A Framework for Integrating Solution Methods

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These slides are available at

http://ba.gsia.cmu.edu/jnh

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Main Goals

- View different solution methods as special cases of a **single, general method**.
- In particular, unify **constraint programming**, **integer programming**, and **local search**.
- Do this by identify **common strategies** in these methods:
  - **Search** over problem restrictions
  - **Inference** of valid constraints
  - Solution of **relaxations** that guide the search

Outline

- Knapsack illustration: Combining CP and IP.
- Common elements of CP, IP and local search.
- An integrated solver: putting it together
- Examples
  - Knapsack problem
  - Traveling salesman
  - Processing network design
  - Local search
  - Digression: relaxation and inference dualities
  - Benders decomposition
  - Machine scheduling – illustrates how Benders-based hybrid methods are a special case of present framework

Illustration: Combining CP and IP

$$
\begin{array}{ll}
\min & 5x_1 + 8x_2 + 4x_3 \\
\text{subject to} & 3x_1 + 5x_2 + 2x_3 \geq 30 \\
 & \text{all-different}\{x_1, x_2, x_3\} \\
 & x_j \in \{1, \ldots, 4\}
\end{array}
$$

We will illustrate how search, inference and relaxation may be combined to solve this problem by:

- constraint programming
- integer programming
- a hybrid approach

Solve as a constraint programming problem

Search: Domain splitting
Inference: Domain reduction
Relaxation: Constraint store (set of current variable domains)

$$\begin{aligned}
&5x_1 + 8x_2 + 4x_3 \le z \\
&3x_1 + 5x_2 + 2x_3 \ge 30 \\
&\text{all-different}\{x_1, x_2, x_3\} \\
&x_j \in \{1,\ldots,4\}
\end{aligned}$$

Start with z = ∞.
Will decrease as feasible solutions are found.

Global constraint

Constraint store can be viewed as consisting of in-domain constraints $x_j \in D_j$, which form a relaxation of the problem.

Domain reduction for inequalities

- Bounds propagation on $$\begin{aligned} 5x_1 + 8x_2 + 4x_3 &\le z \\ 3x_1 + 5x_2 + 2x_3 &\ge 30 \end{aligned}$$

For example, $3x_1 + 5x_2 + 2x_3 \ge 30$ implies

$$x_2 \ge \frac{30 - 3x_1 - 2x_3}{5} \ge \frac{30 - 12 - 8}{5} = 2$$

So the domain of x_2 is reduced to $\{2,3,4\}$.

Domain reduction for all-different *(e.g., Régin)*

- Maintain hyperarc consistency on

$$\text{all-different}\{x_1, x_2, x_3\}$$

Suppose for example:

1	2		
1	2		
1	2	3	4

Domain of x_1

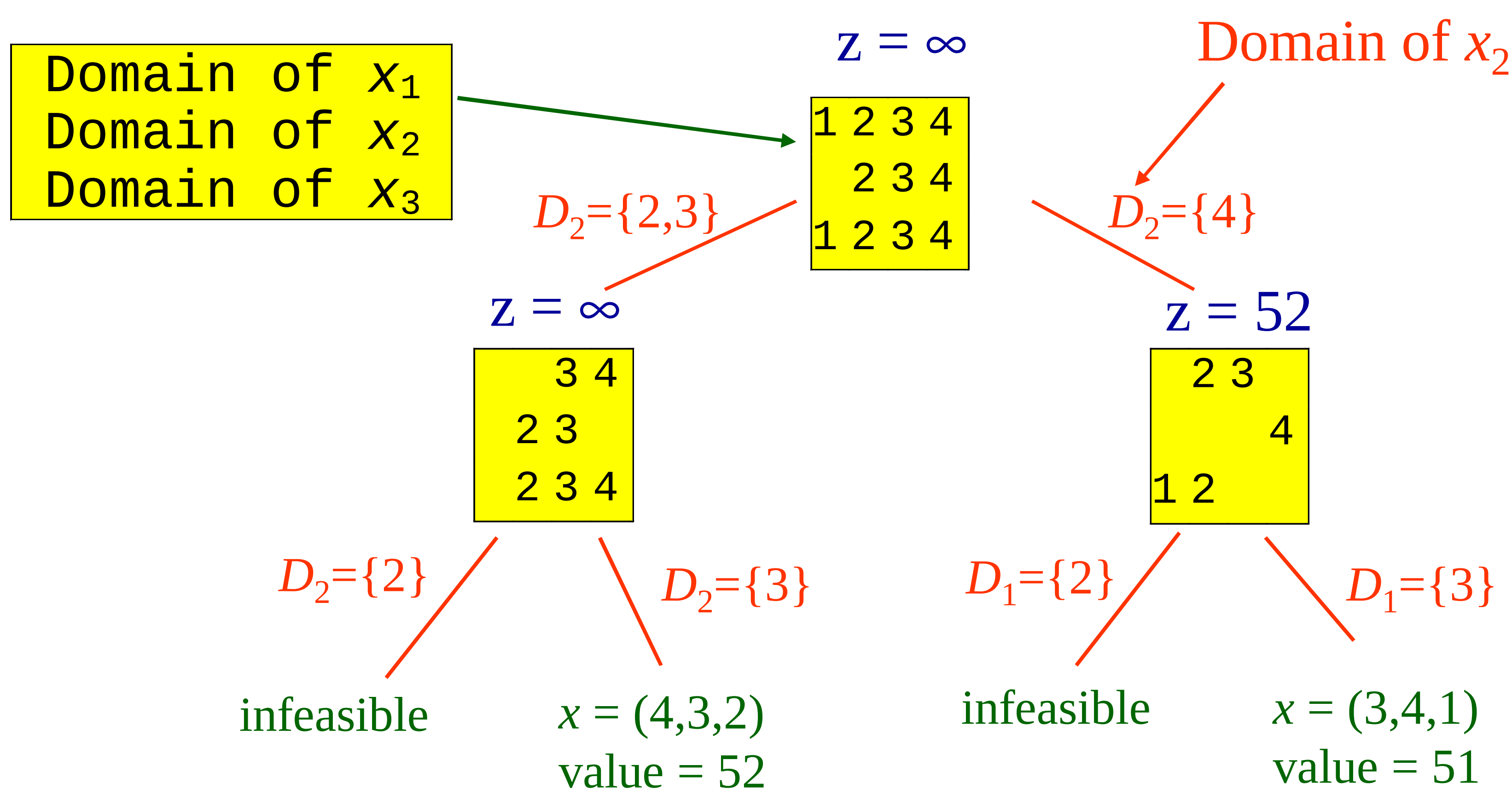
Domain of x_2

Domain of x_3

Then one can reduce the domains:

1	2		
1	2		
		3	4

- In general, solve a maximum cardinality matching problem and apply a theorem of Berge



Domain of x1
Domain of x2
Domain of x3
z = ∞
1 2 3 4
2 3 4
1 2 3 4
Domain of x2
D2={2,3}
D2={4}
z = ∞
3 4
2 3
2 3 4
z = 52
2 3
4
1 2
D2={2}
D2={3}
D1={2}
D1={3}
infeasible
x = (4,3,2)
value = 52
infeasible
x = (3,4,1)
value = 51

Solve as an integer programming problem

Search: Branch on variables with fractional values in solution of continuous relaxation.

Inference: Generate cutting planes (covering inequalities).

Relaxation: Continuous (LP) relaxation.

Rewrite problem using integer programming model:

Let y_{ij} be 1 if $x_i = j$, 0 otherwise.

$$\begin{array}{ll}
\text{min} & 5x_1 + 8x_2 + 4x_3 \\
\text{subject to} & 3x_1 + 5x_2 + 2x_3 \geq 30 \\
 & x_i = \sum_{j=1}^{5} j y_{ij}, \quad i = 1,2,3 \\
 & \sum_{j=1}^{4} y_{ij} = 1, \quad i = 1,2,3 \\
 & \sum_{i=1}^{3} y_{ij} \leq 1, \quad j = 1,\ldots,4 \\
 & y_{ij} \in \{0,1\}, \quad \text{all } i,\ j
\end{array}$$

Continuous relaxation

$$
\begin{array}{ll}
\text{min} & 5x_1 + 8x_2 + 4x_3 \\
\text{subject to} & 3x_1 + 5x_2 + 2x_3 \ge 30 \\
& x_i = \sum_{j=1}^{4} j y_{ij}, \quad i = 1,2,3 \\
& \sum_{j=1}^{4} y_{ij} = 1, \quad i = 1,2,3 \\
& \sum_{i=1}^{3} y_{ij} \le 1, \quad j = 1,\ldots,4 \\
& x_1 + x_2 \ge 5 \\
& x_1 + x_3 \ge 4 \\
& x_2 + x_3 \ge 4 \\
& x_1 + x_2 + x_3 \ge 8 \\
& 0 \le y_{ij} \le 1, \quad \text{all } i, j
\end{array}
$$

Covering inequalities

Relax integrality

Branch and bound (Branch and relax)

The *incumbent solution* is the best feasible solution found so far.

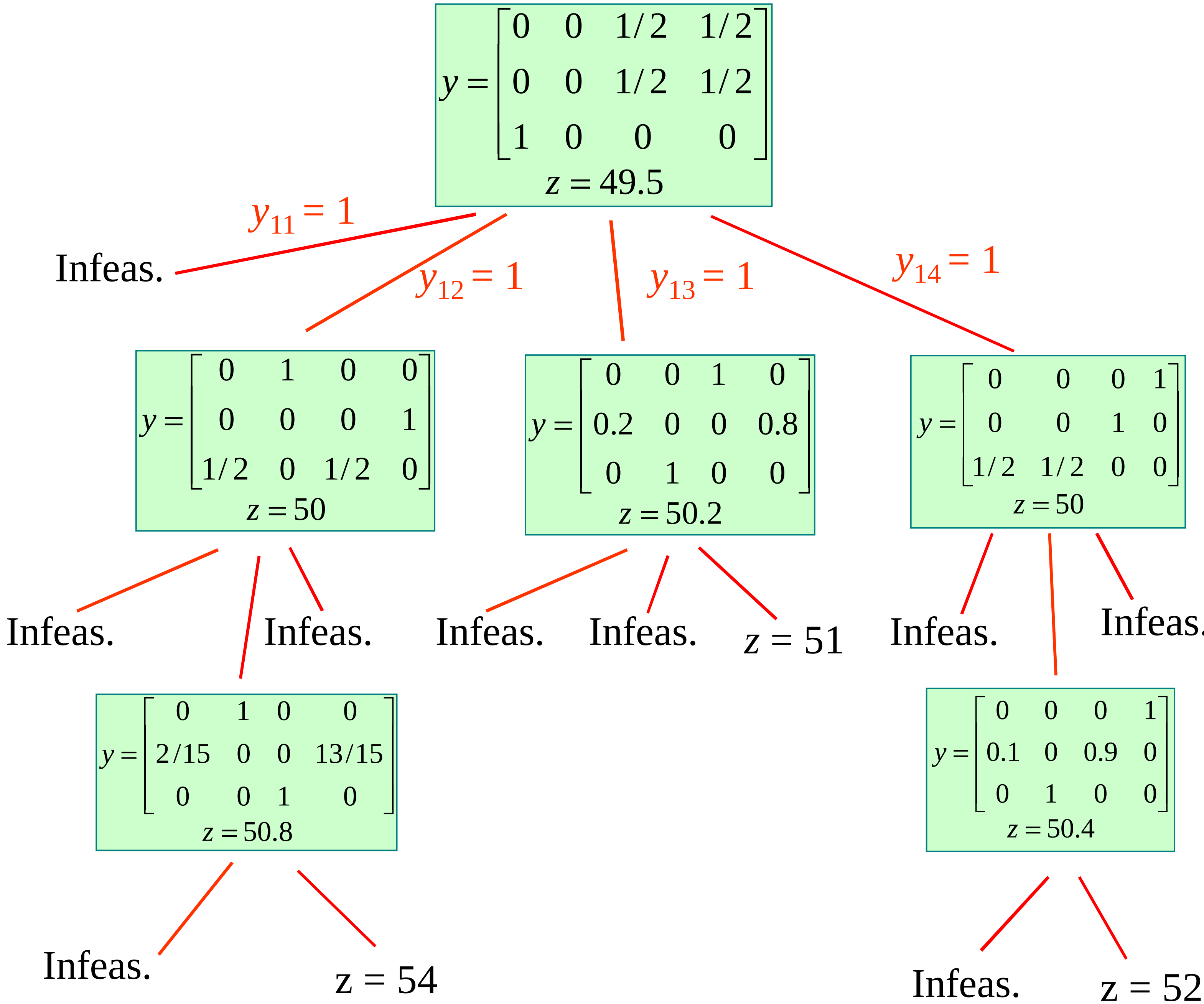
At each node of the branching tree:

- If Optimal value of relaxation $\geq$ Value of incumbent solution

There is no need to branch further.

- No feasible solution in that subtree can be better than the incumbent solution.

- Use SOS-1 branching.



y = [0 0 1/2 1/2; 0 0 1/2 1/2; 1 0 0 0]
z = 49.5
y11 = 1
y12 = 1
y13 = 1
y14 = 1
Infeas.
y = [0 1 0 0; 0 0 0 1; 1/2 0 1/2 0]
z = 50
y = [0 0 1 0; 0.2 0 0 0.8; 0 1 0 0]
z = 50.2
y = [0 0 0 1; 0 0 1 0; 1/2 1/2 0 0]
z = 50
Infeas.
Infeas.
Infeas.
Infeas.
z = 51
Infeas.
Infeas.
y = [0 1 0 0; 2/15 0 0 13/15; 0 0 1 0]
z = 50.8
y = [0 0 0 1; 0.1 0 0.9 0; 0 1 0 0]
z = 50.4
Infeas.
z = 54
Infeas.
z = 52

Solve using a hybrid approach

Search:

- Branch on fractional variables in solution of relaxation.
  - Drop constraints with y_{ij}'s. This makes relaxation too large without much improvement in quality.
  - If variables are all integral, branch by splitting domain.
- Use branch and bound.

Inference:

- Use bounds propagation for all inequalities.
- Maintain hyperarc consistency for all-different constraints.

Relaxation:

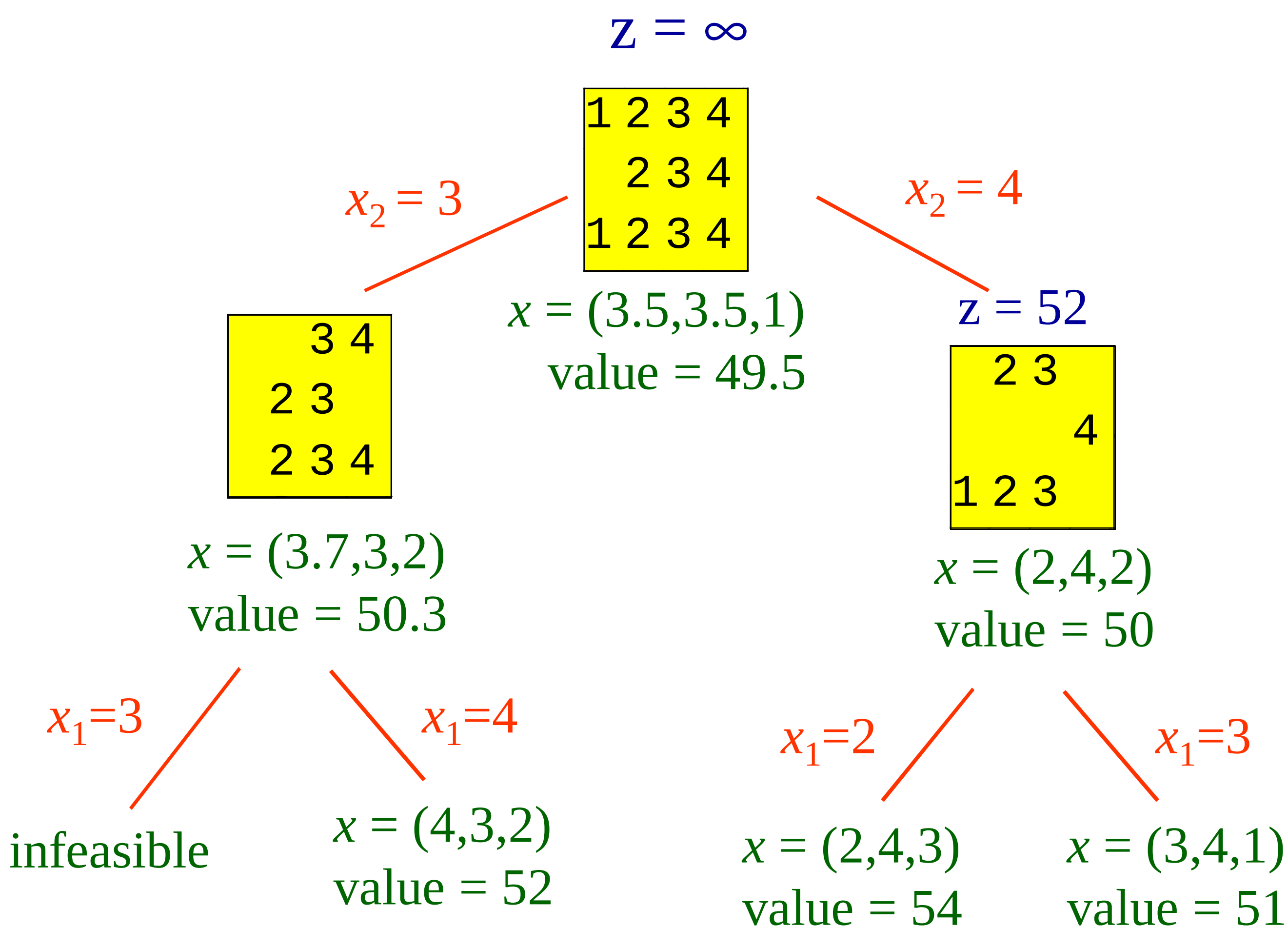
- Put knapsack constraint in LP.
- Put covering inequalities based on knapsack/all-different into LP.

Model for hybrid approach

$$
\begin{array}{ll}
\min & 5x_1 + 8x_2 + 4x_3 \le z \\
\text{s.t.} & 3x_1 + 5x_2 + 2x_3 \ge 30 \\
& \text{all-different}\{x_1, x_2, x_3\} \\
& x_1 + x_2 \ge 5 \\
& x_1 + x_3 \ge 4 \\
& x_2 + x_3 \ge 4 \\
& x_1 + x_2 + x_3 \ge 8 \\
& x_j \in \{1, \ldots, 4\}
\end{array}
$$

Covering inequalities

Generate and propagate covering inequalities at each node of search tree



z = ∞
1 2 3 4
2 3 4
1 2 3 4
x = (3.5,3.5,1)
value = 49.5
x2 = 3
x2 = 4
3 4
2 3
2 3 4
x = (3.7,3,2)
value = 50.3
z = 52
2 3
4
1 2 3
x = (2,4,2)
value = 50
x1=3
x1=4
infeasible
x = (4,3,2)
value = 52
x1=2
x1=3
x = (2,4,3)
value = 54
x = (3,4,1)
value = 51

Advantages of Hybrid Approach

- CP brings:
  - Succinct and natural models
  - Ability to exploit “horizontal” structure (structure of subsets of constraints) by using *global* constraints
  - Constraint propagation technology
    - Particularly useful when constraints contain few variables (or objective is minmax)

Advantages of Hybrid Approach

- IP brings:
  - Ability to exploit "vertical" structure (structure of special classes of problems)
  - Relaxation technology (cutting planes, Lagrangean relaxation, etc.)
    - Particularly useful when constraints contain many variables (or objective is min cost)
  - Duality theory (Benders, Lagrangean dual, etc.)

Common Elements of IP, CP and Local Search

- Search over problem restrictions
  - CP: Search over partial solutions
    - Branch on variable domains
    - Domains can be finite or continuous intervals
  - IP: Search over restrictions
    - Branch on fractional variables
  - Local search: Search over neighborhoods (which represent problem restrictions).
    - Move to a neighboring solution

Common Elements of IP, CP and Local Search

- Inference of new constraints
  - CP: Domain reduction / consistency maintenance
    - In-domain constraints for finite domains ($x_j \in D_j$)
      - Ideal: *hyperarc consistency* (remove all elements of domain that are not part of some solution)
    - Interval arithmetic
      - Ideal: *bounds consistency* (shrink interval domains as much as possible)
  - IP: Cutting planes
    - Covering inequalities, etc., etc.
      - Ideal: facet-defining cuts
  - Local search: research topic

Common Elements of IP, CP and Local Search

- Relaxation
  - CP: constraint store
    - Primarily in-domain constraints
    - Constraint store is not “solved” but propagates constraints and guides branching
  - IP: continuous relaxation
    - Inequalities in continuous variables
    - Solution of relaxation provides bounds and guides branching
  - Local Search: research topic

An Integrated Solver

- A *recursion* specifies **search** over problem restrictions.
- An *inference* engine **infers** valid constraints for each problem restriction.
- **Relaxations** and *special-purpose solvers* provide search guidance and perhaps bounds to prune the search.

Exploiting Structure

- The **search** process can be guided to suit the problem by setting parameters in canned recursive procedures.
- Specialized **domain reduction** and **cutting plane** procedures can be designed for global constraints (which represent specially-structured subsets of constraints)
    - This makes it convenient to use existing cutting-plane technology
- Specialized **relaxations** can be designed for global constraints and assembled into one or more relaxations of the entire problem.

Putting It Together

- Model consists of
  - *declaration window* (variables, initial domains)
  - *relaxation windows* (initialize relaxations & solvers)
  - *constraint windows* (each with its own syntax)
  - *objective function* (optional)
  - *search window* (invokes propagation, branching, relaxation, etc.)
- Basic algorithm searches over problem restrictions, drawing inferences and solving relaxations for each.

Putting It Together

- Relaxations may include:
  - Constraint store (with domains)
  - Linear programming relaxation, etc.
- The relaxations link the windows.
  - Propagation (e.g., through constraint store).
  - Search decisions (e.g., nonintegral solutions of linear relaxation).

Putting It Together

- A generic algorithm:
  - Process constraints.
    - Infer new constraints, reduce domains & propagate, generate relaxations.
  - Solve relaxations.
    - Check for empty domains, solve LP, etc.
  - Continue search (recursively).
    - Create new problem restrictions if desired (e.g, new tree branches).
    - Select problem restriction to explore next (e.g., backtrack or move deeper in the tree).

Example

$$
\begin{array}{ll}
\min & 5x_1 + 8x_2 + 4x_3 \\
\text{s.t.} & 3x_1 + 5x_2 + 2x_3 \geq 30 \\
& \text{all-different}\{x_1, x_2, x_3\} \\
& x_j \in \{1,2,3,4\}
\end{array}
$$

Declaration Window

$x_j \in \{1,2,3,4\}, \; j = 1,\ldots,4$

Variables and initial domains

Objective Function Window

$$\min\ 5x_1 + 8x_2 + 4x_3$$

Relaxation Window

Type: **Constraint store**, consisting of variable domains.

Objective function: None.

Solver: None.

Relaxation Window

Type: **Linear programming.**

Objective function: Same as original problem.

Solver: LP solver.

Constraint Window

Type: **Linear (in)equalities.**

$3x_1 + 5x_2 + 2x_3 \geq 30$

Inference: Bounds consistency maintenance.

Inference: Covering inequalities.

Relaxation: Add reduced bounds to constraint store.

Relaxation: Add original inequality and covering inequalities to LP relaxation.

Constraint Window

Type: **All-different**

all-different(x_1,x_2,x_3,x_4)

Inference: Régin's hyperarc consistency maintenance.

Relaxation: None. (A relaxation exists but is not helpful in this case.)

Search Window

Procedure BandBsearch(*P*,*R*,*S*,CustomBranch)
(canned branch & bound search using CustomBranch as branching rule)

User-Defined Window

Procedure CustomBranch(*P*,*R*,*S*,*i*)

[*Take the i-th branch for problem restriction P, whose relaxation R has solution S*]

If there is a variable with nonintegral value in LP relaxation then
    Perform **BranchOnFraction**(*P*,*R*,*S*,*i*) [*standard B&B branching*]
Else perform **FirstFail**(*P*,*R*,*S*,*i*) [*Standard domain splitting*]

Example: Traveling Salesman

$$\begin{aligned} \min \quad & \sum_j c_{y_j y_{j+1}} \\ \text{s.t.} \quad & \text{all-different}\{y_1, \cdots, y_n\} \\ & y_j \in \{1, \ldots, n\} \end{aligned}$$

j-th city in tour

or

$$\begin{aligned} \min \quad & \sum_j c_{j y_j} \\ \text{s.t.} \quad & \text{cycle}\{y_1, \cdots, y_n\} \\ & y_j \in \{1, \ldots, n\} \end{aligned}$$

enforces Hamiltonian cycle

city following *j* in tour

Element global constraint

To implement a variably indexed constant a_y

Replace a_y with z and add constraint $\text{element}\big(y,(a_1,\ldots,a_n),z\big)$

Domain reduction is trivial.

Convex hull relaxation of element constraint is simply

$$\min_{j\in D_y}\{a_j\} \le z \le \max_{j\in D_y}\{a_j\}$$

Current domain of y

Extension of *element*

To implement variably indexed variable y_x

Replace y_x with z and add constraint $\text{element}\big(y, (x_1, \ldots, x_n), z\big)$
which posts constraint $\bigvee_{j \in D_y} (z = x_j)$

Domain reduction is fairly straightforward.

Relaxation is based on relaxation of disjunctive constraint:

If $0 \le x_j \le m_0$ for each j, there is a simple convex hull relaxation *(JNH)*:

$$\sum_{j \in D_y} x_j - \left(\left|D_y\right| - 1\right) m_0 \le z \le \sum_{j \in D_y} x_j$$

If $0 \le x_j \le m_j$ for each j, another relaxation is

$$\frac{\displaystyle\sum_{j \in D_y} \frac{x_j}{m_j} - \left|D_y\right| + 1}{\displaystyle\sum_{j \in D_y} \frac{1}{m_j}} \le z \le \frac{\displaystyle\sum_{j \in D_y} \frac{x_j}{m_j} + \left|D_y\right| - 1}{\displaystyle\sum_{j \in D_y} \frac{1}{m_j}}$$

Declaration Window for TSP

$y_j \in D_j = \{2,\ldots,n\},\ j = 2,\ldots,n$ — *j*th city in tour

$y_1 \in D_1 = \{1\}$

$z_j \in R$ for $j = 1,\ldots,n$ — cost of *j*th link in tour

Objective Function Window

$$\min \sum_{j=1}^{n} z_j$$

Relaxation Window

Type: **Constraint store**, consisting of domains of $y_1, \ldots, y_n$

Objective function: None.

Solver: None.

Relaxation Window

Type: **Linear programming.**

Objective function: $\min \sum_{jk} c_{jk} x_{jk}$

Solver: LP solver.

Constraint Window

Type: **Element.**

element(y_j, ($c_{j1},\ldots,c_{jn}$), z_j) for $j = 1,\ldots, n$

Inference: Hyperarc consistency maintenance.

Relaxation: Add reduced bounds to constraint store.

Relaxation: Add disjunctive relaxation to LP.

Constraint Window

Type: **cycle**

cycle($y_1,\ldots,y_n$)

Inference: Domain reduction (research topic).

Relaxation: Add reduced domains to constraint store.

Relaxation: Standard IP relaxation and cutting planes for TSP. Also fix $x_{jk} = 0$ if $k \notin D_j$ and $x_{jk} = 1$ if $D_j = \{k\}$.

Search Window

Procedure BandBsearch(*P*,*R*,*S*,TSPBranch)
(canned B&B search using TSPBranch as branching rule)

User-Defined Window

Procedure TSPBranch(*P*,*R*,*S*,*i*)

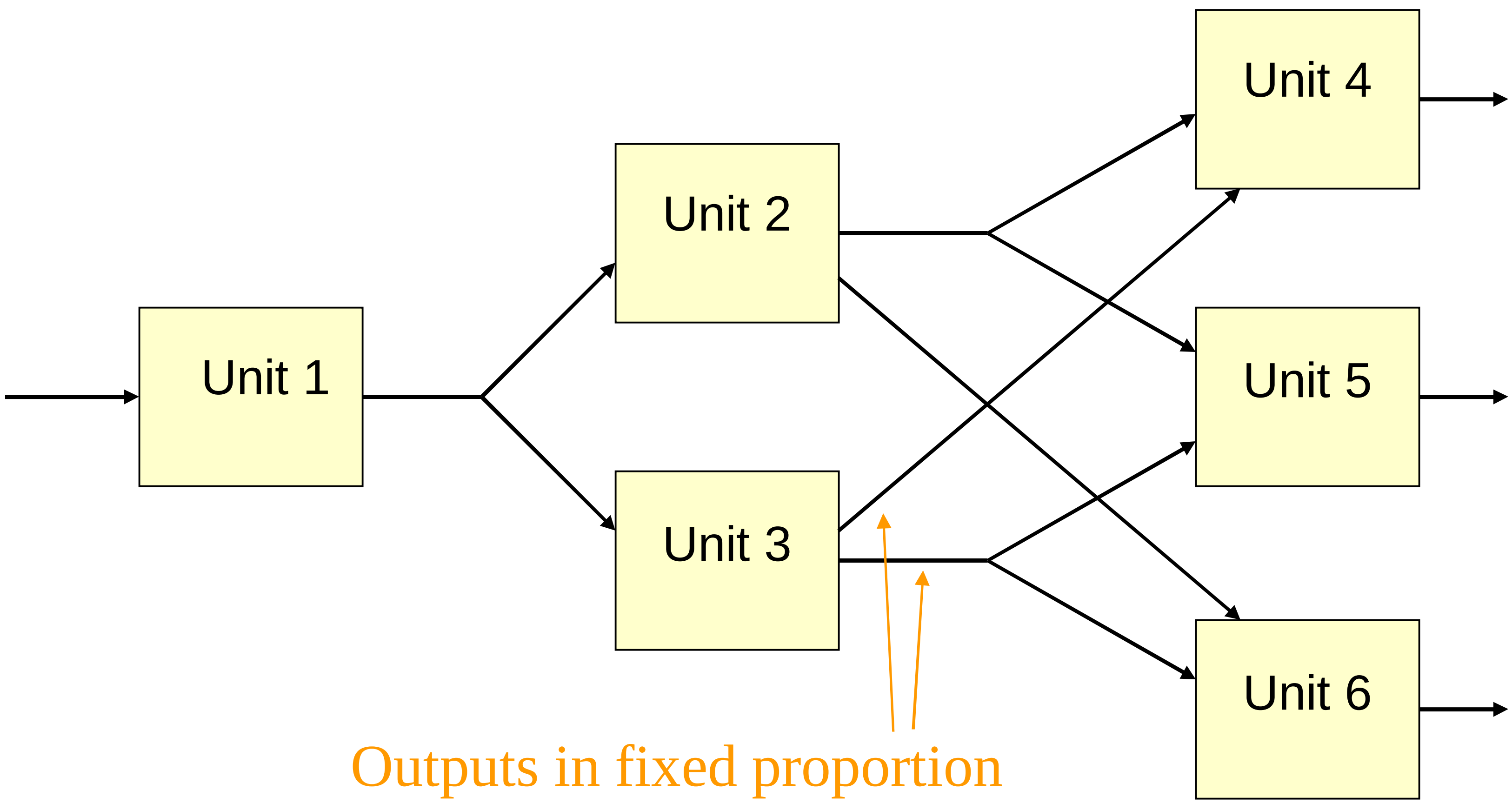
[*Take the i-th branch*]

If there is a variable x_{jk} with a nonintegral value in LP relaxation then
  If $i = 1$ then create P' from P by letting $D_j = \{k\}$ and return P'.
  If $i = 2$ then create P' from P by letting $D_j = D_j \setminus \{k\}$ and return P'.

Example: Processing Network Design

- Find optimal design of processing network.
  - A “superstructure” (largest possible network) is given, but not all processing units are needed.
  - Internal units generate negative profit.
  - Output units generate positive profit.
  - Installation of units incurs fixed costs.
  - Objective is to maximize net profit.

Sample Processing Superstructure



Declaration Window

$u_i \in [0, c_i]$	flow through unit i
$x_{ij} \in [0, c_{ij}]$	flow on arc (i,j)
$z_i \in [0, \infty]$	fixed cost of unit i
$y_i \in D_i = \{\text{true}, \text{false}\}$	presence or absence of unit i

Objective Function Window

$$\max \sum_i (r_i u_i - z_i)$$

Net revenue generated by unit *i* per unit flow

Relaxation Window

Type: **Constraint store**, consisting of variable domains.

Objective function: None.

Solver: None.

Relaxation Window

Type: **Linear programming.**

Objective function: Same as original problem.

Solver: LP solver.

Constraint Window

Type: **Linear (in)equalities.**

$\boldsymbol{Ax} + \boldsymbol{Bu} = \boldsymbol{b}$ (flow balance equations)

Inference: Bounds consistency maintenance.

Relaxation: Add reduced bounds to constraint store.

Relaxation: Add equations to LP relaxation.

Constraint Window

Type: **Disjunction of linear inequalities.**

$$\begin{pmatrix} y_i \\ z_i \ge d_i \end{pmatrix} \vee \begin{pmatrix} \neg y_i \\ u_i \le 0 \end{pmatrix} \quad \text{for each } i$$

Inference: None.

Relaxation: Add convex hull relaxation to LP.

Constraint Window

Type: **Propositional logic.**

Don't-be-stupid constraints:

$$
\begin{array}{ll}
y_1 \rightarrow (y_2 \vee y_3) & y_3 \rightarrow y_4 \\
y_2 \rightarrow y_1 & y_3 \rightarrow (y_5 \vee y_6) \\
y_2 \rightarrow (y_4 \vee y_5) & y_4 \rightarrow (y_2 \vee y_3) \\
y_2 \rightarrow y_6 & y_5 \rightarrow (y_2 \vee y_3) \\
y_3 \rightarrow y_1 & y_6 \rightarrow (y_2 \vee y_3)
\end{array}
$$

Inference: Resolution (add resolvents to constraint set).

Relaxation: Add reduced domains of y_i's to constraint store.

Relaxation (optional): Add 0-1 inequalities representing propositions to LP.

Search Window

Procedure BandBsearch(*P*,*R*,*S*,NetBranch) (canned branch & bound search using NetBranch as branching rule)

User-Defined Window

Procedure NetBranch(*P*,*R*,*S*,*i*)

Let i be a unit for which $u_i > 0$ and $z_i < d_i$.
If $i = 1$ then create P' from P by letting $D_i = \{T\}$
 and return P'.
If $i = 2$ then create P' from P by letting $D_i = \{F\}$
 and return P'.

Example: Local Search

Allocation Problem (Williams)

Each retailer j is served by division 1 ($x_j = 1$) or 2 ($x_j = 0$).

Division 1 delivers a_{ij} units of product i to retailer j.

Assign divisions so as to match division 1 quotas b_i as closely as possible.

$$
\begin{array}{ll}
\min & \sum_i |s_i| \\
\text{s.t.} & \sum_j a_{ij} x_j + s_i = b_i, \quad \text{all } i \\
 & x_j \in \{0,1\}, \quad s_i \in R
\end{array}
$$

Known to be very hard for IP (Cornuejols & Dawande 1999)

Declaration Window

$x_j \in D_j = \{0\}$	1 when division 1 is assigned to retailer *j*
$s_i \in R$	error in meeting goal for product *i*

Objective Function Window

$$\min \quad f(x) = \sum_i |s_i|$$

Relaxation Window

Type: **Linear.**

Objective function: Same as original problem.

Solver: Direct computation (with efficient updating).

Constraint Window

Type: **Linear equations.**

$$s_i = b_i - \sum_j a_{ij} x_j$$

Inference: None.

Relaxation: Compute s_i as above, with each x_j fixed to the single value in D_j.

Search Window

Procedure AnnealingSearch(*P*,*R*,*S*)

```
Return if search has run long enough.
Let random(p) be a random variable that has value 1 with
  probability p.
To flip a domain D_j = {t} is to change it to {1 – t}.
"Solve" relaxation to get objective value v.
Do forever:
  Randomly select j and change P by flipping D_j.
  Solve relaxation to get new objective value v'.
  If v' < v or random(p) = 1 then
    Perform AnnealingSearch(P,R,S) and return.
  Restore P by flipping D_j.
```

Example: Benders Decomposition

- Benders decomposition (and generalizations of it) fit into the framework.
- The master problem becomes a relaxation.
- The search is over subproblems, which are restrictions of the original problems.
- Will apply a generalized Benders to a machine scheduling problem.

The problem:

$$\begin{aligned} \min \quad & cx + dy \\ \text{s.t.} \quad & Ax + By \ge a \\ & x \in R^n,\ y \in Z^m \end{aligned}$$

The subproblem:

$$\begin{aligned} \min \quad & cx + d\bar{y} \\ \text{s.t.} \quad & Ax \ge a - B\bar{y} \quad (u) \end{aligned}$$

$$\bar{y} = \text{solution of previous master problem}$$

The master problem:

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & z \ge \bar{u}(a - By) + dy \\ & + \text{other Benders cuts} \end{aligned}$$

Assume subproblem is feasible and bounded.

Declaration Window

$x_i \in R$	subproblem variables
$y_j \in Z$	master problem variables

Objective Function Window

$$\boxed{\min \quad cx + dy}$$

Relaxation Window

Type: **MILP** (master problem).

Objective function: minimize z

Solver: IP solver

Constraint Window

Type: **Linear inequalities.**

$$Ax + By \geq a$$

Inference: Generation of Benders cuts from subproblem dual.

Relaxation: Add Benders cuts to IP relaxation (master problem).

Search Window

Procedure BendersSearch(*P*,*R*,*S*)

Generate Benders cut from P (subproblem).
Find optimal value v of relaxation (master problem).
If v is equal to optimal value of P, stop.
Define next subproblem.
  Obtain P' from P by setting each D_j to $\{\bar{y}_j\}$
Perform **BendersSearch(*P'*,*R*,*S*)**.

Benders-based Combination of CP and IP

- One promising scheme for combining CP and IP is based on generalized Benders decomposition.
- IP solves master problem, CP solves subproblem.
- Subproblem “dual” is inference dual, which generalizes LP dual.
- This scheme is a special case of the framework presented here.
- Will apply it to a machine scheduling problem.

Digression: Dualities

Two related dualities tend to occur in optimization.

- **Relaxation duality** (relaxations are parameterized).
  - Dual problem is to search parameter space for strongest relaxation.
  - Search parameter space for tightest relaxation.
  - Examples: LP, Lagrangean and surrogate duality.
- **Inference duality**
  - Dual problem is to infer valid constraints.
  - Examples: LP dual, dual used in scheduling problem below.

Digression: Relaxation Duality

The problem: $\min_{x \in S} \{ f(x) \}$

Parameterized relaxation:

$$\theta(\lambda) = \min_{x \in S(\lambda)} \{ f(x, \lambda) \}$$

Dual problem:

$$\max_{\lambda \in \Lambda} \{ \theta(\lambda) \}$$

Example: Lagrangean (& LP) duality $\min_{x \in S} \{ f(x) \mid g(x) \le 0 \}$

Parameterized relaxation:

$$\theta(\lambda) = \min_{x \in S} \left\{ f(x) + \lambda^T g(x) \right\}$$

Dual problem:

$$\max_{\lambda \ge 0} \{ \theta(\lambda) \}$$

Digression: Inference Duality

The problem: $\min_{x \in S} \{ f(x) \}$

Dual problem:

$$\max_{\text{proofs} \Rightarrow} \{ z \mid x \in S \Rightarrow f(x) \geq z \}$$

Example: LP duality

$\min\{cx \mid Ax \geq b\}$

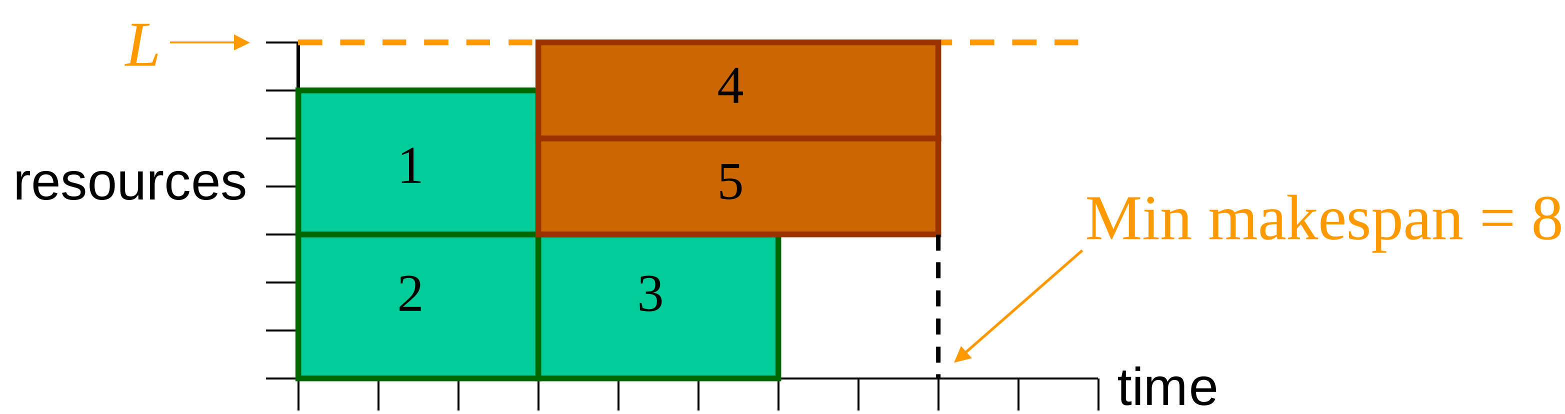
$Ax \geq b \Rightarrow cx \geq z$ when
$uAx \geq ub$ dominates $cx \geq z$;
that is, when $uA \leq c$ and $ub \geq$ z

Dual problem:

$$\max_{\text{proofs} \Rightarrow} \{ z \mid Ax \geq b \Rightarrow cx \geq z \} = \max_{u \geq 0} \{ ub \mid uA \leq c \}$$

Cumulative Global Constraint

Minimize makespan (no deadlines, all release times = 0):



$$
\begin{array}{ll}
\min & z \\
\text{s.t.} & \text{cumulative}\big((t_1,\ldots,t_5),(3,3,3,5,5),(3,3,3,2,2),7\big) \\
& z \geq t_1 + 3 \\
& \vdots \\
& z \geq t_5 + 2
\end{array}
$$

L

Resources used

Durations

Job start times

Machine scheduling

Schedule jobs on parallel machines.

- Machines run at different speeds and incur different costs per job.
- Each job has a release date and a due date.
- Master problem assigns jobs to machines.
- Subproblem schedules jobs on assigned machines.

Domain reduction

Highly developed; based on edge finding.

Relaxation (JNH & Yan, 2001)

If some subset of jobs $\{j_1,\ldots,j_k\}$ are identical (same release time a_0, duration d_0, and resource consumption rate r_0), then

$$t_{j_1} + \cdots + t_{j_k} \geq (P+1)a_0 + \frac{1}{2}P\left[2k - (P+1)Q\right]d_0$$

is a valid cut and is facet-defining if there are no deadlines, where

$$Q = \left\lfloor \frac{L}{r_0} \right\rfloor, \qquad P = \left\lceil \frac{k}{Q} \right\rceil - 1$$

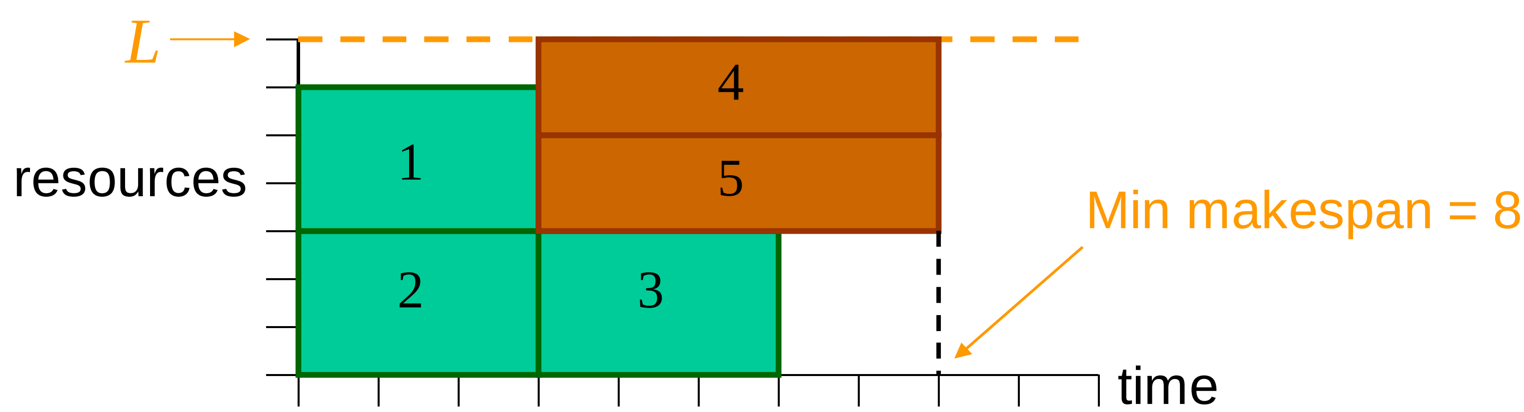
The following cut is valid for any subset of jobs $\{j_1,\ldots,j_k\}$

$$t_{j_1} + \cdots + t_{j_k} \geq \sum_{i=1}^{k} \left((k - i + \tfrac{1}{2}) \frac{r_i}{L} - \tfrac{1}{2} \right) d_i$$

Where the jobs are ordered by nondecreasing $r_j d_j$.

Analogous cuts can be based on deadlines.

Example



Relaxation:

$$
\begin{array}{ll}
\min & z \\
\text{s.t.} & z \geq t_1 + 3,\ t_2 + 3,\ t_3 + 3,\ t_4 + 5,\ t_5 + 5 \\
 & t_1 + t_2 + t_3 \geq 3 \\
 & t_1 + t_2 + t_3 + t_4 \geq 3\frac{5}{14} \\
 & t_2 + t_3 + t_4 + t_5 \geq 2\frac{4}{7} \\
 & t_1 + t_2 + t_3 + t_4 + t_5 \geq 6\frac{6}{7} \\
 & t_j \geq 0
\end{array}
$$

Facet defining

Resulting bound:

$z = \text{makespan} \geq 5.17$

A model for the machine scheduling problem:

Cost of assigning machine x_j to job j

Release date for job j

Job duration

Deadline

$$\begin{aligned} \min \quad & \sum_j C_{x_j j} \\ \text{s.t.} \quad & t_j \geq R_j, \quad \text{all } j \\ & t_j + D_{x_j j} \leq S_j, \quad \text{all } j \\ & \text{cumulative}\big((t_j \mid x_j = i), (D_{ij} \mid x_j = i), e, 1\big), \quad \text{all } i \end{aligned}$$

Start time for job j

Machine assigned to job j

Start times of jobs assigned to machine i

For a given set of assignments $\bar{x}$ the subproblem is the set of 1-machine problems,

$$\begin{aligned} \min \quad & 0 \\ \text{s.t.} \quad & \text{cumulative}\big((t_j \mid \bar{x}_j = i), (D_{ij} \mid \bar{x}_j = i), e, 1\big), \quad \text{all } i \end{aligned}$$

Feasibility of each problem is checked by constraint programming. One or more infeasible problems results in an optimal value ∞. Otherwise the value is zero.

Suppose there is no feasible schedule for machine i. Then some subset $J_i(\bar{x})$ of jobs cannot be assigned to machine i. We have a Benders cut

$$x_j \neq i \text{ for some } j \in J_i(\bar{x})$$

This yields the master problem,

$$\begin{aligned} \min \quad & \sum_j C_{x_j j} \\ \text{s.t.} \quad & t_j \geq R_j, \quad \text{all } j \\ & t_j + D_{x_j j} \leq S_j, \quad \text{all } j \\ & x_j \neq i \text{ for some } j \in J_i(x^k), \ \text{all } i, \ k = 1,\ldots,K \end{aligned}$$

This problem can be written as a mixed 0-1 problem:

$$
\begin{aligned}
\min \quad & \sum_{ij} C_{ij} y_{ij} \\
\text{s.t.} \quad & t_j \ge R_j, \quad \text{all } j \\
& t_j + \sum_i D_{ij} y_{ij} \le S_j, \quad \text{all } j \\
& \sum_i y_{ij} \ge 1, \quad \text{all } j \\
& \sum_{\substack{j \\ x_j^k = i}} (1 - y_{ij}) \ge 1, \quad \text{all } i, \; k = 1, \ldots, K \\
& \sum_j D_{ij} y_{ij} \le \max_j \{S_j\} - \min_j \{R_j\}, \quad \text{all } i \\
& y_{ij} \in \{0,1\}
\end{aligned}
$$

Valid constraint added to improve performance $\longrightarrow$ $\sum_j D_{ij} y_{ij} \le \max_j \{S_j\} - \min_j \{R_j\}$