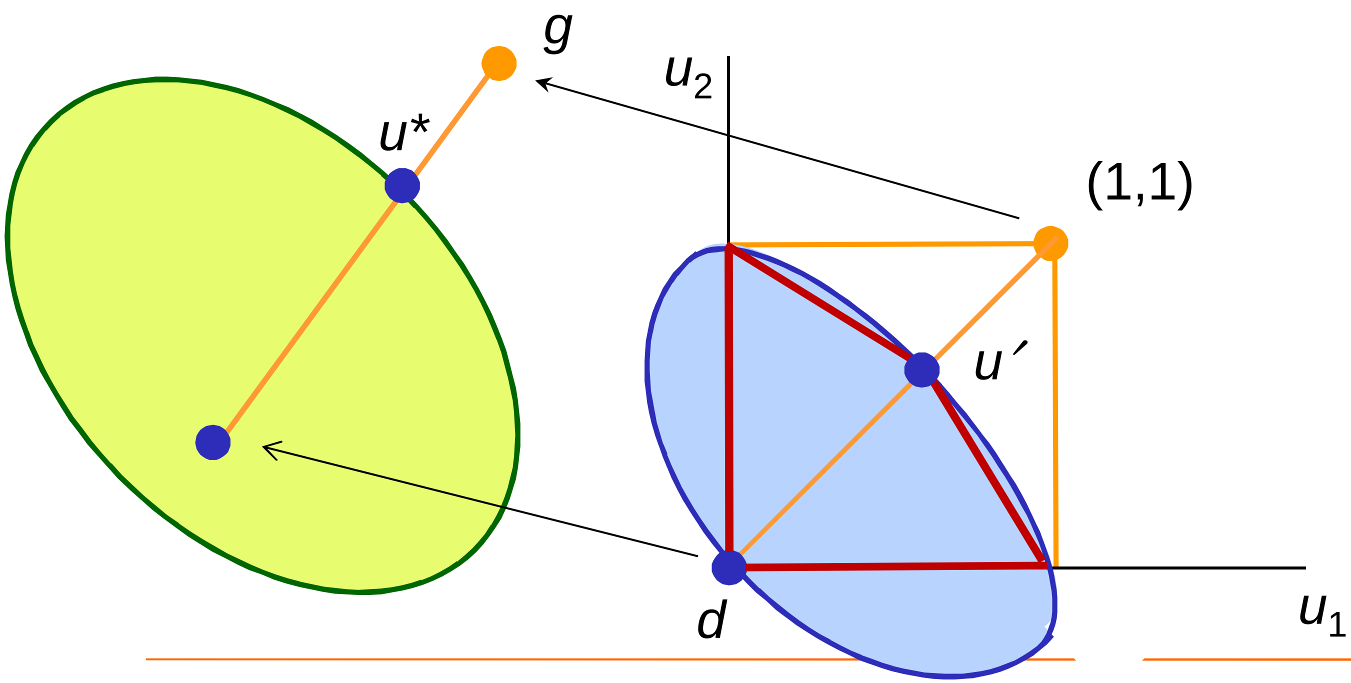


By Axiom 1, u^* is the only bargaining solution in the original problem.

Axiomatic Justification

- **Problems** with axiomatic justification.
  - **Axiom 1** is still in effect.
  - It denies **interpersonal comparability**.
  - Dropping Axiom 4 sacrifices optimization of a social welfare function.
  - This may not be necessary if Axiom 1 is rejected.
  - Needs modification for > 2 players (more on this shortly).

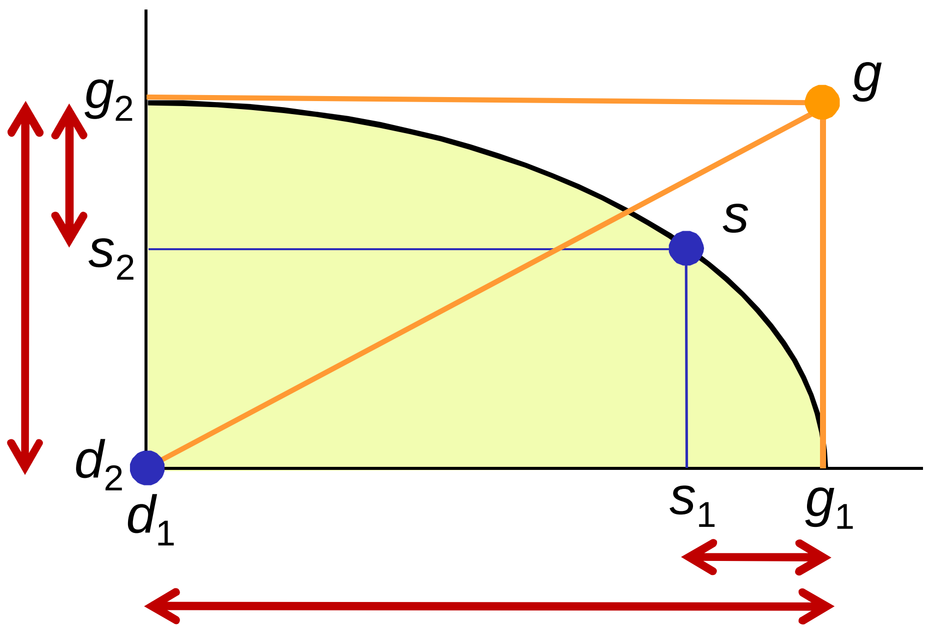
Bargaining Justification

Resistance to an agreement *s* depends on sacrifice relative to sacrifice under no agreement. Here, player 2 is making a larger relative sacrifice:

$$\frac{g_1 - s_1}{g_1 - d_1} \le \frac{g_2 - s_2}{g_2 - d_2}$$

Minimizing resistance to agreement requires minimizing

$$\max_i \left\{ \frac{g_i - s_i}{g_i - d_i} \right\}$$



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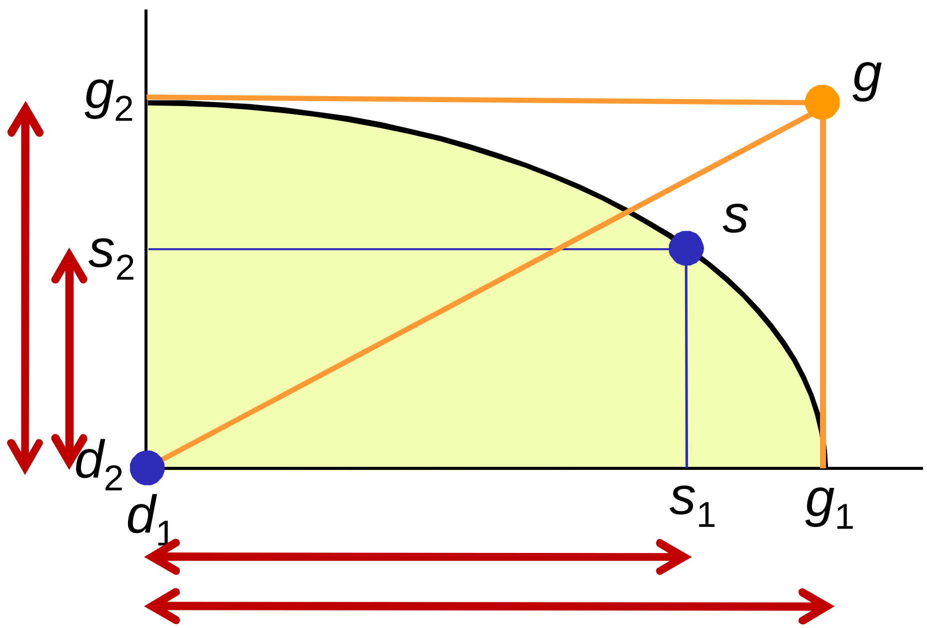


Minimizing resistance to agreement requires minimizing

$$\max_i \left\{ \frac{g_i - s_i}{g_i - d_i} \right\}$$

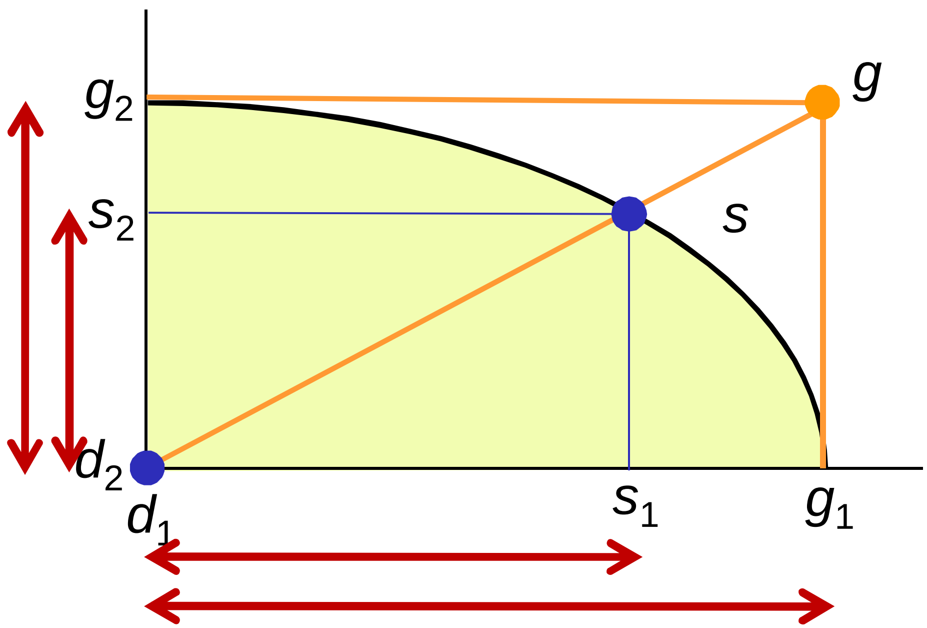
or equivalently, maximizing

$$\min_i \left\{ \frac{s_i - d_i}{g_i - d_i} \right\}$$

Bargaining Justification

Resistance to an agreement *s* depends on sacrifice relative to sacrifice under no agreement. Here, player 2 is making a larger relative sacrifice:

$$\frac{g_1 - s_1}{g_1 - d_1} \le \frac{g_2 - s_2}{g_2 - d_2}$$



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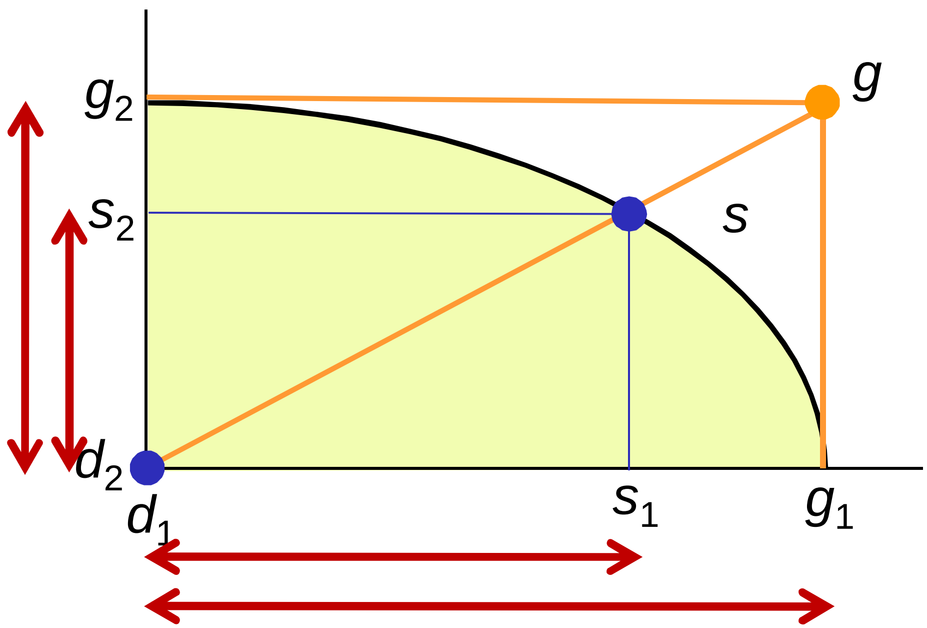
which is achieved by RKS point.

Bargaining Justification

This is the **Rawlsian social contract** argument applied to **gains relative to the ideal**.



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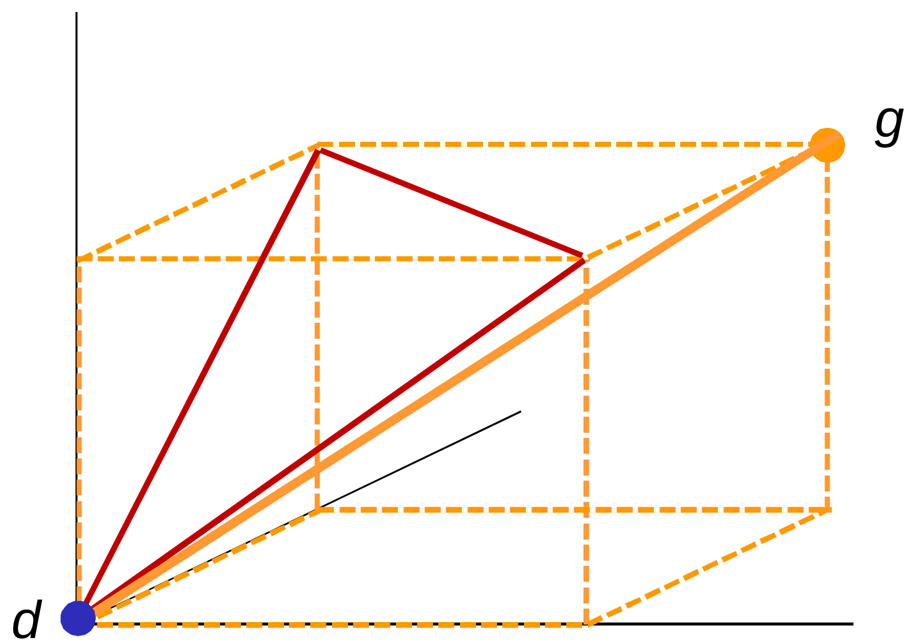
Problem with RKS Solutioon

- However, the RKS solution is Rawlsian only for **2 players**.
  - In fact, RKS leads to counterintuitive results for 3 players.



Red triangle is feasible set.

RKS point is *d* !

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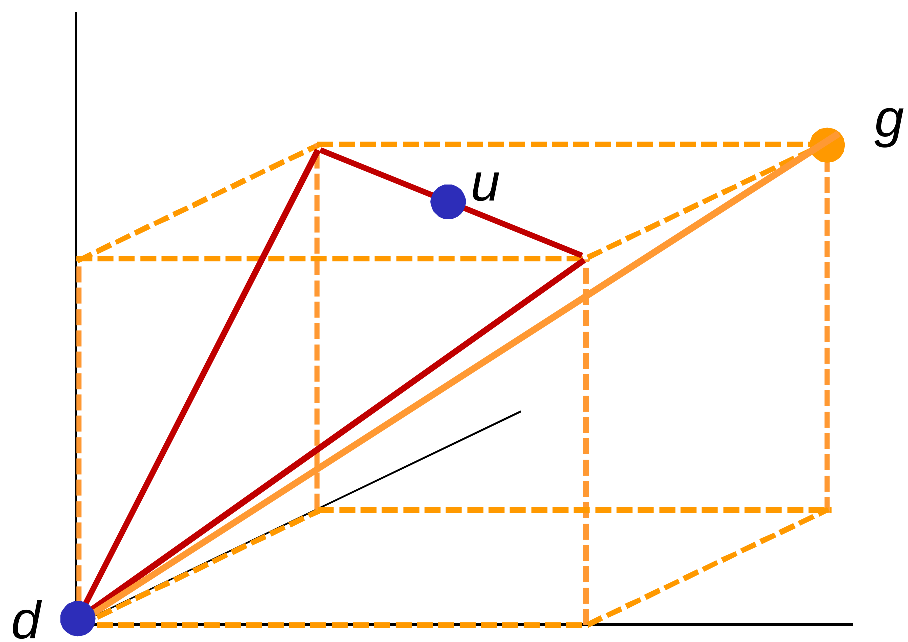
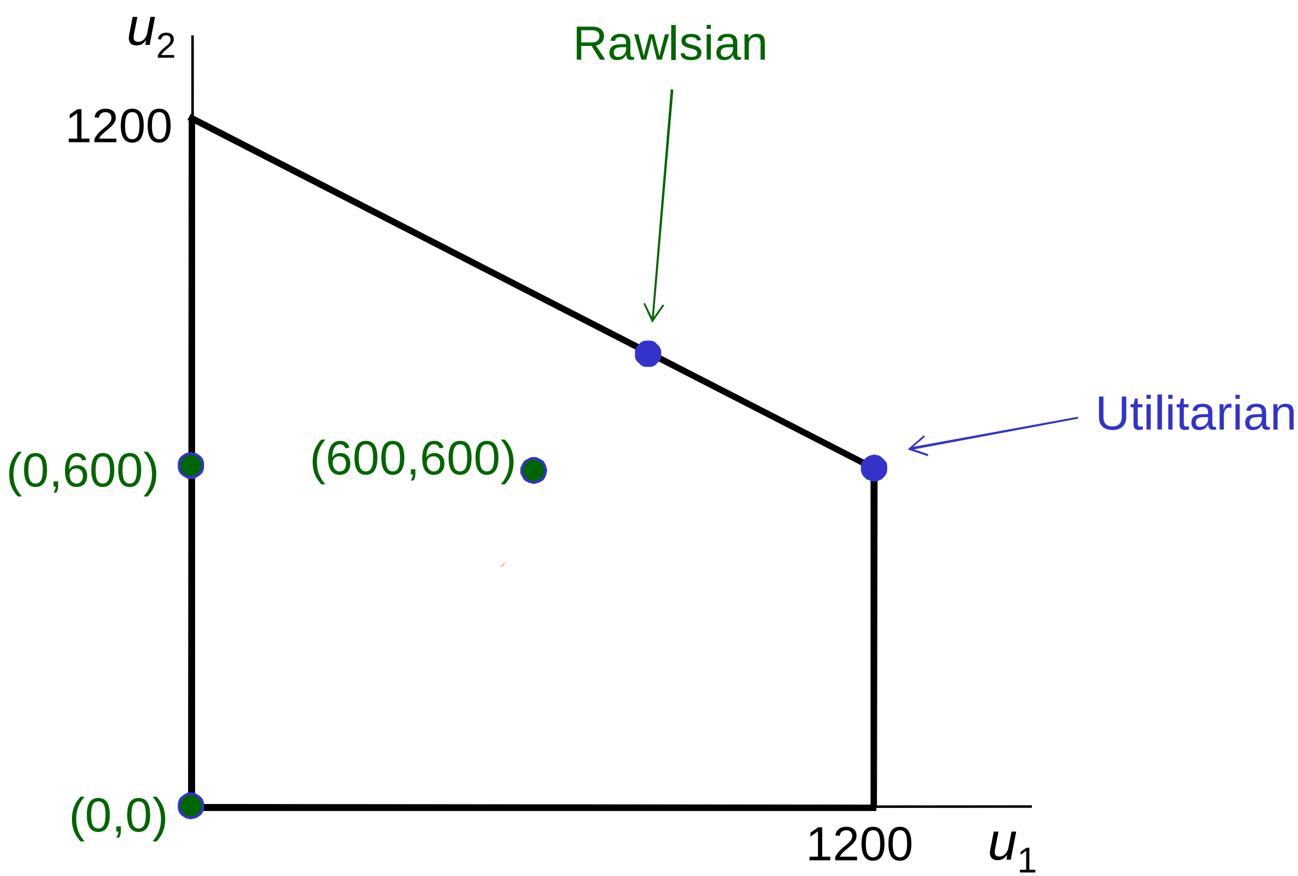


Red triangle is feasible set.

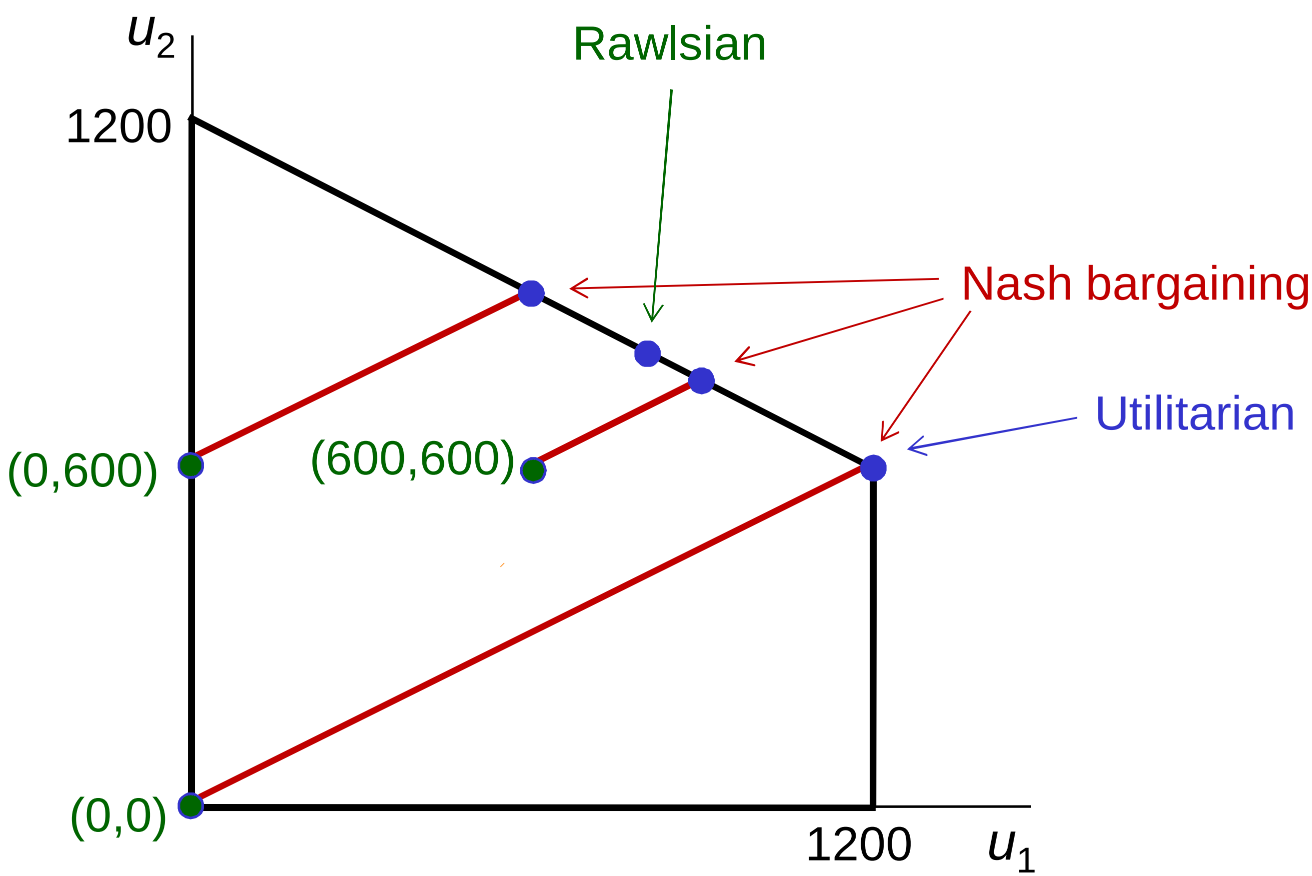
RKS point is *d* !

Rawlsian point is *u*.

Summary

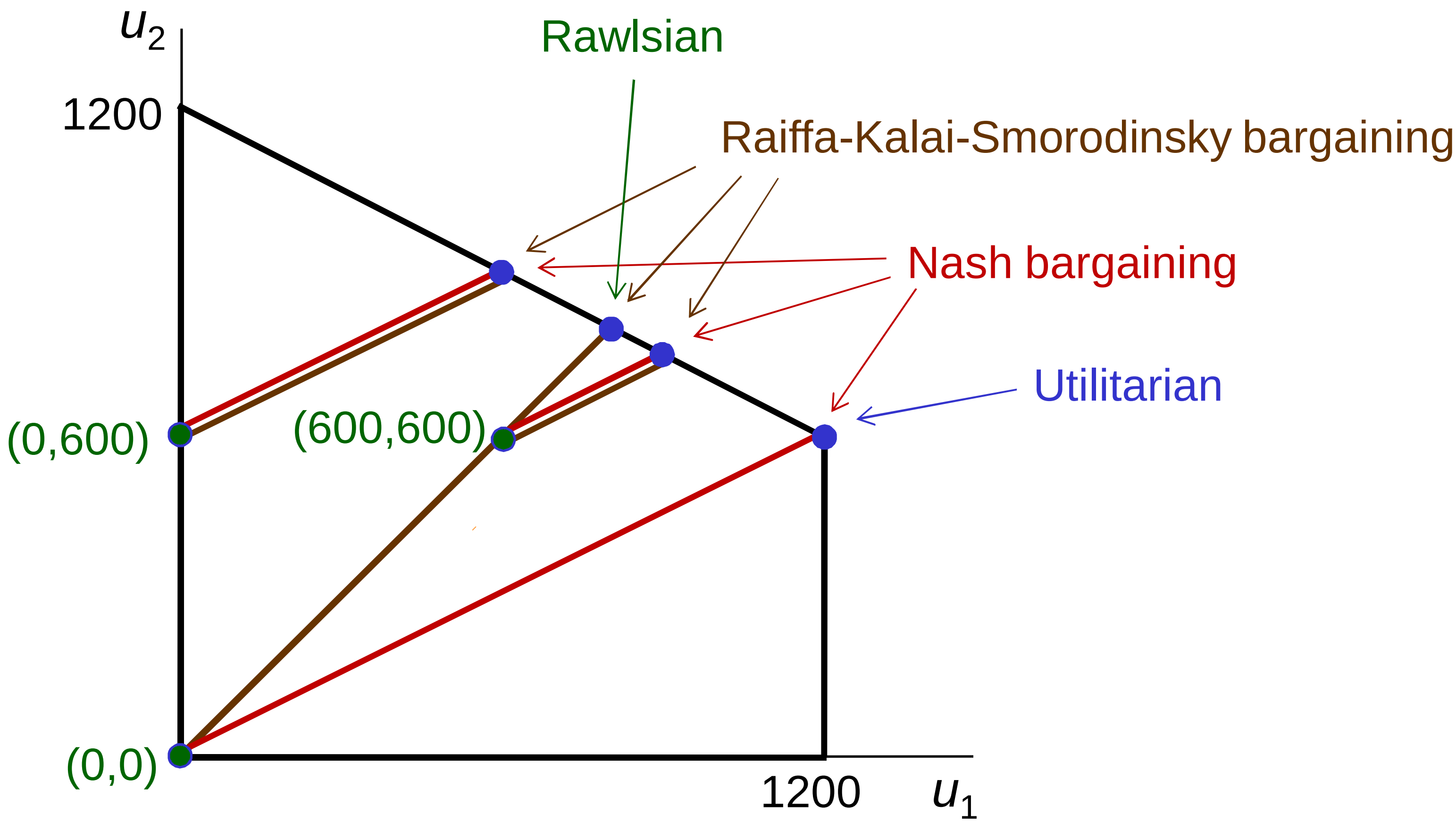


Summary



u_2
Rawlsian
1200
Nash bargaining
Utilitarian
(0,600)
(600,600)
(0,0)
1200
u_1

Summary



u_2
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(0,600)
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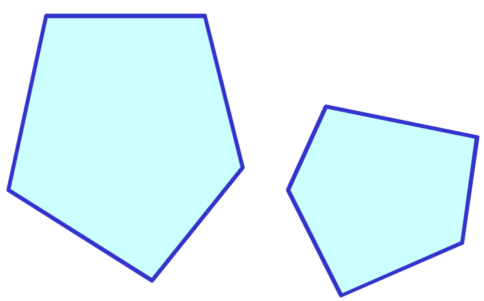
Mixed Integer Linear Modeling

- MILP modeling is basically **disjunctive modeling**.
- A problem has an MILP model if and only if it represents a **union of polyhedra** with the same recession cone.
- One can always write an MILP model by expressing the problem as a **disjunction of linear systems** that describe polyhedra with the same recession cone.
- In fact, one can write a **convex hull** (sharp) MILP model in this fashion.

Disjunctions of linear systems

A disjunction of linear systems represents a union of polyhedra.

$$\min \; cx$$
$$\bigvee_k \left(A^k x \geq b^k \right)$$

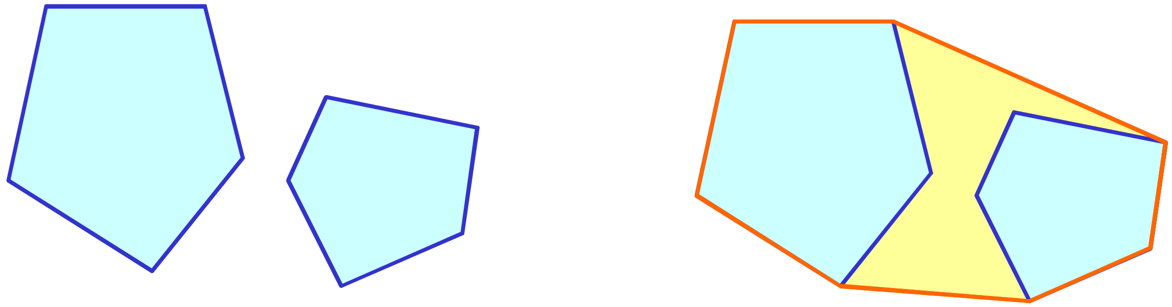


Disjunction of linear systems

A disjunction of linear systems represents a union of polyhedra.

We want a model with a convex hull relaxation (tightest linear relaxation).

$$\min \; cx$$
$$\bigvee_k \left(A^k x \geq b^k \right)$$



Disjunction of linear systems

The closure of the convex hull of

$$\min\ cx$$
$$\bigvee_k \left(A^k x \ge b^k \right)$$

…is described by

$$\begin{aligned} &\min\ cx \\ &A^k x^k \ge b^k y_k,\ \text{all } k \\ &\sum_k y_k = 1 \\ &x = \sum_k x^k \\ &0 \le y_k \le 1 \end{aligned}$$

Why?

To derive convex hull relaxation of a disjunction…

Write each solution as a convex combination of points in the polyhedron

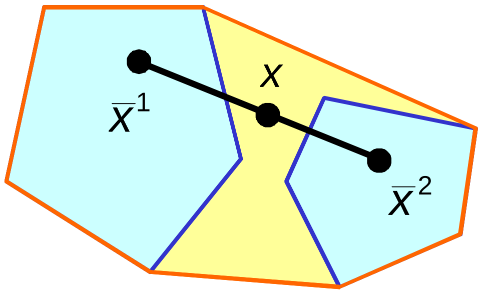
$$\min \; cx$$

$$A^k \bar{x}^k \geq b^k, \text{ all } k$$

$$\sum_k y_k = 1$$

$$x = \sum_k y_k \bar{x}^k$$

$$0 \leq y_k \leq 1$$



Convex hull relaxation
(tightest linear relaxation)

Why?

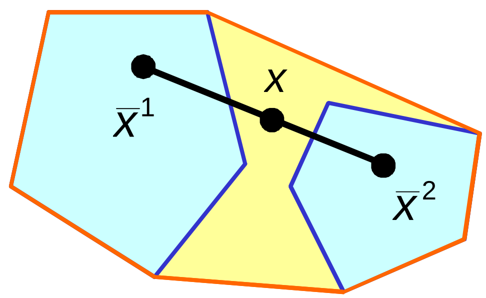
To derive convex hull relaxation of a disjunction…

Write each solution as a convex combination of points in the polyhedron

$$\begin{aligned} &\min\ cx \\ &A^k \bar{x}^k \geq b^k, \text{ all } k \\ &\sum_k y_k = 1 \\ &x = \sum_k y_k \bar{x}^k \\ &0 \leq y_k \leq 1 \end{aligned}$$

Change of variable $x = y_k \bar{x}^k$

$$\begin{aligned} &\min\ cx \\ &A^k x^k \geq b^k y_k, \text{ all } k \\ &\sum_k y_k = 1 \\ &x = \sum_k x^k \\ &0 \leq y_k \leq 1 \end{aligned}$$



Convex hull relaxation (tightest linear relaxation)

MILP Representability

A subset S of $\mathbb{R}^n$ is MILP representable if it is the projection onto x of some MILP constraint set of the form

$$Ax + Bu + Dy \geq b$$

$$x, y \geq 0$$

$$x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ y_k \in \{0,1\}$$

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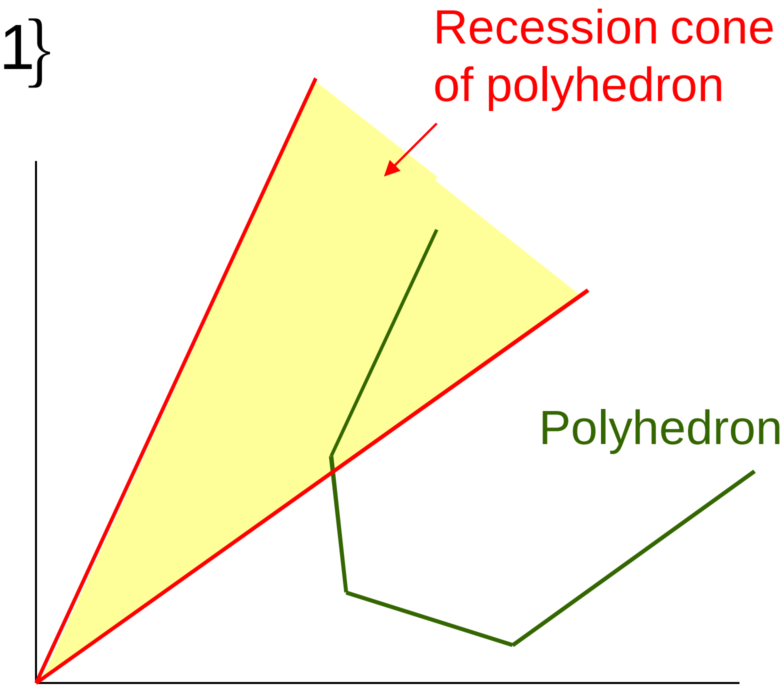
$$Ax + Bu + Dy \geq b$$
$$x, y \geq 0$$
$$x \in \mathbb{R}^n,\ \ u \in \mathbb{R}^m,\ \ y_k \in \{0,1\}$$

Theorem. $S \subset \mathbb{R}^n$ is MILP representable if and only if S is the union of finitely many polyhedra having the same recession cone.



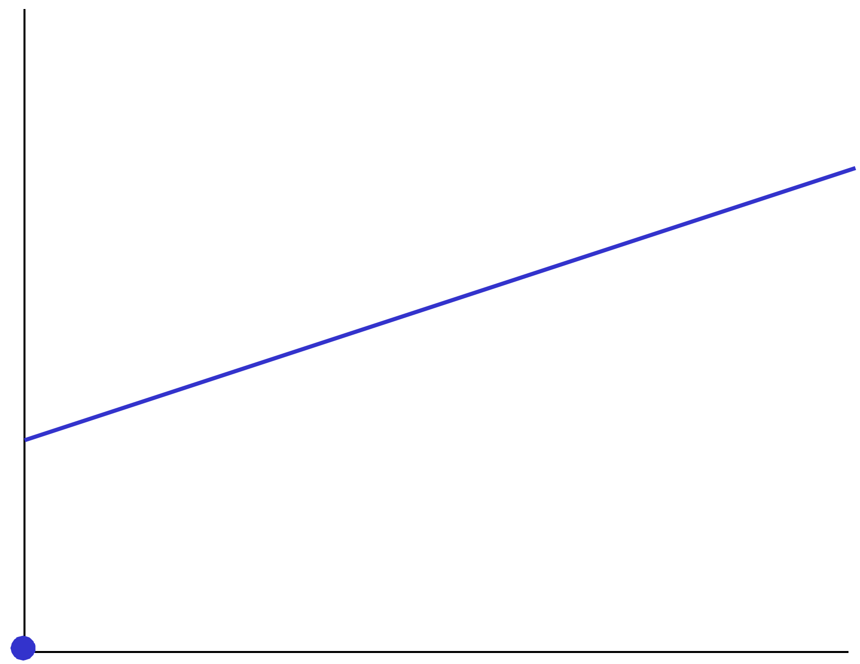
Example: Fixed charge function

Minimize a fixed charge function:

$$\min \; x_2$$

$$x_2 \geq \begin{Bmatrix} 0 & \text{if } x_1 = 0 \\ f + cx_1 & \text{if } x_1 > 0 \end{Bmatrix}$$

$$x_1 \geq 0$$



Example

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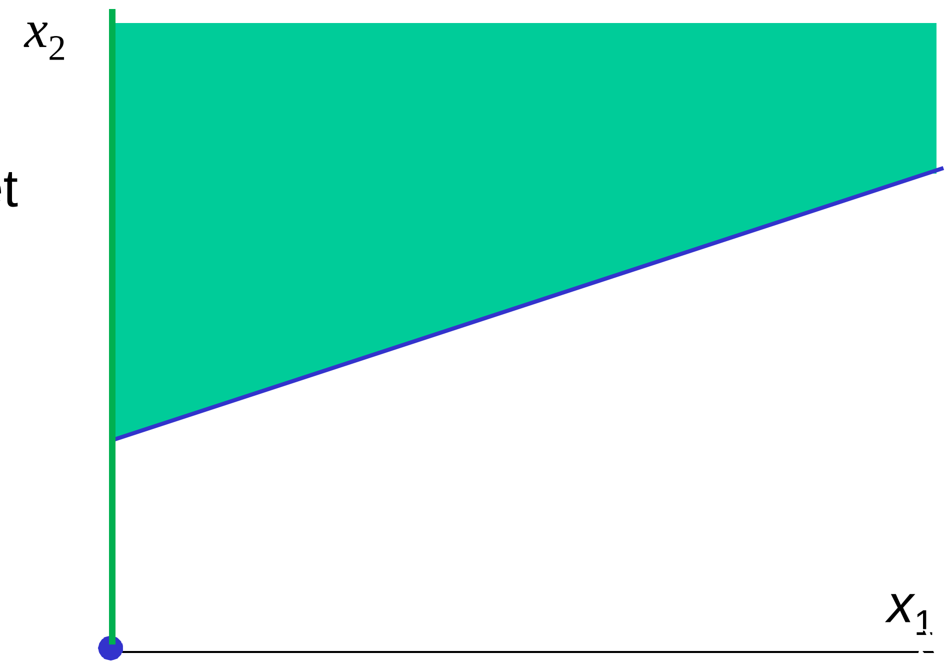
Feasible set



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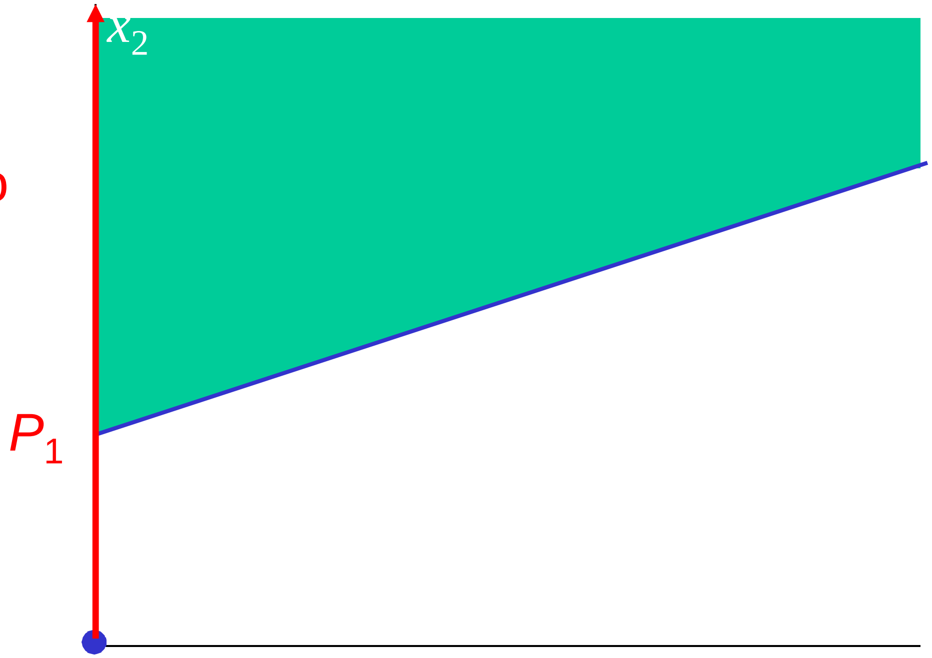
Union of two polyhedra P_1, P_2



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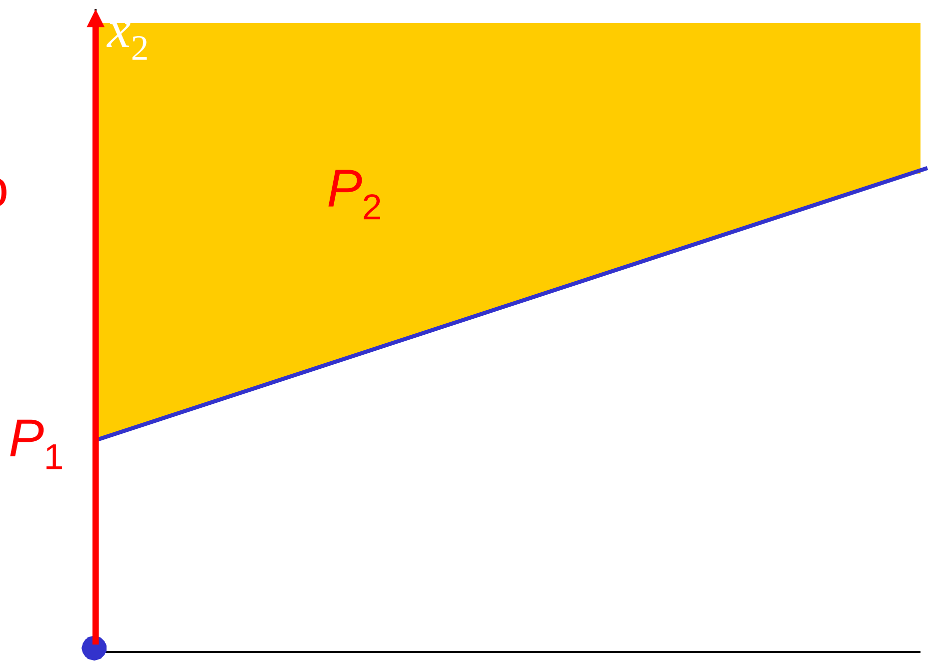
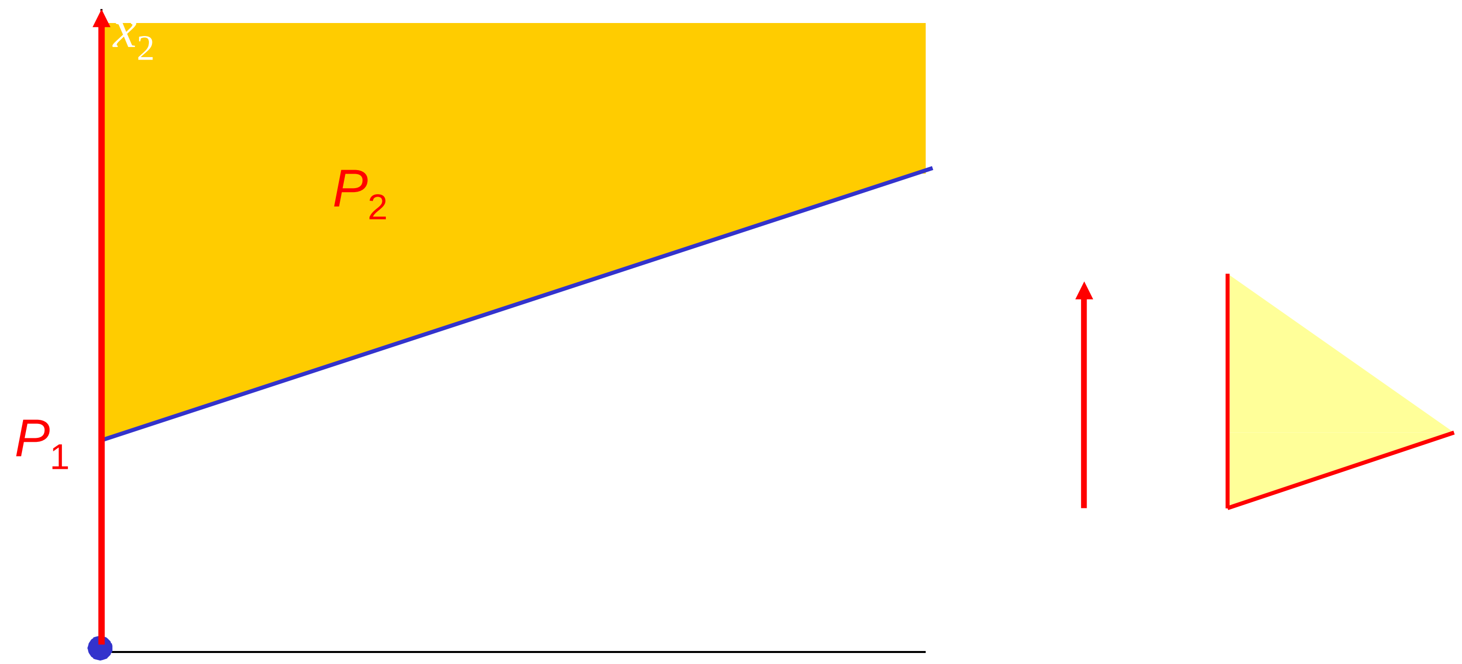


Example

Minimize a fixed charge function:

$$
\begin{aligned}
&\min \quad x_2 \\
&x_2 \geq \begin{Bmatrix} 0 & \text{if } x_1 = 0 \\ f + cx_1 & \text{if } x_1 > 0 \end{Bmatrix} \\
&x_1 \geq 0
\end{aligned}
$$

The polyhedra have different recession cones.



Example

Minimize a fixed charge function:

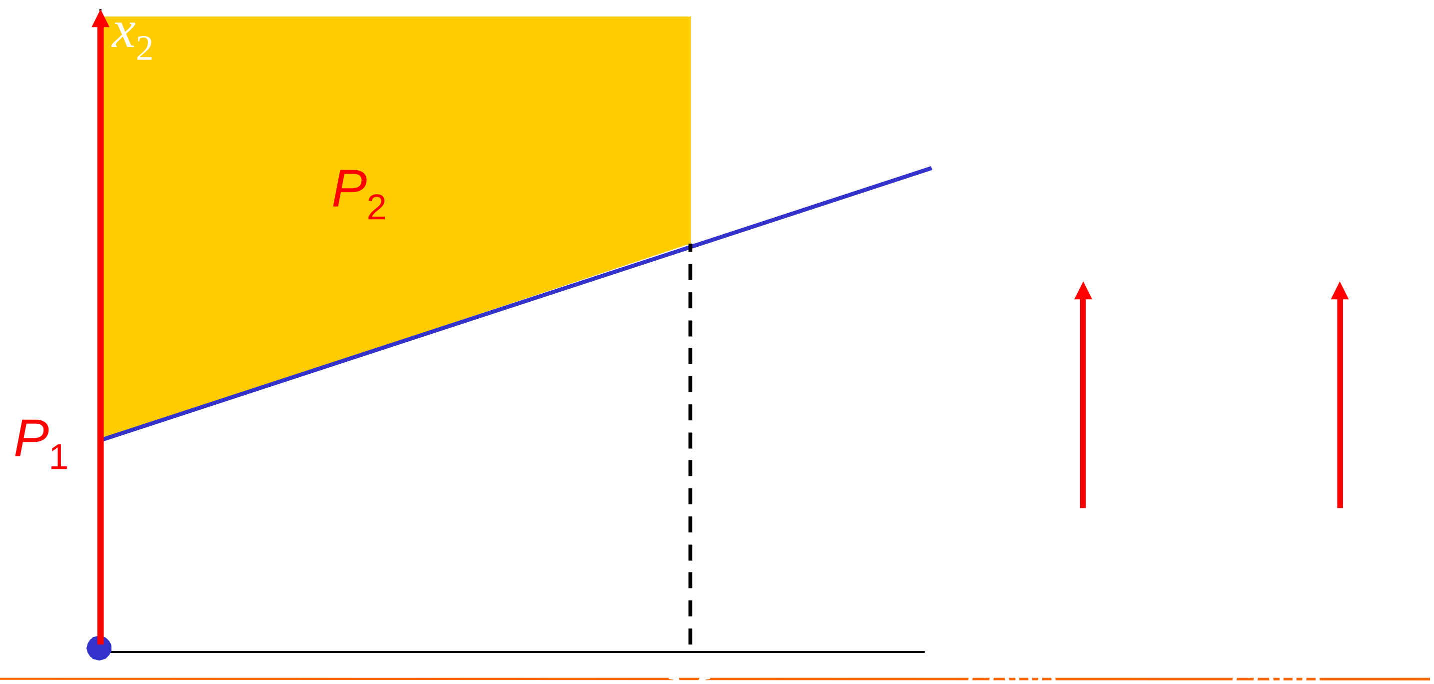
$$\min \; x_2$$

$$x_2 \geq \begin{cases} 0 & \text{if } x_1 = 0 \\ f + cx_1 & \text{if } x_1 > 0 \end{cases}$$

Add an upper bound on x_1

$$0 \leq x_1 \leq M$$

The polyhedra have the same recession cone.



Modeling a union of polyhedra

Start with a disjunction of linear systems to represent the union of polyhedra.

The kth polyhedron is $\{x \mid A^k x \geq b\}$

$$\min \; cx$$
$$\bigvee_k \left(A^k x \geq b^k \right)$$

Introduce a 0-1 variable y_k that is 1 when x is in polyhedron $\underline{k}$.

Disaggregate x to create an x^k for each k.

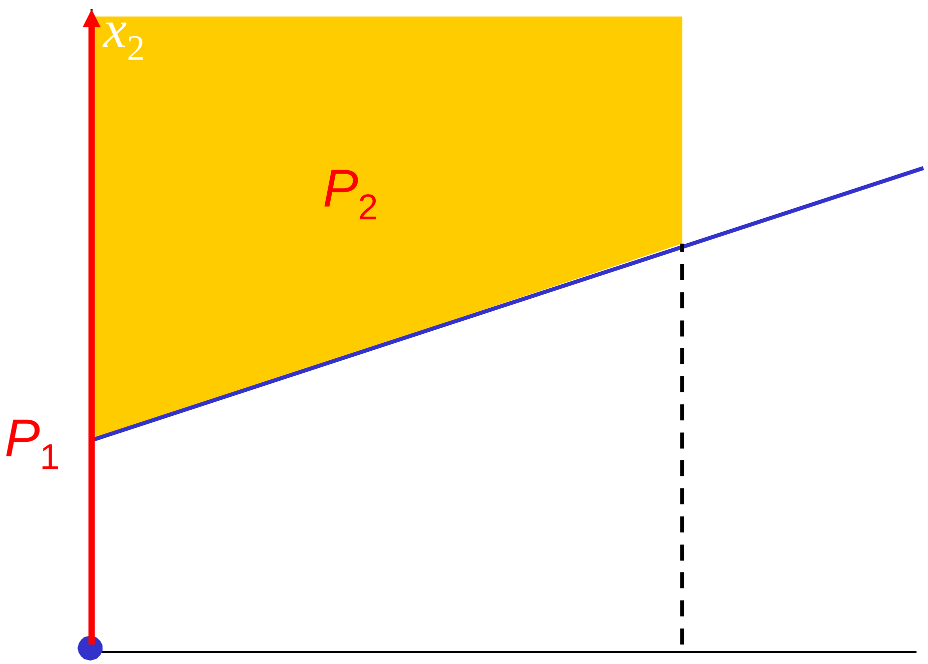
$$\min \; cx$$
$$A^k x^k \geq b^k y_k, \text{ all } k$$
$$\sum_k y_k = 1$$
$$x = \sum_k x^k$$
$$y_k \in \{0,1\}$$

Example

Start with a disjunction of linear systems to represent the union of polyhedra

$$\min \; x_2$$

$$\begin{pmatrix} x_1 = 0 \\ x_2 \geq 0 \end{pmatrix} \vee \begin{pmatrix} 0 \leq x_1 \leq M \\ x_2 \geq f + cx_1 \end{pmatrix}$$



Example

Start with a disjunction of linear systems to represent the union of polyhedra

$$\min \; x_2$$
$$\begin{pmatrix} x_1 = 0 \\ x_2 \ge 0 \end{pmatrix} \vee \begin{pmatrix} 0 \le x_1 \le M \\ x_2 \ge f + cx_1 \end{pmatrix}$$

Introduce a 0-1 variable y_k that is 1 when x is in polyhedron $\underline{k}$.

Disaggregate x to create an x^k for each k.

$$\min \; cx$$
$$x_1^1 = 0, \;\; x_2^1 \ge 0$$
$$0 \le x_1^2 \le My_2, \quad -cx_1^2 + x_2^2 \ge fy_2$$
$$y_1 + y_2 = 1, \;\; y_k \in \{0,1\}$$
$$x = x^1 + x^2$$

Example

To simplify:

Replace x_1^2 with x_1.

Replace x_2^2 with x_2.

Replace y_2 with y.

$$\min \ x_2$$
$$x_1^1 = 0, \ \ x_2^1 \geq 0$$
$$0 \leq x_1^2 \leq My_2, \quad -cx_1^2 + x_2^2 \geq fy_2$$
$$y_1 + y_2 = 1, \ \ y_k \in \{0,1\}$$
$$x = x^1 + x^2$$

This yields

$$\min \ x_2$$
$$0 \leq x_1 \leq My$$
$$x_2 \geq fy + cx_1$$
$$y \in \{0,1\}$$

or

$$\min \ fy + cx$$
$$0 \leq x \leq \boxed{M}y$$
$$y \in \{0,1\}$$

"Big *M* "

Combining Equity and Efficiency

- Utilitarian and Rawlsian distributions seem **too extreme** in practice.
  - How to combine them?

Combining Equity and Efficiency

- Utilitarian and Rawlsian distributions seem **too extreme** in practice.
  - How to combine them?

- **One proposal:**
  - Maximize welfare of **worst off** (Rawlsian)...
  - …until this requires **undue sacrifice** from others
  - Seems appropriate in **health care** allocation.
  - Joint work with H. P. Williams, to appear in *Management Science*.

Combining Equity and Efficiency

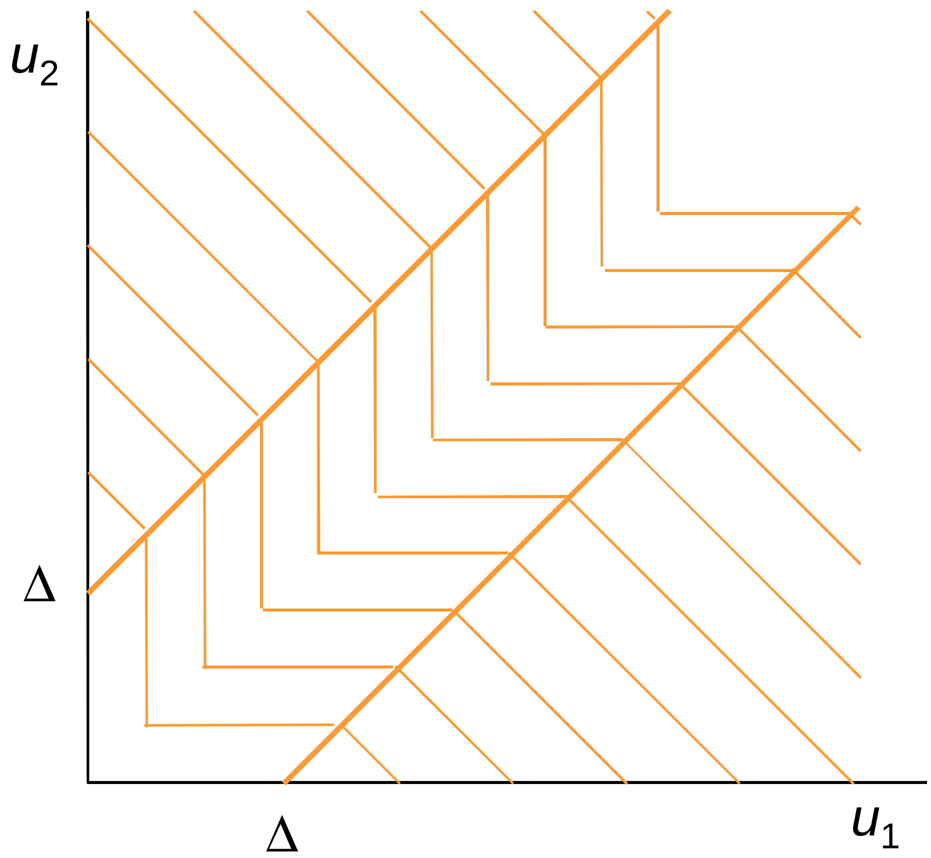
- In particular:
  - Switch from **Rawlsian** to **utilitarian** when **inequality** exceeds Δ.

Combining Equity and Efficiency

- In particular:
  - Switch from **Rawlsian** to **utilitarian** when **inequality** exceeds Δ.
  - Build mixed integer programming model.
  - Let u_i = utility allocated to person i
- For 2 persons:
  - Maximize $\min_i \{u_1, u_2\}$ (Rawlsian) when $|u_1 - u_2| \leq \Delta$
  - Maximize $u_1 + u_2$ (utilitarian) when $|u_1 - u_2| > \Delta$

Two-person Model

Contours of **social welfare function** for 2 persons.



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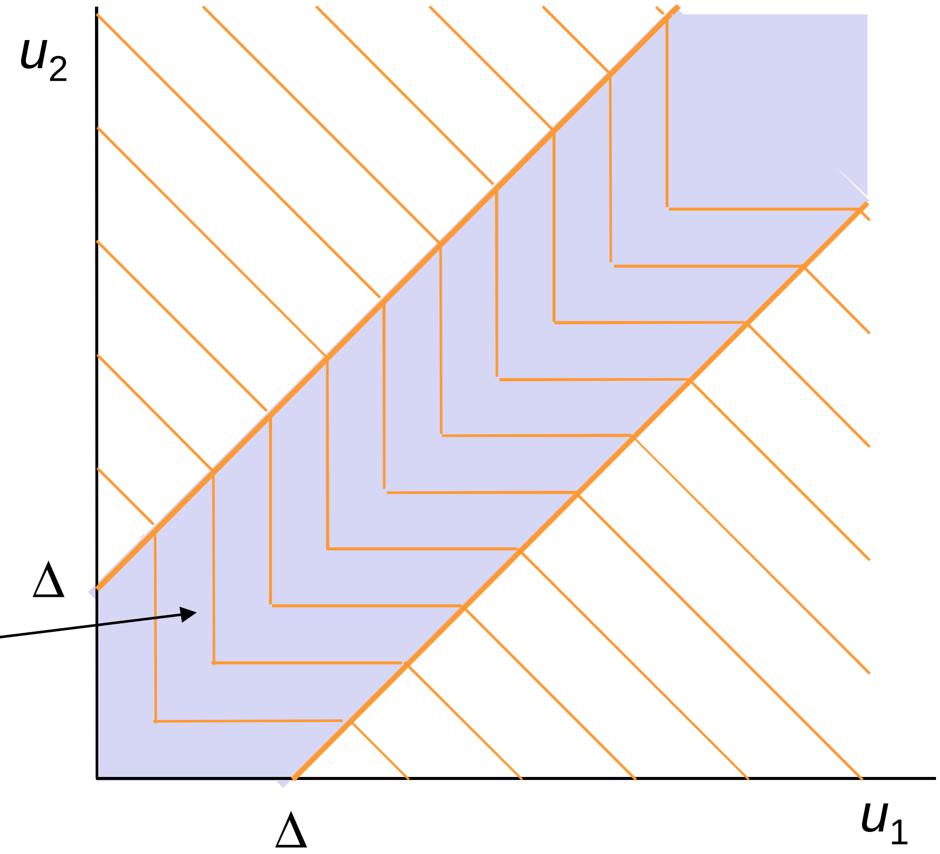


Rawlsian region

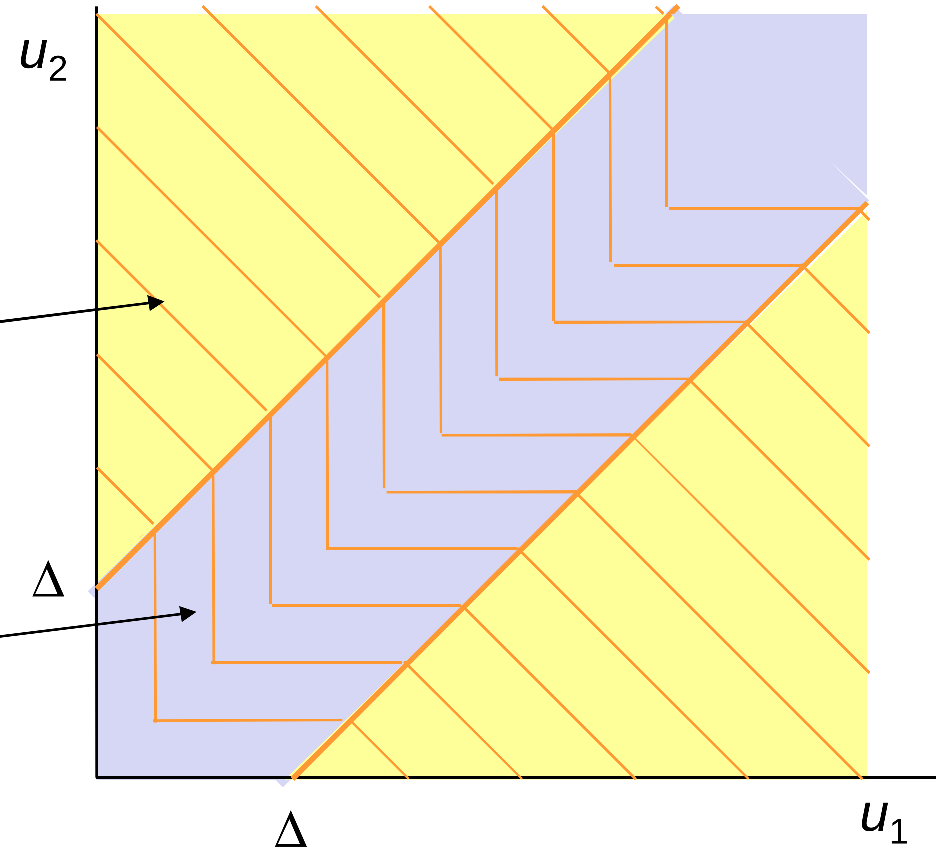
$\min\{u_1, u_2\}$

Two-person Model

Contours of **social welfare function** for 2 persons.

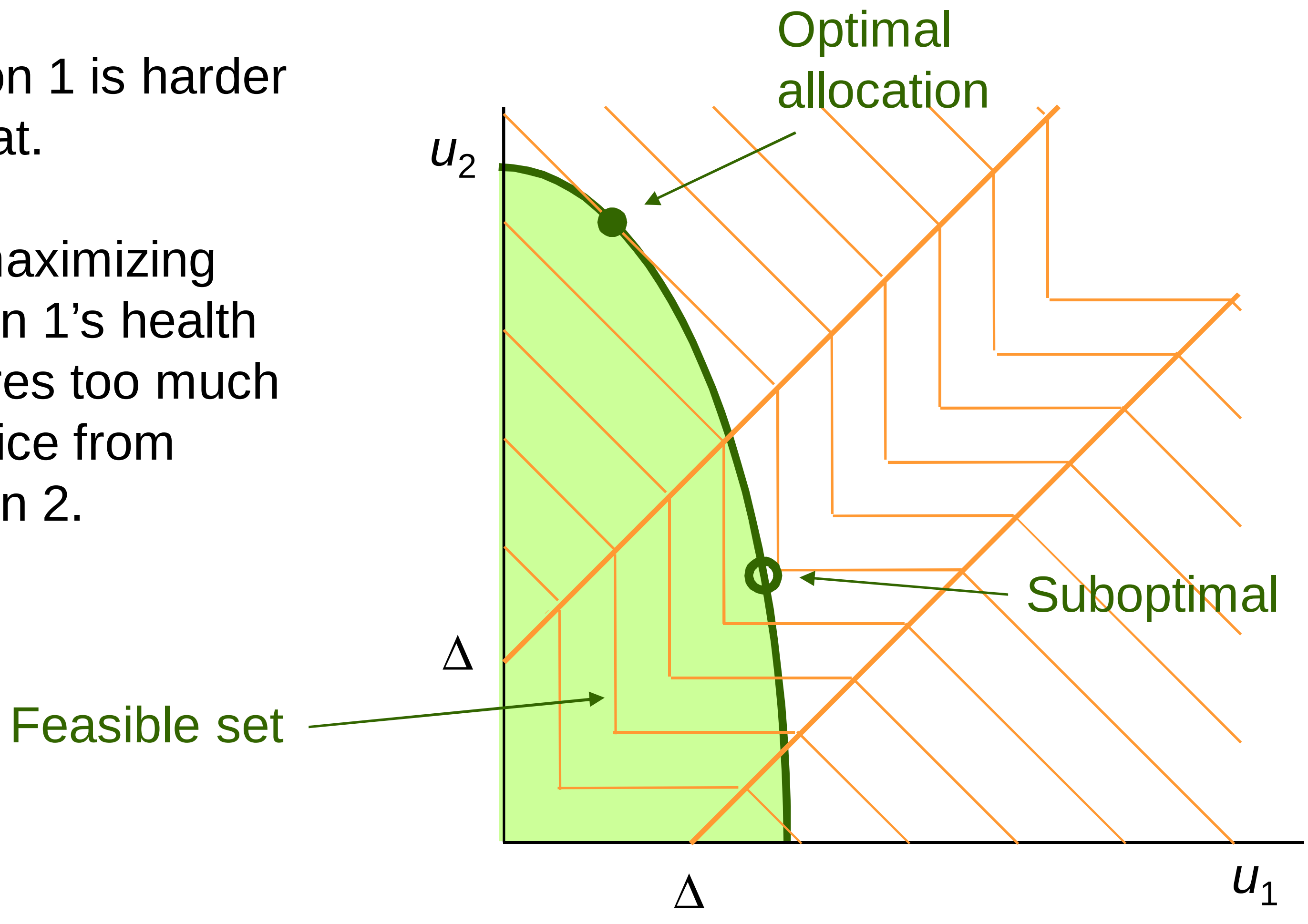
Utilitarian region
$u_1 + u_2$

Rawlsian region
$\min\{u_1, u_2\}$



Person 1 is harder to treat.

But maximizing person 1's health requires too much sacrifice from person 2.

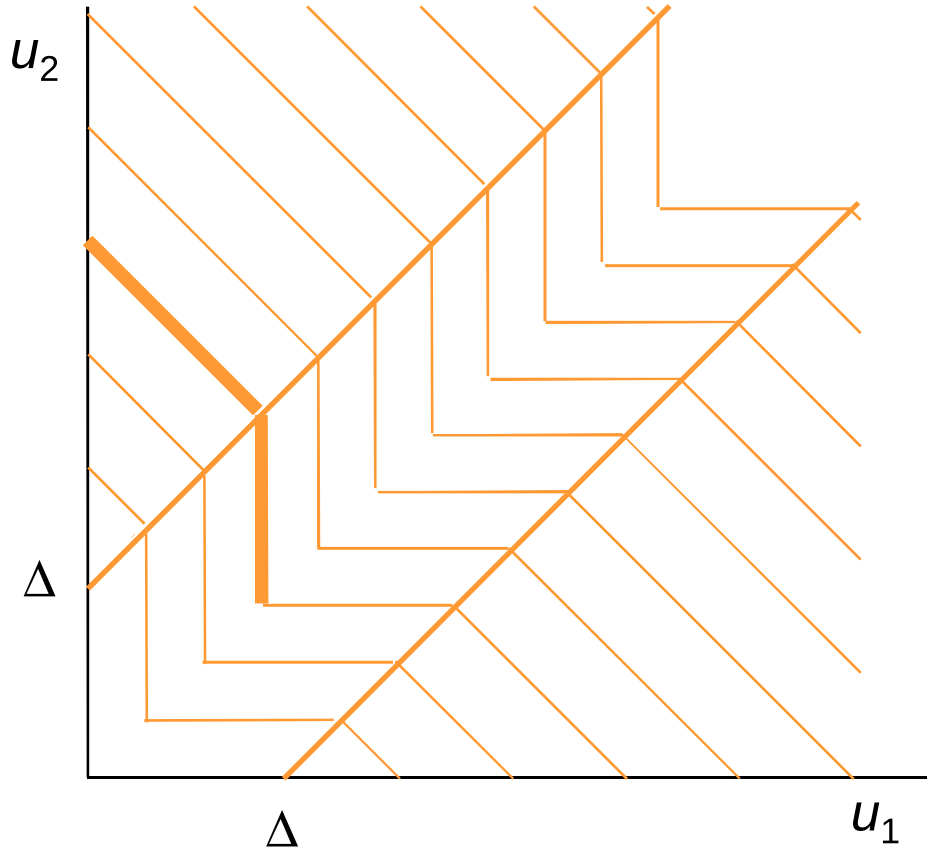


Advantages

- Only **one parameter** Δ
  - **Focus** for debate.
  - Δ has **intuitive meaning** (unlike weights)
  - Examine **consequences** of different settings for Δ
  - Find **least objectionable** setting
  - Results in a **consistent** policy

Social Welfare Function

We want continuous contours…

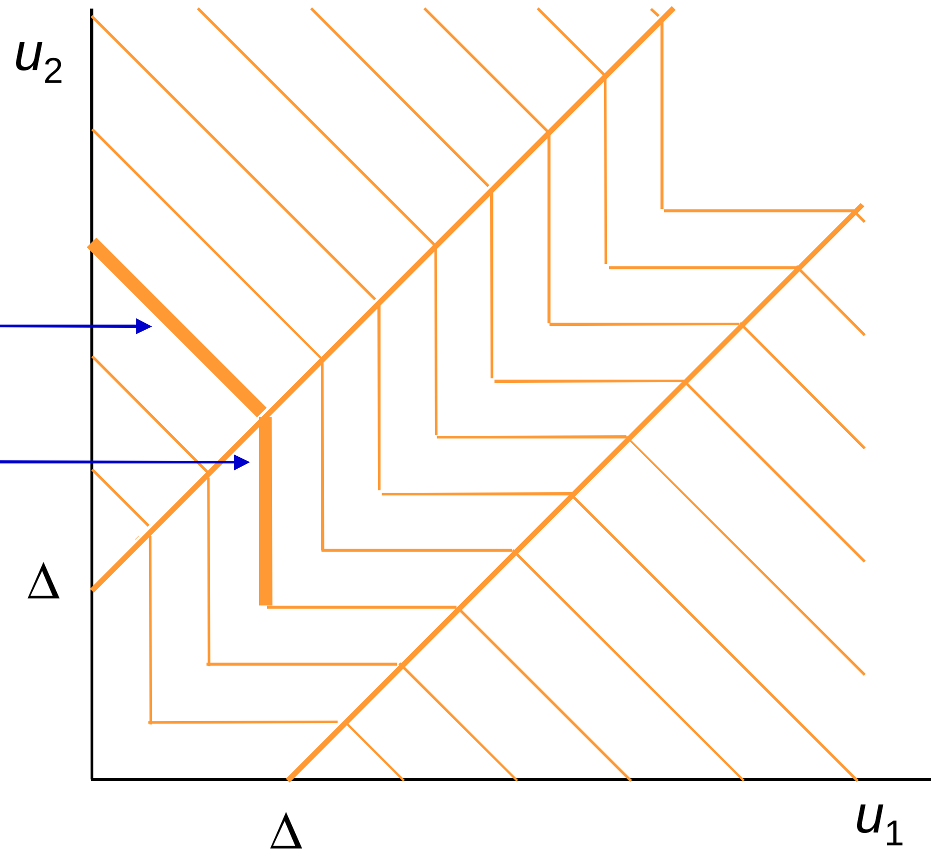


Social Welfare Function

We want continuous contours…

$u_1 + u_2$

$2\min\{u_1, u_2\} + \Delta$



So we use affine transform of Rawlsian criterion

Social Welfare Function

The social welfare problem becomes

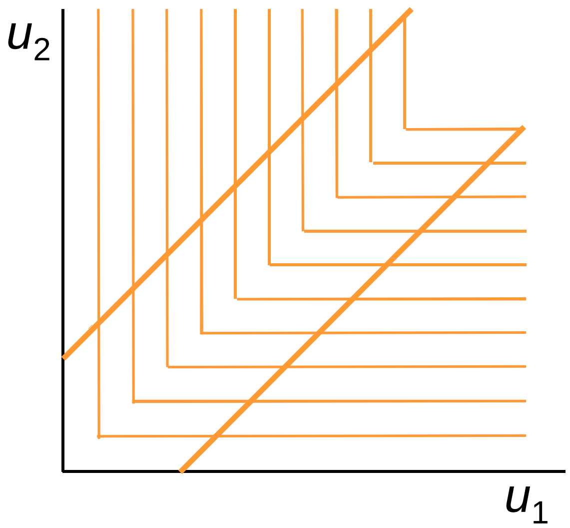
$$\max \ z$$

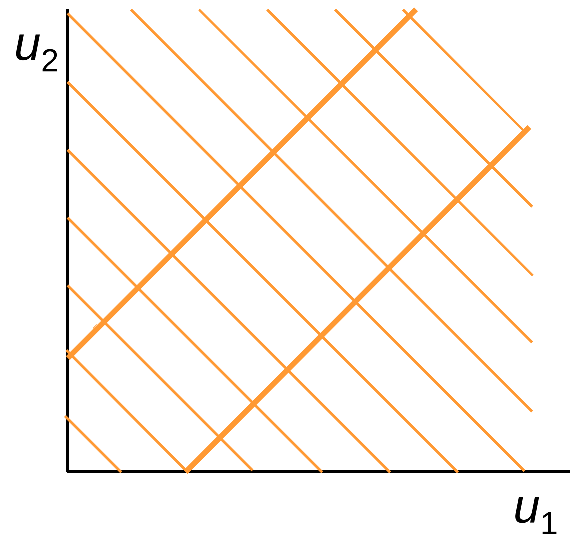
$$z \le \left\{ \begin{array}{ll} 2\min\{u_1, u_2\} + \Delta, & \text{if } \ |u_1 - u_2| \le \Delta \\ u_1 + u_2, & \text{otherwise} \end{array} \right\}$$

$$\text{constraints on feasible set}$$

MILP Model

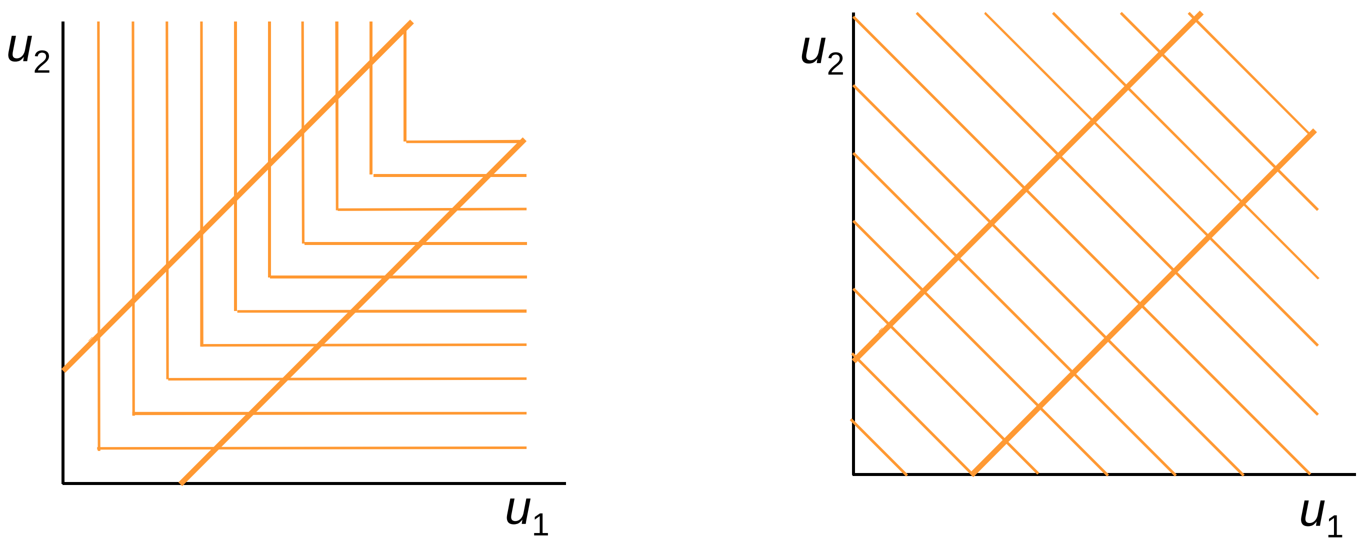
Epigraph is union of 2 polyhedra.





MILP Model

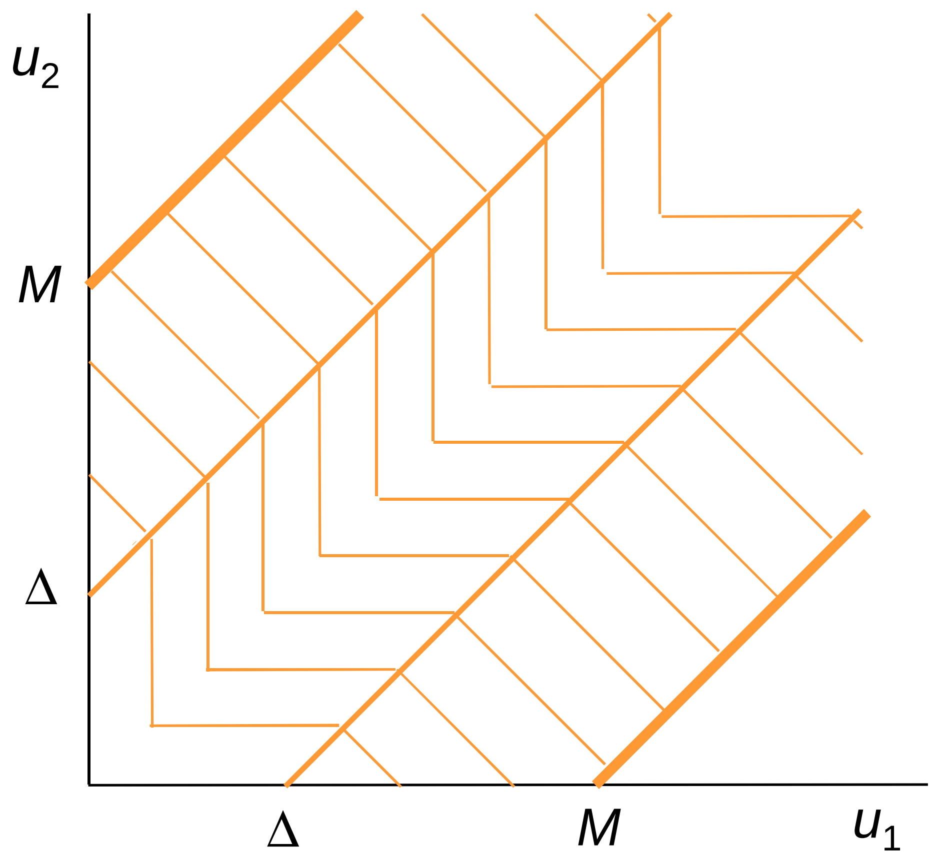
Epigraph is union of 2 polyhedra.
Because they have **different recession cones**, there is no MILP model.



Recession directions (u_1,u_2,z)

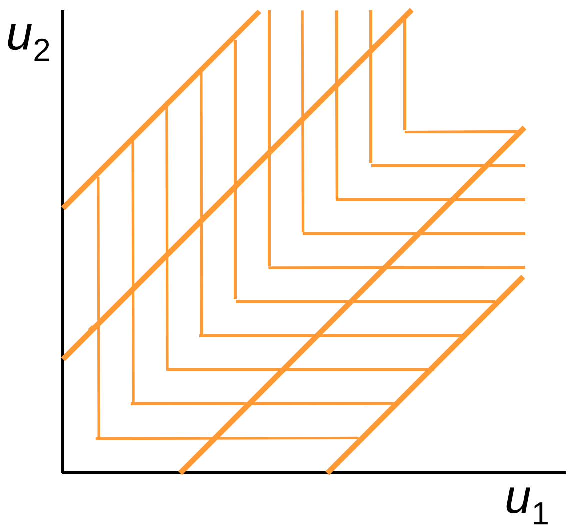
MILP Model

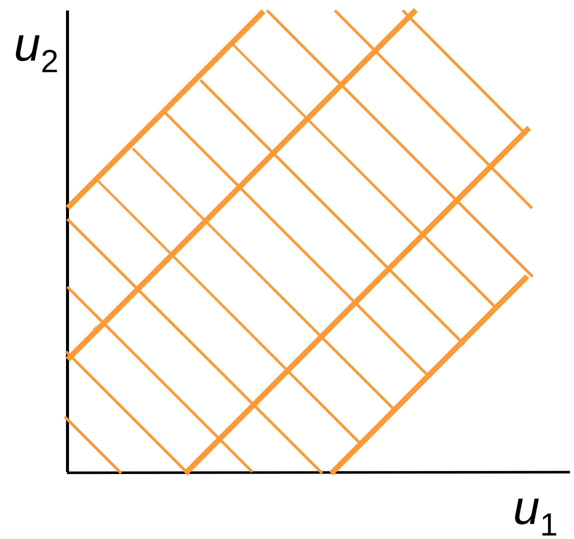
Impose constraints $|u_1 - u_2| \leq M$



MILP Model

This equalizes recession cones.





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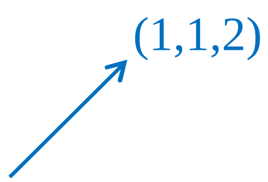

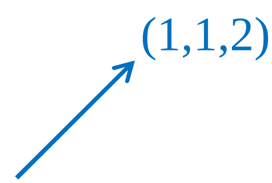

MILP Model

We have the model…

$$\max\ z$$

$$z \le 2u_i + \Delta + (M - \Delta)\delta, \quad i = 1,2$$

$$z \le u_1 + u_2 + \Delta(1 - \delta)$$

$$u_1 - u_2 \le M, \quad u_2 - u_1 \le M$$

$$u_1, u_2 \ge 0$$

$$\delta \in \{0,1\}$$

constraints on feasible set

u_1

MILP Model

We have the model…

$$
\begin{aligned}
&\max\ z \\
&z \le 2u_i + \Delta + (M-\Delta)\delta, \quad i = 1,2 \\
&z \le u_1 + u_2 + \Delta(1-\delta) \\
&u_1 - u_2 \le M, \quad u_2 - u_1 \le M \\
&u_1, u_2 \ge 0 \\
&\delta \in \{0,1\}
\end{aligned}
$$

u_1

This is a **convex hull** formulation.

n-person Model

Rewrite the 2-person social welfare function as…

$$\Delta + 2u_{\min} + \left(u_1 - u_{\min} - \Delta\right)^+ + \left(u_2 - u_{\min} - \Delta\right)^+$$

$\min\{u_1, u_2\}$

$\alpha^+ = \max\{0, \alpha\}$

n-person Model

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This can be generalized to *n* persons:

$$(n-1)\Delta + nu_{\min} + \sum_{j=1}^{n} \left(u_j - u_{\min} - \Delta\right)^+$$

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This can be generalized to *n* persons:

$$(n-1)\Delta + nu_{\min} + \sum_{j=1}^{n}\left(u_j - u_{\min} - \Delta\right)^+$$

Epigraph is a union of $n!$ polyhedra with same recession direction $(u,z) = (1,\ldots,1,n)$ if we require $|u_i - u_j| \le M$

So there is an MILP model…

n-person MILP Model

To avoid $n!$ 0-1 variables, add auxiliary variables w_{ij}

$$\max \; z$$

$$z \le u_i + \sum_{j \ne i} w_{ij}, \text{ all } i$$

$$w_{ij} \le \Delta + u_i + \delta_{ij}(M - \Delta), \text{ all } i, j \text{ with } i \ne j$$

$$w_{ij} \le u_j + (1 - \delta_{ij})\Delta, \text{ all } i, j \text{ with } i \ne j$$

$$u_i - u_j \le M, \text{ all } i, j$$

$$u_i \ge 0, \text{ all } i$$

$$\delta_{ij} \in \{0,1\}, \text{ all } i, j \text{ with } i \ne j$$

n-person MILP Model

To avoid $n!$ 0-1 variables, add auxiliary variables w_{ij}

$$
\begin{aligned}
&\max\ z \\
&z \le u_i + \sum_{j \ne i} w_{ij},\ \text{all } i \\
&w_{ij} \le \Delta + u_i + \delta_{ij}(M - \Delta),\ \text{all } i, j \text{ with } i \ne j \\
&w_{ij} \le u_j + (1 - \delta_{ij})\Delta,\ \text{all } i, j \text{ with } i \ne j \\
&u_i - u_j \le M,\ \text{all } i, j \\
&u_i \ge 0,\ \text{all } i \\
&\delta_{ij} \in \{0,1\},\ \text{all } i, j \text{ with } i \ne j
\end{aligned}
$$

u_1

Theorem. The model is correct (not easy to prove).

n-person MILP Model

To avoid $n!$ 0-1 variables, add auxiliary variables w_{ij}

$$
\begin{aligned}
&\max\ z \\
&z \le u_i + \sum_{j \ne i} w_{ij},\ \text{all } i \\
&w_{ij} \le \Delta + u_i + \delta_{ij}(M - \Delta),\ \text{all } i, j \text{ with } i \ne j \\
&w_{ij} \le u_j + (1 - \delta_{ij})\Delta,\ \text{all } i, j \text{ with } i \ne j \\
&u_i - u_j \le M,\ \text{all } i, j \\
&u_i \ge 0,\ \text{all } i \\
&\delta_{ij} \in \{0,1\},\ \text{all } i, j \text{ with } i \ne j
\end{aligned}
$$

u_1

Theorem. The model is correct (not easy to prove).

Theorem. This is a convex hull formulation (not easy to prove).

n-group Model

In practice, funds may be allocated to groups of different sizes

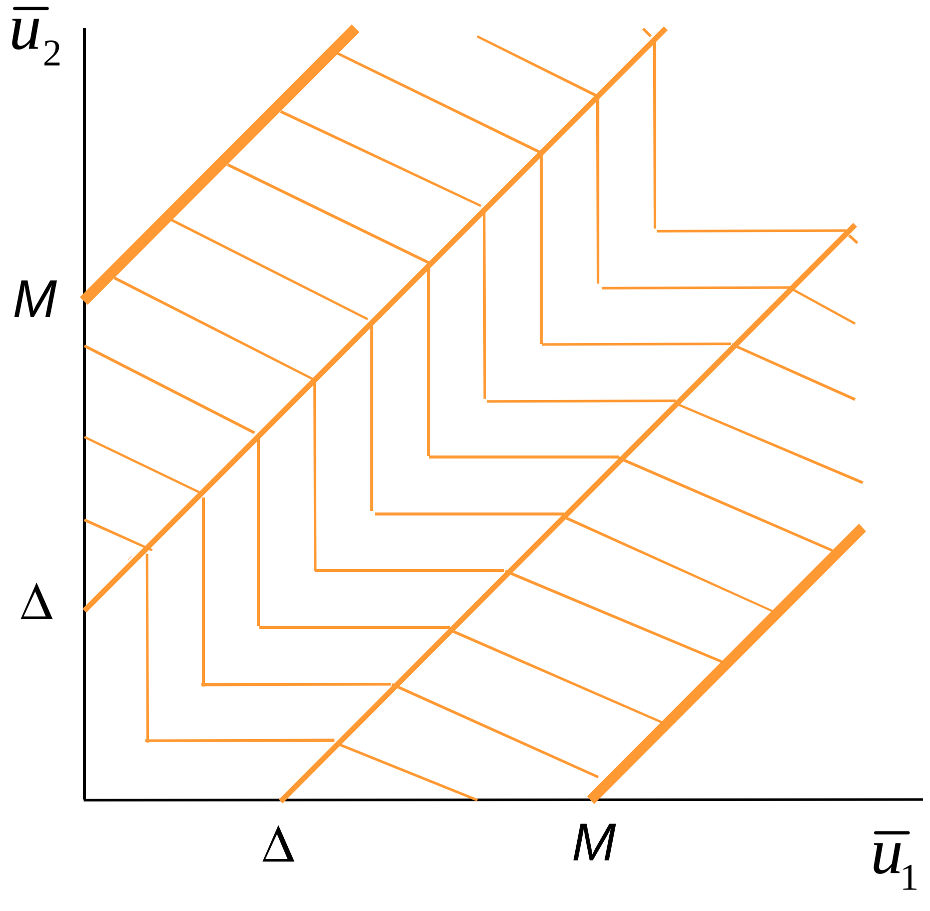
For example, disease/treatment categories.

Let $\bar{u}_i$ = average utility gained by a person in group *i*

n_i = size of group *i*

n-group Model

2-person case with $n_1 < n_2$. Contours have slope $-n_1/n_2$



n-group MILP Model

Again add auxiliary variables w_{ij}

$$\max\ z$$

$$z \le (n_i - 1)\Delta + n_i \bar{u}_i + \sum_{j \ne i} w_{ij},\ \text{all } i$$

$$w_{ij} \le n_j(\bar{u}_i + \Delta) + \delta_{ij} n_j (M - \Delta),\ \text{all } i, j \text{ with } i \ne j$$

$$w_{ij} \le \bar{u}_j + (1 - \delta_{ij}) n_j \Delta,\ \text{all } i, j \text{ with } i \ne j$$

$$\bar{u}_i - \bar{u}_j \le M,\ \text{all } i, j$$

$$\bar{u}_i \ge 0,\ \text{all } i$$

$$\delta_{ij} \in \{0,1\},\ \text{all } i, j \text{ with } i \ne j$$

u_1

Theorem. The model is correct.

Theorem. This is a convex hull formulation.

Health Example

Measure utility in QALYs (quality-adjusted life years).

QALY and cost data based on Briggs & Gray, (2000) etc.

Each group is a disease/treatment pair.

Treatments are discrete, so group funding is all-or-nothing.

Divide groups into relatively homogeneous subgroups.

u_1

Health Example

Add constraints to define feasible set…

$$\max\ z$$

$$z \le (n_i - 1)\Delta + n_i \bar{u}_i + \sum_{j \ne i} w_{ij},\ \text{all } i$$

$$w_{ij} \le n_j(\bar{u}_i + \Delta) + \delta_{ij} n_j (M - \Delta),\ \text{all } i, j \text{ with } i \ne j$$

$$w_{ij} \le \bar{u}_j + (1 - \delta_{ij}) n_j \Delta,\ \text{all } i, j \text{ with } i \ne j$$

$$\bar{u}_i - \bar{u}_j \le M,\ \text{all } i, j$$

$$\bar{u}_i \ge 0,\ \text{all } i$$

$$\delta_{ij} \in \{0,1\},\ \text{all } i, j \text{ with } i \ne j$$

$$\boxed{\begin{array}{l} \bar{u}_i = q_i y_i + \alpha_i \\ \sum_i n_i c_i y_i \le \text{budget} \\ y_i \in \{0,1\},\ \text{all } i \end{array}}$$

u_1

y_i indicates whether group i is funded

QALY & cost data

Part 1

Intervention	Cost per person c_i (£)	QALYs gained q_i	Cost per QALY (£)	QALYs without intervention α_i	Subgroup size n_i
Pacemaker for atrioventricular heart block					
Subgroup A	3500	3	1167	13	35
Subgroup B	3500	5	700	10	45
Subgroup C	3500	10	350	5	35
Hip replacement					
Subgroup A	3000	2	1500	3	45
Subgroup B	3000	4	750	4	45
Subgroup C	3000	8	375	5	45
Valve replacement for aortic stenosis					
Subgroup A	4500	3	1500	2.5	20
Subgroup B	4500	5	900	3	20
Subgroup C	4500	10	450	3.5	20
CABG[1] for left main disease					
Mild angina	3000	1.25	2400	4.75	50
Moderate angina	3000	2.25	1333	3.75	55
Severe angina	3000	2.75	1091	3.25	60
CABG for triple vessel disease					
Mild angina	3000	0.5	6000	5.5	50
Moderate angina	3000	1.25	2400	4.75	55
Severe angina	3000	2.25	1333	3.75	60
CABG for double vessel disease					
Mild angina	3000	0.25	12,000	5.75	60
Moderate angina	3000	0.75	4000	5.25	65
Severe angina	3000	1.25	2400	4.75	70

QALY & cost data

Part 2

Intervention	Cost per person c_i (£)	QALYs gained q_i	Cost per QALY (£)	QALYs without intervention α_i	Subgroup size n_i
	22,500	4.5	5000	1.1	2
Kidney transplant					
Subgroup A	15,000	4	3750	1	8
Subgroup B	15,000	6	2500	1	8
Kidney dialysis					
Less than 1 year survival					
Subgroup A	5000	0.1	50,000	0.3	8
1-2 years survival					
Subgroup B	12,000	0.4	30,000	0.6	6
2-5 years survival					
Subgroup C	20,000	1.2	16,667	0.5	4
Subgroup D	28,000	1.7	16,471	0.7	4
Subgroup E	36,000	2.3	15,652	0.8	4
5-10 years survival					
Subgroup F	46,000	3.3	13,939	0.6	3
Subgroup G	56,000	3.9	14,359	0.8	2
Subgroup H	66,000	4.7	14,043	0.9	2
Subgroup I	77,000	5.4	14,259	1.1	2
At least 10 years survival					
Subgroup J	88,000	6.5	13,538	0.9	2
Subgroup K	100,000	7.4	13,514	1.0	1
Subgroup L	111,000	8.2	13,537	1.2	1

Results

Total budget £3 million

Δ	Pace-	Hip	Aortic	CABG			Heart	Kidney	Kidney dialysis				
range	maker	repl.	valve	L	3	2	trans.	trans.	< 1	1-2	2-5	5-10	> 10
0–3.3	111	111	111	111	111	111	1	11	0	0	000	0000	000
3.4–4.0	111	111	111	111	111	111	0	11	1	0	000	0000	000
4.0–4.4	111	111	111	111	111	111	0	01	1	0	000	0000	001
4.5–5.01	111	011	111	111	111	111	1	01	1	0	000	0000	011
5.02–5.55	111	011	011	111	111	111	0	01	1	0	000	0001	011
5.56–5.58	111	011	011	111	111	011	0	01	1	0	000	0001	111
5.59	111	011	011	110	111	111	0	01	1	0	000	0001	111
5.60–13.1	111	111	111	101	000	000	1	11	1	0	111	1111	111
13.2–14.2	111	011	111	011	000	000	1	11	1	1	111	1111	111
14.3–15.4	111	111	111	011	000	000	1	11	1	1	101	1111	111
15.5–up	111	011	111	011	001	000	1	11	1	0	011	1111	111

Results

Utilitarian solution

Δ range	Pace-maker	Hip repl.	Aortic valve	CABG L	CABG 3	CABG 2	Heart trans.	Kidney trans.	Kidney dialysis < 1	1-2	2-5	5-10	> 10
0–3.3	111	111	111	111	111	111	1	11	0	0	000	0000	000
3.4–4.0	111	111	111	111	111	111	0	11	1	0	000	0000	000
4.0–4.4	111	111	111	111	111	111	0	01	1	0	000	0000	001
4.5–5.01	111	011	111	111	111	111	1	01	1	0	000	0000	011
5.02–5.55	111	011	011	111	111	111	0	01	1	0	000	0001	011
5.56–5.58	111	011	011	111	111	011	0	01	1	0	000	0001	111
5.59	111	011	011	110	111	111	0	01	1	0	000	0001	111
5.60–13.1	111	111	111	101	000	000	1	11	1	0	111	1111	111
13.2–14.2	111	011	111	011	000	000	1	11	1	1	111	1111	111
14.3–15.4	111	111	111	011	000	000	1	11	1	1	101	1111	111
15.5–up	111	011	111	011	001	000	1	11	1	0	011	1111	111

Results

Rawlsian solution

Δ range	Pace-maker	Hip repl.	Aortic valve	CABG L	CABG 3	CABG 2	Heart trans.	Kidney trans.	Kidney dialysis < 1	1-2	2-5	5-10	> 10
0–3.3	111	111	111	111	111	111	1	11	0	0	000	0000	000
3.4–4.0	111	111	111	111	111	111	0	11	1	0	000	0000	000
4.0–4.4	111	111	111	111	111	111	0	01	1	0	000	0000	001
4.5–5.01	111	011	111	111	111	111	1	01	1	0	000	0000	011
5.02–5.55	111	011	011	111	111	111	0	01	1	0	000	0001	011
5.56–5.58	111	011	011	111	111	011	0	01	1	0	000	0001	111
5.59	111	011	011	110	111	111	0	01	1	0	000	0001	111
5.60–13.1	111	111	111	101	000	000	1	11	1	0	111	1111	111
13.2–14.2	111	011	111	011	000	000	1	11	1	1	111	1111	111
14.3–15.4	111	111	111	011	000	000	1	11	1	1	101	1111	111
15.5–up	111	011	111	011	001	000	1	11	1	0	011	1111	111

Results

Fund for all Δ

Δ range	Pace-maker	Hip repl.	Aortic valve	CABG L	CABG 3	CABG 2	Heart trans.	Kidney trans.	Kidney dialysis < 1	Kidney dialysis 1-2	Kidney dialysis 2-5	Kidney dialysis 5-10	Kidney dialysis > 10
0–3.3	111	111	111	111	111	111	1	11	0	0	000	0000	000
3.4–4.0	111	111	111	111	111	111	0	11	1	0	000	0000	000
4.0–4.4	111	111	111	111	111	111	0	01	1	0	000	0000	001
4.5–5.01	111	011	111	111	111	111	1	01	1	0	000	0000	011
5.02–5.55	111	011	011	111	111	111	0	01	1	0	000	0001	011
5.56–5.58	111	011	011	111	111	011	0	01	1	0	000	0001	111
5.59	111	011	011	110	111	111	0	01	1	0	000	0001	111
5.60–13.1	111	111	111	101	000	000	1	11	1	0	111	1111	111
13.2–14.2	111	011	111	011	000	000	1	11	1	1	111	1111	111
14.3–15.4	111	111	111	011	000	000	1	11	1	1	101	1111	111
15.5–up	111	011	111	011	001	000	1	11	1	0	011	1111	111

Results

More dialysis with larger Δ, beginning with longer life span

Δ range	Pace-maker	Hip repl.	Aortic valve	CABG L	CABG 3	CABG 2	Heart trans.	Kidney trans.	Kidney dialysis < 1	Kidney dialysis 1-2	Kidney dialysis 2-5	Kidney dialysis 5-10	Kidney dialysis > 10
0–3.3	111	111	111	111	111	111	1	11	0	0	000	0000	000
3.4–4.0	111	111	111	111	111	111	0	11	1	0	000	0000	000
4.0–4.4	111	111	111	111	111	111	0	01	1	0	000	0000	001
4.5–5.01	111	011	111	111	111	111	1	01	1	0	000	0000	011
5.02–5.55	111	011	011	111	111	111	0	01	1	0	000	0001	011
5.56–5.58	111	011	011	111	111	011	0	01	1	0	000	0001	111
5.59	111	011	011	110	111	111	0	01	1	0	000	0001	111
5.60–13.1	111	111	111	101	000	000	1	11	1	0	111	1111	111
13.2–14.2	111	011	111	011	000	000	1	11	1	1	111	1111	111
14.3–15.4	111	111	111	011	000	000	1	11	1	1	101	1111	111
15.5–up	111	011	111	011	001	000	1	11	1	0	011	1111	111

Results

Abrupt change at Δ = 5.60

Δ	Pace-	Hip	Aortic	CABG			Heart	Kidney	Kidney dialysis				
range	maker	repl.	valve	L	3	2	trans.	trans.	< 1	1-2	2-5	5-10	> 10
0–3.3	111	111	111	111	111	111	1	11	0	0	000	0000	000
3.4–4.0	111	111	111	111	111	111	0	11	1	0	000	0000	000
4.0–4.4	111	111	111	111	111	111	0	01	1	0	000	0000	001
4.5–5.01	111	011	111	111	111	111	1	01	1	0	000	0000	011
5.02–5.55	111	011	011	111	111	111	0	01	1	0	000	0001	011
5.56–5.58	111	011	011	111	111	011	0	01	1	0	000	0001	111
5.59	111	011	011	110	111	111	0	01	1	0	000	0001	111
5.60–13.1	111	111	111	101	000	000	1	11	1	0	111	1111	111
13.2–14.2	111	011	111	011	000	000	1	11	1	1	111	1111	111
14.3–15.4	111	111	111	011	000	000	1	11	1	1	101	1111	111
15.5–up	111	011	111	011	001	000	1	11	1	0	011	1111	111

Results

Come and go together

Δ range	Pace-maker	Hip repl.	Aortic valve	CABG L	CABG 3	CABG 2	Heart trans.	Kidney trans.	Kidney dialysis < 1	1-2	2-5	5-10	> 10
0–3.3	111	111	111	111	111	111	1	11	0	0	000	0000	000
3.4–4.0	111	111	111	111	111	111	0	11	1	0	000	0000	000
4.0–4.4	111	111	111	111	111	111	0	01	1	0	000	0000	001
4.5–5.01	111	011	111	111	111	111	1	01	1	0	000	0000	011
5.02–5.55	111	011	011	111	111	111	0	01	1	0	000	0001	011
5.56–5.58	111	011	011	111	111	011	0	01	1	0	000	0001	111
5.59	111	011	011	110	111	111	0	01	1	0	000	0001	111
5.60–13.1	111	111	111	101	000	000	1	11	1	0	111	1111	111
13.2–14.2	111	011	111	011	000	000	1	11	1	1	111	1111	111
14.3–15.4	111	111	111	011	000	000	1	11	1	1	101	1111	111
15.5–up	111	011	111	011	001	000	1	11	1	0	011	1111	111

Results

In-out-in

Δ range	Pace-maker	Hip repl.	Aortic valve	CABG L	CABG 3	CABG 2	Heart trans.	Kidney trans.	Kidney dialysis < 1	1-2	2-5	5-10	> 10
0–3.3	111	111	111	111	111	111	1	11	0	0	000	0000	000
3.4–4.0	111	111	111	111	111	111	0	11	1	0	000	0000	000
4.0–4.4	111	111	111	111	111	111	0	01	1	0	000	0000	001
4.5–5.01	111	011	111	111	111	111	1	01	1	0	000	0000	011
5.02–5.55	111	011	011	111	111	111	0	01	1	0	000	0001	011
5.56–5.58	111	011	011	111	111	011	0	01	1	0	000	0001	111
5.59	111	011	011	110	111	111	0	01	1	0	000	0001	111
5.60–13.1	111	111	111	101	000	000	1	11	1	0	111	1111	111
13.2–14.2	111	011	111	011	000	000	1	11	1	1	111	1111	111
14.3–15.4	111	111	111	011	000	000	1	11	1	1	101	1111	111
15.5–up	111	011	111	011	001	000	1	11	1	0	011	1111	111

Results

Most rapid change. Possible range for politically acceptable compromise

Δ	Pace-	Hip	Aortic	CABG			Heart	Kidney	Kidney dialysis				
range	maker	repl.	valve	L	3	2	trans.	trans.	< 1	1-2	2-5	5-10	> 10
0–3.3	111	111	111	111	111	111	1	11	0	0	000	0000	000
3.4–4.0	111	111	111	111	111	111	0	11	1	0	000	0000	000
4.0–4.4	111	111	111	111	111	111	0	01	1	0	000	0000	001
4.5–5.01	111	011	111	111	111	111	1	01	1	0	000	0000	011
5.02–5.55	111	011	011	111	111	111	0	01	1	0	000	0001	011
5.56–5.58	111	011	011	111	111	011	0	01	1	0	000	0001	111
5.59	111	011	011	110	111	111	0	01	1	0	000	0001	111
5.60–13.1	111	111	111	101	000	000	1	11	1	0	111	1111	111
13.2–14.2	111	011	111	011	000	000	1	11	1	1	111	1111	111
14.3–15.4	111	111	111	011	000	000	1	11	1	1	101	1111	111
15.5–up	111	011	111	011	001	000	1	11	1	0	011	1111	111

Results

32 groups, 1089 integer variables
Solution time (CPLEX 12.2) is negligible

Δ range	Pace-maker	Hip repl.	Aortic valve	CABG L	CABG 3	CABG 2	Heart trans.	Kidney trans.	Kidney dialysis < 1	Kidney dialysis 1-2	Kidney dialysis 2-5	Kidney dialysis 5-10	Kidney dialysis > 10
0–3.3	111	111	111	111	111	111	1	11	0	0	000	0000	000
3.4–4.0	111	111	111	111	111	111	0	11	1	0	000	0000	000
4.0–4.4	111	111	111	111	111	111	0	01	1	0	000	0000	001
4.5–5.01	111	011	111	111	111	111	1	01	1	0	000	0000	011
5.02–5.55	111	011	011	111	111	111	0	01	1	0	000	0001	011
5.56–5.58	111	011	011	111	111	011	0	01	1	0	000	0001	111
5.59	111	011	011	110	111	111	0	01	1	0	000	0001	111
5.60–13.1	111	111	111	101	000	000	1	11	1	0	111	1111	111
13.2–14.2	111	011	111	011	000	000	1	11	1	1	111	1111	111
14.3–15.4	111	111	111	011	000	000	1	11	1	1	101	1111	111
15.5–up	111	011	111	011	001	000	1	11	1	0	011	1111	111

Results

Table 3: Solution times in seconds for m groups and different values of Δ. Instances with more than a few hundred groups seem very unlikely to occur in practice.

	Δ							
m	0	1	2	3	4	5	6	∞
330	0.02	1.2	0.67	0.56	0.50	0.30	0.03	0.02
660	0.03	4.1	1.6	1.6	0.92	0.80	0.05	0.02
990	0.02	5.2	3.1	3.6	1.5	1.5	0.08	0.02
1320	0.00	15	4.3	4.2	2.7	3.0	0.09	0.02
1980	0.02	24	11	11	11	5.4	0.14	0.02
2640	0.00	32	19	14	8.6	8.8	0.19	0.02
3300	0.17	51	43	44	34	13	0.25	0.02