A Search-Infer-and-Relax Framework for Integrating Solution Methods

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  - One solver does it all.

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- Computational speedup.
  - A selection of results…

Computational Advantage of Integrating CP and MILP

Using CP + relaxation from MILP

	Problem	*Speedup*
Focacci, Lodi, Milano (1999)	Lesson timetabling	2 to 50 times faster than CP
Refalo (1999)	Piecewise linear costs	2 to 200 times faster than MILP
Hooker & Osorio (1999)	Flow shop scheduling, etc.	4 to 150 times faster than MILP.
Thorsteinsson & Ottosson (2001)*	Product configuration	30 to 40 times faster than CP, MILP

*Will discuss

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	Problem	*Speedup*
Sellmann & Fahle (2001)	Automatic recording	1 to 10 times faster than CP, MILP
Van Hoeve (2001)	Stable set problem	Better than CP in less time
Bollapragada, Ghattas & Hooker (2001)	Structural design (nonlinear)	Up to 600 times faster than MILP
Beck & Refalo (2003)	Scheduling with earliness & tardiness costs	Solved 67 of 90, CP solved only 12

Computational Advantage of Integrating CP and MILP

Using CP-based Branch and Price

	Problem	*Speedup*
Yunes, Moura & de Souza (1999)	Urban transit crew scheduling	Optimal schedule for 210 trips, vs. 120 for traditional branch and price
Easton, Nemhauser & Trick (2002)	Traveling tournament scheduling	First to solve 8-team instance

Computational Advantage of Integrating CP and MILP

Using CP/MILP Benders methods

	Problem	*Speedup*
Jain & Grossmann (2001)*	Min-cost planning & scheduing	20 to 1000 times faster than CP, MILP
Thorsteinsson (2001)	Min-cost planning & scheduling	10 times faster than Jain & Grossmann
Timpe (2002)	Polypropylene batch scheduling at BASF	Solved previously insoluble problem in 10 min

*Will discuss

Computational Advantage of Integrating CP and MILP

Using CP/MILP Benders methods

	Problem	*Speedup*
Benoist, Gaudin, Rottembourg (2002)	Call center scheduling	Solved twice as many instances as traditional Benders
Hooker (2004)	Min-cost, min-makespan planning & cumulative scheduling	100-1000 times faster than CP, MILP
Hooker (2005)	Min tardiness planning & cumulative scheduling	10-1000 times faster than CP, MILP

Proposal:

- View solution methods as **special cases** of the same basic algorithm.

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- “Hybrid” methods can then be viewed as **other** special cases of the same basic algorithm.

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  - **Selection function** determines which solution of relaxation to use.
  - Use **post-relaxation inference** if desired.
  - Solution of relaxation guides choice of next restriction.

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  - Let $s(R_i)$ guide choice of P_{i+1}.

Underlying idea:

"Primal-dual" algorithm
exploits duality
of problem **restriction** and **relaxation**.

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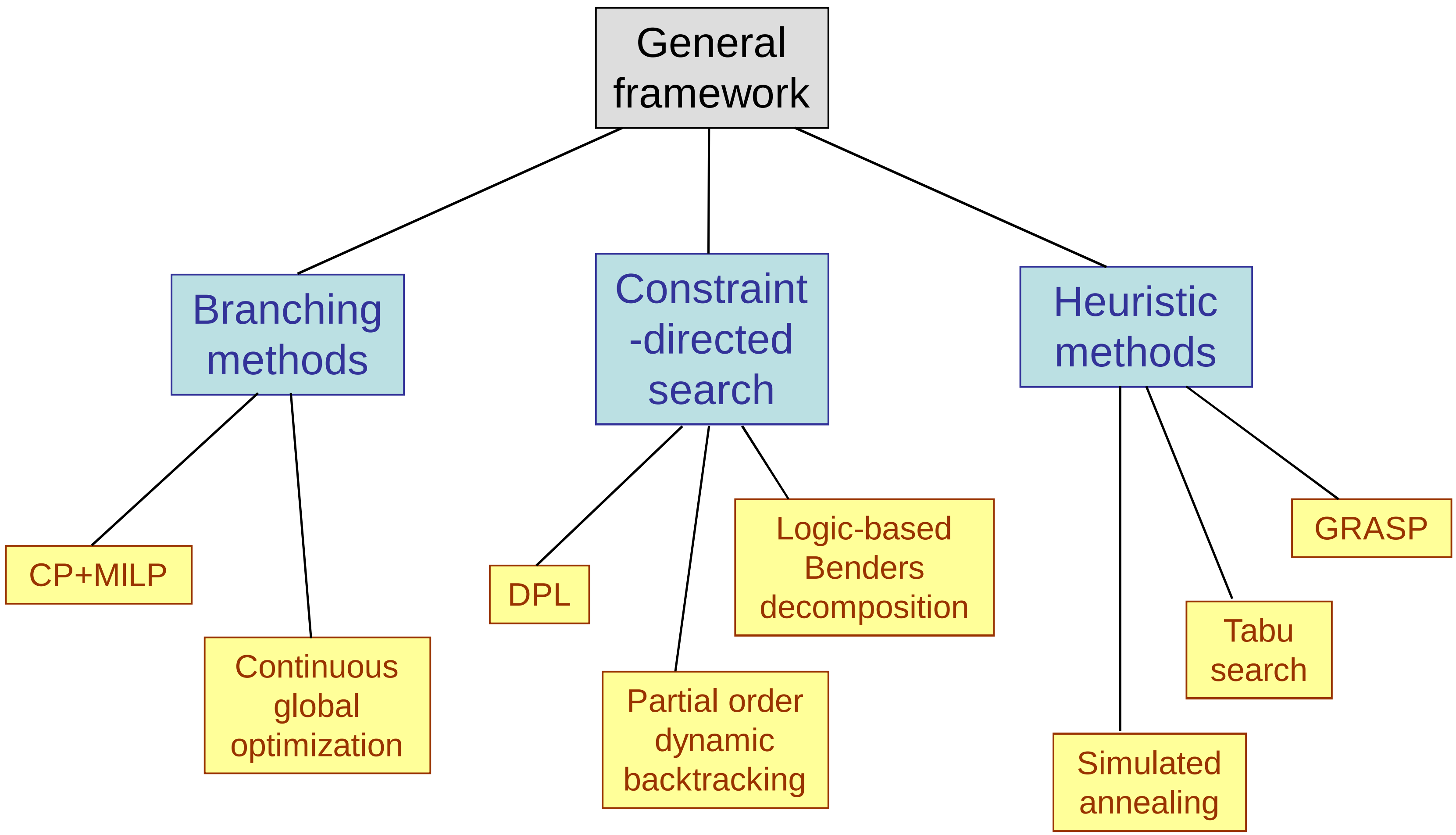
- After looking at the examples, you can judge whether this framework is helpful or artificial.

General Framework

Solution Method	*Restriction P_k*	*Relaxation R_k*	*Selection Function $s(R_k)$*	*Inference*
Branching CP, MILP, global optimization	Node of search tree	LP, NLP, domains	Any optimal (feasible) solution of R_k	Domain filtering, cutting planes
Constraint directed DPL, dynamic backtracking, Benders	Add nogoods generated so far	Processed nogoods generated so far	Solution that results in easy R_k	Nogood generation
Heuristics Local search, GRASP	Neighborhood of current solution	Research topic	Random, best, etc.	?

Structure of Talk

Depth-first traversal of tree:



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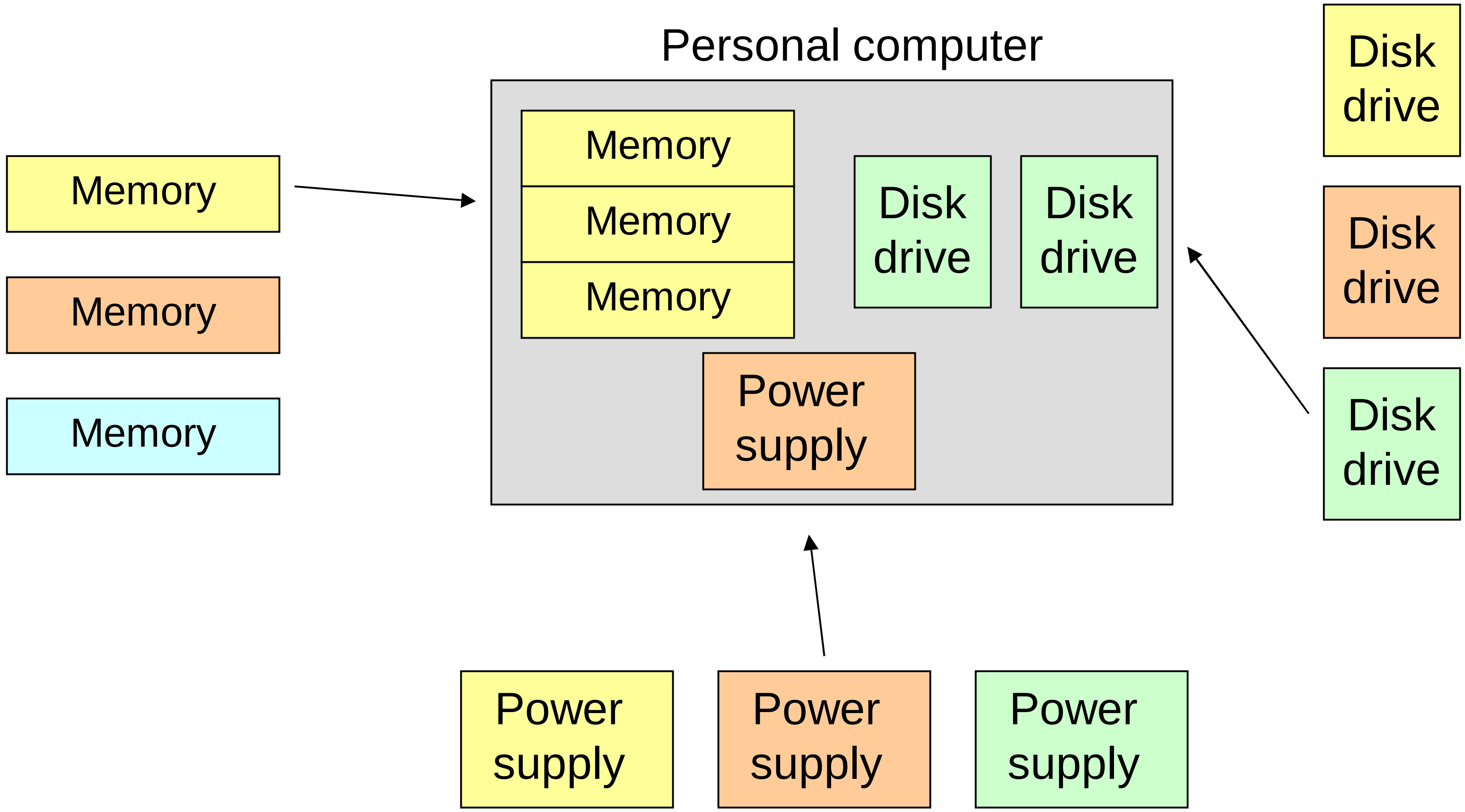
Branching Search:
Product Configuration by CP/MILP

Choose what type of each component,
and how many



This example illustrates
how a “hybrid” method
is a special case
of the general algorithm.

Model of the problem

Amount of attribute *j* produced
(< 0 if consumed):
memory, heat, power, weight, etc.

$$\min \sum_j c_j v_j$$

$$v_j = \sum_{ik} q_i A_{ijt_i}, \quad \text{all } j$$

$$L_j \le v_j \le U_j, \quad \text{all } j$$

Quantity of component *i* installed

Amount of attribute *j* produced by type t_i of component *i*

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t_i is a **variable index**

Quantity of component *i* installed

Unit cost of producing attribute *j*

Amount of attribute *j* produced by type t_i of component *i*

Amount of attribute *j* produced (< 0 if consumed): memory, heat, power, weight, etc.

$$\min \sum_j c_j v_j$$

$$v_j = \sum_{ik} q_i A_{ijt_i}, \quad \text{all } j$$

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Infer (propagate)

$$\min \sum_j c_j v_j$$

$$v_j = \sum_{ik} q_i A_{ijt_i}, \text{ all } j$$

$$\boxed{L_j \le v_j \le U_j, \text{ all } j}$$

This is propagated in the usual way

Infer (propagate)

$$v_j = \sum_i z_i, \ \text{all } j$$

$$\text{element}\big(t_i, (q_i A_{ij1}, \ldots, q_i A_{ijn}), z_i\big), \text{ all } i, j$$

$$\min \sum_j c_j v_j$$

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(a) using specialized **filters** for *element* constraints of this form…

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This is propagated by
(a) using specialized **filters** for *element* constraints of this form,
(b) adding **knapsack cuts** for the valid inequalities:

$$\sum_i \max_{k \in D_{t_i}} \{A_{ijk}\} q_i \geq \underline{v}_j, \ \text{all } j$$
$$\sum_i \min_{k \in D_{t_i}} \{A_{ijk}\} q_i \leq \overline{v}_j, \ \text{all } j$$

$[\underline{v}_j, \overline{v}_j]$ is current domain of v_j

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and (c) propagating the knapsack cuts.

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Relax

$$\min \sum_j c_j v_j$$

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This is relaxed as

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This is relaxed by relaxing *this* and adding the knapsack cuts.

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$$\text{element}\big(t_i, (q_i A_{ij1}, \ldots, q_i A_{ijn}), z_i\big), \text{ all } i, j$$

This is relaxed by replacing each *element* constraint with a **convex hull** relaxation of a disjunctive programming constraint:

$$z_i = \sum_{k \in D_{t_i}} A_{ijk} q_{ik}, \quad q_i = \sum_{k \in D_{t_i}} q_{ik}$$

Relax

So the following LP relaxation is solved at each node of the search tree to obtain a lower bound:

$$
\begin{aligned}
&\min \sum_j c_j v_j \\
&v_j = \sum_i \sum_{k \in D_{t_i}} A_{ijk} q_{ik}, \ \text{all } j \\
&q_i = \sum_{k \in D_{t_i}} q_{ik}, \ \text{all } i \\
&\underline{v}_j \le v_j \le \overline{v}_j, \ \text{all } j \\
&\underline{q}_i \le q_i \le \overline{q}_i, \ \text{all } i \\
&\text{knapsack cuts for } \sum_i \max_{k \in D_{t_i}} \{A_{ijk}\} q_i \ge \underline{v}_j, \ \text{all } j \\
&\text{knapsack cuts for } \sum_i \min_{k \in D_{t_i}} \{A_{ijk}\} q_i \le \overline{v}_j, \ \text{all } j \\
&q_{ik} \ge 0, \ \text{all } i, k
\end{aligned}
$$

Branching Methods

Solution Method	*Restriction P_k*	*Relaxation R_k*	*Selection Function $s(R_k)$*	*Inference*
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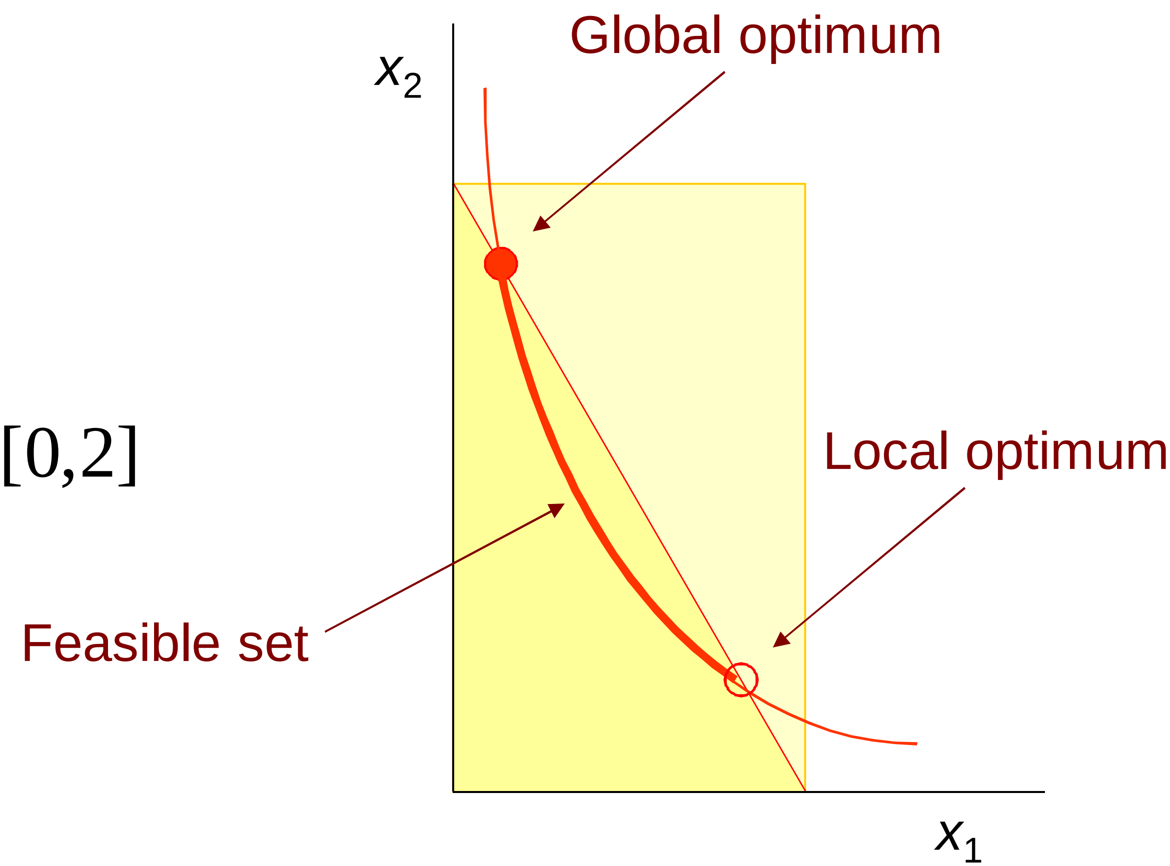
Continuous global optimization	**Created by splitting intervals**	**LP or convex NLP relaxation**	**Optimal solution of R_k** .	**Interval propagation, Lagrangean bounding**

Branching Search: Continuous Global Optimization

$$\max x_1 + x_2$$
$$4x_1 x_2 = 1$$
$$2x_1 + x_2 \le 2$$
$$x_1 \in [0,1],\ x_2 \in [0,2]$$



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  - Post-relaxation inference: Use **Lagrange multipliers** to infer valid inequality for propagation.
  - **Reduced-cost variable** fixing is a special case.

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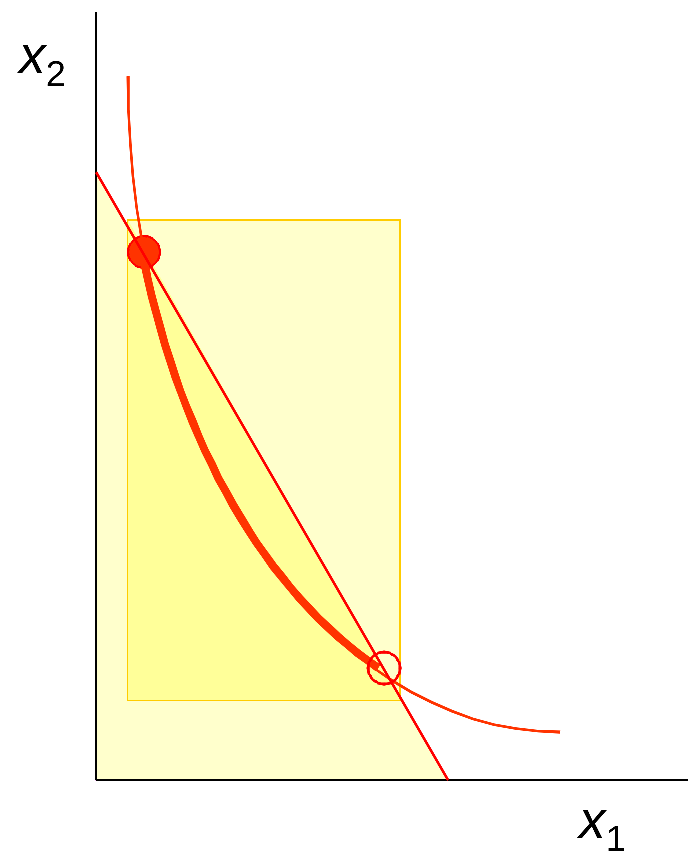
  - **Reduced-cost variable** fixing is a special case.
- **Relax:** Use function **factorization** to obtain linear continuous relaxation.
  - **Selection function:** Any optimal solution of relaxation.

Infer (interval propagation)

Propagate intervals
[0,1], [0,2]
through constraints
to obtain
[1/8,7/8], [1/4,7/4]



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Factor complex functions into elementary functions that have known linear relaxations.

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This factors $4x_1x_2$ into linear function $4y$ and bilinear function x_1x_2.

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Bilinear function $y = x_1x_2$ has relaxation:

$$\underline{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \underline{x}_2 \le y \le \underline{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1 \underline{x}_2$$

$$\bar{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1 \bar{x}_2 \le y \le \bar{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \bar{x}_2$$

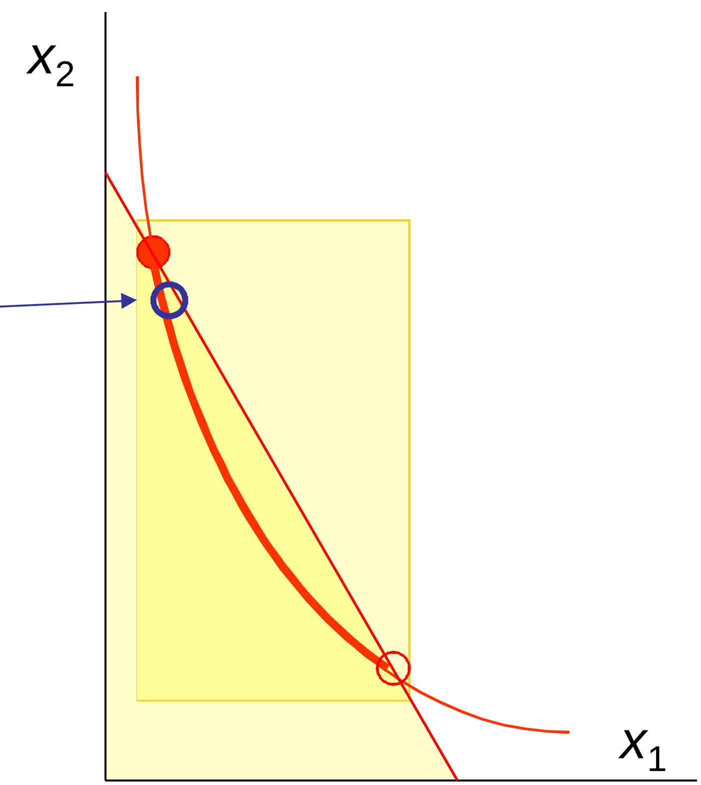
Where domain of x_j is $[\underline{x}_j, \bar{x}_j]$

Relax

The linear relaxation becomes:

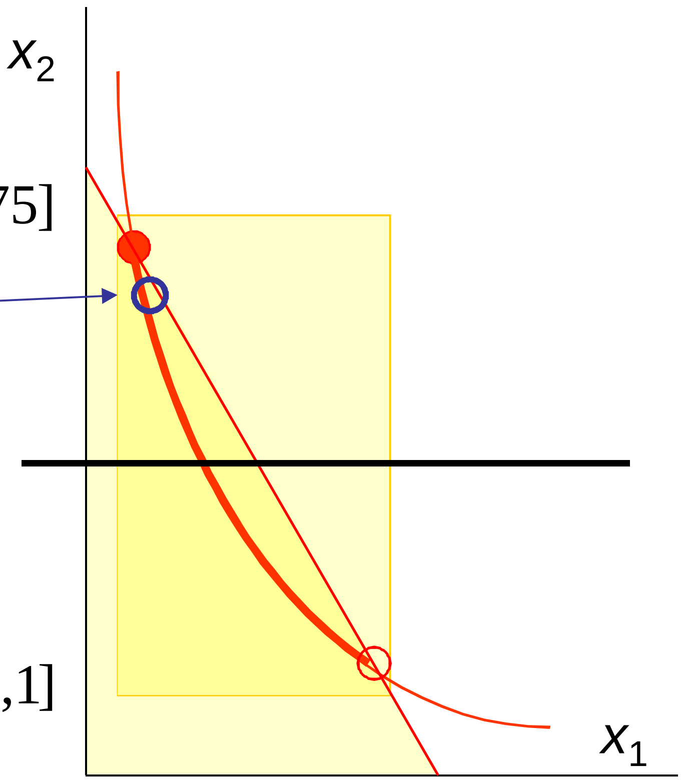
$$
\begin{aligned}
&\min x_1 + x_2 \\
&4y = 1 \\
&2x_1 + x_2 \le 2 \\
&\underline{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \underline{x}_2 \le y \le \underline{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1 \underline{x}_2 \\
&\bar{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1 \bar{x}_2 \le y \le \bar{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \bar{x}_2 \\
&\underline{x}_j \le x_j \le \bar{x}_j, \quad j = 1,2
\end{aligned}
$$

Relax



Solve linear relaxation.

Relax

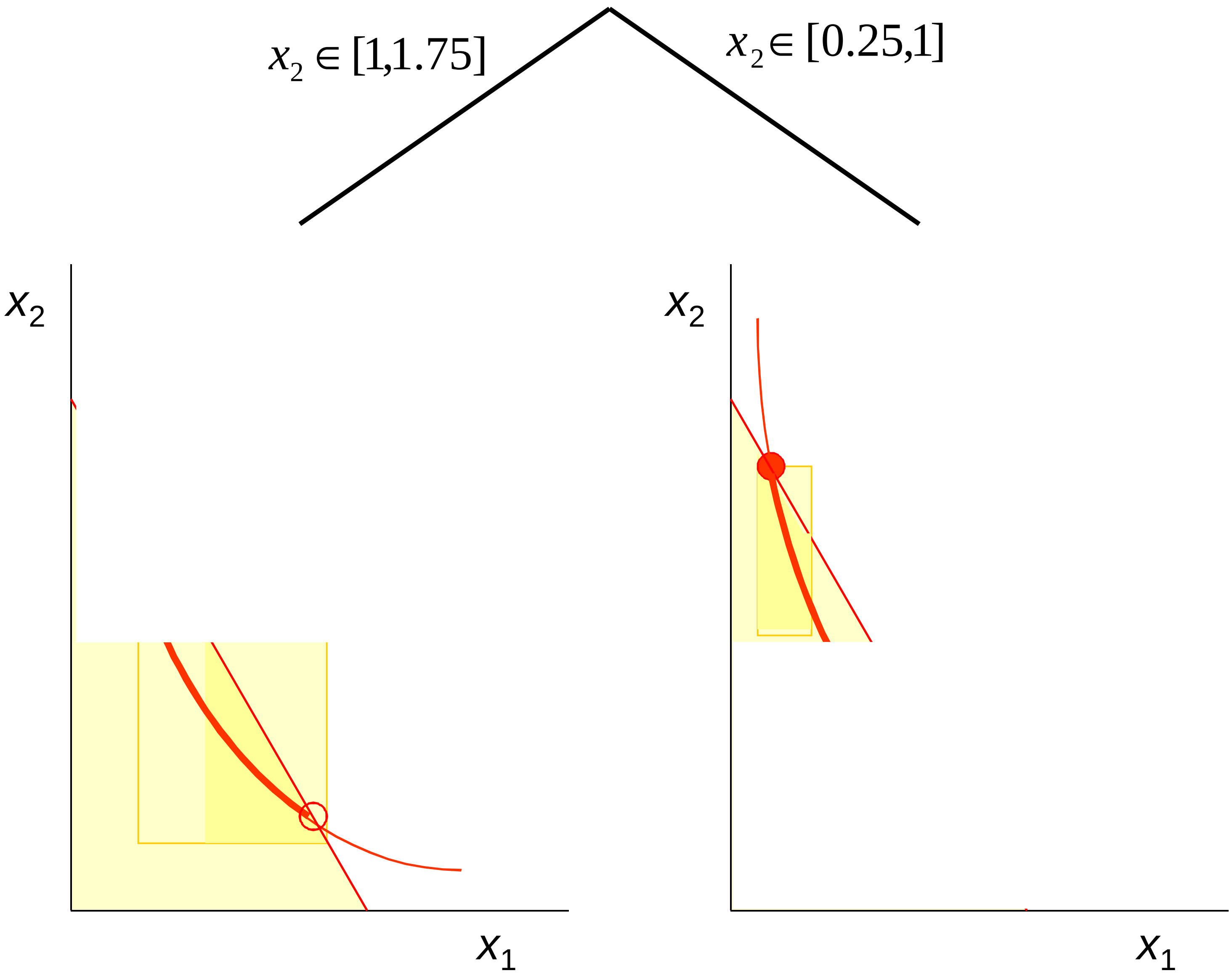


$x_2 \in [1,1.75]$

Solve linear relaxation.

Since solution is infeasible,
split an interval and branch.

$x_2 \in [0.25,1]$



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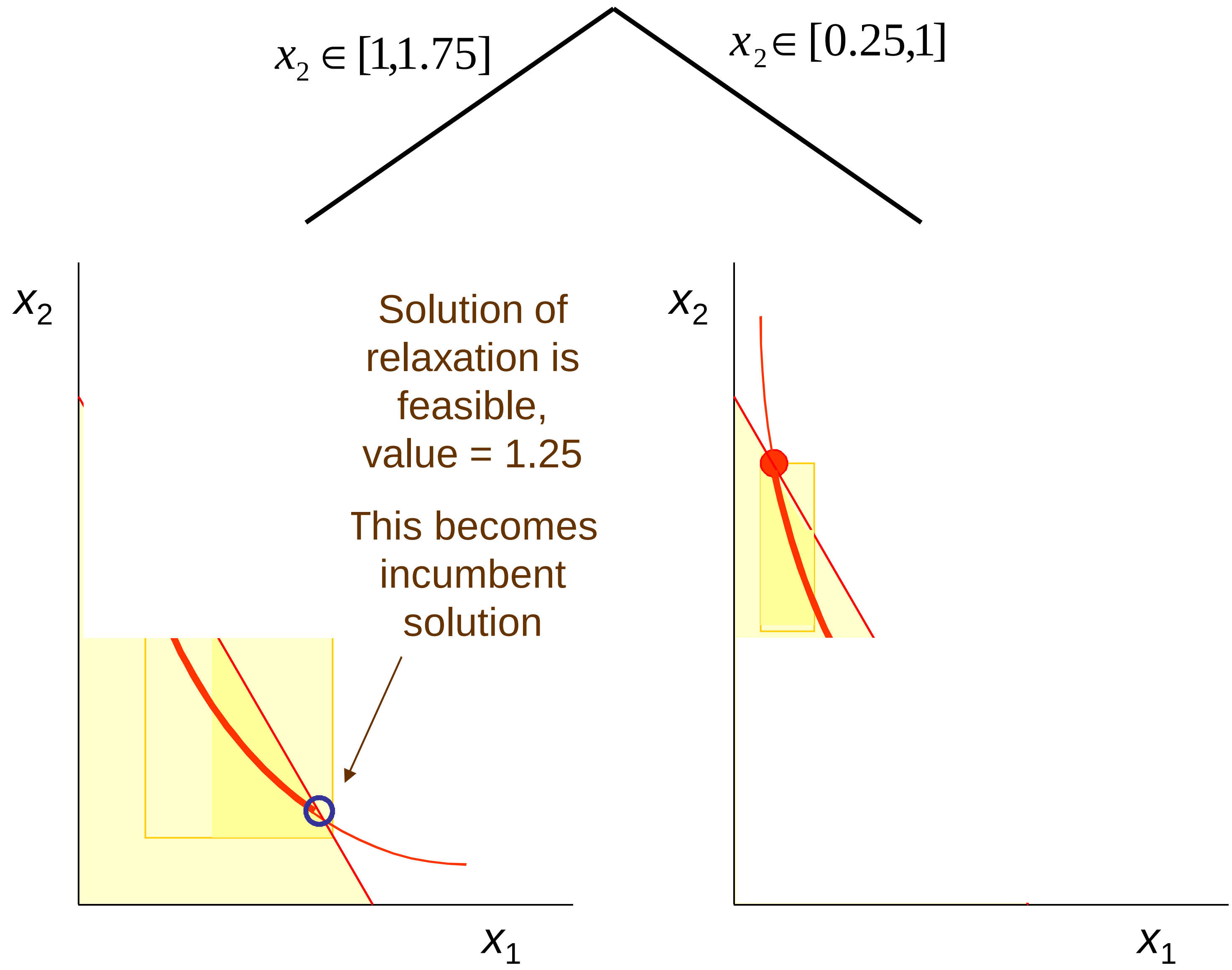
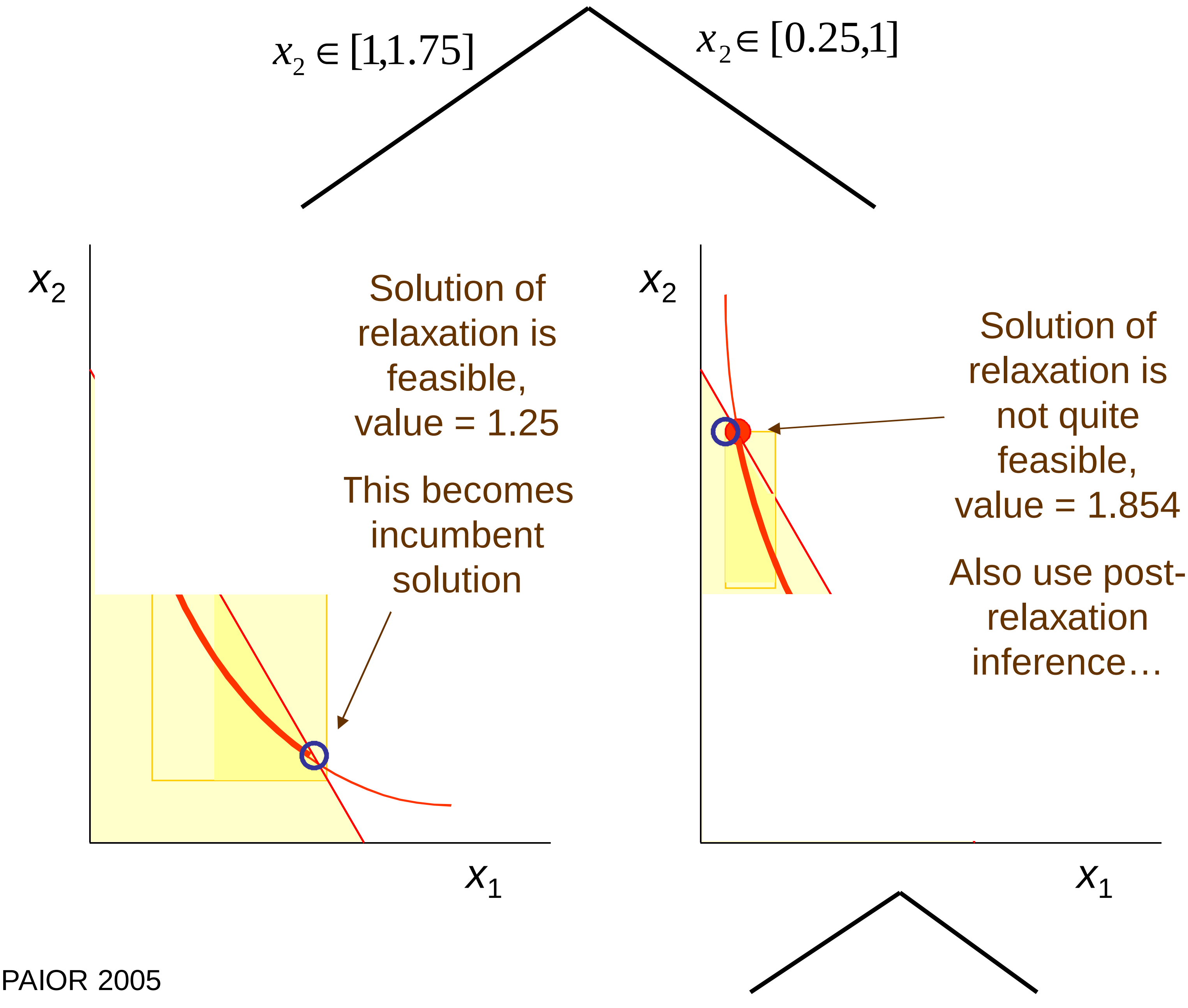
$x_2 \in [0.25,1]$
x_2
x_1
x_2
x_1



$x_2 \in [1,1.75]$
$x_2 \in [0.25,1]$
x_2
x_1
Solution of relaxation is feasible, value = 1.25
This becomes incumbent solution
x_2
x_1



$x_2 \in [1,1.75]$
$x_2 \in [0.25,1]$
x_2
x_1
Solution of relaxation is feasible, value = 1.25
This becomes incumbent solution
x_2
x_1
Solution of relaxation is not quite feasible, value = 1.854
Also use post-relaxation inference…

Post-Relaxation Inference

$$
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&\min x_1 + x_2 \\
&4y = 1 \\
&2x_1 + x_2 \le 2 \\
&\underline{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \underline{x}_2 \le y \le \underline{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1 \underline{x}_2 \\
&\bar{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1 \bar{x}_2 \le y \le \bar{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \bar{x}_2 \\
&\underline{x}_j \le x_j \le \bar{x}_j, \quad j = 1,2
\end{aligned}
$$

Associated Lagrange multiplier in solution of relaxation is 1.1

Post-Relaxation Inference

$$\min x_1 + x_2$$

$$4y = 1$$

$$2x_1 + x_2 \le 2$$

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$$\underline{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \underline{x}_2 \le y \le \underline{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1 \underline{x}_2$$

$$\bar{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1 \bar{x}_2 \le y \le \bar{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \bar{x}_2$$

$$\underline{x}_j \le x_j \le \bar{x}_j, \quad j = 1,2$$

This yields a valid inequality for propagation:

$$2x_1 + x_2 \ge 2 - \frac{1.854 - 1.25}{1.1} = 1.451$$

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Value of relaxation

Post-Relaxation Inference

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Value of relaxation

Lagrange multiplier

Post-Relaxation Inference

$$
\begin{aligned}
&\min x_1 + x_2 \\
&4y = 1 \\
&2x_1 + x_2 \le 2 \\
&\underline{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \underline{x}_2 \le y \le \underline{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1 \underline{x}_2 \\
&\bar{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1 \bar{x}_2 \le y \le \bar{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \bar{x}_2 \\
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$$

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$$2x_1 + x_2 \ge 2 - \frac{\boxed{1.854} - \boxed{1.25}}{\boxed{1.1}} = 1.451$$

Value of relaxation

Lagrange multiplier

Value of incumbent solution

Post-Relaxation Inference

- **Reduced-cost variable fixing** is a special case.

Post-Relaxation Inference

- **Reduced-cost variable fixing** is a special case.
- **Separating cuts** represent another form of post-relaxation inference.

General Framework

Solution Method	*Restriction P_k*	*Relaxation R_k*	*Selection Function $s(R_k)$*	*Inference*
Branching CP, MILP, global optimization	Node of search tree	LP, NLP, domains	Any optimal (feasible) solution of R_k	Domain filtering, cutting planes
Constraint directed **DPL, dynamic backtracking, Benders**	**Add nogoods generated so far**	**Processed nogoods generated so far**	**Solution that results in easy R_k**	**Nogood generation**
Heuristics Local search, GRASP	Neighborhood of current solution	Research topic	Random, best, etc.	?

Constraint-Directed Search

Solution Method	*Restriction P_k*	*Relaxation R_k*	*Selection Function $s(R_k)$*	*Inference*
DPL for SAT	Add conflict clauses	Processed conflict clauses	Unit clause rule + greedy solution of R_k	Parallel resolution & absorption
Partial order dynamic backtracking	Add nogoods	Processed nogoods	Greedy, consistent with partial order	Parallel resolution & absorption
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Constraint-Directed Search: DPL for Propositional Satisfiability

DPL
(Davis-Putnam-Loveland)
with clause learning
can be interpreted as
constraint-directed search

$$x_1 \vee x_5 \vee x_6$$
$$x_1 \vee x_5 \vee \bar{x}_6$$
$$x_2 \vee \bar{x}_5 \vee x_6$$
$$x_2 \vee \bar{x}_5 \vee \bar{x}_6$$
$$\bar{x}_1 \vee x_3 \vee x_4$$
$$\bar{x}_2 \vee x_3 \vee x_4$$
$$\bar{x}_1 \vee \bar{x}_3$$
$$\bar{x}_1 \vee \bar{x}_4$$
$$\bar{x}_2 \vee \bar{x}_3$$
$$\bar{x}_2 \vee \bar{x}_4$$

To solve it by branching:

- **Search**: branch on x_j.
  - Each **node** of search tree is a problem restriction.

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- **Search**: branch on x_j.
  - Each **node** of search tree is a problem restriction.
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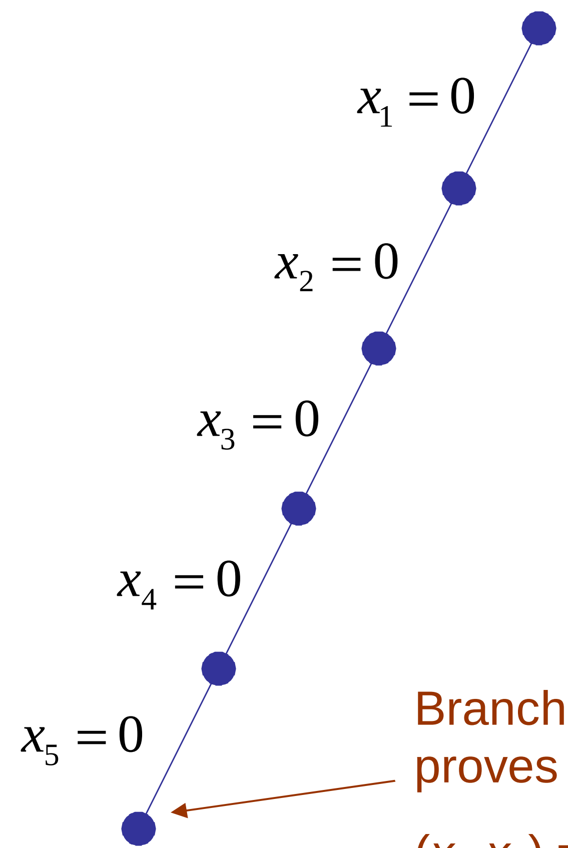
- **Relax:** not used.

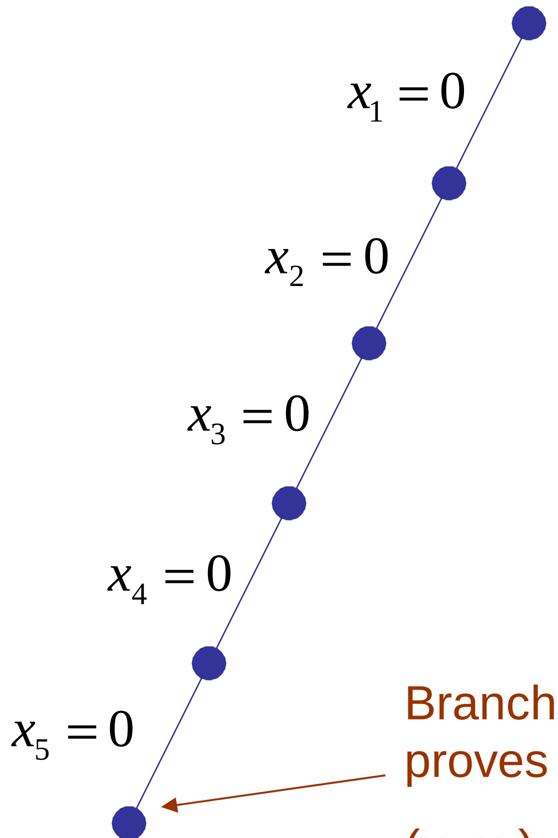
Branching



Branch to here. Unit clause rule proves infeasibility.

$(x_1, x_5) = (0,0)$ creates the conflict.

Branching



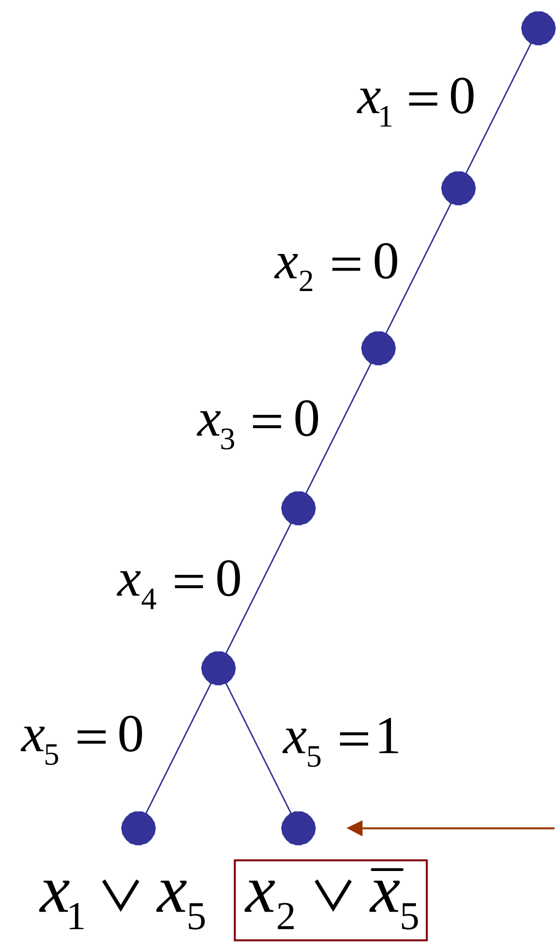
$\boxed{x_1 \vee x_5}$

Branch to here. Unit clause rule proves infeasibility.

$(x_1,x_5) = (0,0)$ creates the conflict.

Add conflict clause to constraint set.

Branching

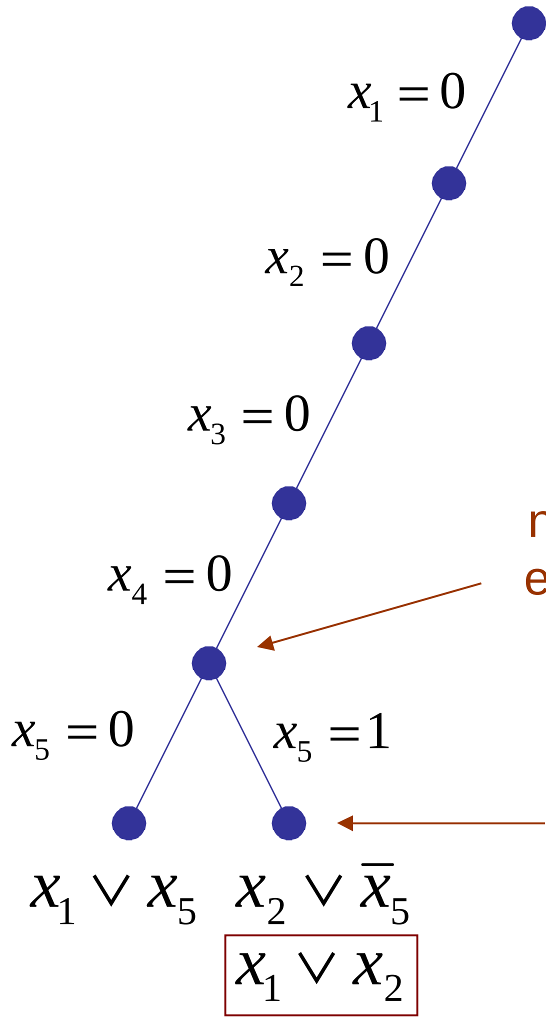


Backtrack and branch to here.

$(x_1,x_5) = (0,1)$ creates the conflict.

Generate conflict clause

Branching



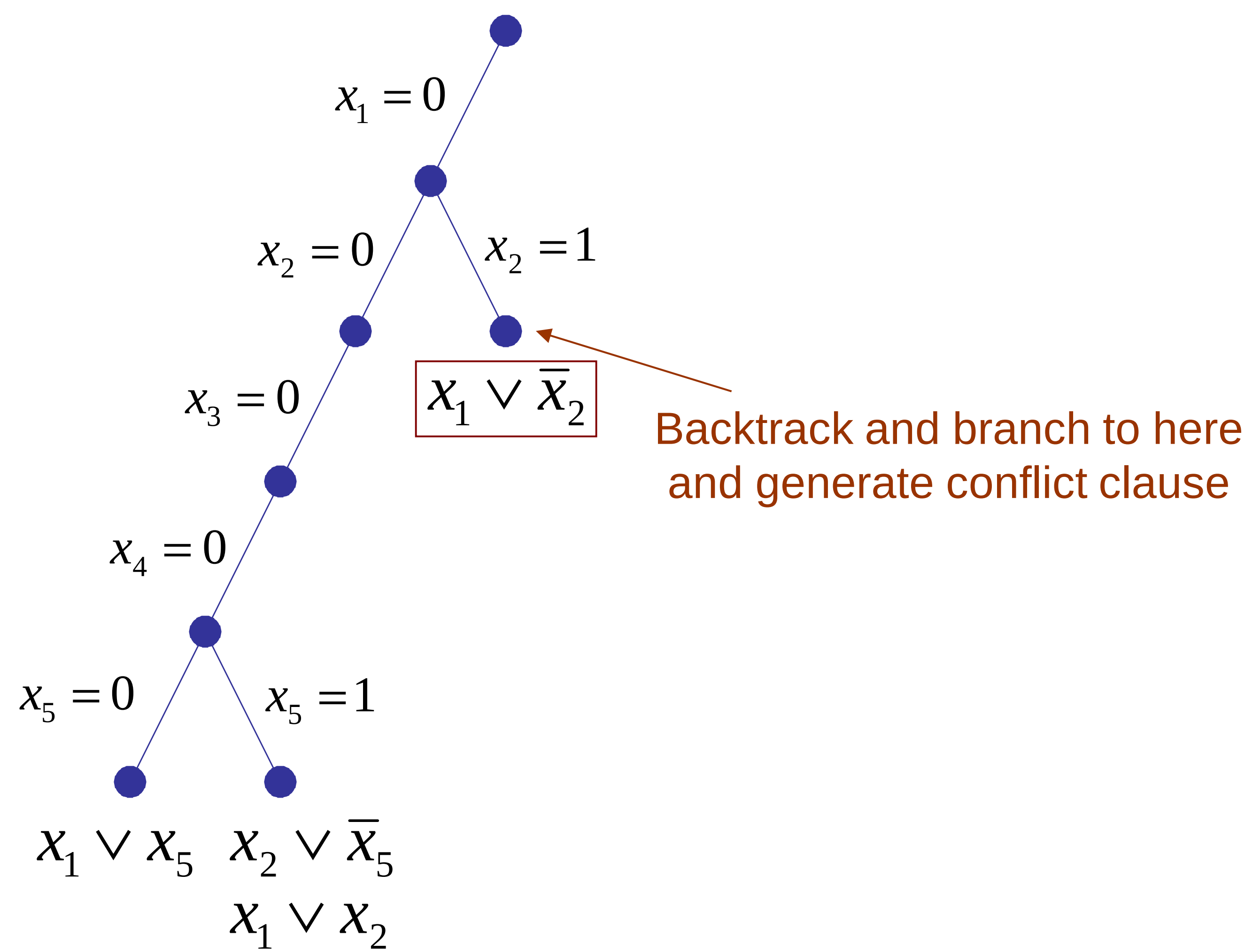
Backtrack to here and note that $(x_1, x_2) = (0,0)$ is enough to create conflict. Generate new conflict clause

Backtrack and branch to here.

$(x_1, x_5) = (0,1)$ creates the conflict.

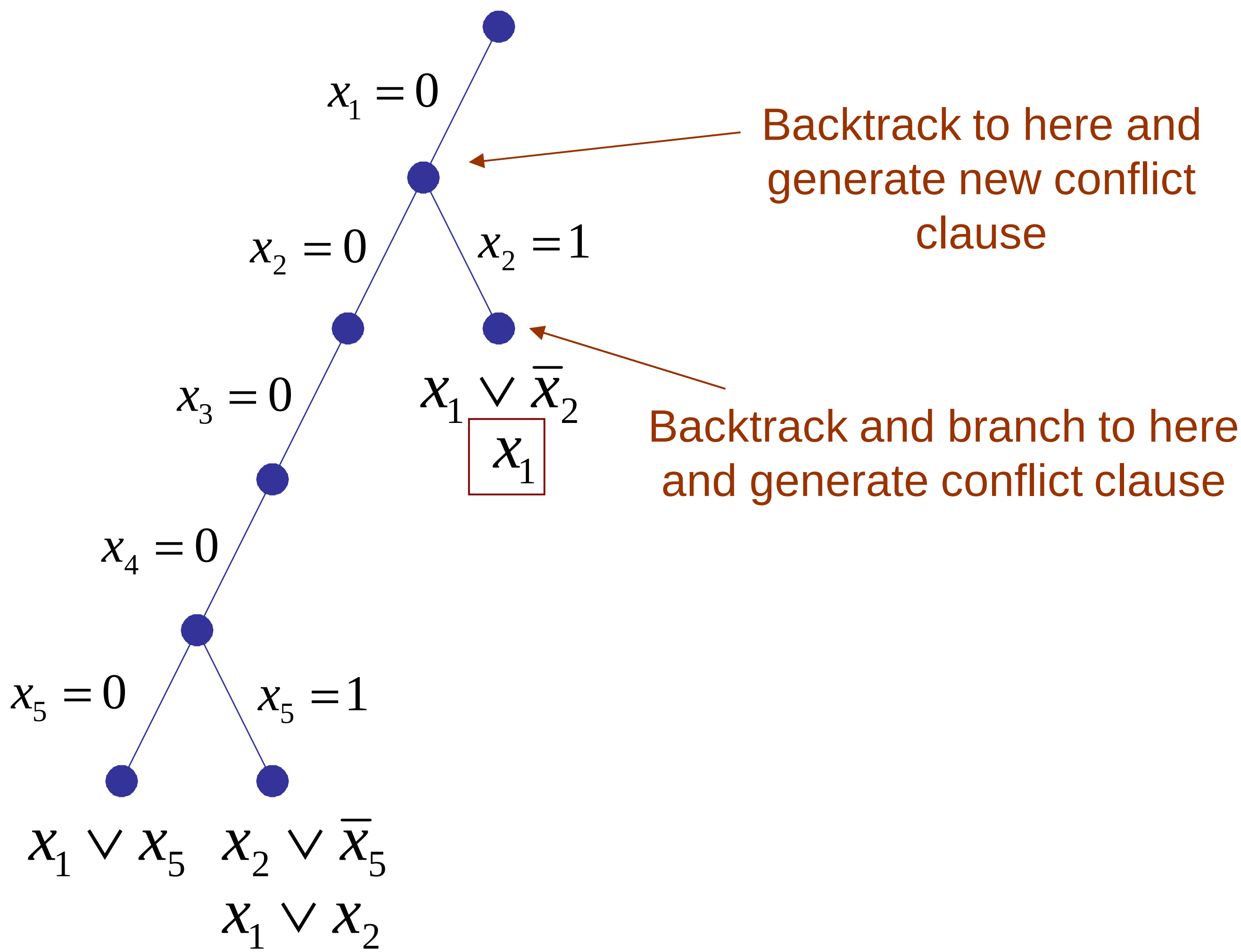
Generate conflict clause

Branching



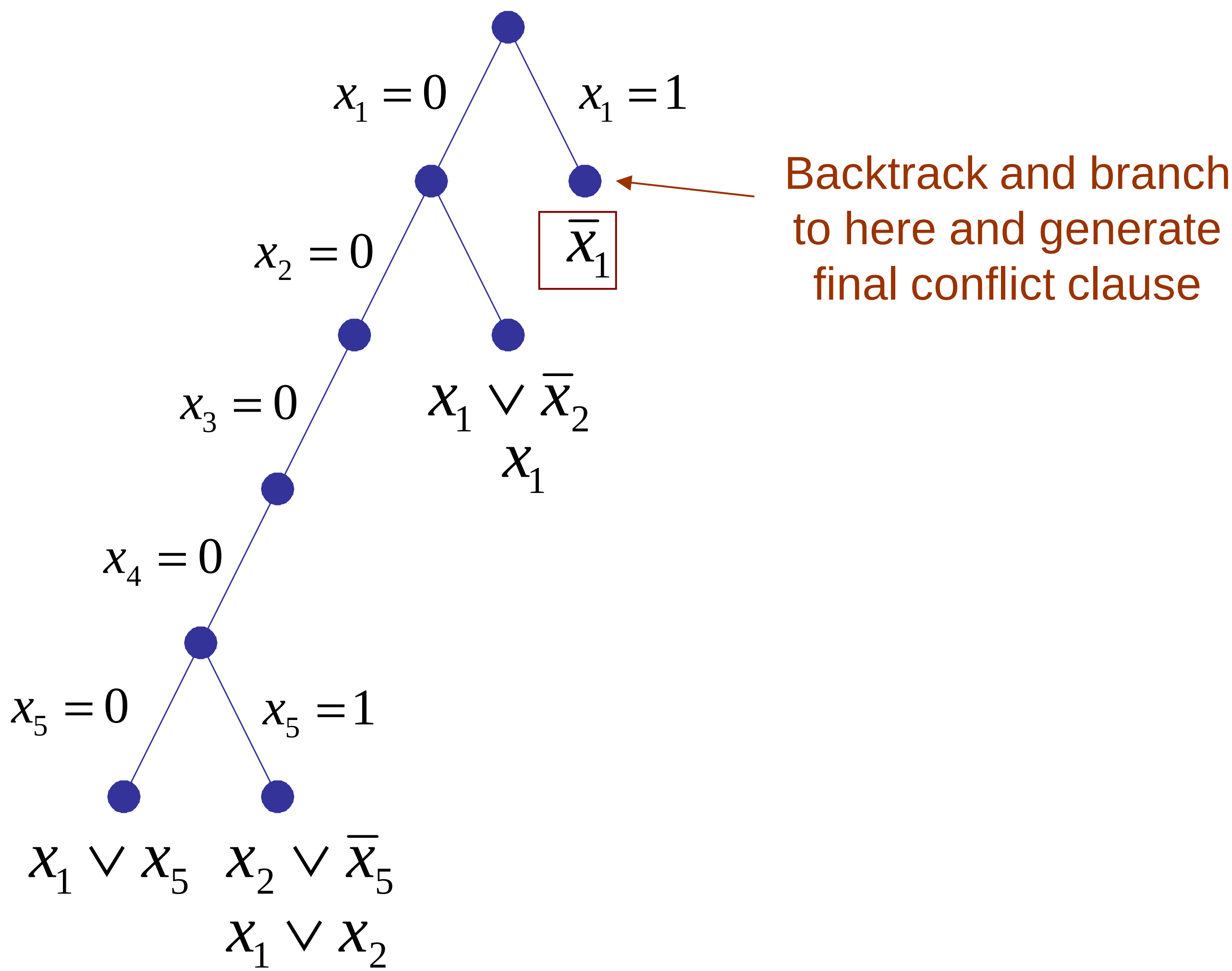
$x_1 = 0$
$x_2 = 0$
$x_2 = 1$
$x_3 = 0$
$x_4 = 0$
$x_5 = 0$
$x_5 = 1$
$\boxed{x_1 \vee \bar{x}_2}$
$x_1 \vee x_5$
$x_2 \vee \bar{x}_5$
$x_1 \vee x_2$
Backtrack and branch to here
and generate conflict clause

Branching



$x_1 = 0$
$x_2 = 0$
$x_2 = 1$
$x_3 = 0$
$x_4 = 0$
$x_5 = 0$
$x_5 = 1$
$x_1 \vee \bar{x}_2$
$\boxed{x_1}$
$x_1 \vee x_5$
$x_2 \vee \bar{x}_5$
$x_1 \vee x_2$
Backtrack to here and generate new conflict clause
Backtrack and branch to here and generate conflict clause

Branching



$x_1 = 0$
$x_1 = 1$
$\boxed{\bar{x}_1}$
Backtrack and branch to here and generate final conflict clause
$x_2 = 0$
$x_3 = 0$
$x_1 \vee \bar{x}_2$
x_1
$x_4 = 0$
$x_5 = 0$
$x_5 = 1$
$x_1 \vee x_5$
$x_2 \vee \bar{x}_5$
$x_1 \vee x_2$

To solve it by constraint-directed search:

- **Search**: generate problem restrictions.
  - Each **leaf node** of search tree is a problem restriction.

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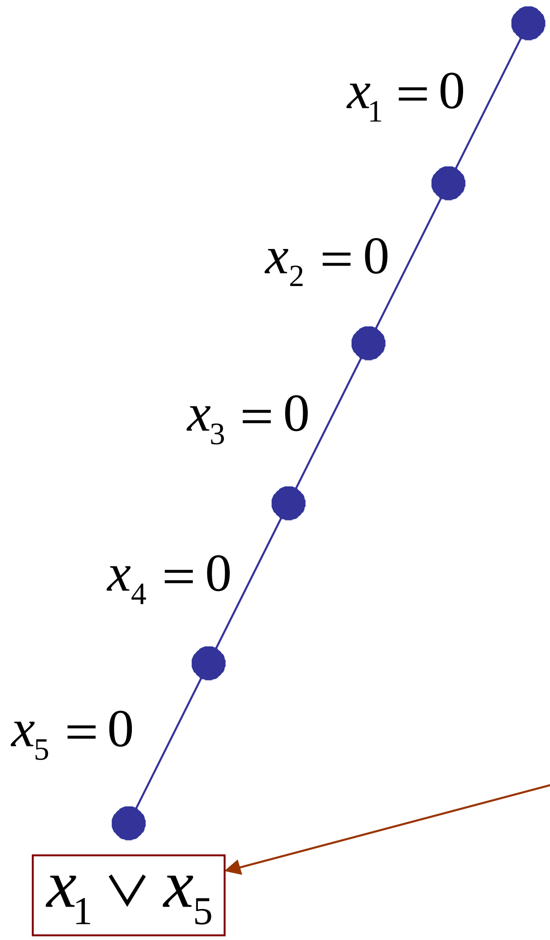
  - Process nogoods in current relaxation with **parallel resolution**.
- **Relax:** Relaxation R_k consists of current processed nogoods.
  - **Selection function**: Mimic chronological backtracking; apply unit clause rule.

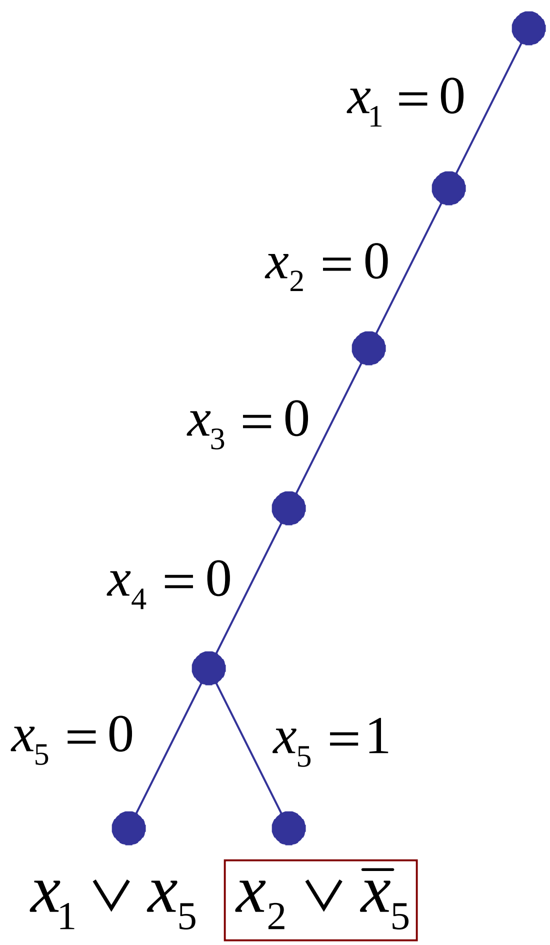
Constraint-Directed Search



k	Relaxation R_k	Solution of R_k	Nogoods
0		$(0,0,0,0,0,\cdot)$	$x_1 \vee x_5$
1	$x_1 \vee x_5$		

Conflict clause appears as nogood induced by solution of R_k.

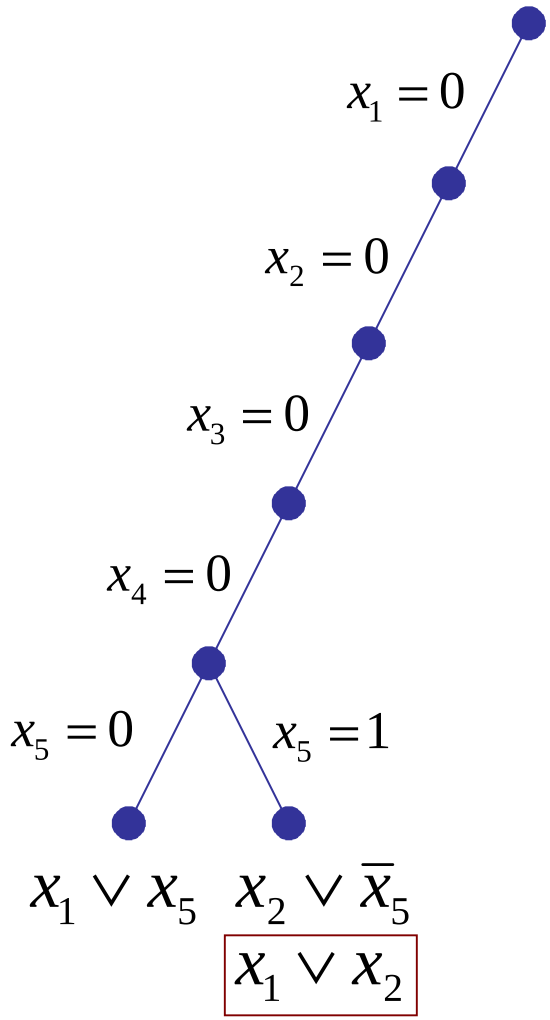
Constraint-Directed Search



Consists of **processed** nogoods

k	Relaxation R_k	Solution of R_k	Nogoods
0		$(0,0,0,0,0,\cdot)$	$x_1 \vee x_5$
1	$x_1 \vee x_5$	$(0,0,0,0,1,\cdot)$	$x_2 \vee \bar{x}_5$

Constraint-Directed Search



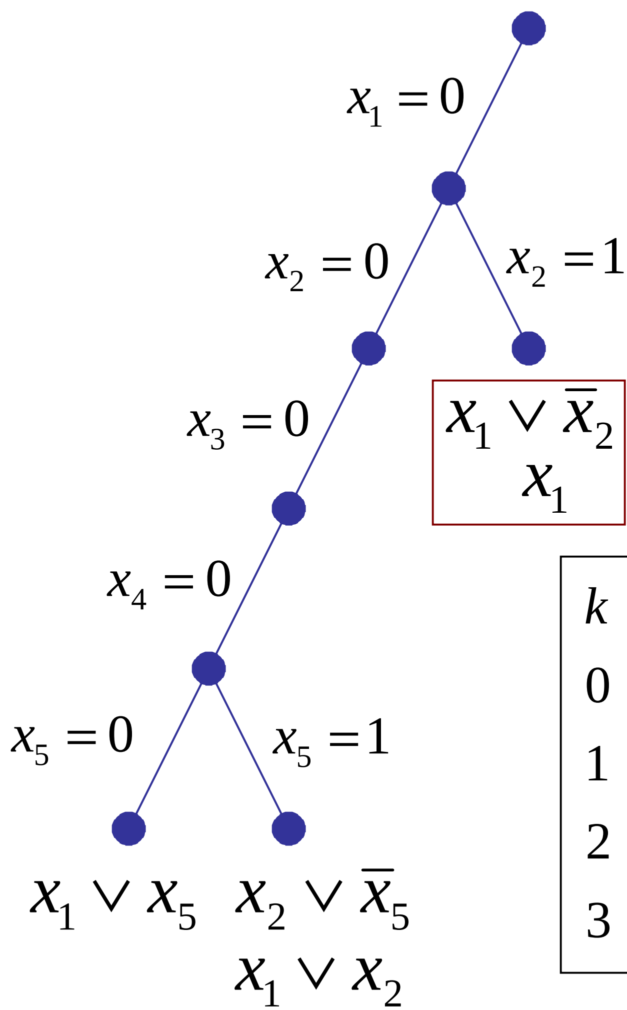
Consists of **processed** nogoods

k	Relaxation R_k	Solution of R_k	Nogoods
0		$(0,0,0,0,0,\cdot)$	$x_1 \vee x_5$
1	$x_1 \vee x_5$	$(0,0,0,0,1,\cdot)$	$x_2 \vee \bar{x}_5$
2	$\boxed{x_1 \vee x_2}$		

$x_1 \vee x_5$, $x_2 \vee \bar{x}_5$ **parallel-resolve** to yield $x_1 \vee x_2$

$x_1 \vee x_2$ **parallel-absorbs** $x_1 \vee x_5$, $x_2 \vee \bar{x}_5$

Constraint-Directed Search

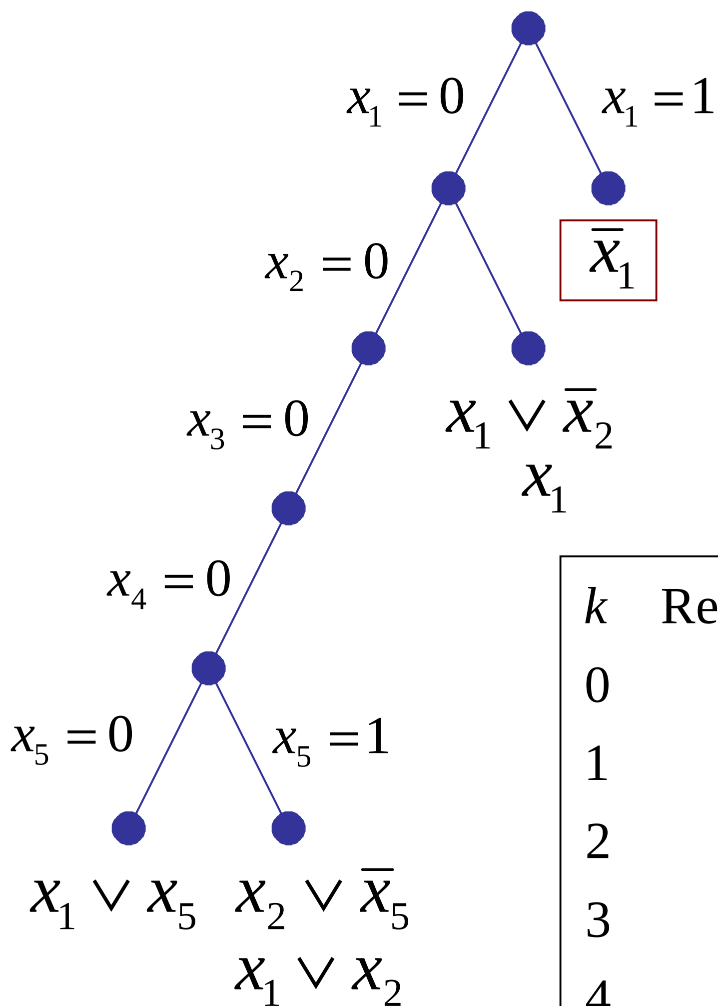


k	Relaxation R_k	Solution of R_k	Nogoods
0		$(0,0,0,0,0,\cdot)$	$x_1 \vee x_5$
1	$x_1 \vee x_5$	$(0,0,0,0,1,\cdot)$	$x_2 \vee \bar{x}_5$
2	$x_1 \vee x_2$	$(0,1,\cdot,\cdot,\cdot,\cdot)$	$x_1 \vee \bar{x}_2$
3	$\boxed{x_1}$		

$x_1 \vee x_2$

$x_1 \vee \bar{x}_2$ parallel-resolve to yield x_1

Constraint-Directed Search



k	Relaxation R_k	Solution of R_k	Nogoods
0		$(0,0,0,0,0,\cdot)$	$x_1 \vee x_5$
1	$x_1 \vee x_5$	$(0,0,0,0,1,\cdot)$	$x_2 \vee \bar{x}_5$
2	$x_1 \vee x_2$	$(0,1,\cdot,\cdot,\cdot,\cdot)$	$x_1 \vee \bar{x}_2$
3	x_1	$(1,\cdot,\cdot,\cdot,\cdot,\cdot)$	$\bar{x}_1$
4	$\boxed{\varnothing}$		

Search terminates

Constraint-Directed Search

Solution Method	*Restriction P_k*	*Relaxation R_k*	*Selection Function $s(R_k)$*	*Inference*
DPL for SAT	Add conflict clauses	Processed conflict clauses	Unit clause rule + greedy solution of R_k	Parallel resolution & absorption
Partial order dynamic backtracking	**Add nogoods**	**Processed nogoods**	**Greedy, consistent with partial order**	**Parallel resolution & absorption**
Logic-based Benders	Subproblem defined by solution of master	Master problem (Benders cuts)	Optimal solution of master	Benders cuts (nogoods)

Constraint-Directed Search: Partial Order Dynamic Backtracking

Solve same problem as before

$$x_1 \vee x_5 \vee x_6$$
$$x_1 \vee x_5 \vee \bar{x}_6$$
$$x_2 \vee \bar{x}_5 \vee x_6$$
$$x_2 \vee \bar{x}_5 \vee \bar{x}_6$$
$$\bar{x}_1 \vee x_3 \vee x_4$$
$$\bar{x}_2 \vee x_3 \vee x_4$$
$$\bar{x}_1 \vee \bar{x}_3$$
$$\bar{x}_1 \vee \bar{x}_4$$
$$\bar{x}_2 \vee \bar{x}_3$$
$$\bar{x}_2 \vee \bar{x}_4$$

To solve it:

- **Search**: generate problem restrictions.

To solve it:

- **Search**: generate problem restrictions.
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To solve it:

- **Search**: generate problem restrictions.
- **Infer:** generate nogoods.
  - Process nogoods in current relaxation with **parallel resolution**.
- **Relax:** Relaxation R_k consists of current processed nogoods.
  - **Selection function**: Solution of R_k must **conform to** current nogoods. Also apply unit clause rule.

Partial Order Dynamic Backtracking

k	Relaxation R_k	Solution of R_k	Nogoods
0		$(0,0,0,0,0,\cdot)$	$x_5 \vee \boxed{x_1}$
1	$x_1 \vee x_5$		

Arbitrarily choose one variable to be **last**

Partial Order Dynamic Backtracking

k	Relaxation R_k	Solution of R_k	Nogoods
0		$(0,0,0,0,0,\cdot)$	$\boxed{x_5} \vee \boxed{x_1}$
1	$x_1 \vee x_5$		

Other variables are **penultimate**

Arbitrarily choose one variable to be **last**

Partial Order Dynamic Backtracking

k	Relaxation R_k	Solution of R_k	Nogoods
0		$(0,0,0,0,0,\cdot)$	$x_5 \vee x_1$
1	$x_1 \vee x_5$	$(1,\cdot,\cdot,\cdot,0,\cdot)$	
2			

Since x_5 is **penultimate** in at least one nogood, it must **conform** to nogoods.

It must take value **opposite** its sign in the nogoods.

x_5 will have the **same sign** in all nogoods where it is penultimate.

This allows **more freedom** than chronological backtracking.

Partial Order Dynamic Backtracking

k	Relaxation R_k	Solution of R_k	Nogoods
0		$(0,0,0,0,0,\cdot)$	$x_5 \vee x_1$
1	$x_1 \vee x_5$	$(1,\cdot,\cdot,\cdot,0,\cdot)$	$x_5 \vee \boxed{\bar{x}_1}$
2			

Choice of **last** variable is arbitrary but must be consistent with **partial order** implied by previous choices.

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Partial Order Dynamic Backtracking

k	Relaxation R_k	Solution of R_k	Nogoods
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1	$x_1 \vee x_5$	$(1,\cdot,\cdot,\cdot,0,\cdot)$	$x_5 \vee \boxed{\bar{x}_1}$
2	$\boxed{x_5}$		

Choice of **last** variable is arbitrary but must be consistent with **partial order** implied by previous choices.

$$x_5 \vee x_1$$
$$x_5 \vee \bar{x}_1$$

Parallel-resolve to yield x_5

Since x_5 is **penultimate** in at least one nogood, it must **conform** to nogoods.

It must take value **opposite** its sign in the nogoods.

x_5 will have the **same sign** in all nogoods where it is penultimate.

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Partial Order Dynamic Backtracking

k	Relaxation R_k	Solution of R_k	Nogoods
0		$(0,0,0,0,0,\cdot)$	$x_5 \vee x_1$
1	$x_1 \vee x_5$	$(1,\cdot,\cdot,\cdot,0,\cdot)$	$x_5 \vee \bar{x}_1$
2	x_5	$(\cdot,0,\cdot,\cdot,1,\cdot)$	$\bar{x}_5 \vee x_2$
3	$\boxed{\left\{\begin{matrix} x_5 \\ \bar{x}_5 \vee x_2 \end{matrix}\right\}}$		

x_5 does not parallel-resolve with $\bar{x}_5 \vee x_2$

because x_5 is not last in both clauses

Partial Order Dynamic Backtracking

k	Relaxation R_k	Solution of R_k	Nogoods
0		$(0,0,0,0,0,\cdot)$	$x_5 \vee x_1$
1	$x_1 \vee x_5$	$(1,\cdot,\cdot,\cdot,0,\cdot)$	$x_5 \vee \bar{x}_1$
2	x_5	$(\cdot,0,\cdot,\cdot,1,\cdot)$	$\bar{x}_5 \vee x_2$
3	$\left\{ \begin{matrix} x_5 \\ \bar{x}_5 \vee x_2 \end{matrix} \right\}$	$(\cdot,1,\cdot,\cdot,1,\cdot)$	$\bar{x}_2$
4	$\varnothing$		

Search terminates

Must conform

Constraint-Directed Search

Solution Method	*Restriction P_k*	*Relaxation R_k*	*Selection Function $s(R_k)$*	*Inference*
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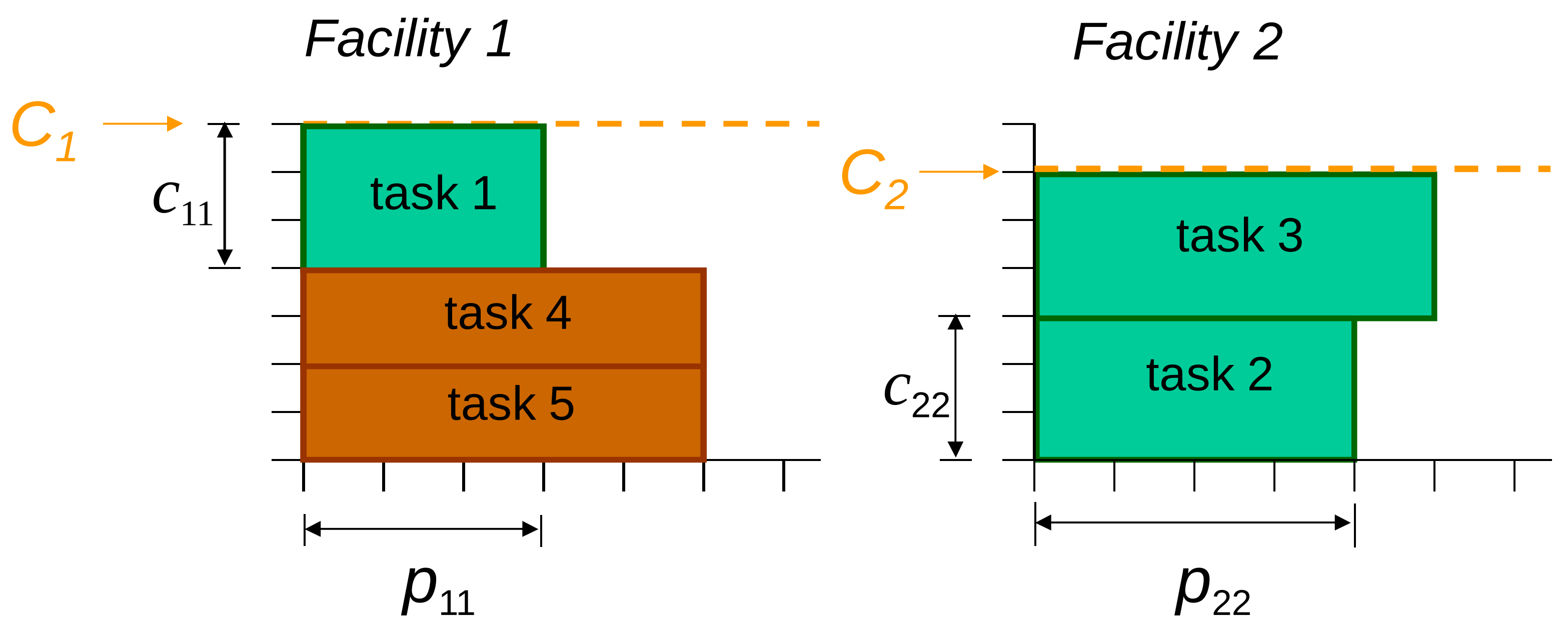
Constraint-Directed Search: Logic-Based Benders Decomposition

Planning and scheduling problem:

- Allocate tasks to facilities.
- Schedule tasks assigned to each facility.
  - Subject to deadlines.
  - Facilities may run at different speeds and incur different costs.
- Cumulative scheduling
  - Several tasks may run simultaneously on a facility.
  - But total resource consumption must never exceed limit.

Planning and Scheduling

p_{ij} = processing time of task j on facility i
c_{ij} = resource consumption of task j on facility i
C_i = resources available on facility i



Total resource consumption $\leq C_i$ at all times.

Planning and Scheduling

y_j = facility assigned to task j

$$\min \sum_j c_{y_j j}$$

$$\text{cumulative}\begin{pmatrix} (t_j \mid y_j = i) \\ (p_{ij} \mid y_j = i) \\ (c_{ij} \mid y_j = i) \\ C_i \end{pmatrix}, \quad \text{all } i$$

$$0 \le t_j \le d_j - p_{y_j j}, \quad \text{all } j$$

Planning and Scheduling

Cost of processing task *j* on facility *i*

y_j = facility assigned to task *j*

$$\min \sum_j c_{y_j j}$$

$$\text{cumulative}\begin{pmatrix} (t_j \mid y_j = i) \\ (p_{ij} \mid y_j = i) \\ (c_{ij} \mid y_j = i) \\ C_i \end{pmatrix}, \quad \text{all } i$$

$$0 \le t_j \le d_j - p_{y_j j}, \quad \text{all } j$$

Planning and Scheduling

Cost of processing task *j* on facility *i*

y_j = facility assigned to task *j*

start times of tasks assigned to facility *i*

$$\min \sum_j c_{y_j j}$$

$$\text{cumulative}\begin{pmatrix} (t_j \mid y_j = i) \\ (p_{ij} \mid y_j = i) \\ (c_{ij} \mid y_j = i) \\ C_i \end{pmatrix}, \quad \text{all } i$$

$$0 \le t_j \le d_j - p_{y_j j}, \quad \text{all } j$$

Observe resource limit on each facility

Planning and Scheduling

Cost of processing task *j* on facility *i*

y_j = facility assigned to task *j*

$$\min \sum_j c_{y_j j}$$

start times of tasks assigned to facility *i*

$$\text{cumulative}\begin{pmatrix} (t_j \mid y_j = i) \\ (p_{ij} \mid y_j = i) \\ (c_{ij} \mid y_j = i) \\ C_i \end{pmatrix}, \quad \text{all } i$$

Observe resource limit on each facility

$$0 \le t_j \le d_j - p_{y_j j}, \quad \text{all } j$$

Observe time windows

To solve it:

- **Search**: enumerate assignments of tasks to facilities.
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- **Relax:** Relaxation R_k consists of Benders cuts generated so far .
  - **Selection function**: Any optimal solution of R_k (= **master problem**).

Restrict (Subproblem)

For a given assignment of tasks to facilities,
find a feasible schedule

Solve by **constraint programming**

Given assignment

$$\left\{ \begin{array}{l} \text{cumulative}\begin{pmatrix} (t_j \mid \bar{y}_j = i) \\ (p_{ij} \mid \bar{y}_j = i) \\ (c_{ij} \mid \bar{y}_j = i) \\ C_i \end{pmatrix} \\ 0 \le t_j \le d_j \end{array} \right\}, \quad \text{all } i$$

Infer (Benders cut = nogood)

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Let J_{ih} = {tasks assigned to facility i in iteration h}.

If there is no feasible schedule, create Benders cut:

$$y_j \neq i \text{ for some } j \in J_{hi}$$

Relax (Master problem)

Solve by MILP.

$$\min \sum_{ij} c_{ij} x_{ij}$$

$$\sum_i x_{ij} = 1, \quad \text{all } j$$

$$\sum_{j \in J_{hi}} (1 - x_{ij}) \geq 1, \quad \text{all } h, i$$

$$x_{ij} \in \{0,1\}$$

Let $x_{ij} = 1$ when $y_j = i$

Task j is assigned to one facility

Benders cuts: $x_{ij} = 0$ for some $j \in J_{hi}$

Stop when subproblem is feasible (original problem is feasible)…

…**or** when master problem is infeasible (original problem is infeasible).

General Framework

Solution Method	*Restriction P_k*	*Relaxation R_k*	*Selection Function $s(R_k)$*	*Inference*
Branching CP, MILP, global optimization	Node of search tree	LP, NLP, domains	Any optimal (feasible) solution of R_k	Domain filtering, cutting planes
Constraint directed DPL, dynamic backtracking, Benders	Add nogoods generated so far	Processed nogoods generated so far	Solution that results in easy R_k	Nogood generation
Heuristics **Local search, GRASP**	**Neighborhood of current solution**	**Research topic**	**Random, best, etc.**	**?**

Heuristic Methods

Solution Method	*Restriction* P_k	*Relaxation* R_k	*Selection Function* $s(R_k)$	*Inference*
Simulated annealing	Neighborhood of current solution	P_k	Random solution from nbhd	None
Tabu search	Neighborhood minus tabu list	P_k	Best solution in nbhd	Items in tabu list
GRASP	Neighborhood of partial solution	Problem specific	Solve R_k only at leaf node	None

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Simulated Annealing

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- **Relax:** Same as restriction.
  - **Selection function**: Random solution x in neighborhood.
  - Next restriction:
    - neighborhood of x if x is better than previous solution;
    - otherwise neighborhood of x with probability p,
      current neighborhood with probability $1 - p$.

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  - **Selection function**: Best solution x in neighborhood that is consistent with tabu list.
  - Next restriction: neighborhood of x.

Heuristic Methods

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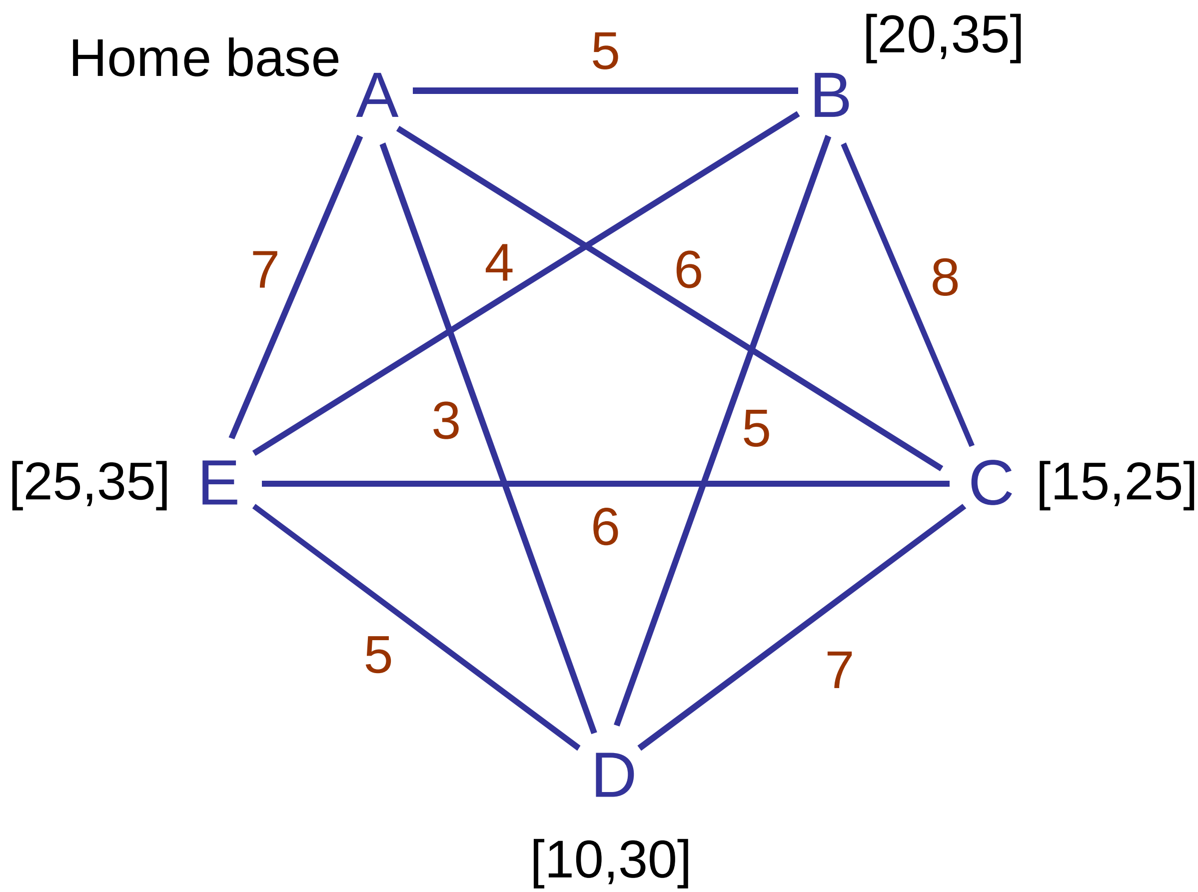
Heuristics:
Generalized GRASP

Greedy Randomized Adaptive Search Procedure

TSP with Time Windows



Heuristics: Generalized GRASP

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    - **Local search phase**: Randomly backtrack and select next customer in random fashion.

Generalized GRASP

Sequence
of customers
visited

A · · · · A

Basically,

GRASP =
greedy solution +
local search

Begin with greedy
assignments that
can be viewed as
creating
“branches”

Generalized GRASP

Greedy phase

A · · · · A

AD · · · A

Visit customer than can be served earliest from A

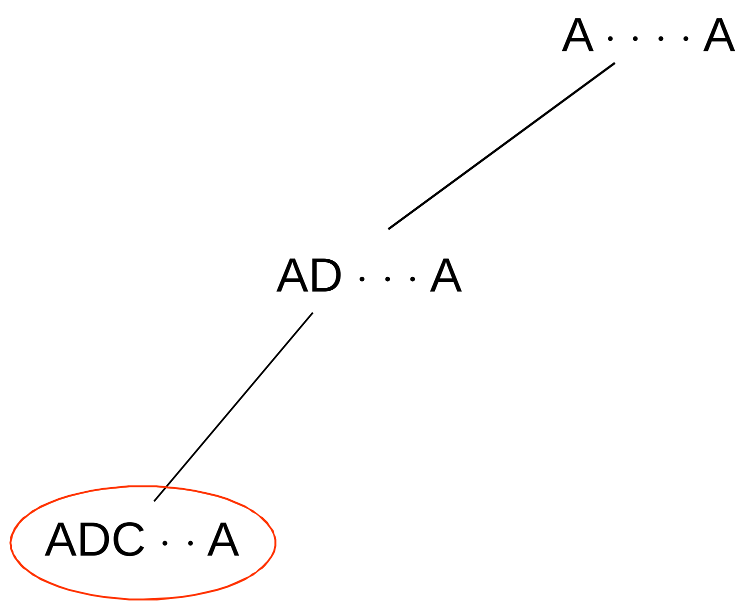
Basically,

GRASP = greedy solution + local search

Begin with greedy assignments that can be viewed as creating "branches"

Generalized GRASP

Greedy
phase



Next, visit
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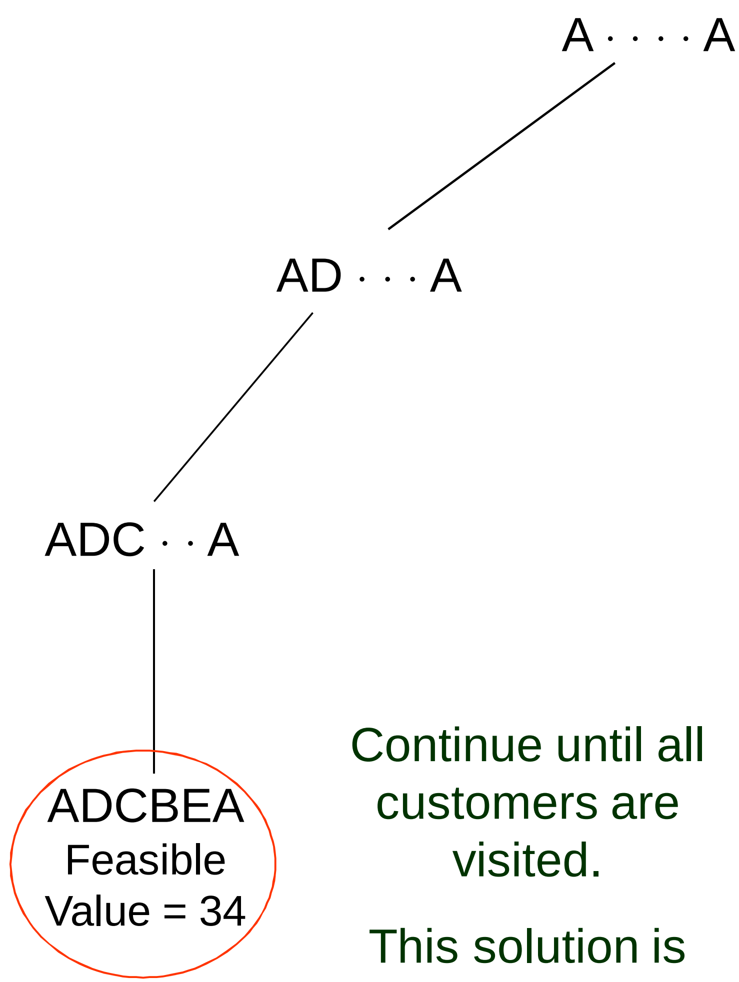
Generalized GRASP

Greedy phase



Continue until all customers are visited.

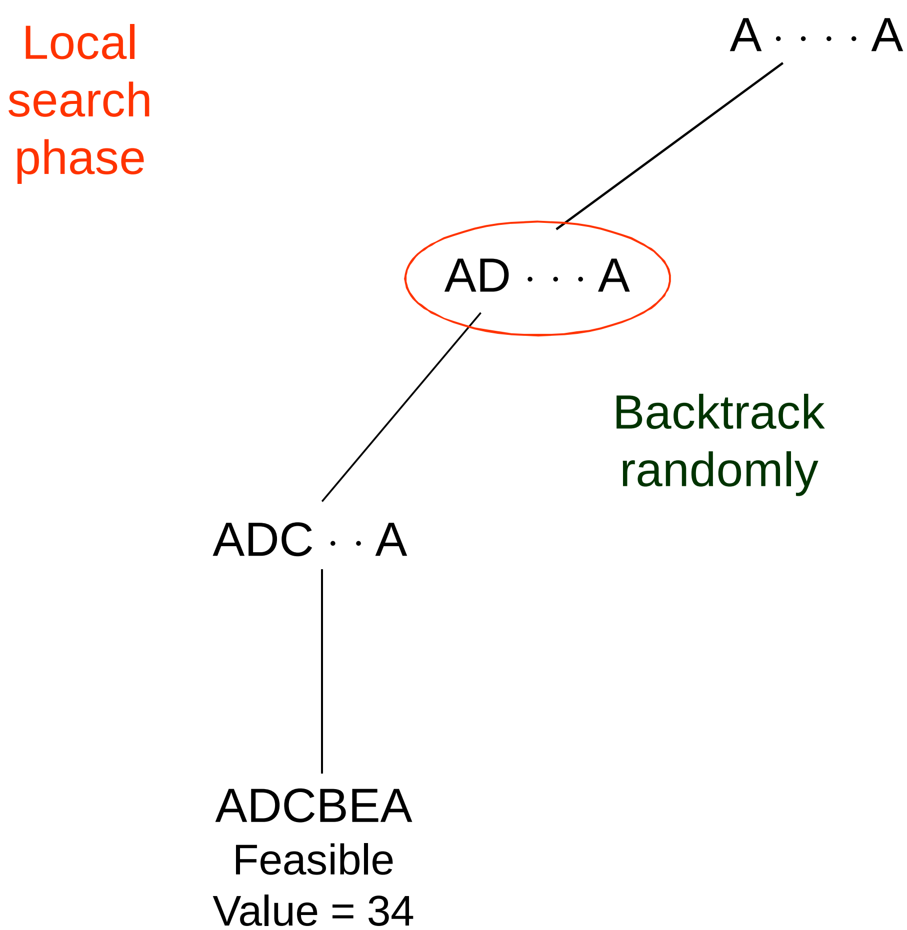
This solution is feasible. Save it.

Basically,

GRASP = greedy solution + local search

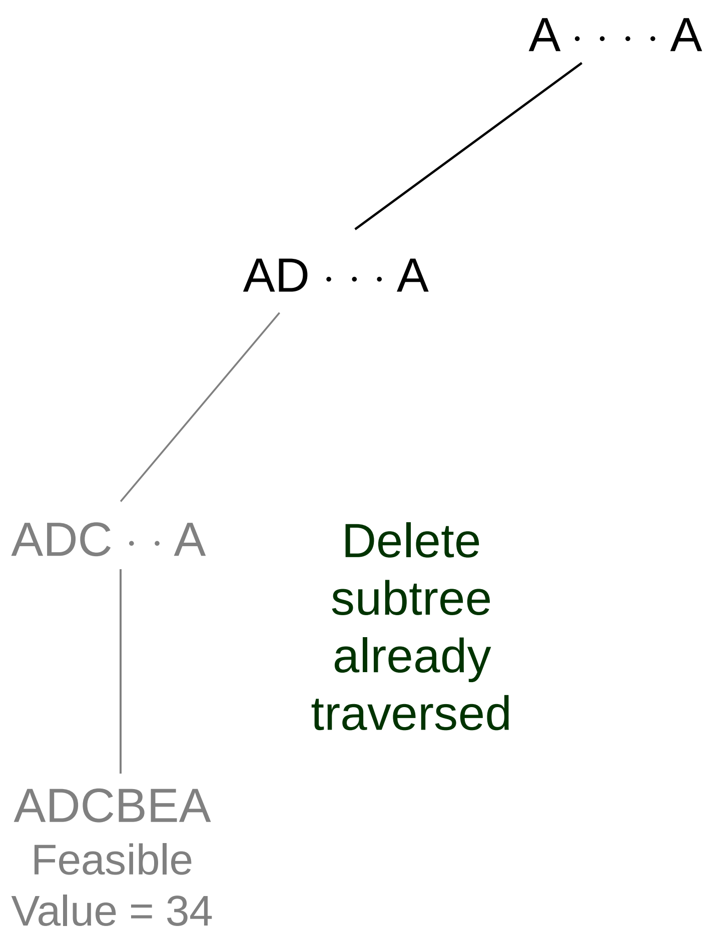
Begin with greedy assignments that can be viewed as creating “branches”

Generalized GRASP



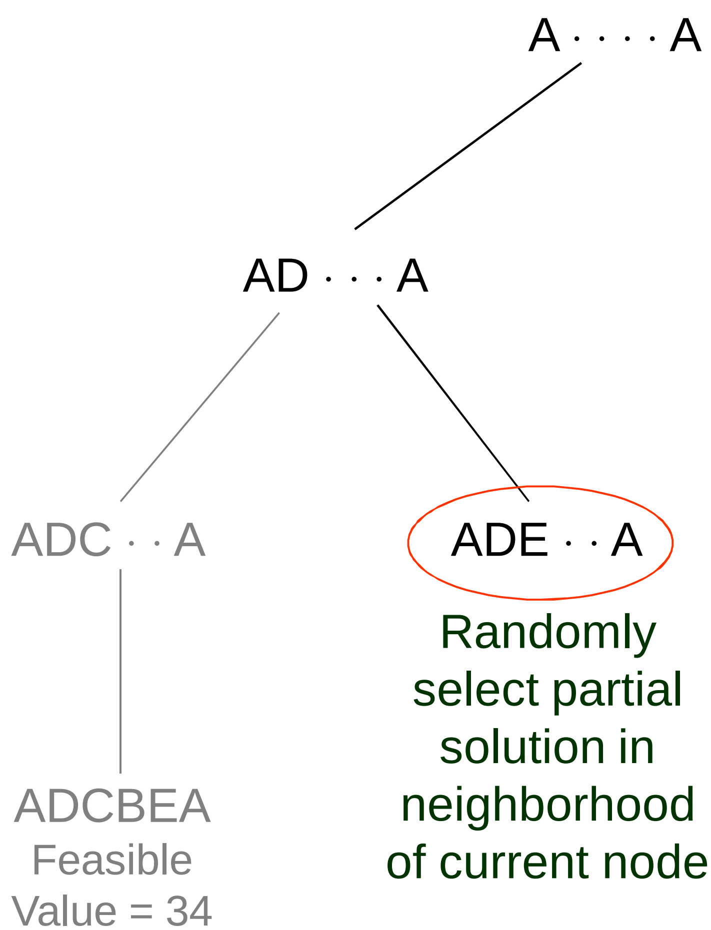
Generalized GRASP

Local
search
phase

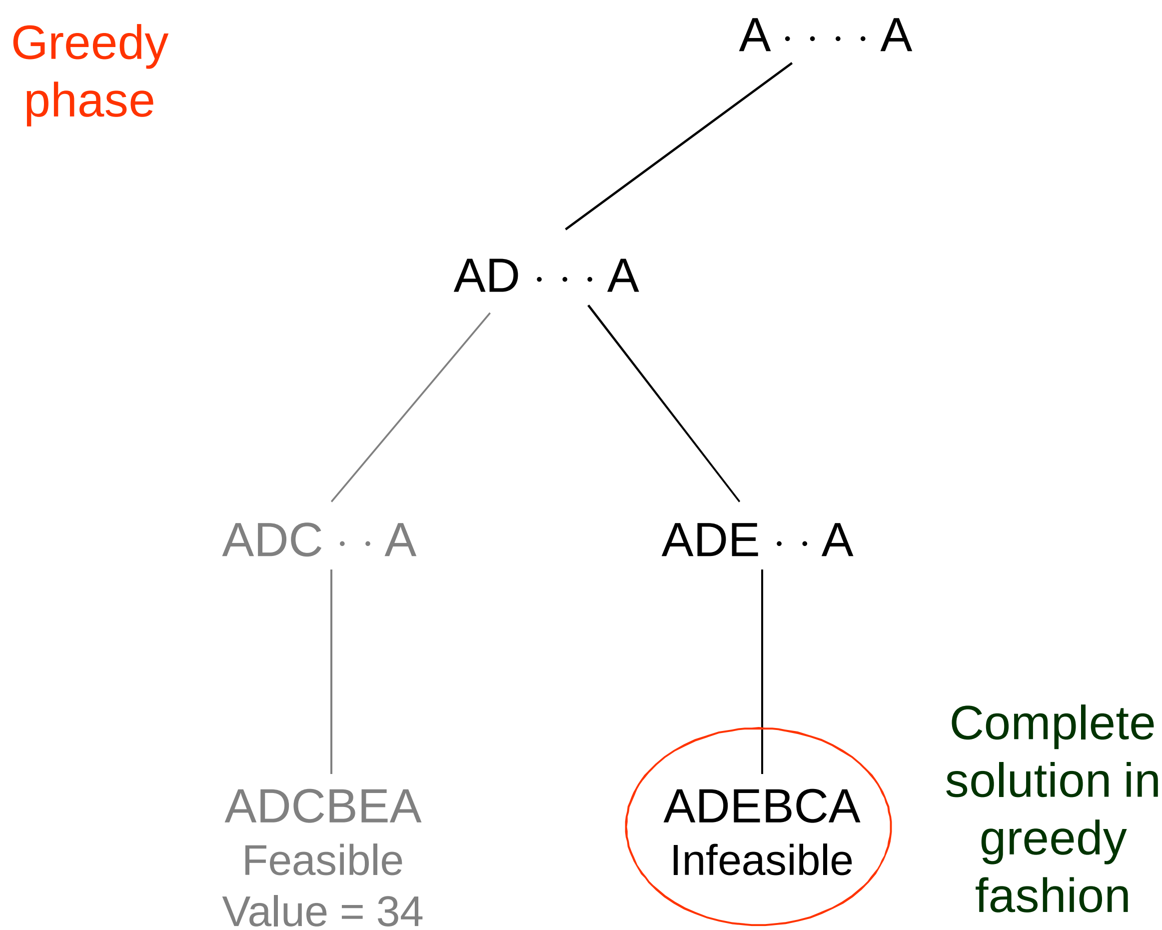


Generalized GRASP

Local search phase



Generalized GRASP



Generalized GRASP

Local
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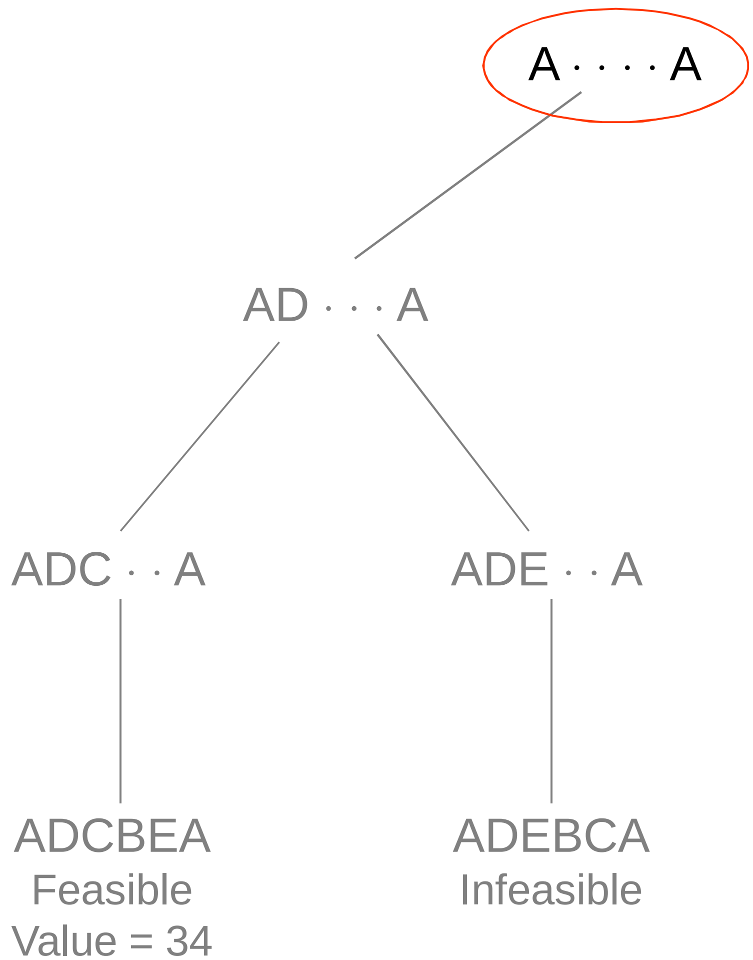
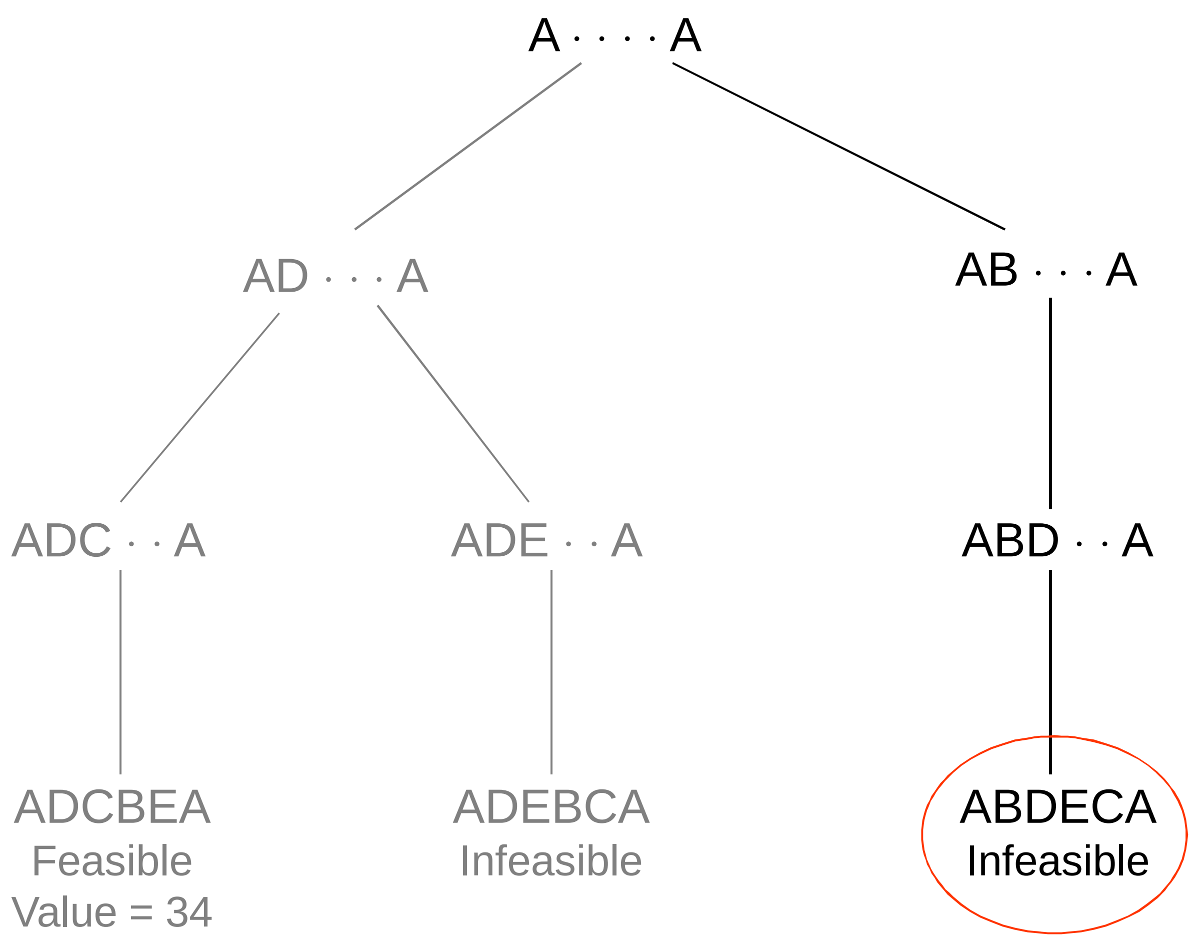


Randomly
backtrack

Generalized GRASP



Continue in similar fashion

Local search
is generalized GRASP
in which the “branching” tree
has 2 levels

To add a relaxation to generalized GRASP:

Suppose that customers $x_0, x_1, \ldots, x_k$ have been visited so far.

Let t_{ij} = travel time from customer i to j.

Then total travel time of completed route is bounded below by

$$\boxed{T} + \sum_{j \notin \{x_0,\ldots,x_k\}} \boxed{\min\left\{ t_{x_k j},\ \min_{i \notin \{j, x_0,\ldots,x_k\}} \{t_{ij}\} \right\}} + \boxed{\min_{j \notin \{x_0,\ldots,x_k\}} \{t_{j0}\}}$$

Earliest time vehicle can leave customer *k*

Min time from customer *j*'s predecessor to *j*

Min time from last customer back to home

Heuristics: Generalized GRASP with relaxation

To solve it:

- **Search**: enumerate neighborhoods of partial solutions
  - Neighboring solution is created by selecting customer to visit next.
- **Infer:** None.
- **Relax:** As just described.
  - **Selection function**:
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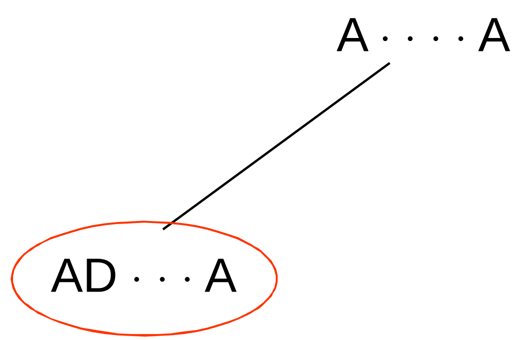
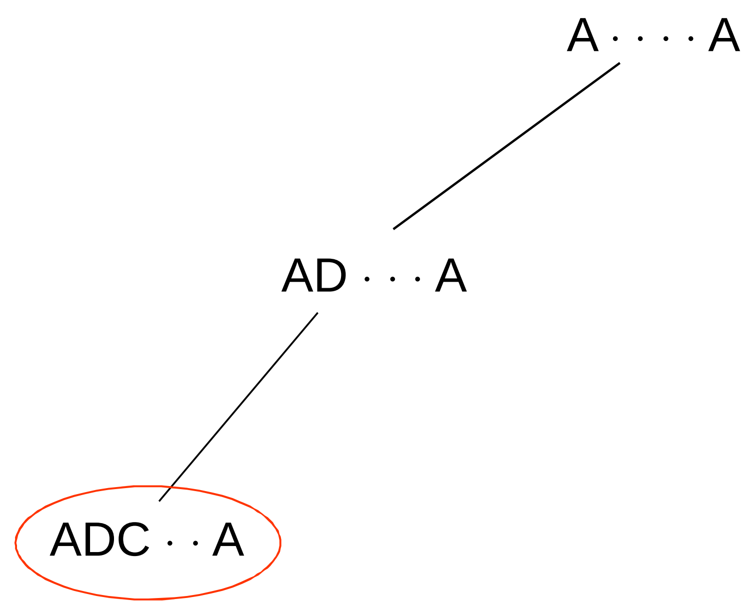
Generalized GRASP with relaxation

Greedy phase



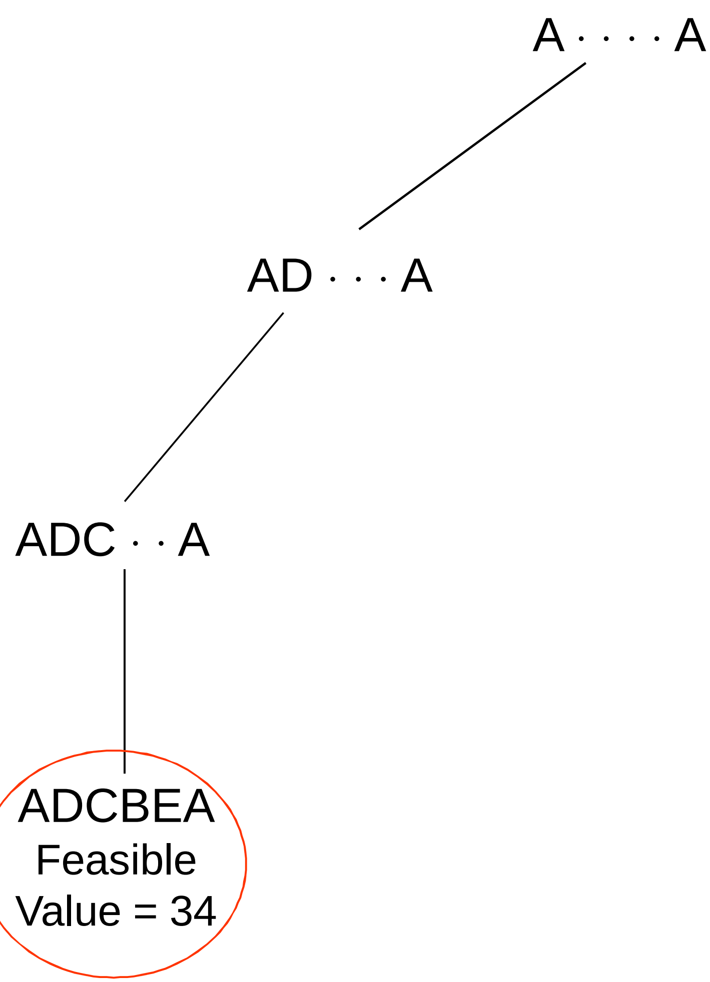
Generalized GRASP

Greedy
phase

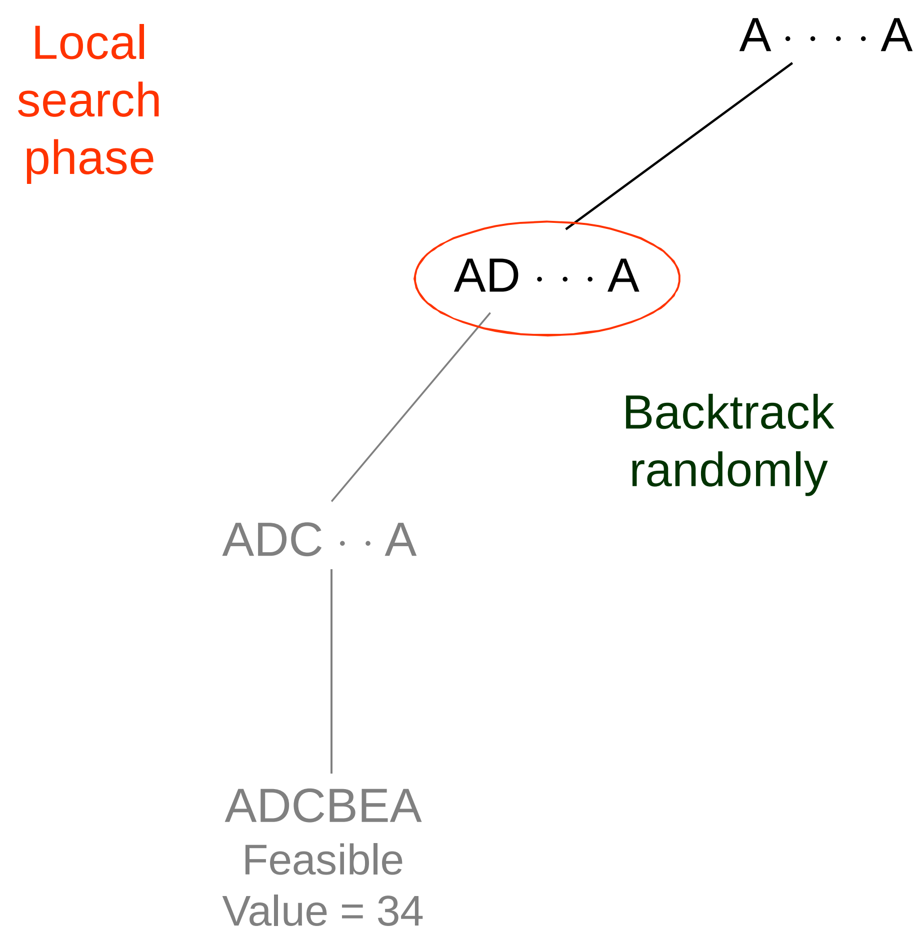


Generalized GRASP

Greedy phase

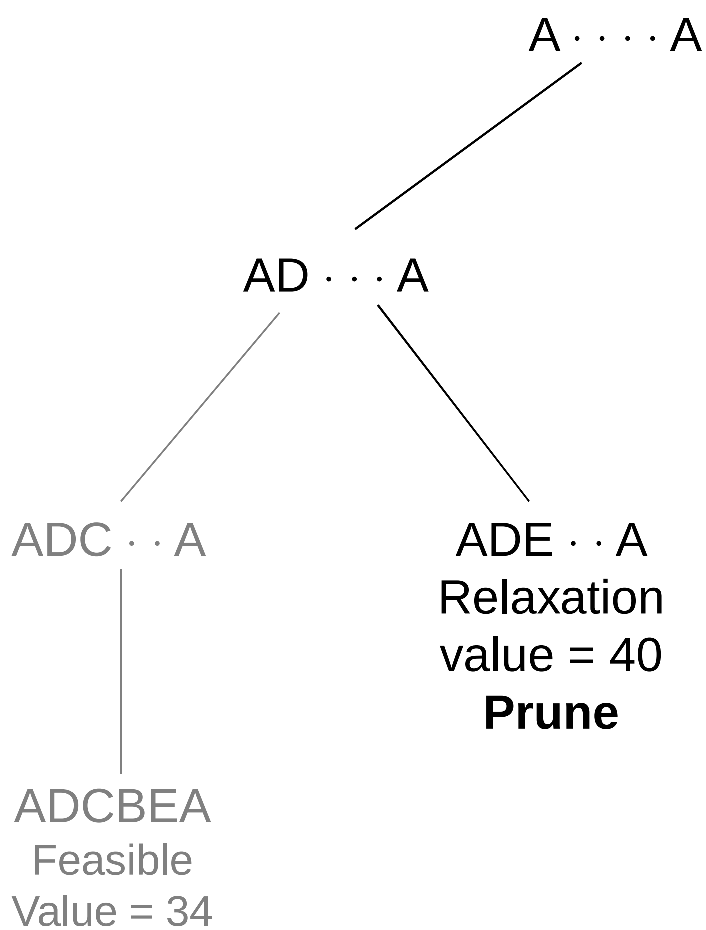


Generalized GRASP

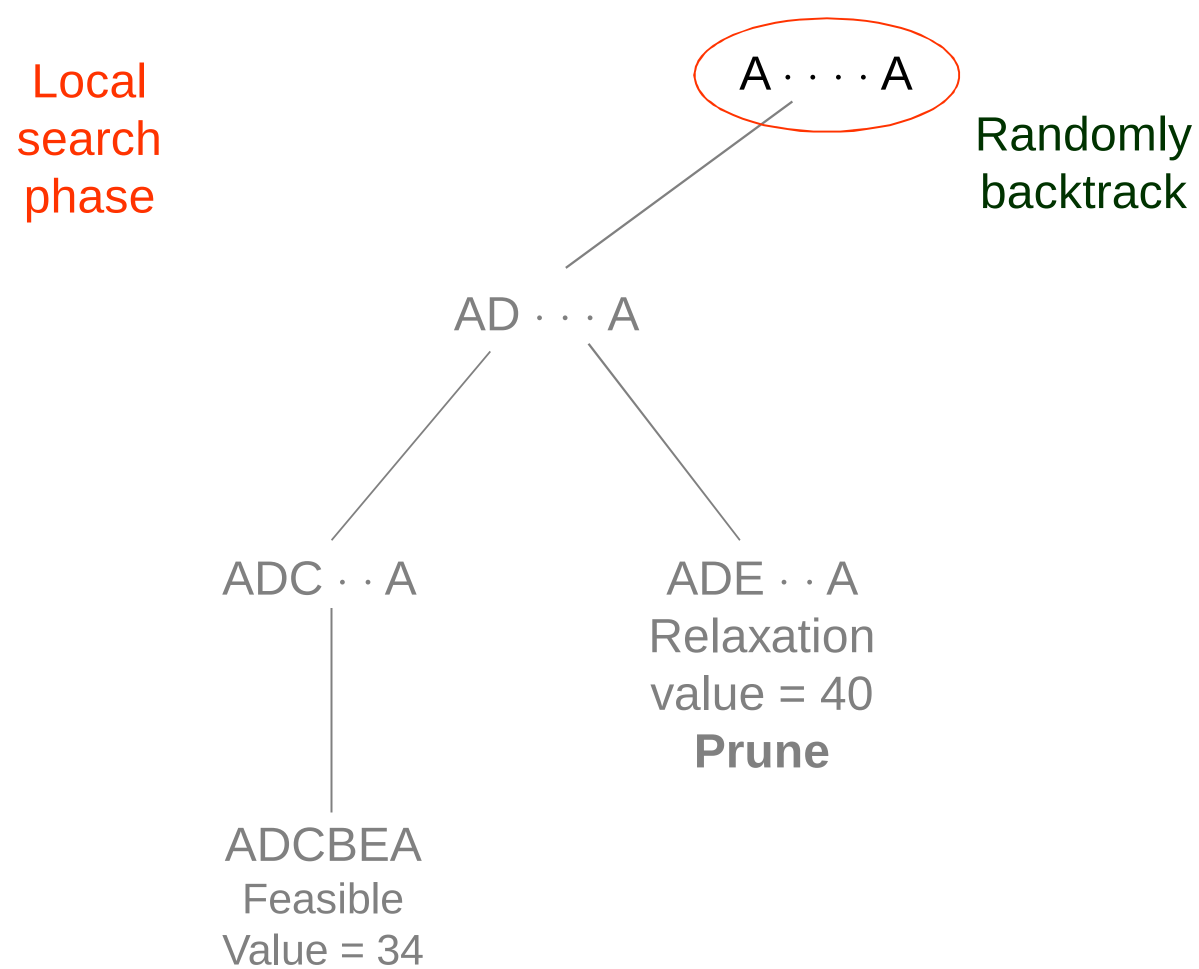


Generalized GRASP

Local search phase

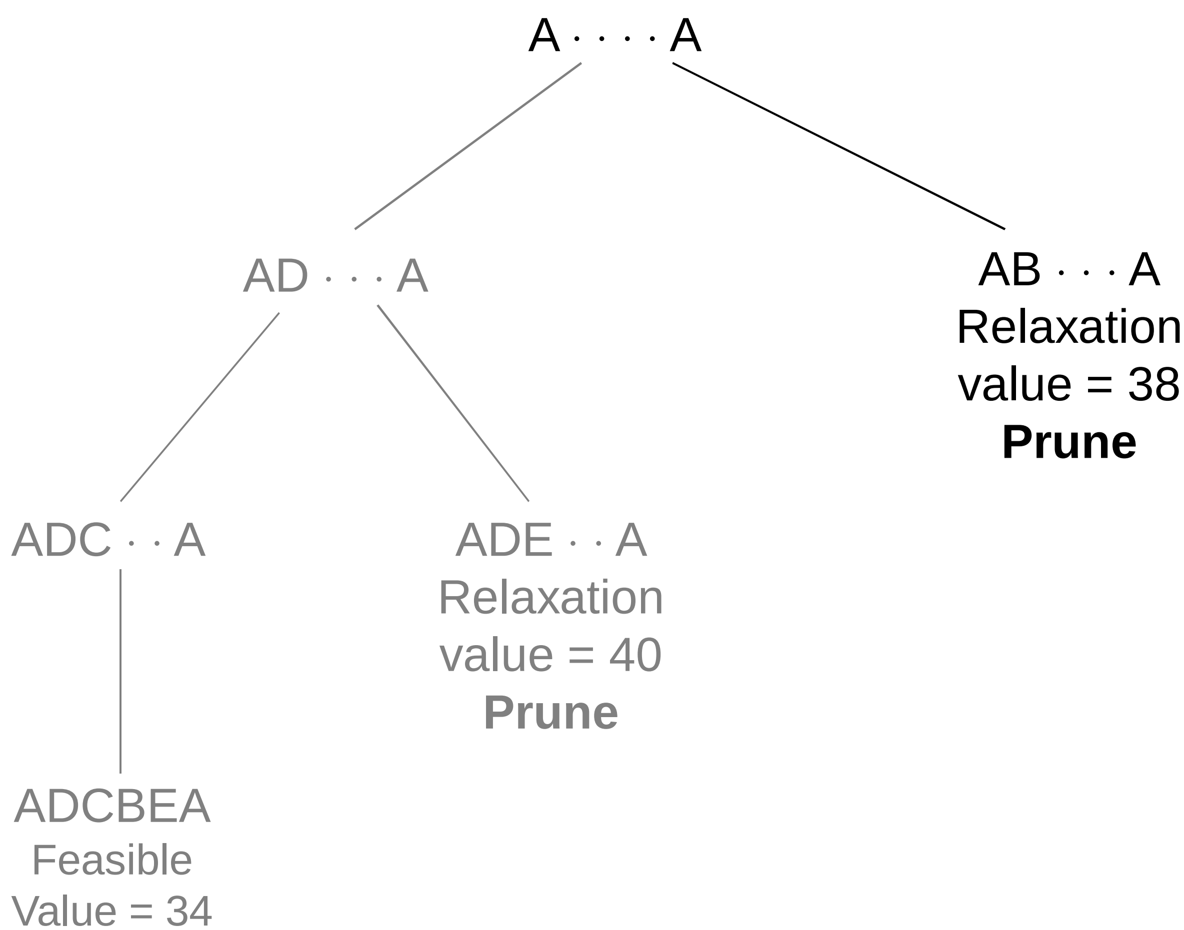


Generalized GRASP



Generalized GRASP

Local
search
phase



Product endorsement:

SIMPL
is a partial implementation of this approach
(CP-AI-OR 2004)

We expect to post on the web this fall.