

Finite-Domain Cuts for Graph Coloring*

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Abstract

We explore the idea of obtaining valid inequalities from a finite-domain formulation of a problem, rather than a 0-1 formulation. A finite-domain model represents discrete choices with variables that have several possible values, as is frequently done in constraint programming. We apply the idea to graph coloring and identify facet-defining cuts that, when converted to cuts in a 0-1 model of the problem, provide tighter bounds on the chromatic number than known 0-1 cuts. In particular, we show that finite-domain cuts for webs and odd holes are superior to standard cuts, and that two cuts for an odd cycle (a generalization of an odd hole) yield substantially tighter bounds, in much less time, than hundreds or thousands of standard cuts. We also identify a large family of facet-defining cuts for intersecting systems, for which there are apparently no previously known 0-1 cuts.

1 Introduction

In integer programming models, a choice from several alternatives is typically encoded by a set of binary variables. For example, the job assigned to a particular worker might be represented by 0-1 variables y_{ij} , where $\sum_j y_{ij} = 1$ for each worker i , and $y_{ij} = 1$ indicates that job j is assigned to worker i . Valid inequalities can then be generated in terms of the 0-1 variables, so as to strengthen the continuous relaxation of the model.

An alternative approach is to formulate such a choice directly in terms of finite-domain variables. For example, variable x_i might indicate which job is assigned to worker i . The value of x_i need not be a number, but if we choose to denote jobs by numbers, we can analyze the convex hull of feasible solutions and write valid inequalities in terms of the variables x_i . These inequalities can then be mapped into a 0-1 model of the problem using a simple change of variable. The resulting 0-1 inequalities may be different from and more effective than known cutting planes for the 0-1 model.

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This paper explores the idea of using a finite-domain formulation of a problem as a source of new valid inequalities for the 0-1 model. We will refer to such inequalities as *finite-domain cuts*. We apply the idea to the vertex coloring problem on graphs, which has a natural finite-domain formulation in terms of *all-different* constraints. Such “global” constraints frequently appear in constraint programming models, where finite-domain variables are often used rather than 0-1 variables to encode discrete choices.

We employ a common strategy for generating problem-specific cuts: the identification of facet-defining cuts for special types of induced subgraphs, such as odd holes, webs, and paths. We identify cuts that bound the objective function (which we call *z-cuts*) as well as cuts that exclude infeasible solutions (*x-cuts*).

We find that for coloring problems, finite-domain cuts for several subgraph structures (when mapped into 0-1 space) provide tighter bounds than known 0-1 cuts for those subgraphs. Furthermore, we identify more general structures for which finite-domain cuts are substantially more effective than known cuts, or for which no known cuts exist.

We present here our results for webs, odd cycles, paths, and intersecting systems, because they illustrate four possible outcomes:

- Finite-domain *web cuts*, when mapped into the 0-1 model, yield tighter bounds than standard web cuts. This means, in particular, that if an existing algorithm identifies separating web cuts, we can replace them with more effective finite-domain web cuts at no additional computational cost.
- *Odd cycles* are a generalization of odd holes. We show that in the special case of odd holes, finite-domain cuts provide tighter bounds than standard odd hole and clique cuts. We can therefore replace known separating odd hole cuts with more effective cuts, at no additional cost. In the general case of odd cycles, only two finite-domain cuts for a given cycle provide a substantially tighter bound than hundreds or thousands of odd hole and clique cuts that can be generated for that cycle. We provide a polynomial-time algorithm that identifies all separating finite-domain cuts for a given odd cycle.
- By contrast, finite-domain *path cuts* do not improve existing bounds. When mapped into 0-1 space, they have no effect on the bound provided by the standard 0-1 model.
- *Intersecting systems* illustrate how a finite-domain perspective can yield facet-defining cuts for novel structures. To our knowledge, no 0-1 cuts have previously been identified for this general class of subgraphs. We also present a polynomial-time separation algorithm.

Mapping finite-domain cuts into 0-1 space has the advantage that finite-domain cuts can be combined with standard 0-1 constraints as well as previously known families of 0-1 cuts. However, bounds can also be obtained directly from

the finite-domain model by solving its relaxation, which is much smaller than the 0-1 model. We investigate both approaches computationally.

We begin below with a problem statement and brief literature review. We then describe the mapping of finite-domain cuts into 0-1 space and prove some of its elementary properties. We next derive facet-defining inequalities for odd cycles, webs, paths, and intersecting systems, and study their properties when mapped into 0-1 space. In particular, we show that a family of facet-defining x -cuts gives rise to a family of facet-defining z -cuts in a canonical way, a result that is crucial for obtaining good bounds. A section on computational results compares the strength of finite-domain cuts and known 0-1 cuts on odd cycles and webs. It also demonstrates the advantages of odd cycle cuts on a set of benchmark instances. The paper concludes with a summary and suggestions for future research.

2 The Problem

Given an undirected graph G with vertex set V and edge set E , the vertex coloring problem is to assign a color x_i to each vertex $i \in V$ so that $x_i \neq x_j$ for each $(i, j) \in E$. We seek a solution with the minimum number of colors; that is, a solution that minimizes $|\{x_i \mid i \in V\}|$.

The vertex coloring problem can be formulated as a system of all-different constraints. An all-different constraint $\text{alldiff}(X)$ requires that the variables in set X take pairwise distinct values. Let $\{V_k \mid k \in K\}$ be the vertex sets of the maximal cliques of G , and let X_k be the set of variables x_i with $i \in V_k$. Let the colors be denoted by distinct nonnegative numbers v_j for $j \in J$, so that each variable x_i has the finite domain $D = \{v_j \mid j \in J\}$. Then the problem of minimizing the number of colors is

$$\begin{aligned}
 & \min z \\
 & z \geq x_i, \quad i \in V \\
 & \text{alldiff}(X_k), \quad k = 1, \dots, K \\
 & x_i \in D = \{v_j \mid j \in J\}, \quad i \in V
 \end{aligned} \tag{1}$$

Here we use maximal cliques V_k , but any clique cover $\{V_k \mid k \in K\}$ suffices to formulate the coloring problem.

It is convenient assume that $|V| = n$ colors v_0, \dots, v_{n-1} are available. We also assume $v_0 < \dots < v_{n-1}$. An initial question is how to select numerical domain values v_0, \dots, v_n , and how polyhedral structure depends on the selection. We note that this same question arises in 0-1 programming, because the numerical domain of a boolean variable need not be $\{0, 1\}$. In the boolean case, polyhedral results are valid for any binary domain, modulo appropriate adjustments in the coefficients and right-hand sides of valid inequalities. The issue is more complicated for general finite domains, but we find that the x -cuts identified here are valid for arbitrary nonnegative domain values, while z -cuts are valid for any domain of the form $D_\delta = \{0, \delta, 2\delta, \dots, (n-1)\delta\}$, where $\delta > 0$. In

practice, it is convenient to use domain D_1 , because in this case the minimum color number z is one less than the chromatic number.

A standard 0-1 model for the coloring problem uses binary variables y_{ij} to denote whether vertex i receives color j , and binary variables w_j that indicate whether color j is used. The model is

$$\begin{aligned} \min \quad & \sum_{j \in J} w_j \\ & \sum_{j \in J} y_{ij} = 1, \quad i \in V & (a) \\ & \sum_{i \in V_k} y_{ij} \leq w_j, \quad j \in J, k \in K & (b) \\ & y_{ij} \in \{0, 1\}, \quad i \in V, j \in J \end{aligned} \tag{2}$$

3 Previous Work

All facets for a single all-different constraint $\text{alldiff}(X)$ are given in [6, 15]. If $X = \{x_1, \dots, x_m\}$ and each x_i has domain $\{v_1, \dots, v_m\}$ with $n \leq m$, they are

$$\sum_{j=1}^{|J|} v_j \leq \sum_{i \in J} x_i \leq \sum_{j=m-|J|+1}^m v_j, \quad \text{all nonempty } J \subseteq \{1, \dots, n\} \tag{3}$$

where again $v_1 < \dots < v_m$. If $m = n$, (3) defines the affine hull when $J = \{1, \dots, n\}$. The facial structure of a system of two all-different constraints is studied in [1, 2].

Facets for general all-different systems are derived for combs in [8, 9, 11] and for odd holes and webs in [10]. To our knowledge, the cuts we describe here for cycles, paths, and intersecting systems have not been previously identified. We also generalize the web cuts in [10] and introduce z -cuts for webs.

It is natural to ask when all facets of an all-different system are facets of individual constraints in the system. It is shown in [11] that this occurs if and only if the all-different system has an *inclusion* property, which means that pairwise intersections of sets V_k in the alldiff constraints are ordered by inclusion. The structures studied here lack the inclusion property and therefore generate new classes of facets.

Known facets for the 0-1 graph coloring model are discussed in [4, 12, 13, 14]. These include cuts based on odd holes, webs, anti-webs, cliques, and paths.

Finite-domain cuts have been developed for a few global constraints other than alldiff systems. These include the element constraint [6], the circuit constraint [5], the cardinality constraint [7], cardinality rules [16], the sum constraint [17], and disjunctive and cumulative constraints [7].

In a conference paper [3], we presented the cycle cuts described here and mapped them into 0-1 space. The present paper extends the computational tests to benchmark instances, introduces additional families of cuts, and studies the

properties of the mapping. Aside from [3], the strategy of mapping finite-domain cuts into 0-1 space has, to our knowledge, not been previously investigated.

4 Mapping into 0-1 Space

We now specify what it means to map a valid finite-domain cut into 0-1 space. We first discuss cuts for (1) involving only the variables x_i , which we call x -cuts. We then consider bounds on the largest color number z , which we call z -cuts.

We convert a valid x -cut $ax \geq b$ to a 0-1 inequality simply by replacing each x_i with $\sum_j v_j y_{ij}$. The inequality $ax \geq b$ therefore becomes

$$\sum_{i=1}^n a_i \sum_{j=0}^{n-1} v_j y_{ij} \geq b \quad (4)$$

We refer to this as a 0 -1 x -cut. It is important to analyze this conversion carefully, to ensure that valid cuts are mapped to valid cuts and to study their strength in the 0-1 model.

The domain D^n of the coloring problem is the set of all tuples (x_1, \dots, x_n) with each $x_i \in D$. A bijection ϕ maps each $x \in D^n$ to a point $y = \phi(x)$ given by

$$y_{ij} = \begin{cases} 1 & \text{if } x_i = v_j \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

For $S \subset D^n$, let $\phi(S) = \{\phi(x) \mid x \in S\}$. All $y \in \phi(S)$ satisfy

$$\sum_{j=0}^{n-1} y_{ij} = 1, \quad i = 1, \dots, n \quad (6)$$

An inequality $ax \geq b$ is *valid* for $S \subset D^n$ if $ax \geq b$ for all $x \in S$. The inequality (4) is valid for $\phi(S)$ if it is satisfied by all $y \in \phi(S)$. Valid cuts map to valid cuts:

Lemma 1 *If $ax \geq b$ is valid for $S \subset D^n$, then (4) is valid for $\phi(S)$.*

Proof. Supposing $\bar{y} \in \phi(S)$, we wish to show that \bar{y} satisfies (4). Because $\bar{y} \in \phi(S)$, we have $\bar{y} = \phi(\bar{x})$ for some $\bar{x} \in S$. Thus $a\bar{x} \geq b$, which implies that \bar{y} satisfies (4) because $\bar{x} = \sum_j v_j \bar{y}_{ij}$ from the definition of ϕ . \square

An important issue is the strength of cuts mapped into 0-1 space. In particular, we may wish to know whether a 0-1 x -cut (4) is redundant of a system $Ay \geq c$ of known 0-1 cuts. To make this precise, we will say that (4) is *redundant* of system $Ay \geq c$ if all $y \in [0, 1]^{n \times n}$ satisfying (6) and $Ay \geq c$ also satisfy (4). Cut (4) is simply *redundant* if all $y \in [0, 1]^{n \times n}$ satisfying (6) also satisfy (4).

We now consider z -cuts, or bounds $z \geq ax + b$ on the largest color number z . If we suppose that the color numbers are $0, \dots, n-1$, minimizing z is equivalent

to minimizing the number of colors minus 1. Because the number of colors is $\sum_j w_j$, we map $z \geq ax + b$ to the 0-1 inequality

$$\sum_{j=0}^{n-1} w_j - 1 \geq \sum_{i=1}^n a_i \sum_{j=0}^{n-1} j y_{ij} + b \quad (7)$$

which we call a 0-1 z -cut. This inequality may be added to the 0-1 model because some optimal solution satisfies it. Yet (7) is not valid, because it can be violated by solutions that use larger color numbers but the same number of colors. Thus 0-1 z -cuts have the advantage of excluding symmetric solutions. We can ensure that (7) is formally valid by adding symmetry breaking constraints

$$w_j \geq w_{j+1}, \quad j = 0, \dots, n-2 \quad (8)$$

to the 0-1 model (2).

We can now define validity as follows. Given $S \subset D_1^n$, we say that the z -cut $z \geq ax + b$ is valid for S if $\max_i \{x_i\} \geq ax + b$ for all $x \in S$. Inequality (7) is valid for $\phi(S)$ when it is satisfied by all $y \in \phi(S)$ and $w \in \{0, 1\}^n$ that satisfy (6), (8), and

$$w_j \geq y_{ij}, \quad \text{all } i, j \quad (9)$$

Then valid bounds map to valid bounds:

Lemma 2 *If $z \geq ax + b$ is valid for $S \in D_1^n$, then (7) is valid for $\phi(S)$.*

Proof. Suppose that $\bar{y} \in \phi(S)$ and $\bar{w} \in \{0, 1\}^n$ satisfy (6), (8) and (9). We wish to show that (\bar{y}, \bar{w}) satisfies (7). Because $\bar{y} = \phi(\bar{x})$ for some $\bar{x} \in S$, we have

$$\max_i \{\bar{x}_i\} \geq a\bar{x} + b = \sum_{i=1}^n a_i \sum_{j=0}^{n-1} j \bar{y}_{ij} + b$$

It therefore suffices to show that

$$\sum_{j=0}^{n-1} \bar{w}_j - 1 \geq \max_i \{\bar{x}_i\} \quad (10)$$

Due to (8), we can suppose $\bar{w}_j = 1$ for $j \leq k$ and $\bar{w}_j = 0$ for $j > k$. Then from (9) we have $\bar{y}_{ij} = 0$ for all i and all $j > k$. Thus (6) implies $\sum_{j=0}^k \bar{y}_{ij} = 1$ for all i , which implies

$$\sum_{j=0}^k j \bar{y}_{ij} \leq k, \quad \text{all } i \quad (11)$$

Now we have

$$\sum_{j=0}^{n-1} \bar{w}_j - 1 = (k+1) - 1 \geq \max_i \left\{ \sum_{j=0}^k j \bar{y}_{ij} \right\} = \max_i \left\{ \sum_{j=0}^{n-1} j \bar{y}_{ij} \right\} = \max_i \{\bar{x}_i\}$$

where the inequality is due to (11). This establishes (10), as desired. \square

It is an interesting question whether facets map to facets. In general, they do not. Consider the feasible set S for the single constraint $\text{alldiff}(x_1, x_2, x_3)$ with $x_i \in \{0, 1, 2\}$. Then $x_1 + x_2 \geq 1$ is one of the facet-defining inequalities (3). It maps to

$$y_{11} + 2y_{12} + y_{21} + 2y_{22} \geq 1 \quad (12)$$

This is not facet-defining because the convex hull of $\phi(S)$ has dimension 4, while only 2 points satisfy (12) at equality:

$$\begin{bmatrix} y_{10} & y_{11} & y_{12} \\ y_{20} & y_{21} & y_{22} \\ y_{30} & y_{31} & y_{32} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The finite-domain cuts we obtain below do not in general map to facet-defining cuts in 0-1 space. They can nonetheless provide substantially tighter bounds on the chromatic number than known cuts, which themselves may not be facet-defining, as in the case of odd hole cuts.

5 Cycles

We first investigate valid inequalities that correspond to odd cycles. We define a cycle in graph G to be a subgraph of G induced by the vertices in $V_1, \dots, V_q \in V$ (for $q \geq 3$), where the subgraph induced by each V_k is a clique, and the only overlapping V_k 's are adjacent ones in the cycle V_1, \dots, V_q, V_1 . Thus,

$$V_k \cap V_\ell = \begin{cases} S_k & \text{if } k+1 = \ell \text{ or } (k, \ell) = (q, 1) \\ \emptyset & \text{otherwise} \end{cases}$$

where $S_k \neq \emptyset$. A feasible vertex coloring on G must therefore satisfy

$$\text{alldiff}(X_k), \quad k = 1, \dots, q \quad (13)$$

where again $X_k = \{x_i \mid i \in V_k\}$. The cycle is *odd* if q is odd. If $|V_k| = 2$ for each k , an odd cycle is an *odd hole*.

Figure 1 illustrates an odd cycle with $q = 5$. Each solid oval corresponds to a constraint $\text{alldiff}(X_k)$. Thus $V_1 = \{0, 1, 2, 3, 10, 11\}$, and similarly for V_2, \dots, V_5 . All the vertices in a given V_k are connected by edges in G .

5.1 Valid Inequalities

We first identify valid inequalities that correspond to a given cycle. In the next section, we show that they are facet-defining.

Lemma 3 *Let V_1, \dots, V_q induce a cycle, and let $\bar{S}_k \subseteq S_k$ and $|\bar{S}_k| = s \geq 1$ for $k = 1, \dots, q$. If q is odd and $\bar{S} = \bar{S}_1 \cup \dots \cup \bar{S}_q$, the following inequality is valid for (1):*

$$\sum_{i \in \bar{S}} x_i \geq \beta(q, s) \quad (14)$$

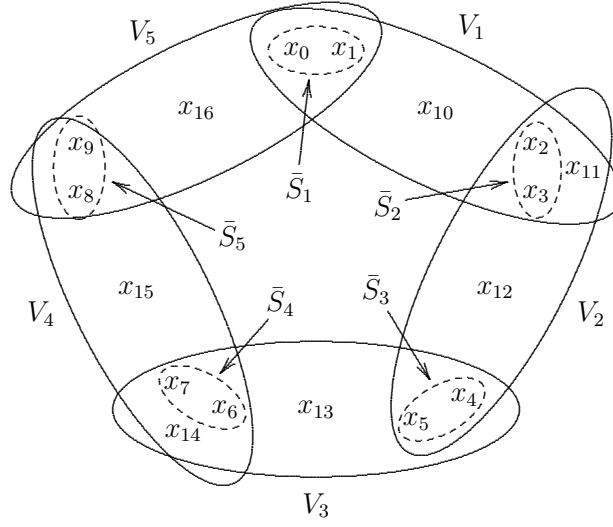


Figure 1: A 5-cycle. The solid ovals correspond to constraints $\text{alldiff}(X_k)$ for $k = 1, \dots, 5$. The sets $\bar{S}_1, \dots, \bar{S}_5$ provide the basis for one possible valid cut with $s = 2$.

where

$$\beta(q, s) = \frac{q-1}{2} \sum_{j=0}^{L-2} v_j + \left(sq - \frac{q-1}{2}(L-2) \right) v_{L-1}$$

and

$$L = \left\lceil \frac{sq}{(q-1)/2} \right\rceil$$

Proof. Because q is odd, each color can be assigned to at most $(q-1)/2$ vertices in the cycle. This means that the vertices must receive at least L distinct colors, and the variables in (13) must take at least L different values. Because $v_0 < \dots < v_{n-1}$, we have

$$\sum_{i \in \bar{S}} x_i \geq \frac{q-1}{2} (v_0 + v_1 + \dots + v_{L-2}) + \left(sq - \frac{q-1}{2}(L-2) \right) v_{L-1} = \beta(q, s)$$

where the coefficient of v_{L-1} is the number of vertices remaining to receive color v_L after colors v_0, \dots, v_{L-2} are assigned to $(q-1)/2$ vertices each. \square

If the cycle is an odd hole, each $|S_k| = 1$ and $L = 3$. So (14) becomes

$$\sum_{i \in \bar{S}} x_i \geq \frac{q-1}{2} (v_0 + v_1) + v_2 \quad (15)$$

If the domain $\{v_0, \dots, v_{n-1}\}$ of each x_i is $D_\delta = \{0, \delta, 2\delta, \dots, (n-1)\delta\}$ for some $\delta > 0$, inequality (14) becomes

$$\sum_{i \in \bar{S}} x_i \geq \left(sq - \frac{q-1}{4}L \right) (L-1)\delta \quad (16)$$

for a general cycle and

$$\sum_{i \in \bar{S}} x_i \geq \frac{q+3}{2}\delta$$

for an odd hole.

An example with $q = 5$ appears in Fig. 1. By setting $s = 2$ we can obtain 9 valid inequalities by selecting 2-element subsets \bar{S}_2 and \bar{S}_4 of S_2 and S_4 , respectively. Here $L = 5$, and if the colors are $0, \dots, 9$, the right-hand side of the cut is $\beta(5, 2) = 20$. The sets $\bar{S}_1, \dots, \bar{S}_5$ illustrated in the figure give rise to the valid inequality

$$x_0 + \dots + x_9 \geq 20 \quad (17)$$

5.2 Facet-defining Inequalities

We now show that the valid inequalities identified in Lemma 3 are facet-defining. Let the variables x_i for $i \in \bar{S}$ be indexed x_0, \dots, x_{qs-1} . We will say that a partial solution

$$(x_0, x_1, \dots, x_{qs-1}) = (\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{qs-1}) \quad (18)$$

is feasible for (1) if it can be extended to a feasible solution of (1). That is, there is a complete solution (x_1, \dots, x_n) that is feasible in (1) and that satisfies (18). Because $|V|$ colors are available, any partial solution (18) that satisfies (13) can be extended to a feasible solution simply by assigning the remaining vertices distinct unused colors. That is, assign vertices in $V \setminus \{0, \dots, sq-1\}$ distinct colors from the set $J \setminus \{\bar{x}_0, \dots, \bar{x}_{sq-1}\}$.

Theorem 4 *If the graph coloring problem (1) is defined on a graph in which vertex sets V_1, \dots, V_q induce a cycle, where q is odd, then inequality (14) is facet defining for (1).*

Proof. Define

$$F = \{x \text{ feasible for (1)} \mid (x_0, \dots, x_{qs-1}) \text{ satisfies (14) at equality}\}$$

It suffices to show that if $\mu x \geq \mu_{n+1}$ holds for all $x \in F$, then there is a scalar $\lambda > 0$ such that

$$\mu_i = \begin{cases} \lambda & \text{for } i = 0, \dots, qs-1 \\ \beta(q, s)\lambda & \text{for } i = n+1 \\ 0 & \text{otherwise} \end{cases} \quad (19)$$

We will construct a partial solution $(\bar{x}_0, \dots, \bar{x}_{qs-1})$ that is feasible for (1) as follows. Domain values v_0, \dots, v_{L-2} will occur $(q-1)/2$ times in the solution, and domain value v_{L-1} will occur r times, where

$$r = qs - \frac{q-1}{2}(L-1)$$

This will ensure that (14) is satisfied at equality. We form the partial solution by first cycling r times through the values v_0, \dots, v_{L-1} , and then by cycling through the values v_0, \dots, v_{L-2} . Thus

$$\bar{x}_i = \begin{cases} v_{i \bmod L} & \text{for } i = 0, \dots, rL-1 \\ v_{(i-rL) \bmod (L-1)} & \text{for } i = rL, \dots, rs-1 \end{cases} \quad (20)$$

To show that this partial solution is feasible for the odd cycle, we must show

$$\text{alldiff}\{\bar{x}_i, i \in \bar{S}_k \cup \bar{S}_{k+1}\}, \text{ for } k = 1, \dots, q-1 \quad (a)$$

$$\text{alldiff}\{\bar{x}_i, i \in \bar{S}_1 \cup \bar{S}_q\} \quad (b)$$

To show (a), we note that the definition of L implies $L-1 \geq 2s$. Therefore, any sequence of $2s$ consecutive \bar{x}_i 's are distinct, and (a) is satisfied. To show (b), we note that the number of values $\bar{x}_{rL}, \dots, \bar{x}_{rs-1}$ is

$$(rs-1) - rL + 1 = (L-1) \left(\frac{q-1}{2}L - qs \right)$$

from the definition of r . Because the number of values is a multiple of $L-1$, the values \bar{x}_i for $i \in \bar{S}_q$ are $(\bar{x}_{(q-1)s}, \dots, \bar{x}_{qs-1}) = (v_{L-s-1}, \dots, v_{L-2})$, and they are all distinct. The values \bar{x}_i for $i \in \bar{S}_1$ are $(\bar{x}_0, \dots, \bar{x}_{s-1}) = (v_0, \dots, v_{s-1})$ and are all distinct. But $L-1 \geq 2s$ implies $L-s > s$, and (b) follows.

We now construct a partial solution $(\tilde{x}_0, \dots, \tilde{x}_{qs-1})$ from the partial solution in (20) by swapping any two values $\bar{x}_\ell, \bar{x}_{\ell'}$ for $\ell, \ell' \in \bar{S}_k \cup \bar{S}_{k+1}$, for any $k \in \{1, \dots, q-1\}$. That is,

$$\tilde{x}_i = \begin{cases} \bar{x}_{\ell'} & \text{if } i = \ell \\ \bar{x}_\ell & \text{if } i = \ell' \\ \bar{x}_i & \text{otherwise} \end{cases} \quad (21)$$

Extend the partial solutions (20) and (21) to complete solutions \bar{x} and \tilde{x} , respectively, by assigning values with

$$\bar{x}_i = \tilde{x}_i \text{ for } i \notin \{0, \dots, qs-1\}$$

such that the values assigned to \bar{x}_i for $i \notin \{0, \dots, qs-1\}$ are all distinct and do not belong to $\{v_0, \dots, v_{L-1}\}$. Because \bar{x} and \tilde{x} are feasible and satisfy (14) at equality, they satisfy $\mu\bar{x} = \mu\tilde{x}$. So we have $\mu\bar{x} = \mu\tilde{x}$, which implies $\mu_\ell = \mu_{\ell'}$ for $\ell, \ell' \in \bar{S}_k \cup \bar{S}_{k+1}$ for any pair $\ell, \ell' \in \bar{S}_k \cup \bar{S}_{k+1}$ and any $k \in \{1, \dots, q-1\}$. This implies

$$\mu_\ell = \mu_{\ell'} \text{ for any } \ell, \ell' \in \bar{S} \quad (22)$$

Define \bar{x}' by letting $\bar{x}' = \bar{x}$ except that for an arbitrary $\ell \notin \{0, \dots, qs - 1\}$, \bar{x}'_ℓ is assigned a value that does not appear in the tuple \bar{x} . Since \bar{x} and \bar{x}' are feasible and satisfy (14) at equality, we have $\mu\bar{x} = \mu\bar{x}'$. This and $\bar{x}_\ell \neq \bar{x}'_\ell$ imply

$$\mu_i = 0, \quad i \in V \setminus \{0, \dots, qs - 1\} \quad (23)$$

Finally, (22) implies that for some $\lambda > 0$,

$$\mu_i = \lambda, \quad i = 0, \dots, qs - 1 \quad (24)$$

Because $\mu\bar{x} = \mu_{n+1}$, we have from (24) that $\mu_{n+1} = \beta(q, s)\lambda$. This, (23), and (24) imply (19). \square

In the example of Fig. 1, suppose that the vertices in V_1, \dots, V_5 induce a cycle of G . That is, all vertices in each V_k are connected by edges, and there are no other edges of G between vertices in $V_1 \cup \dots \cup V_5$. Then (17) is facet-defining for (1).

5.3 Bounds on the Chromatic Number

We can write a facet-defining inequality involving the objective function variable z if the domain of each x_i is D_δ for $\delta > 0$. To do so we rely on the following:

Theorem 5 *If $ax \geq \beta$ is facet-defining for a graph coloring problem (1) in which each x_i has domain D_δ for $\delta > 0$, then*

$$aez \geq ax + \beta \quad (25)$$

is also facet defining, where $e = (1, \dots, 1)$.

Proof. To show that (25) is valid, note that for any $x \in D_\delta^n$, $z - x_i \in D_\delta$ for all i , where $z = \max_i \{x_i\}$. Because $ax \geq \beta$ is valid for all $x \in D_\delta^n$ and $z - x_i \in D_\delta$, $ax \geq \beta$ holds when $z - x_i$ is substituted for each x_i . This implies (25) because z in (1) satisfies $z \geq x_i$ for each i .

To show that the z -cut (25) is facet-defining, let

$$F = \{(z, x) \text{ feasible for (1)} \mid aez = ax + \beta\}$$

It suffices to show that if $\mu_z z = \mu x + \mu_0$ is satisfied by all $(z, x) \in F$, then there is a $\lambda > 0$ with

$$\begin{aligned} \mu_z &= \lambda a e \\ \mu &= \lambda a \\ \mu_0 &= \lambda \beta \end{aligned} \quad (26)$$

Let $F' = \{x \text{ feasible for (1)} \mid ax = \beta\}$. F' is nonempty because $ax \geq \beta$ is facet defining. F is therefore nonempty, because for any $x \in F'$, we have $(\bar{z}, \bar{x}) \in F$ where $\bar{z} = \max_i \{x_i\}$ and $\bar{x} = ze - x$. But for any point $(z, x) \in F$, we also have $(z + \delta, x + \delta e) \in F$. So $\mu_z z = \mu x + \mu_0$ and $\mu_z(z + \delta) = \mu(x + \delta e) + \mu_0$. Subtracting one equation from the other, we get $\mu_z = \mu e$. We now claim that

any $(ez - x) \in F'$ satisfies $\mu(ez - x) = \mu_0$. This is because $(ez - x) \in F'$ implies $(z, x) \in F$, which implies $\mu ez = \mu x + \mu_0$, which implies $\mu(ex - x) = \mu_0$. But because $ax \geq \beta$ is facet defining, there is a $\lambda > 0$ for which $\mu = \lambda a$ and $\mu_0 = \lambda \beta$. Because $\mu_z = \mu e$, this same λ satisfies (26). \square

Inequality (14) and Theorem 5 imply

Corollary 6 *If the graph coloring problem (1) is defined on a graph in which vertex sets V_1, \dots, V_q induce a cycle, where q is odd and each x_i has domain D_δ with $\delta > 0$, then*

$$z \geq \frac{1}{qs} \sum_{i \in \bar{S}} x_i + \frac{\beta(q, s)}{qs} \quad (27)$$

is facet defining for (1), where

$$\frac{\beta(q, s)}{qs} = \left(1 - \frac{q-1}{4qs}L\right) (L-1)\delta$$

In the case of an odd hole ($s = 1$), the z -cut is

$$z \geq \frac{1}{q} \sum_{i \in \bar{S}} x_i + \frac{q+3}{2q} \delta$$

In the example of Fig. 1, the z -cut is

$$z \geq \frac{1}{10} (x_0 + \dots + x_9) + 2 \quad (28)$$

5.4 Mapping to 0-1 Cuts

The 0-1 model for a coloring problem on a cycle has the following continuous relaxation:

$$\sum_{j \in J} y_{ij} = 1, \quad i = 1, \dots, q \quad (a)$$

$$\sum_{i \in V_k} y_{ij} \leq w_j, \quad j \in J, \quad k = 1, \dots, q \quad (b) \quad (29)$$

$$0 \leq y_{ij}, w_j \leq 1, \quad \text{all } i, j \quad (c)$$

Because constraints (b) appear for each maximal clique, the relaxation implies all clique inequalities $\sum_{i \in V_k} y_{ij} \leq 1$. Nonetheless, we will see that two finite-domain cuts strengthen the relaxation more than the collection of all odd hole cuts.

To simplify discussion, let each x_i have domain $D_1 = \{0, 1, \dots, n-1\}$. The x -cut (16) maps into the cut

$$\sum_{i \in \bar{S}} \sum_{j=1}^{n-1} j y_{ij} \geq \left(sq - \frac{q-1}{4}L\right) (L-1) \quad (30)$$

which is valid by Lemma 1. The z -cut (27) maps into

$$\sum_{j=0}^{n-1} w_j - 1 \geq \frac{1}{q} \sum_{i \in \bar{S}} \sum_{j=1}^{n-1} j y_{ij} + \frac{q+3}{2q} \quad (31)$$

which is valid by Lemma 2.

We will compare cuts (30)–(31) with classical odd hole cuts, which have the form

$$\sum_{i \in H} y_{ij} \leq \frac{q-1}{2} w_j, \quad j = 0, \dots, n-1 \quad (32)$$

where H is the vertex set for an odd hole. The cut (32) is not facet defining in general, although it is facet defining when H contains all vertices of G . This is in contrast with the finite-domain cut (14), which is facet defining in the x -space for any odd hole in G (and more generally, any odd cycle in G).

We first note that when $s = 1$, the 0-1 x -cut (30) is redundant of odd hole cuts.

Lemma 7 *If $s = 1$, the 0-1 x -cut (30) is implied by the 0-1 model (29) with odd hole cuts (32).*

Proof. When $s = 1$, the cut (30) becomes

$$\sum_{i \in \bar{S}} \sum_{j=0}^{n-1} j y_{ij} \geq \frac{q+3}{2} \quad (33)$$

It suffices to show that (33) is dominated by a nonnegative linear combination of (29) and (32), where $H = \bar{S}$ in (32). Assign multiplier 2 to each constraint in (29a); multipliers 2 and 1, respectively, to constraints (32) with $j = 0, 1$; and multipliers $q-1$ and $(q-1)/2$, respectively, to the constraints $w_0 \leq 1$ and $w_1 \leq 1$. The resulting linear combination is

$$\sum_{i \in \bar{S}} y_{i1} + 2 \sum_{j=2}^{n-1} \sum_{i \in \bar{S}} y_{ij} \geq 2q - \frac{q-1}{2} - (q-1) = \frac{q+3}{2}$$

This dominates (33) because the left-hand side coefficients are less than or equal to the corresponding coefficients in (33). \square

However, the two finite-domain cuts (30) and (31), when combined, provide a tighter bound than the n odd hole cuts (32) even when $s = 1$. For example, when $q = 5$, the 10 odd hole cuts provide a lower bound of 2.5 on the chromatic number, while the two finite-domain cuts provide a bound of 2.6. The improvement is modest, but 10 cuts are replaced by only two cuts. Comparisons for larger q appear in the next section.

Furthermore, when $s > 1$, the single 0-1 z -cut (31) provides a tighter bound than the collection of all odd hole cuts, which have no effect in this case. There

are s^q odd hole cuts (32) for each color j , one for every H that selects one element from each S_k , $k = 1, \dots, q$. For example, when $q = 5$ and $s = 2$, there are $ns^q = 320$ odd hole cuts. The lower bound on the chromatic number is 4.0 with or without them. However, the one finite-domain cut (31) yields a bound of 4.5. Addition of the 0-1 x -cut (30) strengthens the bound further, raising it to 5.0. This bound is actually sharp in the present instance, because the chromatic number is 5. Thus two finite-domain cuts significantly improve the bound, while 320 odd hole cuts have no effect on the bound. Further comparisons appear in Section 9.

5.5 Separation

Separating cuts can be identified in either the x -space or the y -space. When a continuous relaxation of the 0-1 model is solved, the resulting values of the y_{ij} s can be used to identify a separating cut directly in 0-1 space. Alternatively, these values can be mapped to values of the x_j s using the transformation $x_i = \sum_j v_j y_{ij}$, and a separation algorithm applied in x -space.

In practice, a solver may apply existing algorithms to identify separating odd hole cuts. The odd holes that give rise to these cuts can trigger the generation of an x -cut and a z -cut. These superior cuts can then replace the odd hole cuts.

If odd cycle cuts for $s > 1$ are desired, a separation algorithm can be applied to the x_i -values by heuristically seeking a cycle that gives rise to separating cuts. We show here that a simple polynomial-time algorithm identifies a separating x -cut and a separating z -cut for a given cycle if such cuts exist.

The algorithm is as follows. We again suppose the colors are $0, 1, \dots, n-1$. Let (13) be an odd q -cycle for which we wish to find a separating cut. Let \bar{y}, \bar{w} be a solution of the continuous relaxation of the 0-1 model, and let

$$\bar{x}_i = \sum_{j=1}^{n-1} j \bar{y}_{ij}, \quad i \in \bigcup_{k=1}^q V_k \quad \bar{z} = \sum_{j=0}^{n-1} \bar{w}_j - 1$$

For each $k = 1, \dots, q$, define the bijection $\pi_k : \{1, \dots, |S_k|\} \rightarrow S_k$ such that $\bar{x}_{\pi_k(\ell)} \leq \bar{x}_{\pi_k(\ell')}$ whenever $\ell < \ell'$. Then for $s = 1, \dots, \min_k |S_k|$, generate a separating x -cut

$$\sum_{k=1}^q \sum_{\ell=1}^s x_{\pi_k(\ell)} \geq \beta(q, s)$$

whenever \bar{x} violates this inequality, and generate a separating z -cut

$$z \geq \frac{1}{qs} \sum_{k=1}^q \sum_{\ell=1}^s x_{\pi_k(|S_k|-\ell+1)} + \frac{\beta(q, s)}{qs}$$

whenever (\bar{x}, \bar{z}) violates this inequality. The running time of the algorithm is $\mathcal{O}(q\bar{s} \log \bar{s})$, where $\bar{s} = \max_k |S_k|$ and $\bar{s} \log \bar{s}$ is the sort time for \bar{s} values.

Lemma 8 *The above algorithms find a separating x -cut and separating z -cut for a given odd q -cycle if such cuts exist.*

Proof. Suppose there is a separating x -cut with $\bar{S}_k \subset S_k$ and $s^* = |\bar{S}_k|$ for $k = 1, \dots, q$. Then

$$\sum_{i \in \bar{S}} \bar{x}_i < \beta(q, s^*) \quad (34)$$

where $\bar{S} = \bigcup_k \bar{S}_k$. Because π_k orders the elements of S_k by size,

$$\sum_{\ell=1}^{s^*} \bar{x}_{\pi_k(\ell)} \leq \sum_{i \in \bar{S}_k} \bar{x}_i, \quad k = 1, \dots, q$$

Summing this over $k = 1, \dots, q$, we get

$$\sum_{k=1}^q \sum_{\ell=1}^{s^*} \bar{x}_{\pi_k(\ell)} \leq \sum_{i \in \bar{S}} \bar{x}_i < \beta(q, s^*)$$

where the strict inequality is due to (34). This means that the algorithm generates the separating cut for $s = s^*$. The proof is similar for z -cuts. \square

6 Webs

We next study cuts that arise from webs. A web $W(q, r)$ is a graph in which vertices can be arranged cyclically so that the edges connect pairs of vertices separated by a distance of at least r on the cycle. More formally, given that $q \geq 2r + 1$ and $r \geq 1$, a web $W(q, r)$ is a graph on vertices $0, \dots, q - 1$ whose edges are all (i, i') such that $0 \leq i \leq q - r - 1$ and $r \leq i' - i \leq q - r$. Thus $W(q, 1)$ is a clique. When q is odd, $W(q, \frac{q-1}{2})$ is an odd hole, and $W(q, 2)$ is an odd anti-hole (the complement of an odd hole). Figure 2 illustrates $W(7, 2)$.

We will focus on webs for which q and r are mutually prime. Odd holes and odd anti-holes are special cases. Such webs give rise to 0-1 finite-domain cuts that provide tighter bounds than known 0-1 cuts.

6.1 Facet-Defining Inequalities

Theorem 9 *Let vertices $0, \dots, q - 1$ of G induce a web $W(q, r)$, where q and r are mutually prime. The following inequality is facet-defining for (1):*

$$\sum_{i=0}^{q-1} x_i \geq \gamma(q, r) \quad (35)$$

where

$$\gamma(q, r) = r \sum_{j=0}^{t-1} v_j + (q - tr)v_t$$

and $t = \lfloor q/r \rfloor$.

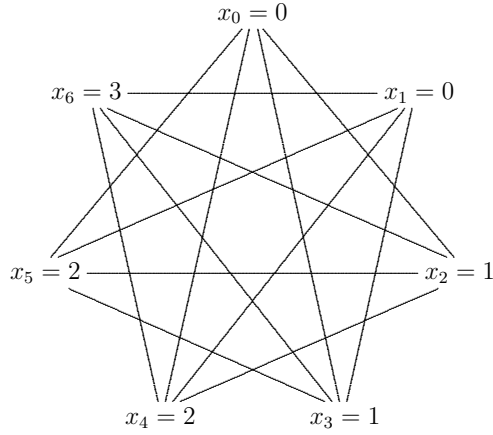


Figure 2: Web $W(7,2)$, which is an odd antihole. Variables connected by an edge appear in a common alldiff constraint. A feasible solution is shown.

Proof. To show that (35) is valid, we observe that each color can be used at most r times. This is because if any set of r vertices receive color j , no two of these vertices can be separated by distance of r or more in the cycle, because any such pair of vertices are connected. The vertices must therefore be adjacent. Because every other vertex is connected to one of them, no other vertex can receive color j , and no more than r vertices can receive color j . This means that at least $t + 1$ colors must be used. Thus

$$\sum_{i=0}^{q-1} x_i \geq r(v_0 + v_1 + \dots + v_{t-1}) + dv_t = \gamma(q, r)$$

where the coefficient $d = q - tr$ is number of vertices remaining after assigning each the colors v_0, \dots, v_{t-1} to r vertices.

To show that (35) is facet defining, let

$$F = \{x \text{ feasible for (1)} \mid (x_0, \dots, x_{q-1}) \text{ satisfies (35) at equality}\}$$

It suffices to show that if $\mu x \geq \mu_n$ holds for all $x \in F$, then there is a scalar $\lambda > 0$ such that

$$\mu_i = \begin{cases} \lambda & \text{for } i = 0, \dots, q-1 \\ \gamma(q, r)\lambda & \text{for } i = n \\ 0 & \text{otherwise} \end{cases} \quad (36)$$

The partial solution

$$(\bar{x}_0, \dots, \bar{x}_{q-1}) = \left(\underbrace{v_0, \dots, v_0}_r, \underbrace{v_1, \dots, v_1}_r, \dots, \underbrace{v_{t-1}, \dots, v_{t-1}}_r, \underbrace{v_t, \dots, v_t}_d \right) \quad (37)$$

is clearly feasible and satisfies (35) at equality. We construct a partial solution $(\tilde{x}_0, \dots, \tilde{x}_{q-1})$ from (37) by swapping the color assignment of the vertices receiving color v_t with that of the last d vertices receiving color v_0 . That is, we let

$$\tilde{x}_i = \begin{cases} v_t & \text{if } i \in \{r-d, \dots, r-1\} \\ v_0 & \text{if } i \in \{q-d, \dots, q-1\} \\ \bar{x}_i & \text{otherwise} \end{cases}$$

Note that $(\tilde{x}_0, \dots, \tilde{x}_{q-1})$ is feasible, because colors v_1, \dots, v_{t-1} are assigned to r adjacent vertices as before, color v_0 is assigned to the r adjacent vertices $q-d, \dots, q-1, 0, \dots, r-d-1$, and color v_t is assigned to the remaining adjacent vertices $r-d, \dots, r-1$. Extend the two partial solutions to feasible solutions \bar{x} and \tilde{x} of (1). Because \bar{x} and \tilde{x} satisfy (35) at equality, we have $\mu\bar{x} = \mu\tilde{x}$, which yields

$$\mu_{r-d} + \dots + \mu_{r-1} = \mu_{q-d} + \dots + \mu_{q-1}$$

By symmetry, we conclude

$$\mu_i + \dots + \mu_{(i+d-1) \bmod q} = \mu_{(i-r) \bmod q} + \dots + \mu_{(i-r+d-1) \bmod q}$$

for $i = 0, \dots, q-1$. Because q and r are mutually prime, this implies that the sums $\mu_i + \dots + \mu_{(i+d-1) \bmod q}$ are equal for all i . Thus, in particular, they are equal for i and $i+1$, which yields $\mu_i = \mu_{(i+d) \bmod q}$ for all i . Because $d \neq q$, this implies that $\mu_0 = \dots = \mu_{q-1}$ and

$$\mu_i = \lambda, \quad i = 0, \dots, q-1 \tag{38}$$

for some $\lambda > 0$.

Define \bar{x}' by letting $\bar{x}' = \bar{x}$ except that for an arbitrary $\ell \notin \{0, \dots, q-1\}$, \bar{x}'_ℓ is assigned a value that does not appear in the tuple \bar{x} . Since $\bar{x}, \bar{x}' \in F$, we have $\mu\bar{x} = \mu\bar{x}'$, which implies

$$\mu_i = 0, \quad i \in V \setminus \{0, \dots, q-1\} \tag{39}$$

Because $\mu\bar{x} = \mu_n$, we have from (38) that $\mu_n = \gamma(q, r)\lambda$. This, (38) and (39) imply (37). \square

For domain D_δ with $\delta > 0$, the cut (35) is

$$\sum_{i=0}^{q-1} x_i \geq \left(q - \frac{1}{2}(t+1)r\right) t\delta$$

For an odd antihole $W(q, 2)$ with domain D_δ , the cut simplifies to

$$\sum_{i=0}^{q-1} x_i \geq \frac{1}{4}(q-1)^2\delta \tag{40}$$

Theorems 5 and 9 imply

Corollary 10 *If the graph coloring problem (1) is defined on a graph in which vertex sets V_r induce a web $W(q, r)$, where q and r are mutually prime, and each x_i has domain D_δ with $\delta > 0$, then*

$$z \geq \frac{1}{q} \sum_{i=1}^q x_i + \left(1 - \frac{r}{2q}(t+1)\right) t\delta \quad (41)$$

is facet defining for (1), where $t = \lfloor q/r \rfloor$.

For example, the antihole of Fig. 2 gives rise to the facet-defining cuts

$$\sum_{i=0}^6 x_i \geq 9, \quad z \geq \frac{1}{7} \sum_{i=0}^6 x_i + \frac{9}{7}$$

6.2 Mapping to 0-1 Cuts

If each x_i has domain $D_1 = \{0, 1, \dots, n-1\}$ the x -cut (16) maps into the cut

$$\sum_{i=0}^{q-1} \sum_{j=1}^{q-1} j y_{ij} \geq \left(q - \frac{1}{2}(t+1)r\right) t \quad (42)$$

which is valid by Lemma 1. The z -cut (41) maps into

$$\sum_{j=0}^{q-1} w_j - 1 \geq \frac{1}{q} \sum_{i=0}^{q-1} \sum_{j=1}^{q-1} j y_{ij} + \left(1 - \frac{r}{2q}(t+1)\right) t \quad (43)$$

We wish to compare these cuts with known cuts for webs. Facet-defining web cuts for a 0-1 model of the coloring problem are given in [14]. These cuts are defined in a space in which the variables correspond to edges and colorings correspond to “admissible star partitions” of the graph. However, the cuts are based on the fact that at most r vertices can be assigned any given color, and we can write analogous web cuts in the y_{ij} -space:

$$\sum_{i=0}^{q-1} y_{ij} \leq r \cdot w_j, \quad \text{all } j \quad (44)$$

As with odd cycle cuts, the finite-domain web cuts provide a tighter bound than known 0-1 cuts. For example, for an antihole $W(7, 2)$, seven 0-1 web cuts (44) give a bound of 3.5, while the two finite-domain cuts provide a bound of 3.5714. For the web $W(8, 3)$, eight 0-1 cuts give the bound 2.6667, while the two finite-domain cuts yield the bound 2.75. This if an existing separation algorithm identifies 0-1 cuts for a given web, they can be replaced (at no additional cost) with two finite-domain cuts that yield a tighter bound. Further comparisons appear in Section 9.

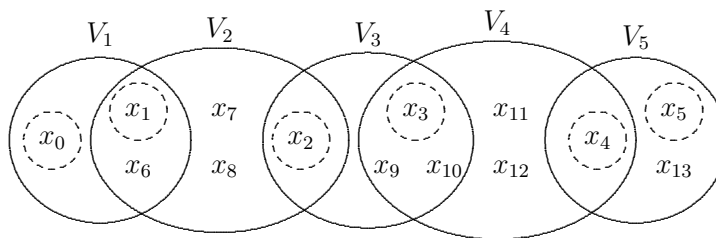


Figure 3: A path with $q = 5$. The variables in dashed circles appear in one possible valid cut.

7 Paths

Paths present an interesting case because they give rise to finite-domain cuts that are redundant in the 0-1 model. That is, they have no effect on the bound when the problem consists entirely of a path system. However, a few finite-domain cuts may replace a large number of inequalities in a more complex coloring problem, allowing a substantial reduction in the size of the 0-1 model. In addition, path cuts are useful in a finite-domain model of the problem.

A path is a subgraph of G induced by the vertices in subsets V_0, \dots, V_q of V , provided the subgraph induced by each V_k is a clique, and only adjacent V_k 's overlap. That is,

$$V_k \cap V_\ell = \begin{cases} S_k & \text{if } k + 1 = \ell \\ \emptyset & \text{otherwise} \end{cases}$$

where $S_k \neq \emptyset$.

7.1 Facet-defining Inequalities

Facet-defining inequalities can be obtained by selecting one variable from each overlap S_k . Valid cuts can be obtained if two or more variables are selected, as with cycles, but they are redundant of the single-variable cuts.

Lemma 11 *Let V_0, \dots, V_q be a path, and let $x_0 \in V_0 \setminus V_1$, $x_q \in V_q \setminus V_{q-1}$, and $x_i \in S_i$ for $i = 1, \dots, q-1$. If q is odd, the following inequality is valid for (1):*

$$(v_2 - v_0)(x_0 + x_q) + (v_1 - v_0) \sum_{i=1}^{q-1} x_i \geq \phi(q) \quad (45)$$

where

$$\phi(q) = \left(\frac{q-1}{2}(v_1 - v_0) + v_2 - v_0 \right) (v_0 + v_1)$$

Proof. Suppose to the contrary (45) is not satisfied, which implies

$$(v_1 - v_0) \sum_{i=1}^{q-1} x_i < \phi(q) - (v_2 - v_0)(x_0 + x_q) \quad (46)$$

Adjacent variables in the list x_0, \dots, x_q must take distinct values. So we have $x_i + x_{i+1} \geq v_0 + v_1$ for $i = 1, 3, \dots, q-2$. Summing these, we obtain

$$\sum_{i=1}^{q-1} x_i \geq \frac{q-1}{2}(v_0 + v_1)$$

This and (46) imply

$$\frac{q-1}{2}(v_1 - v_0)(v_0 + v_1) < \left(\frac{q-1}{2}(v_1 - v_0) + v_2 - v_0 \right) (v_0 + v_1) - (v_2 - v_0)(x_0 + x_q)$$

which implies $x_0 + x_q < v_0 + v_1$. This is possible only if $x_0 = x_q = v_0$, because $v_1 > v_0 \geq 0$. Thus $x_1, x_{q-1} \neq v_0$, which means that at most $(q-3)/2$ of the variables x_1, \dots, x_{q-1} can take the value v_0 , and at least one variable must take a value larger than v_1 . So

$$(v_1 - v_0) \sum_{i=1}^{q-1} x_i \geq (v_1 - v_0) \left(\frac{q-3}{2}v_0 + \frac{q-1}{2}v_1 + v_2 \right) = \phi(q) + (v_1 - v_0)(v_2 - v_0)$$

But this is inconsistent with (46) because $x_0 + x_q \geq 0$ and $(v_1 - v_0)(v_2 - v_0) > 0$.
□

If each x_i has domain D_δ for $\delta > 0$, the cut (45) is

$$2(x_0 + x_q) + \sum_{i=1}^{q-1} x_i \geq \frac{1}{2}(q+3)\delta \quad (47)$$

Theorem 12 *If the graph coloring problem (1) is defined on a graph in which vertex sets V_0, \dots, V_q induce a path, where q is odd, then inequality (45) is facet defining for (1).*

Proof. Let

$$F = \{x \text{ feasible for (1)} \mid (x_0, \dots, x_q) \text{ satisfies (45) at equality}\}$$

It suffices to show that any equation $\mu x \geq \mu_{n+1}$ that holds for all $x \in F$ is a positive scalar multiple of (45). It therefore suffices to show that there is a scalar $\lambda > 0$ such that

$$\mu_i = \begin{cases} (v_1 - v_0)\lambda & \text{for } i = 1, \dots, q-1 \\ (v_2 - v_0)\lambda & \text{for } i = 0, q \\ \phi(q)\lambda & \text{for } i = n+1 \\ 0 & \text{otherwise} \end{cases} \quad (48)$$

The following partial solutions are feasible for (1):

$$\begin{aligned} (\bar{x}_0, \dots, \bar{x}_q) &= (v_1, v_0, v_1, v_0, v_1, v_0, v_1, \dots, v_1, v_0) \\ (\hat{x}_0, \dots, \hat{x}_q) &= (v_0, v_2, v_1, v_0, v_1, v_0, v_1, \dots, v_1, v_0) \\ (\tilde{x}_0, \dots, \tilde{x}_q) &= (v_0, v_1, v_2, v_0, v_1, v_0, v_1, \dots, v_1, v_0) \end{aligned}$$

They can be extended to complete solutions $\bar{x}, \hat{x}, \tilde{x}$ with

$$\bar{x}_i = \hat{x}_i = \tilde{x}_i \text{ for } i \notin \{0, \dots, q\}$$

in the manner described above. Because \hat{x} and \tilde{x} satisfy (45) at equality, they satisfy $\mu\hat{x} = \mu_{n+1}$, and we have $\mu\hat{x} = \mu\tilde{x} = \mu_{n+1}$. This implies $\mu(\hat{x} - \tilde{x}) = 0$, and therefore $\mu_1 = \mu_2$. By symmetry, we have

$$\mu_1 = \dots = \mu_{q-1} \quad (49)$$

Also \bar{x} satisfies (45) at equality, and we have $\mu\bar{x} = \mu\hat{x} = \mu_{n+1}$. This implies $(v_1 - v_0)\mu_0 = (v_2 - v_0)\mu_1$. Thus by (49) and symmetry

$$(v_1 - v_0)\mu_0 = (v_1 - v_0)\mu_q = (v_2 - v_0)\mu_i, \quad i = 1, \dots, q-1 \quad (50)$$

Define \bar{x}' by letting $\bar{x}' = \bar{x}$ except that for an arbitrary $\ell \notin \{0, \dots, q\}$, \bar{x}'_ℓ is assigned a value that does not appear in the tuple \bar{x} . Since \bar{x}, \bar{x}' are feasible and satisfy (45) at equality, we have $\mu\bar{x} = \mu\bar{x}'$. This and $\bar{x}_\ell \neq \bar{x}'_\ell$ imply

$$\mu_i = 0, \quad i \in V \setminus \{0, \dots, q\} \quad (51)$$

Finally, (50) implies that for some $\lambda > 0$,

$$\mu_0 = \mu_q = (v_2 - v_0)\lambda \text{ and } \mu_i = (v_1 - v_0)\lambda, \quad i = 1, \dots, q-1 \quad (52)$$

Because $\mu\bar{x} = \mu_{n+1}$, we have from (52) that $\mu_{n+1} = \phi(q)\lambda$. This, (51), and (52) imply (48). \square

Theorems 5 and 12 imply

Corollary 13 *If the graph coloring problem (1) is defined on a graph in which vertex sets V_0, \dots, V_q induce a path, where q is odd and each x_i has domain D_δ for $\delta > 0$, then*

$$z \geq \frac{1}{q+3} \left(2(x_0 + x_q) + \sum_{i=1}^{q-1} x_i \right) + \frac{\delta}{2} \quad (53)$$

is facet defining for (1).

If the colors are $0, 1, \dots, 4$, the cuts for the path in Fig. 3 are

$$2(x_0 + x_5) + \sum_{i=1}^4 x_i \geq 4, \quad z \geq \frac{1}{4}(x_0 + x_5) + \frac{1}{8} \sum_{i=1}^4 x_i + \frac{1}{2}$$

7.2 Mapping to 0-1 Cuts

Assuming domain D_1 , the x -cut (45) and z -cut (53) respectively map to 0-1 space as follows:

$$2 \sum_{j=1}^{q-1} j(y_{0j} + y_{qj}) + \sum_{i=1}^{q-1} \sum_{j=1}^{q-1} j y_{ij} \geq \frac{1}{2}(q+3) \quad (54)$$

$$\sum_{j=1}^q w_j - 1 \geq \frac{2}{q+3} \sum_{j=1}^{q-1} j(y_{0j} + y_{qj}) + \frac{1}{q+3} \sum_{i=1}^{q-1} \sum_{j=1}^{q-1} j y_{ij} + \frac{1}{2} \quad (55)$$

To simplify notation, we suppose in this section that each $S_k = \{k\}$. The arguments are very similar for the more general case. Given this simplification, the continuous relaxation of the 0-1 path model is

$$\begin{aligned} \sum_{j=0}^{q-1} y_{ij} &= 1, \quad i = 0, \dots, q && (r_i) \\ y_{ij} + y_{i+1,j} &\leq w_j, \quad i = 0, \dots, q-1, j = 0, \dots, q-1 && (s_{ij}) \\ 0 &\leq y_{ij}, w_j \leq 1, \quad \text{all } i, j \end{aligned} \quad (56)$$

It can be shown that if the w_j s are treated as constants equal to 1, the constraint matrix of this model is totally unimodular. The 0-1 x -cut (54) is therefore redundant of (56).

We cannot use a similar argument to show that the 0-1 z -cut (55) is redundant, because the full model (56) with w_j s is not totally unimodular. In fact, the 0-1 z -cut is not redundant, because it is not implied by (56) augmented with symmetry-breaking constraints $w_i \geq w_{i+1}$. The following (extreme point) solution satisfies (56) with $q = 3$ but violates the cut because it results in a left-hand side of $1\frac{2}{7}$ and a right-hand side of $2\frac{5}{14}$.

$$y = \begin{bmatrix} 0 & \frac{4}{7} & 0 & \frac{3}{7} \\ \frac{2}{7} & 0 & \frac{4}{7} & \frac{1}{7} \\ \frac{2}{7} & \frac{4}{7} & 0 & \frac{1}{7} \\ 0 & 0 & \frac{4}{7} & \frac{3}{7} \end{bmatrix}, \quad w = \left(\frac{4}{7}, \frac{4}{7}, \frac{4}{7}, \frac{4}{7}\right)$$

The 0-1 z -cut has no effect on the bound in a problem consisting entirely of a path system, because the relaxation (56) already implies the optimal bound of 2. The sum of the negated constraints (r_{q-1}) and (r_q) with constraints (s_{q-1}, j) for $j = 0, \dots, q$ yields the bound $\sum_j w_j \geq 2$.

On the other hand, if the 0-1 x -cut and z -cut are present, the 0-1 model yields the same bound of 2 even if all of the constraints (s_{ij}) are dropped. This suggests that for a more complex problem, a few finite-domain cuts could replace many constraints in the 0-1 model with little effect on the resulting bound. We leave this issue to future research.

8 Intersecting Systems

Finally, we identify cuts for more general structures for which there are apparently no previously known 0-1 cuts. They illustrate how a finite-domain perspective can lead to facet-defining cuts for a wide variety of situations.

We define an *intersecting system* to be a family of clique-inducing vertex sets such that (a) at least one vertex belongs to all the sets, (b) every set contains

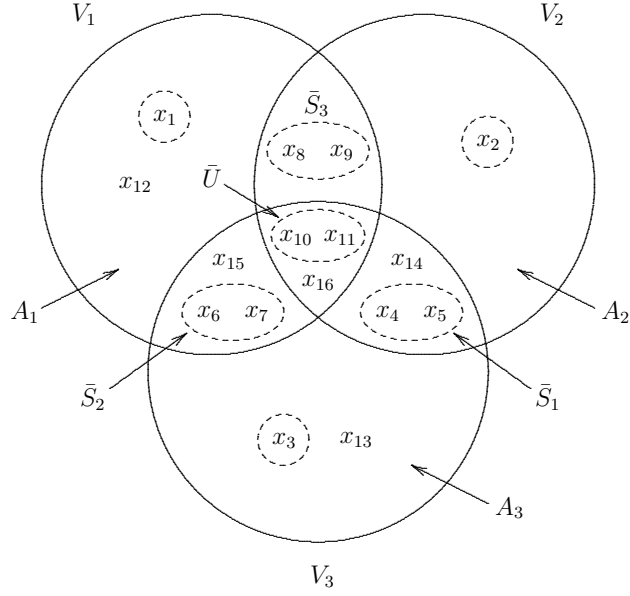


Figure 4: An intersecting system with $q = 3$. The variables x_1, x_2, x_3 and sets $\bar{S}_1, \bar{S}_2, \bar{S}_3, \bar{U}$ provide the basis for two possible facet-defining cuts with $s = u = 2$.

at least one vertex that belongs to no other set, and (c) every set excludes at least one vertex that belongs to all the other sets. Formally, an intersecting system consists of a family V_1, \dots, V_q of clique-inducing vertex sets such that the following sets are nonempty:

$$\begin{aligned}
 U &= \bigcap_{k=1}^q V_k \\
 A_k &= V_k \setminus \bigcup_{\ell \neq k} V_\ell, \quad k = 1, \dots, q \\
 S_k &= \left(\bigcap_{\ell \neq k} V_\ell \right) \setminus V_k, \quad k = 1, \dots, q
 \end{aligned}$$

Figure 4 illustrates an intersecting system with $q = 3$.

8.1 Valid Inequalities

We first establish two families of valid inequalities for an intersecting system.

Lemma 14 *Given an intersecting system V_1, \dots, V_q , suppose $x_k \in A_k$, $\bar{S}_k \subset S_k$, and $|\bar{S}_k| = s$ for $k = 1, \dots, q$. Then if $A = \bigcup_{k=1}^q \{x_k\}$, $S = \bigcup_k \bar{S}_k$, $\bar{U} \subset U$,*

and $|\bar{U}| = u$, the following inequalities are valid for (1):

$$c_1 \sum_{i \in A} x_i + c_2 \sum_{i \in S \cup \bar{U}} x_i \geq \psi(q, s, u) \quad (57)$$

$$d_1 \sum_{i \in A} x_i + d_2 \sum_{i \in S} x_i + d_3 \sum_{i \in \bar{U}} x_i \geq \omega(q, s, u) \quad (58)$$

where

$$c_1 = v_{qs+u} - v_0$$

$$c_2 = \sum_{i=1}^{q-1} v_i - (q-1)v_0$$

$$\psi(q, s, u) = c_1 q v_0 + c_2 \sum_{i=1}^{qs+u} v_i$$

$$d_1 = v_{qs+u} - v_u$$

$$d_2 = \sum_{i=1}^{q+u-1} v_i - q v_u$$

$$d_3 = (v_{qs+u} - v_u)q$$

$$\omega(q, s, u) = d_1 q v_u + d_2 \sum_{i=u+1}^{qs+u} v_i + d_3 \sum_{i=1}^{u-1} v_i$$

Proof. We prove the validity of (57) only, as the proof for (58) is similar. Write (57) as $ax \geq \psi(q, s, u)$, and let \bar{x} be any feasible solution of (1) that minimizes ax . It suffices to show that $a\bar{x} = \psi(q, s, u)$.

We first show that $\bar{x}_i \leq v_{qs+u}$ for all $i \in S \cup \bar{U}$. Suppose to the contrary that $\bar{x}_j = w > v_{qs+u}$ for some $j \in S \cup \bar{U}$. Then at least two values $w_1, w_2 \in \{v_0, \dots, v_{qs+u}\}$ fail to appear in $\{\bar{x}_i \mid i \in S \cup \bar{U} \setminus \{j\}\}$, because the latter set has $qs+u-1$ elements. Suppose $w_1 < w_2$. Then w_2 does not appear in $\{\bar{x}_i \mid i \in A\}$, because if it did, we could define a feasible solution \hat{x} that is identical to \bar{x} except that $\hat{x}_k = w_1$ for some $k \in A$, and we would have $a\hat{x} < a\bar{x}$, contrary to the assumption that \bar{x} minimizes ax . So w_2 appears nowhere in $\{\bar{x}_i \mid i = 1, \dots, q\}$. This means we can define a feasible solution \tilde{x} that is identical to \bar{x} except that $\tilde{x}_j = w_2$ (rather than w), and we have $a\tilde{x} < a\bar{x}$, contrary to assumption.

Now since $\bar{x}_i \leq v_{qs+u}$ for all $i \in S \cup \bar{U}$, and all the variables in $S \cup \bar{U}$ must take different values, the set $\{\bar{x}_i \mid i \in S \cup \bar{U}\}$ must be $\{v_0, \dots, v_{qs+u}\} \setminus \{v_r\}$ for some $r \in \{0, \dots, qs+u\}$. This implies

$$\min_x \{ax\} = c_1 \sum_{i \in A} \bar{x}_i + c_2 \left(\sum_{i=0}^{qs+u} v_i - v_r \right) \quad (59)$$

We consider two cases.

Case 1: $r \geq q$. Here all the values in $\{v_0, \dots, v_{q-1}\}$ appear in $\{\bar{x}_i \mid i \in S \cup \bar{U}\}$. So each value in $\{v_0, \dots, v_{q-1}\}$ can appear at most once in $\{\bar{x}_j \mid j \in A\}$. Thus

from (59) we have

$$\min_x \{ax \mid r \geq q\} = c_1 \sum_{i=0}^{q-1} v_i + c_2 \left(\sum_{i=0}^{qs+u} v_i - v_r \right) = c_1 \sum_{i=0}^{q-1} v_i + c_2 \sum_{i=0}^{qs+u-1} v_i = \psi(q, s, u)$$

where the second equality is due to the fact that we must have $r = qs + u$. Otherwise, $\bar{x}_i = v_{qs+u} > v_r$ for some $i \in S \cup \bar{U}$, and we can create a feasible solution \hat{x} that is identical to \bar{x} except that $\hat{x}_i = v_r$, and we have $a\hat{x} < a\bar{x}$. The third equality follows algebraically from the definitions of c_1, c_2 .

Case 2. $0 \leq r \leq q-1$. We show that the minimum is obtained in this case only if $c_1 \geq c_2$, and that the minimum is again $\psi(q, s, u)$. Because v_r does not appear in $\{\bar{x}_i \mid i \in S \cup \bar{U}\}$, no value greater than v_r appears in $\{\bar{x}_i \mid i \in A\}$. Furthermore, all the values in $\{v_0, \dots, v_{r-1}\}$ appear in $\{\bar{x}_i \mid i \in S \cup \bar{U}\}$. Thus each value in $\{v_0, \dots, v_{r-1}\}$ can appear at most once in $\{\bar{x}_j \mid j \in A\}$. The remaining $q-r$ variables in A can be assigned the value v_r , because it does not appear in $\{\bar{x}_i \mid i \in S \cup \bar{U}\}$. So, from (59) we have

$$\min_x \{ax \mid 0 \leq r \leq q-1\} = c_1 \left(\sum_{i=0}^{r-1} v_i + (q-r)v_r \right) + c_2 \left(\sum_{i=0}^{qs+u} v_i - v_r \right)$$

If $c_1 \geq c_2$, this expression is minimized by setting $r = 0$, whereupon it reduces to $\psi(q, s, u)$. If $c_1 < c_2$, the minimum occurs when $r = q-1$, and the expression becomes

$$c_1 \sum_{i=0}^{q-1} v_i + c_2 \left(\sum_{i=1}^{qs+u} v_i - v_{q-1} \right) > c_1 \sum_{i=0}^{q-1} v_i + c_2 \sum_{i=0}^{qs+u-1} v_i = \psi(q, s, u)$$

Thus if $c_1 < c_2$, the minimum occurs in Case 1, and the lemma follows. \square

When the domains are D_δ , the inequalities (57)–(58) become

$$\begin{aligned} (qs+u) \sum_{i \in A} x_i + \frac{1}{2}q(q-1) \sum_{i \in S \cup \bar{U}} x_i &\geq \psi(q, s, u) \\ s \sum_{i \in A} x_i + \frac{1}{2}(q-1) \sum_{i \in S} x_i + qs \sum_{i \in \bar{U}} x_i &\geq \omega(q, s, u)/q \end{aligned} \tag{60}$$

where

$$\begin{aligned} \psi(s, q, u) &= \frac{1}{4}q(q-1)(qs+u)(qs+u+1)\delta \\ \omega(s, q, u)/q &= \frac{1}{4}qs(2u(u+1) + (q-1)(sq+2u+1))\delta \end{aligned}$$

Given domain D_1 , the cuts for the system in Fig. 4 are

$$\begin{aligned} 8(x_1 + x_2 + x_3) + 3(x_4 + \dots + x_{11}) &\geq 108 \\ 2(x_1 + x_2 + x_3) + (x_4 + \dots + x_9) + 6(x_{10} + x_{11}) &\geq 51 \end{aligned}$$

8.2 Facet-Defining Inequalities

Theorem 15 *If V_1, \dots, V_q is an intersecting system, then inequalities (57) and (58) are facet defining.*

Proof. We prove that (57) is facet defining, as the proof for (58) is similar. We know that (57) is valid, by Lemma 14. Let

$$F = \{x \text{ feasible for (1)} \mid x \text{ satisfies (57) at equality}\}$$

To show that (57) is facet defining, it suffices to show that if $\mu x \geq \mu_0$ holds for all $x \in F$, then there is a scalar $\lambda > 0$ such that

$$\mu_i = \begin{cases} \lambda c_1 & \text{for } i \in A \\ \lambda c_2 & \text{for } i \in S \cup \bar{U} \\ 0 & \text{for } i \in V \setminus (A \cup S \cup \bar{U}) \\ \lambda \psi(q, s, u) & \text{for } i = 0 \end{cases} \quad (61)$$

Let x^A be the tuple of variables x_j for $i \in A$, x^i the tuple of variables x_j for $j \in S_i$, and x^U the tuple of variables x_j for $j \in \bar{U}$. The partial solution

$$\begin{aligned} \bar{x}^A &= (v_0, \dots, v_0) \\ \bar{x}^i &= (v_i, v_{i+q}, v_{i+2q}, \dots, v_{i+(s-1)q}), \text{ for } i = 1, \dots, q \\ \bar{x}^U &= (v_{sq+1}, \dots, v_{sq+u}) \end{aligned}$$

is feasible and satisfies (57) at equality. As in previous proofs, it is straightforward to show that $\mu_i = 0$ for $i \in V \setminus (A \cup S \cup \bar{U})$. Also, by symmetry, μ_i and μ_j take the same value λ_A (or λ_S or λ_U) for any pair i, j in A (or S or \bar{U}). So if $x^S = (x^1, \dots, x^q)$, the equation $\mu x = \mu_0$ reduces to

$$\lambda_A \sum_{i \in A} x_i + \lambda_S \sum_{i \in S} x_i + \lambda_U \sum_{i \in \bar{U}} x_i = \mu_0 \quad (62)$$

Now let $(\tilde{x}^A, \tilde{x}^S, \tilde{x}^U) = (\bar{x}^A, \bar{x}^S, \bar{x}^U)$ except that $\tilde{x}_j = \bar{x}_k$ for arbitrary $j \in S$, $k \in \bar{U}$. Extend these two partial solutions to feasible solutions \bar{x}, \tilde{x} of (1). Since \bar{x}, \tilde{x} satisfy (62), we have $\lambda_S = \lambda_U$. So (62) reduces to

$$\lambda_A \sum_{i \in A} x_i + \lambda_S \sum_{i \in S \cup \bar{U}} x_i = \mu_0 \quad (63)$$

By definition of \bar{x} , we have that $\bar{x}_j = v_q$ for some $j \in S_q$, and $\bar{x}_k = qs + u$ for some $k \in \bar{U}$. Let $(\hat{x}^A, \hat{x}^S, \hat{x}^U) = (\bar{x}^A, \bar{x}^S, \bar{x}^U)$ except that $\hat{x}_j = v_0$ and $\hat{x}_k = v_q$, and extend this partial solution to a feasible solution \hat{x} . Then since \bar{x}, \hat{x} satisfy (63), we have

$$\lambda_A q v_0 + \lambda_S \sum_{i=1}^{qs+u} v_i = \lambda_A \sum_{i=1}^{q-1} v_i + \lambda_S \left(\sum_{i=1}^{q-1} v_i + v_0 + \sum_{i=q+1}^{qs+u-1} v_i + v_q \right)$$

This yields

$$\lambda_A \left(\sum_{i=1}^{q-1} x_i - (q-1)v_0 \right) = \lambda_S (v_{qs+u} - v_0)$$

or $\lambda_A c_2 = \lambda_S c_1$. Thus $\mu_j/\mu_i = c_2/c_1$ for any $i \in A$ and any $j \in S \cup \bar{U}$. Also since $\mu \bar{x} = \mu_0$, we have

$$\mu_0/\mu_i = \sum_{i \in A} \bar{x}_i + \frac{c_2}{c_1} \sum_{i \in S \cup \bar{U}} \bar{x}_i = qv_0 + \frac{c_2}{c_1} \sum_{i=1}^{qs+u} v_i = \psi(q, s, u)/c_1$$

So there exists a λ satisfying (61). \square

8.3 Bounds on the Chromatic Number

Theorems 5 and 15 imply

Corollary 16 *If V_1, \dots, V_q is an intersecting system and each x_i has domain D_δ with $\delta > 0$, then the inequalities*

$$\begin{aligned} z &\geq \frac{2}{q(q+1)} \sum_{i \in A} x_i + \frac{q-1}{(q+1)(qs+u)} \sum_{i \in S \cup \bar{U}} x_i + \frac{1}{2} \frac{q-1}{q+1} (qs+u+1) \\ z &\geq \frac{2}{qh} \sum_{i \in A} x_i + \frac{q-1}{qsh} \sum_{i \in S} x_i + \frac{2}{h} \sum_{i \in \bar{U}} x_i + \frac{1}{2h} (2u(u+1) + (q-1)(qs+2u+1)) \end{aligned} \quad (64)$$

are facet defining for (1), where $h = q + 2u + 1$.

Given domain D_1 , the bounds for the system in Fig. 4 are

$$\begin{aligned} z &\geq \frac{1}{6}(x_1 + x_2 + x_3) + \frac{1}{16}(x_4 + \dots + x_{11}) + \frac{9}{4} \\ z &\geq \frac{1}{12}(x_1 + x_2 + x_3) + \frac{1}{24}(x_2 + \dots + x_9) + \frac{1}{4}(x_{10} + x_{11}) + \frac{17}{8} \end{aligned}$$

As with other cuts, the inequalities (57), (58), and (64) can be mapped to valid 0-1 inequalities by replacing each x_i with $\sum_j v_j y_{ij}$ and z with $\sum_j w_j - 1$.

8.4 Separation

A polynomial-time separation algorithm for (57), given domain D_1 , is as follows. Let V_1, \dots, V_q define an intersecting system for which we wish to find a separating cut, and let (\bar{x}, \bar{z}) be a solution of the current continuous relaxation (perhaps mapped from the 0-1 model). For each $k = 1, \dots, q$, let $\bar{x}_{a(i)} = \min_{i \in A} \{\bar{x}_i\}$ and $\bar{x}_{b(i)} = \max_{i \in A} \{\bar{x}_i\}$, and define the bijection $\pi_k : \{1, \dots, |S_k|\} \rightarrow S_k$ such that $\bar{x}_{\pi_k(i)} \leq \bar{x}_{\pi_k(i')}$ whenever $i < i'$. Also define the bijection $\pi : \{1, \dots, |U|\}$ such that $\bar{x}_{\pi(i)} \leq \bar{x}_{\pi(i')}$ whenever $i < i'$. Then for $s = 1, \dots, \min_k |S_k|$ and $u = 1, \dots, |U|$, generate a separating x -cut

$$(qs+u) \sum_{i=1}^q x_{a(i)} + \frac{1}{2} q(q-1) \left(\sum_{k=1}^q \sum_{i=1}^s x_{\pi_k(i)} + \sum_{i=1}^u x_{\pi(i)} \right) \geq \frac{1}{4} q(q-1)(qs+u)(qs+u+1)$$

Table 1: Lower bounds on the chromatic number in a 0-1 clique formulation of problem instances consisting of one q -cycle with overlap of s .

q	s	Without cuts	Odd hole cuts only	x -cut only	z -cut only	x -cut & z -cut	Optimal value	No. of odd hole cuts
5	1	2.00	2.50	2.00	2.30	2.60	3	5
	2	4.00	4.00	4.00	4.50	5.00	5	320
	3	6.00	6.00	6.00	6.77	7.53	8	3645
	4	8.00	8.00	8.00	9.00	10.00	10	20,480
	5	10.00	10.00	10.00	11.26	12.52	13	78,125
7	1	2.00	2.33	2.00	2.21	2.43	3	7
	2	4.00	4.00	4.00	4.36	4.71	5	1792
	3	6.00	6.00	6.00	6.50	7.00	7	45,927
	4	8.00	8.00	8.00	8.68	9.36	10	458,752
9	1	2.00	2.25	2.00	2.17	2.33	3	9
	2	4.00	4.00	4.00	4.28	4.56	5	9216
	3	6.00	6.00	6.00	6.39	6.78	7	531,441

whenever \bar{x} violates this inequality, and generate a separating z -cut

$$z \geq \frac{2}{q(q+1)} \sum_{i=1}^q x_{b(i)} + \frac{q-1}{(q+1)(qs+u)} \left(\sum_{k=1}^q \sum_{i=1}^s x_{|S_k|-i+1} + \sum_{i=1}^u x_{|U|-i+1} \right) + \frac{1}{2} \frac{q-1}{q+1} (qs+u+1)$$

whenever (\bar{x}, \bar{z}) violates this inequality.

This procedure obtains a separating cut whenever one exists. Separation for (58) is similar.

9 Computational Results

9.1 Cycles

We generated instances of the cycle problem parameterized by s and q . All of the overlap sets S_k have size s , and vertex set $V_k = S_k \cup S_{k+1}$ for $k = 1, \dots, q-1$ (with $V_k = S_k \cup S_1$). For each instance, we solved the linear programming relaxation that minimizes $\sum_j w_j$ subject to (29) and various classes of cuts. Clique inequalities are always present.

We generated the instances indicated in Table 1, which shows the resulting bounds, the optimal value (chromatic number), and the number of odd hole cuts.

For $s = 1$, the table confirms that 0-1 x -cuts are redundant of odd hole cuts. However, the combination of one x -cut and one z -cut yields a tighter bound

than n odd hole cuts. The improvement is modest, but it is obtained at no additional cost. It is therefore advantageous to replace any set of standard cuts generated for an odd hole with these two cuts.

For $s > 1$, neither odd hole cuts nor 0-1 x -cuts alone have any effect on the bound when clique inequalities are present. However, a single 0-1 z -cut significantly improves the bound. Combining the z -cut with the x -cut raises the bound still further, substantially reducing the integrality gap, sometimes to zero. Two finite-domain cuts therefore provide a much tighter relaxation than a large set of standard clique inequalities and odd hole cuts.

We also investigated whether the finite-domain cuts are equally effective in the x -space, where they take their original form (14) and (27). We formulated a linear relaxation of the finite domain model that minimizes $z + 1$ subject to $z \geq x_i$ for all i , plus cuts. It is easy to show that the LP bound subject to the x -cut alone, or to the z -cut alone, is $\beta(q, s)/qs + 1$. The bound subject to both cuts is $2\beta(q, s)/qs + 1$. These bounds appear in the left half of Table 2. The two cuts, when combined, yield the same bound as in the 0-1 model.

One might obtain a fairer comparison if clique inequalities are added to the finite-domain model, because they appear in the 0-1 model. In the finite-domain model, clique inequalities correspond to the individual alldiff constraints. We know from [6, 15] that for domain D_1 , the following is facet-defining for $\text{alldiff}(X_k)$:

$$\sum_{i \in V_k} x_i \geq \frac{1}{2}|V_k|(|V_k| - 1)$$

In the test instances, $|V_k| = 2s$. We therefore added the following cuts:

$$\sum_{i \in V_k} x_i \geq s(2s - 1), \quad k = 1, \dots, q$$

Using Theorem 5, we also added the cuts:

$$z \geq \frac{1}{qs} \sum_{i \in V_k} x_i + \frac{2s - 1}{q}, \quad k = 1, \dots, q$$

The results appear in the right half of Table 2. The x -cut performs as before, but now the z -cut provides the same bound as in the 0-1 model. When combined, the x -cut and y -cut again deliver the same bound as in the 0-1 model.

Thus two odd cycle cuts yield the same bound in the very small finite-domain relaxation (even without clique inequalities) as in the much larger 0-1 relaxation. The finite-domain relaxation contains n variables x_i and $n + 2$ constraints, while the 0-1 relaxation contains $n^2 + n$ variables y_{ij}, w_j and $n^2 + n + 2$ constraints (dropping odd hole cuts). It may therefore be advantageous to obtain bounds from a finite-domain model rather than a 0-1 model.

9.2 Webs

We generated the webs $W(q, r)$ shown in Table 3. We omitted clique cuts (when they exist) because they have no effect on the bound. The table shows that the

Table 2: Lower bounds on the chromatic number in the finite-domain model of problem instances consisting of one q -cycle with overlap of s and color set $\{0, 1, \dots, n - 1\}$.

q	s	No cuts	x -cut only	z -cut only	x -cut & z -cut	Clique cuts	Plus x -cut	Plus z -cut	Plus x -cut & z -cut
5	1	1.00	1.80	1.80	2.60	1.50	1.80	2.30	2.60
	2	1.00	3.00	3.00	5.00	2.50	3.00	4.50	5.00
	3	1.00	4.27	4.27	7.53	3.50	4.27	6.77	7.53
	4	1.00	5.50	5.50	10.00	4.50	5.50	9.00	10.00
	5	1.00	6.76	6.76	12.52	5.50	6.76	11.26	12.52
7	1	1.00	1.71	1.71	2.43	1.50	1.71	2.21	2.43
	2	1.00	2.86	2.86	4.71	2.50	2.86	4.36	4.71
	3	1.00	4.00	4.00	7.00	3.50	4.00	6.50	7.00
	4	1.00	5.18	5.18	9.36	4.50	5.18	8.68	9.36
9	1	1.00	1.67	1.67	2.33	1.50	1.67	2.17	2.33
	2	1.00	2.78	2.78	4.56	2.50	2.78	4.28	4.56
	3	1.00	3.89	3.89	6.78	3.50	3.89	6.39	6.78

0-1 x -cut and z -cut, when used together, yield a tighter bound than the known 0-1 cuts discussed above. The improvement is modest, but the finite-domain cuts can replace known web cuts and tighten the bound at no additional cost.

The LP bound given by the finite-domain model is $\gamma(q, r)/q + 1$ when the x -cut alone, or the z -cut alone, is present. The bound subject to both cuts is $2\gamma(q, r)/q + 1$. The latter bound is the same as shown in Table 3 for the combined cuts in the 0-1 model.

9.3 Benchmark Instances

We tested the strength of odd cycle cuts on benchmark instances of the vertex coloring problem taken from the DIMACS library. Table 4 displays the odd cycle bounds computed for instances with fewer than 100 variables. Larger instances almost always resulted in an out-of-memory error when the odd hole cuts were added.

We searched a given graph $G = (V, E)$ for cycles with $s = 1, 2, 3$ using the following greedy algorithm. Let the *co-neighborhood* of a set K of vertices be the intersection of the neighborhoods of the individual vertices in K . For each $s \in \{1, 2, 3\}$ we proceed as follows. We first select the clique \bar{S}_1 of size s with the largest co-neighborhood (breaking ties arbitrarily). We then progressively build a path $\bar{S}_1, \bar{S}_2, \dots$ by adding cliques \bar{S}_ℓ . For each ℓ , if ℓ is odd, we examine cliques of size s that have vertices in $V \setminus (\bar{S}_1 \cup \dots \cup \bar{S}_{\ell-1})$ and that continue the path, and select from these a clique \bar{S}_ℓ with the largest co-neighborhood. A clique K continues the path if all pairs $(i, j) \in \bar{S}_{\ell-1} \times K$ are edges in E . If ℓ is even, we check, for each clique K that continues the path, whether it allows

Table 3: Lower bounds on the chromatic number in a 0-1 formulation of problem instances consisting of a web $W(q, r)$. The bound given by the finite-domain formulation is the same as shown below when both cuts are used.

q	r	Without cuts	0-1 cuts only	x -cut only	z -cut only	x -cut & z -cut	Optimal value	No. of 0-1 cuts
5	2	2	2.50	2.00	2.30	2.60	3	5
7	2	2	3.50	2.29	2.79	3.57	4	7
	3	2	2.33	2.00	2.21	2.43	3	7
8	3	2	2.67	2.00	2.38	2.75	3	8
9	2	2	4.50	2.78	3.28	4.56	5	9
	4	2	2.25	2.00	2.17	2.33	3	9
10	3	2	3.33	2.20	2.70	3.40	4	10
11	2	2	5.50	3.27	3.77	5.55	6	11
	3	2	3.67	2.36	2.86	3.73	4	11
	4	2	2.75	2.00	2.41	2.82	3	11
	5	2	2.20	2.00	2.14	2.27	3	11

completion of the cycle; that is, whether each pair $(i, j) \in \bar{S}_1 \times K$ is an edge in E . If so, we generate the cycle. (The vertices in $\bar{S}_1 \cup \dots \cup \bar{S}_\ell$ may induce edges that are not in a cycle, but the odd cycle cuts are still valid.) We then let \bar{S}_ℓ be a clique that continues the path and has the largest co-neighborhood. The process terminates when no clique continues the path.

Table 4 compares the bounds obtained from the 0-1 model after adding all odd hole cuts for the cycles found with the bounds obtained after adding all 0-1 x -cuts and z -cuts on these same cycles.

The results depend on the problem structure, but the finite-domain odd cycle cuts obtained tighter bounds in most instances, in some cases substantially tighter. As one might expect, the advantage is greater when there are cycles with $s > 1$. The time required to solve the LP relaxation was also consistently less for the finite-domain cuts (because there are only two of them per cycle), in some cases dramatically less.

10 Conclusion

We explored the idea of obtaining valid inequalities from a finite-domain formulation of a problem that is normally given a 0-1 formulation. We showed that in the case of graph coloring, this approach yields valid inequalities that provide tighter bounds on the chromatic number than known 0-1 cuts for the problem. In particular, we identified facet-defining inequalities for webs and odd holes that, when mapped into a 0-1 model, yield a tighter bound than standard 0-1 cuts. Furthermore, two finite-domain cuts for an odd cycle can yield substantially tighter bounds, in much less time, than hundreds or thousands of odd hole

Table 4: Lower bounds on the chromatic number in a 0-1 formulation of benchmark instances with n vertices and m edges, based on odd hole cuts and finite-domain odd cycle cuts. Number of cycles found (for $s = 1, 2, 3$) and LP solution time in seconds are also shown.

Instance	n	m	Bound			No. cycles found			LP Time (sec)	
			Odd hole	Odd cycle	Opt	$s = 1$	$s = 2$	$s = 3$	Odd hole	Odd cycle
1-Fulllns_3	30	100	2.00	2.00	3	18	0	0	0.4	0.4
1-Fulllns_4	93	593	2.00	2.00	4	61	0	0	208.0	0.4
1-Insertions_4	67	232	1.33	1.43	4	48	0	0	30.3	2.4
2-Fulllns_3	52	201	2.00	2.00	4	18	0	0	0.9	0.7
2-Insertions_3	37	72	1.25	1.33	3	8	0	0	2.9	0.2
3-Fulllns_3	80	346	2.00	2.00	5	25	0	0	25.8	0.2
3-Insertions_3	56	110	1.20	1.27	4	10	0	0	11.5	1.0
4-Insertions_3	79	156	1.17	1.23	3	12	0	0	12.1	6.0
david	87	406	2.00	8.00	10	103	48	10	11.0	0.8
huck	74	301	2.00	8.00	10	71	28	4	7.2	0.3
jean	80	254	2.00	8.00	10	26	0	8	10.2	1.8
mug88_1	88	146	2.00	2.00	3	2	0	0	7.8	2.7
mug88_25	88	146	2.00	2.00	3	4	0	0	5.3	1.7
myciel3	11	20	1.50	1.60	3	4	0	0	0.0	0.0
myciel4	23	71	1.50	1.60	4	18	0	0	0.6	0.0
myciel5	47	236	1.50	1.60	5	45	0	0	7.9	0.1
myciel6	95	755	1.50	1.60	6	153	0	0	1754.7	0.6
queen5_5	25	160	2.00	2.00	4	47	0	0	0.4	0.0
queen6_6	36	290	2.00	5.00	6	65	1	0	1.5	0.1
queen7_7	49	476	2.00	3.71	6	105	1	0	10.6	0.2
queen8_8	64	728	*	3.38	8	133	5	0	*	3.4
queen8_12	96	1368	2.00	8.00	11	229	31	20	439.6	1.7
queen9_9	81	1056	2.00	8.00	9	193	2	1	212.4	1.3

*LP solver ran out of memory.

cuts. We also described a family of facet-defining cuts for intersecting systems, for which no 0-1 cuts seem to have been previously identified.

In addition, we discovered that web and odd cycle cuts provide the same tight bound in a relaxation of the finite-domain model as in a relaxation of the 0-1 model. If other families of finite-domain cuts follow this pattern, there could be advantage in obtaining bounds from a finite-domain relaxation that is much smaller than the 0-1 model. Given that some benchmark instances result in 0-1 models that are too large even to load into a linear solver [13], this could provide a viable alternative for solving large graph coloring and related problems.

The alternate polyhedral perspective afforded by the finite-domain formulation therefore seems beneficial, at least in the case of graph coloring. The next step is to seek additional families of finite-domain cuts for graph coloring, perhaps corresponding to combs, anti-webs, and more general structures. The general strategy of obtaining valid inequalities and tight bounds from finite-domain formulations can be investigated for other problem classes.

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