The Hamiltonian Circuit Polytope

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Abstract

The hamiltonian circuit polytope is the convex hull of feasible solutions for the circuit constraint, which provides a succinct formulation of the traveling salesman and other sequencing problems. We study the polytope by establishing its dimension, developing tools for the identification of facets, and using these tools to derive several families of facets. The tools include necessary and sufficient conditions for an inequality to be facet defining, and an algorithm for generating all undominated circuits. We use a novel approach to identifying families of facet-defining inequalities, based on the structure of variable indices rather than on subgraphs such as combs or subtours. This leads to our main result, a hierarchy of families of facet-defining inequalities and polynomial-time separation algorithms for them.

Keywords: hamltonian circuit, circuit constraint, polyhedral theory

1. Introduction

The *circuit constraint* [1, 2, 3] requires that a sequence of vertices in a directed graph define a hamiltonian circuit. Given a directed graph G on vertices $1, \ldots, n$, the constraint is written

$$\text{circuit}(x_1, \ldots, x_n) \tag{1}$$

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where variable x_i denote the vertex that follows vertex i in the sequence. The constraint requires that $x = (x_1, \ldots, x_n)$ describe a hamiltonian circuit of G. For brevity, we will say that an x satisfying (1) is a *circuit*.

We define the *hamiltonian circuit polytope* to be the convex hull of the feasible solutions of (1) when G is a complete graph. Thus if the *domain* D_i of variable x_i is the set of values x_i can take, we suppose that each domain is *complete*; that is, each $D_i = \{1, \ldots, n\}$.

To our knowledge, this polytope has not been studied. Our objective is to establish its basic properties and provide tools for identifying classes of facets of the polytope. We use these tools to describe several families of facets. In particular, we identify a hierarchy of families of facets, along with polynomial-time separation algorithms.

A circuit should be distinguished from a permutation. Although a circuit $x = (x_1, \ldots, x_n)$ is always a permutation of $(1, \ldots, n)$, a permutation is not necessarily a circuit. For example, $(x_1, x_2, x_3, x_4) = (3, 4, 2, 1)$ is a circuit that goes from 1 to 3 to 2 to 4, and back to 1. However, the permutation $(x_1, x_2, x_3, x_4) = (3, 4, 1, 2)$ is not a circuit because it contains two subtours (1 to 3 to 1, 2 to 4 to 2). If the domain of each x_i is $\{1, \ldots, n\}$, then $n!$ values of x are permutations but only $(n-1)!$ of these are circuits.

The convex hull of permutations of $1, \ldots, n$ is the *permutohedron*, which has been studied for at least a century [4]. The permutohedron is well understood and quite different from the hamiltonian circuit polytope, although we will see that they have some facets in common.

The paper is organized as follows. We begin by noting how the circuit constraint can succinctly formulate the traveling salesman and other sequencing problems. We then introduce general variable domains and establish the dimension of the hamiltonian circuit polytope for an arbitrary domain. Following this, we develop two tools for identifying facets of the polytope: (a) necessary and sufficient conditions for an inequality with at most $n - 4$ variables to be facet-defining, stated in terms of *undominated* circuits; and (b) a simple greedy algorithm that generates all undominated circuits, along with a proof of its completeness.

We then apply these tools to analyze the structure of the hamiltonian circuit polytope. A key element of the analysis is a novel approach to identifying families of facets. Rather than associate facet-defining inequalities with graphical substructures such as combs and subtours, we associate them with the position of their variables in the sequence $x_1, \ldots, x_n$. Different patterns of variable indices give rise to different classes of facets.

We first describe a family of inequalities that are facet defining for both the permutohedron and the hamiltonian circuit polytope, and we provide an exhaustive list of two-term facets. We then proceed to our main result, which is a hierarchy of facets of increasing combinatorial complexity. We explicitly describe the facets on levels 0, 1 and 2 of the hierarchy and show how similar analysis can identify facets on higher levels. We conclude by presenting polynomial-time separation algorithms for all families of facets identified here. The algorithms yield a separating cut for each family whenever one exists.

2. Sequencing Problems

The circuit constraint is useful for formulating combinatorial problems that involve permutations or sequencing. One of the best known such problems is the *traveling salesman problem* (TSP), which may be very succinctly written

$$\begin{aligned} &\min \sum_{i=1}^{n} c_{ix_i} \\ &\text{circuit}(x_1, \ldots, x_n), \;\; x_i \in D_i, \; i = 1, \ldots n \end{aligned} \tag{2}$$

where c_{ij} is the distance from city i to city j. The objective is to visit each city once, and return to the starting city, in such a way as to minimize the total travel distance. In other sequencing problems, such as task sequencing problems, x_i might be the task that immediately follows task i.

A TSP defined on an incomplete graph can be formulated by letting the domain D_i be the set of cities that can be visited immediately after city i. Sequencing problems in general can be formulated in this fashion to restrict which tasks can immediately follow others. The removal of elements from some domains often allows the removal of additional domain elements that cannot be part of any feasible solution. Methods for this kind of "domain filtering" are developed in [2, 5, 3].

If a sequencing problem of this kind has complete domains, the convex hull of its feasible set is the hamiltonian circuit polytope. If one or more domains are incomplete, facet-defining inequalities for the hamiltonian circuit polytope remain valid for the problem, but they do not necessarily define facets of the convex hull.

The TSP is frequently formulated with 0–1 variables y_{ij}, where $y_{ij} = 1$ if vertex j immediately follows vertex i in the hamiltonian circuit. The standard formulation [6] consists of assignment constraints and exponentially many subtour elimination constraints. The convex hull of feasible solutions has a very different polyhedral structure than the hamiltonian circuit polytope studied here, as it is defined in a completely different space. It has been intensively analyzed but only partially described by identifying a few families of facets based on subtours, "combs," and other types of subgraphs. Surveys of this work may be found in [6, 7, 8].

3. General Domains

A peculiar characteristic of the circuit constraint is that the values of its variables are indices of other variables. Because the vertex immediately after x_i is x_{x_i}, the value of x_i must index a variable. The numbers $1, \ldots, n$ are normally used as indices, but this is an arbitrary choice. One could just as well use any other set of distinct numbers, which would give rise to a different polytope. Thus the hamiltonian circuit polytope cannot be fully understood unless it is characterized for general numerical domains, and not just for $1, \ldots, n$.

We therefore generalize the circuit constraint so that each domain D_i is drawn from an arbitrary set $\{v_1, \ldots, v_n\}$ of nonnegative real numbers. The constraint is written

$$\text{circuit}(x_{v_1}, \ldots, x_{v_n}) \tag{3}$$

It is convenient to assume $v_1 < \cdots < v_n$. Thus $\text{circuit}(x_0, x_{2.3}, x_{3.1})$ is a well-formed circuit constraint if the variable domains are subsets of $\{0, 2.3, 3.1\}$. The nonnegativity of the v_is does not sacrifice generality when the domains are finite, since one can always translate the origin so that the feasible points lie in the nonnegative orthant.

Most of the results stated here are valid for a general finite domain. However, to simplify notation we develop the facets in the hierarchy mentioned earlier only for $\{1, \ldots, n\}$.

To avoid an additional layer of subscripts, we will consistently abuse notation by writing x_{v_i} as x_i. We therefore write the constraint (3) as (1), with the understanding that $x = (x_1, \ldots, x_n)$ satisfies (1) if and only if $\pi_1, \ldots, \pi_n$ is a permutation of $1, \ldots, n$, where $\pi_1 = 1$ and $v_{\pi_i} = x_{\pi_{i-1}}$ for $i = 2, \ldots n$.

We define the hamiltionian circuit polytope $H_n(v)$ with respect to $v = (v_1, \ldots, v_n)$ to be the convex hull of the feasible solutions of (1) for complete domains; that is, each domain D_i is $\{v_1, \ldots, v_n\}$. All of the facet-defining inequalities we identify for complete domains are valid inequalities for smaller domains, even if they may not define facets of the convex hull.

4. Dimension of the Polytope

We begin by establishing the dimension of the hamiltonian circuit polytope.

Theorem 1. *The dimension of $H_n(v)$ is $n-2$ for $n = 2, 3$ and $n-1$ for $n \geq 4$.*

Proof. The polytope $H_n(v)$ is a point (v_2, v_1) for $n = 2$ and the line segment from (v_2, v_3, v_1) to (v_3, v_1, v_2) for $n = 3$. In either case the dimension is $n-2$.

To prove the theorem for $n \geq 4$, note first that all feasible points for (1) satisfy

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} v_i \tag{4}$$

(Recall that x_i is shorthand for x_{v_i}.) Thus, $H_n(v)$ has dimension at most $n-1$. To show it has dimension exactly $n-1$, it suffices to exhibit n affinely independent points in $H_n(v)$. Consider the following n permutations of $v_1, \ldots, v_n$, where the first $n-1$ permutations consist of v_1 followed by cyclic permutations of $v_2, \ldots, v_n$. The last permutation is obtained by swapping v_{n-1} and v_n in the first permutation:

$$\begin{array}{llllllll}
v_1, & v_2, & v_3, & \ldots, & v_{n-2}, & v_{n-1}, & v_n \\
v_1, & v_3, & v_4, & \ldots, & v_{n-1}, & v_n, & v_2 \\
v_1, & v_4, & v_5, & \ldots, & v_n, & v_2, & v_3 \\
 & & & \vdots & & & \\
v_1, & v_{n-1}, & v_n, & \ldots, & v_{n-4}, & v_{n-3}, & v_{n-2} \\
v_1, & v_n, & v_2, & \ldots, & v_{n-3}, & v_{n-2}, & v_{n-1} \\
v_1, & v_2, & v_3, & \ldots, & v_{n-2}, & v_n, & v_{n-1}
\end{array} \tag{5}$$

The rows of the following matrix correspond to circuit representations of the above permutations. Thus row i contains the values $x_1, \ldots, x_n$ for the ith permutation in (5).

$$\begin{bmatrix} v_2 & v_3 & v_4 & \cdots & v_{n-1} & v_n & v_1 \\ v_3 & v_1 & v_4 & \cdots & v_{n-1} & v_n & v_2 \\ v_4 & v_3 & v_1 & \cdots & v_{n-1} & v_n & v_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ v_{n-1} & v_3 & v_4 & \cdots & v_1 & v_n & v_2 \\ v_n & v_3 & v_4 & \cdots & v_{n-1} & v_1 & v_2 \\ v_2 & v_3 & v_4 & \cdots & v_n & v_1 & v_{n-1} \end{bmatrix} \tag{6}$$

Since each row of (6) is a point in $H_n(v)$, it suffices to show that the rows are affinely independent. Subtract $[v_n \ \ v_3 \ \ v_4 \ \cdots \ v_{n-1} \ \ v_n \ \ v_2]$ from every row of (6) to obtain

$$\begin{bmatrix} v_2 - v_n & 0 & 0 & \cdots & 0 & 0 & v_1 - v_2 \\ v_3 - v_n & v_1 - v_3 & 0 & \cdots & 0 & 0 & 0 \\ v_4 - v_n & 0 & v_1 - v_4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ v_{n-1} - v_n & 0 & 0 & \cdots & v_1 - v_{n-1} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & v_1 - v_n & 0 \\ v_2 - v_n & 0 & 0 & \cdots & v_n - v_{n-1} & v_1 - v_n & v_{n-1} - v_2 \end{bmatrix} \tag{7}$$

The rows of (6) are affinely independent if and only if the rows of (7) are. It now suffices to show that (7) is nonsingular, and we do so through a series of row operations. The first step is to subtract $(v_{n-1} - v_2)/(v_1 - v_2)$ times row 1, $(v_n - v_{n-1})/(v_1 - v_{n-1})$ times row $n-2$, and row $n-1$ from row n to obtain

$$\begin{bmatrix} v_2 - v_n & 0 & 0 & \cdots & 0 & 0 & v_1 - v_2 \\ v_3 - v_n & v_1 - v_3 & 0 & \cdots & 0 & 0 & 0 \\ v_4 - v_n & 0 & v_1 - v_4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ v_{n-1} - v_n & 0 & 0 & \cdots & v_1 - v_{n-1} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & v_1 - v_n & 0 \\ E_n & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

where

$$E_n = -\frac{v_n - v_{n-1}}{v_{n-1} - v_1}(v_n - v_{n-1}) - \frac{v_{n-1} - v_1}{v_2 - v_1}(v_n - v_2)$$

Interchange the first and last rows of (8) to obtain

$$\begin{bmatrix} E_n & 0 & 0 & \cdots & 0 & 0 & 0 \\ v_1 - v_n & v_1 - v_3 & 0 & \cdots & 0 & 0 & 0 \\ v_4 - v_n & 0 & v_1 - v_4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ v_{n-1} - v_n & 0 & 0 & \cdots & v_1 - v_{n-1} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & v_1 - v_n & 0 \\ v_2 - v_n & 0 & 0 & \cdots & 0 & 0 & v_1 - v_2 \end{bmatrix} \tag{9}$$

Note that $E_n < 0$ since $v_1 < \cdots < v_n$. Thus (9) is a lower triangular matrix with nonzero diagonal elements and is therefore nonsingular. □

5. Facet-Defining Inequalities

We now develop necessary and sufficient conditions for an inequality containing at most $n - 4$ variables to be facet defining for the hamiltonian circuit polytope. The following lemma is key.

Lemma 2. *Suppose that the inequality*

$$\sum_{j \in J} a_j x_j \geq \alpha \tag{10}$$

is valid for circuit$(x_1, \ldots, x_n)$ *and is satisfied as an equation by at least one circuit* x*. If* $|J| \leq n - 4$ *and*

$$\sum_{j=1}^{n} d_j x_j = \delta \tag{11}$$

is satisfied by all circuits x *that satisfy (10) as an equation, then* $d_i = d_j$ *for all* $i, j \notin J$*.*

Proof. Because $|J| \leq n - 4$, it suffices to prove that $d_{j_1} = d_{j_2} = d_{j_3} = d_{j_4}$ for any four distinct indices $j_1, \ldots, j_4 \notin J$.

Let x^0 be any circuit that satisfies (10) as an equation, and let the permutation described by x^0 be

$$v_1, \ldots, v_{j_1 - 1}, v_{j_1}, v_{j_1 + 1}, \ldots, v_{j_2 - 1}, v_{j_2}, v_{j_2 + 1}, \ldots, v_{j_3 - 1}, v_{j_3}, v_{j_3 + 1}, \ldots, v_{j_4 - 1}, v_{j_4}$$

Consider the circuits $x^1,\ldots,x^5$ that describe the following permutations, respectively:

$$v_1,\ldots,v_{j_1-1},v_{j_1},v_{j_3+1},\ldots,v_{j_4-1},v_{j_4},v_{j_1+1},\ldots,v_{j_2-1},v_{j_2},v_{j_2+1},\ldots,v_{j_3-1},v_{j_3}$$
$$v_1,\ldots,v_{j_1-1},v_{j_1},v_{j_2+1},\ldots,v_{j_3-1},v_{j_3},v_{j_3+1},\ldots,v_{j_4-1},v_{j_4},v_{j_1+1},\ldots,v_{j_2-1},v_{j_2}$$
$$v_1,\ldots,v_{j_1-1},v_{j_1},v_{j_2+1},\ldots,v_{j_3-1},v_{j_3},v_{j_1+1},\ldots,v_{j_2-1},v_{j_2},v_{j_3+1},\ldots,v_{j_4-1},v_{j_4}$$
$$v_1,\ldots,v_{j_1-1},v_{j_1},v_{j_1+1},\ldots,v_{j_2-1},v_{j_2},v_{j_3+1},\ldots,v_{j_4-1},v_{j_4},v_{j_2+1},\ldots,v_{j_3-1},v_{j_3}$$
$$v_1,\ldots,v_{j_1-1},v_{j_1},v_{j_3+1},\ldots,v_{j_4-1},v_{j_4},v_{j_2+1},\ldots,v_{j_3-1},v_{j_3},v_{j_1+1},\ldots,v_{j_2-1},v_{j_2}$$

We obtain $x^1,\ldots,x^5$ from x^0 by viewing the permutation represented by x^0 as a concatenation of four subsequences, each ending in one of the values v_{j_i}. We fix the first subsequence and obtain x^1 and x^2 by cyclically permuting the remaining three subsequences. We obtain x^3, x^4 and x^5 by interchanging a pair of subsequences.

Note that variables $x_{j_1},\ldots,x_{j_4}$ have the values shown below in each circuit x^i:

x_{j_1}	x_{j_2}	x_{j_3}	x_{j_4}	
v_{j_1+1}	v_{j_2+1}	v_{j_3+1}	v_1	(x^0)
v_{j_3+1}	v_{j_2+1}	v_1	v_{j_1+1}	(x^1)
v_{j_2+1}	v_1	v_{j_3+1}	v_{j_1+1}	(x^2)
v_{j_2+1}	v_{j_3+1}	v_{j_1+1}	v_1	(x^3)
v_{j_1+1}	v_{j_3+1}	v_1	v_{j_2+1}	(x^4)
v_{j_3+1}	v_1	v_{j_1+1}	v_{j_2+1}	(x^5)

and all other variables x_j have value x_j^0 in each circuit x^i. Thus all six circuits $x^0,\ldots,x^5$ satisfy (10) at equality, so that $dx^i=\delta$ for $i=0,\ldots,5$. This implies

$$\frac{1}{2}\begin{bmatrix} (dx^0+dx^1+dx^5)-(dx^2+dx^3+dx^4) \\ (dx^0+dx^2+dx^5)-(dx^1+dx^3+dx^4) \\ (dx^0+dx^3+dx^5)-(dx^1+dx^2+dx^4) \end{bmatrix} = \begin{bmatrix} 0\\0\\0 \end{bmatrix}$$

Substituting the values of $x^0,\ldots,x^5$, we obtain

$$\begin{bmatrix} v_{j_3+1}-v_{j_2+1} & v_{j_2+1}-v_{j_3+1} & 0 & 0 \\ 0 & v_1-v_{j_3+1} & v_{j_3+1}-v_1 & 0 \\ 0 & 0 & v_{j_1+1}-v_1 & v_1-v_{j_1+1} \end{bmatrix} \begin{bmatrix} d_{j_1}\\ d_{j_2}\\ d_{j_3}\\ d_{j_4} \end{bmatrix} = \begin{bmatrix} 0\\0\\0 \end{bmatrix}$$

from which we can conclude that $d_{j_1}=d_{j_2}=d_{j_3}=d_{j_4}$. □

Lemma 2 applies only when $|J| \le n-4$ because its proof relies on the absence of at least four variables from (10). The theorems below are therefore stated only for $|J| \le n-4$. We conjecture that they also hold for the densest facets ($|J| > n-4$), but proof seems to require the analysis of several special cases that substantially complicate the argument. This slightly stronger result would be of little additional value for identifying families of facets.

For a given x, we denote by $x(J)$ the tuple $(x_{j_1}, \ldots, x_{j_m})$ when $J = \{j_1, \ldots, j_m\}$. We say that $x(J)$ is a *J-circuit* if it creates no cycles and is therefore a partial solution of the circuit constraint. That is, $x(J)$ is a J-circuit if there is no subsequence $j_{i_1}, \ldots, j_{i_k}$ of the indices in J such that $x_{j_{i_t}} = v_{j_{i_{t+1}}}$ for $t = 1, \ldots, k-1$ and $x_{j_{i_k}} = v_{j_{i_1}}$. The following lemma is straightforward, but its proof introduces notation we will need later.

Lemma 3. *If $\bar{x}(J)$ is a J-circuit, then there is a circuit x such that $x(J) = \bar{x}(J)$.*

Proof. Let $J = \{j_1, \ldots, j_m\}$, and let $\{v_{i_1}, \ldots, v_{i_r}\}$ be the subset of domain values $v_1, \ldots, v_n$ that occur in neither $\{v_{j_1}, \ldots, v_{j_m}\}$ nor $\{\bar{x}_{j_1}, \ldots, \bar{x}_{j_m}\}$. Consider the directed graph $G_{\bar{x}(J)}$ that contains a vertex v_i for each $i \in \{1, \ldots, n\}$, a directed edge $(v_{j_k}, \bar{x}_{j_k})$ for $k = 1, \ldots, m$, and a directed edge $(v_{i_k}, v_{i_{k+1}})$ for each $k = 1, \ldots, r-1$. The maximal subchains of $G_{\bar{x}(J)}$ have the form

$$\begin{array}{l} v_{j_{k_1}} \to \cdots \to v_{j_{k'_1}} \to \bar{x}_{j_{k'_1}} \\ v_{j_{k_2}} \to \cdots \to v_{j_{k'_2}} \to \bar{x}_{j_{k'_2}} \\ \vdots \\ v_{j_{k_p}} \to \cdots \to v_{j_{k'_p}} \to \bar{x}_{j_{k'_p}} \\ v_{i_1} \to \cdots \to v_{i_r} \end{array}$$

Because maximal subchains are disjoint, we can form a hamiltonian circuit in $G_{\bar{x}(J)}$ by linking the last element of each subchain to the first element of the next, and linking v_{i_r} to v_{k_1}. Let $v_{s_1}, \ldots, v_{s_n}$ be the resulting circuit. Then if x is given by $x_i = v_{s_{((i-1) \bmod n)+1}}$ for $i = 1, \ldots, n$, then x is a circuit and $x(J) = \bar{x}(J)$. □

The concept of *domination* between J-circuits is central to identifying facets of $H_n(v)$, because inequality (10) is valid if and only if it is satisfied by all undominated J-circuits. If (J_+, J_-) is a partition of J, we say that $x(J)$ dominates $y(J)$ with respect to (J_+, J_-) when $x_j \le y_j$ for all $j \in J_+$ and $x_j \ge y_j$ for all $j \in J_-$. A J-circuit $x(J)$ is *undominated* with respect to (J_+, J_-) if no other J-circuit dominates it with respect to (J_+, J_-).

Lemma 4. *Inequality (10) is valid for the hamiltonian circuit polytope if and only if it is satisfied by all undominated J-circuits with respect to (J_+, J_-), where $J_+ = \{j \mid a_j > 0\}$ and $J_- = \{j \mid a_j < 0\}$.*

Proof. A valid inequality must be satisfied by all circuits. This means, due to Lemma 3, that it must be satisfied by all J-circuits and therefore by all undominated J-circuits. For the converse, suppose (10) is satisfied by all undominated J-circuits, and let x be any circuit. Then $x(J)$ is dominated by some undominated J-circuit $x'(J)$ with respect to (J_+, J_-), which means that $a_j(x_j - x'_j) \geq 0$ for all $j \in J$. Thus we have

$$\sum_{j \in J} a_j x_j \geq \sum_{j \in J} a_j x'_j \geq \alpha$$

because $x'(J)$ satisfies (10), and so x satisfies (10). This shows (10) is valid. □

The following theorem provides sufficient conditions under which an inequality is facet defining.

Theorem 5. *Consider any inequality of the form (10). Let S be the set of J-circuits that are undominated with respect to (J_+, J_-), where $J_+ = \{j \mid a_j > 0\}$, $J_- = \{j \mid a_j < 0\}$, and $1 \leq |J| \leq n - 4$. If all J-circuits in S satisfy (10) and at least $|J|$ affinely independent J-circuits satisfy*

$$\sum_{j \in J} a_j x_j = \alpha \tag{12}$$

then (10) defines a facet of $H_n(v)$.

Proof. Inequality (10) is valid by Lemma 4. To show (10) is facet defining, let (11) be any equation satisfied by all circuits x that satisfy (10) at equality. Recall that all circuits satisfy (4). It suffices to show that (11) is a linear combination of (12) and (4).

Let $S = \{x^1(J), \ldots, x^m(J)\}$. Because $|J| \geq 1$ and S is therefore nonempty, at least one J-circuit $x^i(J) \in S$ satisfies (10) at equality. Lemma 3 therefore implies that at least one circuit x^i satisfies (10) at equality. Thus since $|J| \leq n - 4$, we have from Lemma 2 that $d_i = d_j$ for all $i, j \notin J$.

We first suppose that $d_j = 0$ for all $j \notin J$. Then (11) has the form

$$\sum_{j \in J} d_j x_j = \delta \tag{13}$$

Because $|J|$ affinely independent J-circuits satisfy (12) and therefore (**??**), these two equations are the same up to a scalar multiple. Thus (11) is a linear combination of (12) and (4), where the latter has multiplier zero.

We now suppose that $d_j \neq 0$ for $j \notin J$. Because the d_js are equal for all $j \notin J$, we can without loss of generality write (11) as

$$\sum_{j \in J} d_j x_j + \sum_{j \notin J} x_j = \delta$$

This is a linear combination of (12) and (4) if the following is a scalar multiple of (12):

$$\sum_{j \in J} (d_j - 1) x_j = \delta - \sum_{j=1}^{n} v_j \tag{14}$$

But this follows from the fact that $|J|$ affinely independent J-circuits satisfy (12) and (14). □

A simple corollary sometimes suffices to show that inequalities are facet defining.

Corollary 6. *If J is as in Theorem 5, (10) is valid, and at least $|J|$ affinely independent J-circuits satisfy (10) at equality, then (10) is facet defining.*

Proof. If (10) is valid, then it is satisfied by all undominated J-circuits, and the conditions of Theorem 5 apply. □

To apply Theorem 5 (or Corollary 6), one must identify a set of affinely independent J-circuits. However, the number of circuits required is only the number $|J|$ of terms included in the facet-defining inequality, as opposed to n circuits in traditional arguments based on affine independence. The theorem can therefore be regarded as a lifting lemma. It will allow us to exploit patterns in the selection of terms to be included, so as to establish several classes of facets.

Finally, we note that the conditions of Theorem 5 are necessary as well as sufficient for (10) to be facet defining.

For each ordering $j_1, \ldots, j_m$ of the elements of J:
  Let $\bar{J} = \{1, \ldots, n\}$ and $J' = \emptyset$.
  For $i = 1, \ldots, m$:
    Add j_i to J'.
    If $j_i \in J_+$ then let $\bar{x}_{j_i}$ be the minimum value v_k in $\{v_i \mid i \in \bar{J}\}$
      such that $\bar{x}(J')$ is a J'-circuit.
    Else let $\bar{x}_{j_i}$ be the maximum value v_k in $\{v_i \mid i \in \bar{J}\}$
      such that $\bar{x}(J')$ is a J'-circuit.
    Remove k from $\bar{J}$.
  Add $\bar{x}(J)$ to the list of undominated J-circuits.

Figure 1: Greedy procedure for generating undominated J-circuits. Input: tuple v of domain values, index set J, and partition (J_+, J_-) of J. Output: a complete list of J-circuits that are undominated with respect to (J_+, J_-).

Theorem 7. *Consider any inequality (10) that is facet-defining for a hamiltonian circuit polytope $H_n(v)$. Let $J_+ = \{j \mid a_j > 0\}$ and $J_- = \{j \mid a_j < 0\}$. Then (10) is satisfied by all undominated J-circuits with respect to (J_+, J_-), and at least $|J|$ affinely independent J-circuits satisfy (12).*

Proof. Because (10) is valid, Lemma 4 implies that it is satisfied by all undominated J-circuits. Furthermore, because (10) is facet defining, it is satisfied at equality by n affinely independent circuits $\bar{x}^1, \ldots, \bar{x}^n$. Then $\{\bar{x}^1(J), \ldots, \bar{x}^n(J)\}$ contains some subset $\{\bar{x}^{j_1}(J), \ldots, \bar{x}^{j_m}(J)\}$ of $|J| = m$ affinely independent J-circuits, which satisfy (10). $\square$

6. Generating Undominated Circuits

A simple greedy procedure can be used to generate all J-circuits $\bar{x}(J)$ that are undominated with respect to (J_+, J_-). It is applied for each ordering $j_1, \ldots, j_m$ of the elements of J. First, let $\bar{x}_{j_1}$ be the smallest domain value v_i if $j_1 \in J_+$, or the largest if $j_1 \in J_-$. Then let $\bar{x}_{j_2}$ be the smallest (or largest) remaining domain value that does not create a cycle. Continue until all $\bar{x}_j$ for $j \in J$ are defined. The precise algorithm appears in Fig. 1.

To prove that the greedy procedure is correct, it is convenient to write $x_j \prec y_j$ when either $x_j < y_j$ and $j \in J_+$ or $x_j > y_j$ and $j \in J_-$.

Theorem 8. *The greedy procedure of Fig. 1 generates J-circuits that are undominated with respect to (J_+, J_-).*

Proof. Let $\bar{x}(J)$ be a J-circuit generated by the procedure for a given ordering $j_1, \ldots, j_m$. To see that $\bar{x}(J)$ is undominated with respect to (J_+, J_-), assume otherwise. Then there exists a J-circuit $\bar{y}(J)$ that dominates $\bar{x}(J)$ such that $\bar{y}_{j_t} \prec \bar{x}_{j_t}$ for some $t \in \{1, \ldots, m\}$. Let t be the smallest such index, so that $\bar{x}_{j_k} = \bar{y}_{j_k}$ for $k = 1, \ldots, t-1$. This contradicts the greedy construction of $\bar{x}$, because $\bar{y}_{j_t}$ is available when $\bar{x}_{j_t}$ is assigned to x_{j_t}. □

As an example, consider circuit$(x_1, \ldots, x_7)$ where each x_j has domain $\{v_1, \ldots, v_7\}$. The undominated J-circuits of $J = \{1, 3, 4\}$ with respect to $(J, \emptyset)$ can be generated by considering the six orderings of $1, 3, 4$ listed on the left below. The resulting undominated J-circuits appear on the right.

(j_1, j_2, j_3)	(x_1, x_3, x_4)
$(1, 3, 4)$	(v_2, v_1, v_3)
$(1, 4, 3)$	(v_2, v_4, v_1)
$(3, 1, 4)$	(v_2, v_1, v_3)
$(3, 4, 1)$	(v_4, v_1, v_2)
$(4, 1, 3)$	(v_2, v_4, v_1)
$(4, 3, 1)$	(v_3, v_2, v_1)

There is only one undominated J-circuit with respect to $(\{1, 3\}, \{4\})$, because all six orderings result in the same J-circuit (v_2, v_1, v_7).

It remains to show that the greedy procedure finds all undominated J-circuits. We will first prove this for the partition $(J, \emptyset)$ because the argument simplifies considerably in this case. Thus we assume that circuit x dominates circuit x' when $x \leq x'$. The proof for the general case appears in the Appendix.

Theorem 9. *Any undominated J-circuit with respect to $(J, \emptyset)$ can be generated in a greedy fashion for some ordering of the indices in J.*

Proof. Let $\bar{x}(J)$ be a J-circuit that is undominated with respect to $(J, \emptyset)$. Let $J = \{i_1, \ldots, i_m\}$ where $\bar{x}_{i_1} < \cdots < \bar{x}_{i_m}$, and let $y = (y_{i_1}, \ldots, y_{i_m})$ be the greedy solution with respect to the ordering $i_1, \ldots, i_m$. We claim that $\bar{x}_{i_\ell} = y_{i_\ell}$ for $\ell = 1, \ldots, m$, which suffices to prove the theorem.

Supposing to the contrary, let t be the smallest index for which $\bar{x}_{i_t} \neq y_{i_t}$. Clearly $\bar{x}_{i_t} < y_{i_t}$ is inconsistent with the greedy choice, because $\bar{x}_{i_t}$ is available when y_{i_t} is assigned a value. Thus we have $\bar{x}_{i_t} > y_{i_t}$. By hypothesis, $\bar{x}$ is undominated with respect to J. We therefore have $\bar{x}_{i_\ell} < y_{i_\ell}$ for some $\ell \in \{t+1, \ldots, m\}$. Let u be the smallest such index. Finally, let t' be the largest index in $\{t, \ldots, u-1\}$ such that $\bar{x}_{i_{t'}} > y_{i_{t'}}$. We know that t' exists because $\bar{x}_{i_t} > y_{i_t}$. Thus we have two sequences of values related as follows:

$$\begin{array}{ccccccccccccccc}
\bar{x}_{i_1} & < \cdots < & \bar{x}_{i_{t-1}} & < & \bar{x}_{i_t} & < \cdots < & \bar{x}_{i_{t'-1}} & < & \bar{x}_{i_{t'}} & < \cdots < & \bar{x}_{i_{u-1}} & < & \bar{x}_{i_u} \\
\| & & \| & & \vee & & \mathrel{\rotatebox{90}{$\leq$}} & & \vee & & \mathrel{\rotatebox{90}{$\leq$}} & & \wedge \\
y_{i_1} & \cdots & y_{i_{t-1}} & & y_{i_t} & \cdots & y_{i_{t'-1}} & & y_{i_{t'}} & \cdots & y_{i_{u-1}} & & y_{i_u}
\end{array}$$

We first show that value $\bar{x}_{i_u}$ has not yet been assigned in the greedy algorithm when y_{i_u} is assigned a value. That is, we show that $\bar{x}_{i_u} \notin \{y_{i_1}, \ldots, y_{i_{u-1}}\}$. Suppose to the contrary that $\bar{x}_{i_u} = y_{i_w}$ for some $w \in \{1 \ldots, u-1\}$. But this is impossible, because $\bar{x}_{i_u} > \bar{x}_{i_w} \geq y_{i_w}$. We next show that value $\bar{x}_{i_{t'}}$ has not yet been assigned in the greedy algorithm when y_{i_u} is assigned a value. That is, we show that $\bar{x}_{i_{t'}} \notin \{y_{i_1}, \ldots, y_{i_{u-1}}\}$. To begin with, we have that $\bar{x}_{i_{t'}} \notin \{y_{i_1}, \ldots, y_{i_{t'-1}}\}$, by virtue of the same reasoning just applied to $\bar{x}_{i_u}$. Also $\bar{x}_{i_{t'}} \neq y_{i_{t'}}$, since by hypothesis $\bar{x}_{i_{t'}} > y_{i_{t'}}$. To show that $\bar{x}_{i_{t'}} \notin \{y_{i_{t'+1}}, \ldots, y_{i_{u-1}}\}$, suppose to the contrary that $\bar{x}_{i_{t'}} = y_{i_w}$ for some $w \in \{t'+1, \ldots, u-1\}$. Then since $\bar{x}_{i_{t'}} < \bar{x}_{i_w}$, we must have $\bar{x}_{i_w} > y_{i_w}$. But this contradicts the definition of t' $(< w)$ as the largest index in $\{1, \ldots, u-1\}$ such that $\bar{x}_{i_{t'}} > y_{i_{t'}}$. Thus $\bar{x}_{i_{t'}} \neq y_{i_w}$.

Because $\bar{x}_{i_u} < y_{i_u}$ and value $\bar{x}_{i_u}$ has not yet been assigned, setting $y_{i_u} = \bar{x}_{i_u}$ must create a cycle in y, because otherwise setting $y_{i_u} = \bar{x}_{i_u}$ would have been the greedy choice. Also, setting $y_{i_u} = \bar{x}_{i_{t'}}$ was not the greedy choice because $y_{i_u} > \bar{x}_{i_u} > \bar{x}_{i_{t'}}$. Thus setting $y_{i_u} = \bar{x}_{i_{t'}}$ must likewise create a cycle in y, because $\bar{x}_{i_{t'}}$ has not yet been assigned. Now define $G_{y(J)}$ as before and consider the maximal subchain in $G_{y(J)}$ that contains y_{i_u}. Let the segment of the subchain up to y_{i_u} be

$$v_z \rightarrow \cdots \rightarrow v_{i_u} \rightarrow y_{i_u}$$

Because setting $y_{i_u} = \bar{x}_{i_u}$ creates a cycle in y, we must have $\bar{x}_{i_u} = v_z$. Similarly, because setting $y_{i_u} = \bar{x}_{i_{t'}}$ creates a cycle in y, we must have $\bar{x}_{i_{t'}} = v_z$. This implies $\bar{x}_{i_u} = \bar{x}_{i_{t'}}$, which is impossible because $\bar{x}_{i_u} > \bar{x}_{i_{t'}}$. □

Theorem 10. *Any undominated J-circuit with respect to (J_+, J_-) can be generated in a greedy fashion for some ordering of the indices in J.*

Proof. See the Appendix. □

7. Permutation and Two-term Facets

We begin by identifying two special classes of facets of $H_n(v)$, namely, permutation facets and two-term facets.

The *permutohedron* $P_n(v)$ for an arbitrary domain $\{v_1, \ldots, v_n\}$ can be defined as the convex hull of all points whose coordinates are permutations of $v_1, \ldots, v_n$. We refer to the facets of $P_n(v)$ as *permutation facets*. The circuit polytope $H_n(v)$ is contained in $P_n(v)$ because every circuit $(x_1, \ldots, x_n)$ is a permutation of $v_1, \ldots, v_n$. This means that every facet-defining inequality for $P_n(v)$ is valid for circuit but not necessarily facet defining. This raises the question as to which permutation facets are also circuit facets. We will identify a large family of permutation facets that can be immediately recognized as circuit facets.

The permutohedron $P_n(v)$ has dimension $n - 1$, and its affine hull is described by

$$\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} v_j \tag{15}$$

The facets of $P_n(v)$ are identified in [9, 10], and they are defined by

$$\sum_{j \in J} x_j \geq \sum_{j=1}^{|J|} v_j \tag{16}$$

for all $J \subset \{1, \ldots, n\}$ with $1 \leq |J| \leq n - 1$. (Recall that $0 \leq v_1 < \cdots < v_n$.) This result is generalized in [11] to domains with more than n elements.

For example, the permutohedron $P_3(v)$ with $v = (2, 4, 5)$ is defined by

$$\begin{aligned}
&x_1 + x_2 + x_3 = 11 \\
&x_i \geq 2, \text{ for } i = 1, 2, 3 \\
&x_i + x_j \geq 6, \text{ for distinct } i, j \in \{1, 2, 3\}
\end{aligned}$$

We can see at this point that a facet-defining inequality for $P_n(v)$ need not be facet-defining for $H_n(v)$. The inequality $x_1 + x_2 \geq 6$ is facet-defining for $P_3(v)$ but not for $H_3(v)$, which is the line segment from $(4, 5, 2)$ to $(5, 2, 4)$. However, a large family of inequalities are facet defining for both $H_n(v)$ and $P_n(v)$.

Theorem 11. *The inequality (16) defines a facet of $H_n(v)$ if $1 \leq |J| \leq n - 4$ and $j > 2$ for all $j \in J$.*

Proof. Let $J = \{j_1, \ldots, j_m\}$. Inequality (16) is clearly valid because the variables $x_{j_1}, \ldots, x_{j_m}$ must have pairwise distinct values. By Corollary 6, it suffices to exhibit m affinely independent J-circuits that satisfy (16) at equality. Consider the following assignments to $(x_{j_1}, \ldots, x_{j_m})$:

$$\begin{array}{cccccc} x_{j_1} & x_{j_2} & x_{j_3} & \cdots & x_{j_{m-1}} & x_{j_m} \\ \hline v_1 & v_2 & v_3 & \cdots & v_{m-1} & v_m \\ v_2 & v_1 & v_3 & \cdots & v_{m-1} & v_m \\ v_1 & v_3 & v_2 & \cdots & v_{m-1} & v_m \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ v_1 & v_2 & v_3 & \cdots & v_m & v_{m-1} \end{array} \tag{17}$$

The ith assignment is obtained from the first by swapping v_{i-1} and v_i. These assignments obviously satisfy (16) at equality. They are also affinely independent, as can be seen by subtracting the first row from each row. It remains to show that the assignments create no cycles and are therefore J-circuits. For this, it suffices to show that each x_{j_i} is assigned a value v_k with $k < j_i$. The first assignment satisfies this condition because $2 < j_1$ and $j_1 < \cdots < j_m$ imply that $i < j_i - 1$ for $i = 1, \ldots, m$. The ith assignment agrees with the first on the values of all variables except $x_{j_{i-1}}, x_{j_i}$. It sets $(x_{j_{i-1}}, x_{j_i}) = (v_i, v_{i-1})$, which satisfies $i < j_{i-1}$ because $i - 1 < j_{i-1} - 1$, and satisfies $i - 1 < j_i$ because $i < j_i - 1$. The ith assignment therefore satisfies the condition and is a J-circuit for $i = 2, \ldots, m$. □

Another special class of facet-defining inequalities are those containing two terms, which can be listed in closed form.

Corollary 12. *If $n \geq 6$, the two-term facets of $H_n(v)$ are precisely those defined by*

$$x_i + x_j \geq v_1 + v_2, \ \ \textit{for distinct } i, j \in \{3, \ldots, n\} \tag{18}$$

$$(v_3 - v_1)x_1 + (v_3 - v_2)x_2 \geq v_3^2 - v_1 v_2 \tag{19}$$

$$(v_2 - v_1)x_2 + (v_3 - v_1)x_i \geq v_2 v_3 - v_1^2, \ \ \textit{for } i \in \{3, \ldots, n\} \tag{20}$$

$$(v_{n-1} - v_{n-2})x_{n-1} + (v_n - v_{n-2})x_n \leq v_n v_{n-1} - v_{n-2}^2 \tag{21}$$

$$(v_n - v_{n-2})x_i + (v_n - v_{n-1})x_{n-1} \leq v_n^2 - v_{n-1} v_{n-2}, \ \ \textit{for } i \in \{1, \ldots, n-2\} \tag{22}$$

Proof. Consider an arbitrary two-term inequality $a_i x_i + a_j x_j \geq \alpha$. If we suppose $a_i, a_j > 0$, four cases can be distinguished. *Case 1:* $i, j > 2$. The two permutations of i, j generate the two undominated J-circuits (v_1, v_2) and (v_2, v_1), where $J = \{i, j\}$. The only equation satisfied by these two affinely independent J-circuits, up to a positive scalar multiple, is $x_i + x_j = v_1 + v_2$. So by Theorems 5 and 7, all facet-defining inequalities for this case have the form (18). *Case 2:* $(i, j) = (1, 2)$. The undominated J-circuits are (v_2, v_3) and (v_3, v_1), which satisfy only (19) at equality, up to a positive scalar multiple. *Case 3.* $i = 1$, $j > 2$. The two permutations of $1, j$ generate the same undominated J-circuit (v_2, v_3). Thus no two affinely independent J-circuits satisfy $a_1x_1 + a_jx_j = \alpha$, and by Theorem 7 there are no facet-defining inequalities in this case. *Case 4.* $i = 2$, $j > 2$. The undominated J-circuits are (v_1, v_2) and (v_3, v_1), which satisfy only (20) at equality.

Now if we suppose $a_i, a_j < 0$, similar reasoning yields the facets (21)–(22) and

$$x_i + x_j \leq v_{n-1} + v_n, \ \ \text{for distinct } i, j \in \{1, \ldots, n-2\}$$

which is redundant of (18) because it is the sum of (18) and the negation of (15). Finally, if $a_i > 0$ and $a_j < 0$, we consider four cases: $i > 1$ and $j < n$; $i = 1$ and $j < n$; $i > 1$ and $j = n$; and $(i, j) = (1, n)$. The two permutations of i, j generate only one J-circuit in each case, respectively (v_1, v_n), (v_2, v_n), (v_1, v_{n-1}), and (v_2, v_{n-1}). This means by Theorem 7 that there are no additional facets. The situation is similar when $a_i < 0$ and $a_j > 0$. □

8. A Hierarchy of Facets

We now describe a hierarchy of facets of increasing complexity. To simplify discussion, we suppose in this section that each variable has domain $\{v_1, \ldots, v_n\} = \{1, \ldots, n\}$, and we consider only facets defined by inequalities with nonnegative coefficients. We therefore focus on $H_n(u)$, where $u = (1, \ldots, n)$.

The intuition behind the hierarchy is as follows. On level 0 of the hierarchy, the number of variables in an inequality (10) is less than the smallest index in J. The undominated J-circuits are simply the permutations of $1, \ldots, m$, because the greedy algorithm of Section 6 never encounters a cycle. As a result, the only facets on level 0 are permutation facets. In higher levels of the hierarchy, the index of the first variable is smaller than the number of variables in the facet, which increases the combinatorial complexity of undominated J-circuits and yields more complicated facets. We will exhaustively identify facets for levels 0, 1, and 2, although one can in principle use similar methods to identify facets on higher levels.

Let level d of the hierarchy consist of inequalities of the form

$$\sum_{j=m-d+1}^{m} a_j x_j + \sum_{i=d+1}^{m} a_{j_i} x_{j_i} \geq \alpha \tag{23}$$

where each $a_j > 0$, where $m < j_{d+1} < \cdots < j_m$, and where $\{x_{j_{d+1}}, \ldots, x_{j_m}\}$ is any subset of $m-d$ variables in $\{x_{m+1}, \ldots, x_n\}$. Thus (23) contains m variables, and $m-d$ variables are absent before the first variable. Note also that the first d variables are consecutive. We will identify one family of facet-defining inequalities on level 0, two families on level 1, and five families on level 2.

First, we have immediately from Theorem 11 that level 0 contains a class of permutation facets.

Corollary 13. *The following level 0 inequalities are facet defining for $H_n(u)$:*

$$\sum_{i=1}^{m} x_{j_i} \geq \tfrac{1}{2} m(m+1), \quad m = 2, \ldots, n$$

for any set $\{x_{j_1}, \ldots, x_{j_m}\}$ of m variables in $\{x_{m+1}, \ldots, x_n\}$.

For level 1 we have the following.

Theorem 14. *The following level 1 inequalities are facet defining for $H_n(u)$:*

$$x_m + \sum_{i=2}^{m} x_{j_i} \geq \tfrac{1}{2} m(m+1), \quad m = 3, \ldots, \lceil n/2 \rceil \tag{24}$$

$$x_m + 2\sum_{i=2}^{m} x_{j_i} \geq m^2 + 1, \quad m = 2, \ldots, \lceil n/2 \rceil \tag{25}$$

for any subset $\{x_{j_2}, \ldots, x_{j_m}\}$ of $m-1$ variables in $\{x_{m+1}, \ldots, x_n\}$, provided $n - m \geq 4$.

Proof. Here $J = \{m, j_2, \ldots, j_m\}$. Inequality (24) is facet defining due to Theorem 11. To show that (25) is facet defining, it suffices to show that it is satisfied by all undominated J-circuits and is satisfied at equality by m affinely independent J-circuits. From Theorem 9, all undominated J-circuits correspond to permutations of the elements of J, or equivalently, permutations x' of $(x_m, x_{j_2}, \ldots, x_{j_m})$. We distinguish two cases: permutations in which x_m is last, resulting in *type 1* circuits, and permutations in which x_m is not last, resulting in *type 2* circuits. Type 1 J-circuits have the form $x' = (1, \ldots, m-1, m+1)$, because once the first $m-1$ variables in x' are assigned $1, \ldots, m-1$, x_m cannot be assigned the next value m and must be assigned $m+1$. For all such J-circuits, the left-hand side of (25) has value

$$(m+1) + 2\,(1 + 2 + \cdots + (m-1)) = m^2 + 1$$

which satisfies (25). Type 2 J-circuits have the form $x'' = (1, \ldots, m)$ where x'' is any permutation of $(x_m, x_{j_2}, \ldots, x_{j_m})$ in which x_m is not last. Because x_m has the smallest coefficient in (25), the LHS of (25) is minimized over type 2 J-circuits when x_m occurs next to last in x'', in which case the LHS has value

$$(m-1) + 2\,(1 + 2 + \cdots + (m-2) + m) = m^2 + 1$$

Thus (25) is again satisfied.

We now exhibit m affinely independent J-circuits satisfying (25) at equality. The first $m-1$ J-circuits below are type 1, and the last is type 2:

$$\begin{array}{c|cccccc}
x_m & x_{j_2} & x_{j_3} & x_{j_4} & \cdots & x_{j_{m-1}} & x_{j_m} \\
\hline
m-1 & 1 & 2 & 3 & \cdots & m-2 & m \\
m-1 & 2 & 1 & 3 & \cdots & m-2 & m \\
m-1 & 1 & 3 & 2 & \cdots & m-2 & m \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
m-1 & 1 & 2 & 3 & \cdots & m & m-2 \\
 & & & & & & \\
m+1 & 1 & 2 & 3 & \cdots & m-2 & m-1
\end{array} \tag{26}$$

These satisfy (25) at equality, as noted above. The $(m-1)\times(m-1)$ submatrix in the upper right is obtained by swapping pairs of elements in the first row.

After suitable row operations, (26) becomes

$$\begin{array}{c|cccccc} x_m & x_{j_2} & x_{j_3} & x_{j_4} & \cdots & x_{j_{m-1}} & x_{j_m} \\ \hline (m-1)/s & 1 & 0 & 0 & \cdots & 0 & 0 \\ (m-1)/s & 0 & 1 & 0 & \cdots & 0 & 0 \\ (m-1)/s & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ (m-1)/s & 0 & 0 & 0 & \cdots & 0 & 1 \\ & & & & & & \\ m+1 & 1 & 2 & 3 & \cdots & m-2 & m-1 \end{array} \tag{27}$$

where $s = \frac{1}{2}m(m-1)+1$ is the sum of the entries in each row of the submatrix. After further row operations, the last row is reduced to $2+(m-1)/s$ followed by $m-1$ zeros, resulting in a triangular matrix (after rearranging columns) with nonzeros on the diagonal. The matrix (26) is therefore nonsingular, and the rows are affinely independent. □

Finally, we identify five classes of level 2 facets.

Theorem 15. *The following level 2 inequalities are facet defining for* $H_n(u)$*:*

$$x_{m-1} + x_m + \sum_{i=3}^{m} x_{j_i} \geq \tfrac{1}{2}m(m+1), \quad m = 4, \ldots, \lceil (n+1)/2 \rceil \tag{28}$$

$$2x_{m-1} + x_m + 2\sum_{i=3}^{m} x_{j_i} \geq m^2+1, \quad m = 4, \ldots, \lceil (n+1)/2 \rceil \tag{29}$$

$$2x_{m-1} + x_m + 4\sum_{i=3}^{m} x_{j_i} \geq m(2m-3)+5, \quad m = 3, \ldots, \lceil (n+1)/2 \rceil \tag{30}$$

$$3x_{m-1} + 2x_m + 4\sum_{i=3}^{m} x_{j_i} \geq m(2m-1)+4, \quad m = 3, \ldots, \lceil (n+1)/2 \rceil \tag{31}$$

$$3x_{m-1} + 2x_m + 5\sum_{i=3}^{m} x_{j_i} \geq \tfrac{5}{2}m(m-1)+6, \quad m = 3, \ldots, \lceil (n+1)/2 \rceil \tag{32}$$

for any given set $\{x_{j_3}, \ldots, x_{j_m}\}$ *of* $m-2$ *variables in* $\{x_{m+1}, \ldots, x_n\}$*, if* $n-m \geq 4$.

Proof. Proof. Inequality (28) is facet defining due to Theorem 11. For the remaining inequalities we apply Theorem 5. First, we show that (29)–(32)

are satisfied by all undominated J-circuits, where $J = \{m-1, m, j_3, \dots, j_m\}$. This can be shown individually for each inequality, but we can establish the result for all at once by showing that

$$ax_{m-1} + bx_m + c\sum_{i=3}^{m} x_{j_i} \geq \beta \tag{33}$$

is satisfied by all undominated J-circuits, given that

$$\beta = (m-2)a + (m+1)b + \tfrac{1}{2}(m-3)(m-2)c + (m-1)c$$

and

$$2a \geq c, \quad c \geq a, \quad 3a \geq 2b + c, \quad 2a \geq 3b, \quad c \geq 2b \tag{34}$$

Note that the inequalities (29)–(32) have the form (33) and satisfy the relations (34). It can also be checked that β is equal to the right-hand side of each inequality (29)–(32). It therefore suffices to show that all undominated J-circuits satisfy (33).

To show this, we again apply Theorem 9. We partition permutations x' of $x = (x_{m-1}, x_m, x_{j_3}, \dots, x_{j_m})$ into 5 classes, which give rise to 5 types of J-circuits. It suffices to show that J-circuits of all 5 types satisfy (33).

Type 1. x_m occurs last and x_{m-1} next to last in x'. Circuits constructed in a greedy fashion have the form $x' = (1, \dots, m-2, m, m+1)$. This is because once the first $m-2$ variables in x' are assigned $1, \dots, m-2$, variable x_{m-1} cannot be assigned $m-1$ and is therefore assigned m. Now x_m cannot be assigned $m-1$ without creating a cycle with x_{m-1} and is therefore assigned $m+1$. The LHS of (33) is

$$ma + (m+1)b + (m-2)c + \tfrac{1}{2}(m-3)(m-2)c \geq \beta$$

where the inequality follows from the fact that $2a \geq c$. So J-circuits of type 1 satisfy (33).

Type 2. x_m occurs last but x_{m-1} does not occur next to last in x'. The circuits have the form $x' = (1, \dots, m-1, m+1)$. Because $a \leq c$, the LHS of (33) is minimized when x_{m-1} occurs second from last in x' (i.e., in position $m-2$), in which case the LHS has value equal to β. So J-circuits of type 2 satisfy (33).

Type 3. x_{m-1} occurs last and x_m next to last in x'. The circuits have the form $x' = (1, \ldots, m-1, m+1)$, for which the LHS of (33) is

$$(m+1)a + (m-1)b + (m-2)c + \tfrac{1}{2}(m-3)(m-2)c \geq \beta$$

where the inequality follows from the fact that $3a \geq 2b+c$. So J-circuits of type 3 satisfy (33).

Type 4. x_{m-1} occurs last but x_m does not occur next to last in x'. The circuits have the form $x' = (1, \ldots, m)$. Because $b \leq c$, the LHS of (33) is minimized when x_m occurs second from last in x', in which case the LHS has value

$$ma + (m-2)b + (m-1)c + \tfrac{1}{2}(m-3)(m-2)c \geq \beta$$

where the inequality follows from the fact that $2a \geq 3b$. So J-circuits of type 4 satisfy (33).

Type 5. Neither x_{m-1} nor x_m occurs last in x'. The circuits have the form $x' = (1, \ldots, m)$. Because $b \leq a \leq c$, the LHS of (33) is minimized when x_{m-1} is second from last and x_m is next to last in x', in which case the LHS has value

$$(m-2)a + (m-1)b + mc + \tfrac{1}{2}(m-3)(m-2)c \geq \beta$$

where the inequality follows from the fact that $c \geq 2b$. So J-circuits of type 5 satisfy (33).

It remains to exhibit, for each inequality (29)–(32), m affinely independent J-circuits that satisfy it at equality. The scheme for doing so is very similar for (30)–(32), but somewhat different for (29). Beginning with (30), suppose for the moment that $m > 3$. We use circuits of type 1, 2, and 3, which are the only types that can satisfy (33) at equality:

x_{m-1}	x_m	x_{j_3}	x_{j_4}	x_{j_5}	$\cdots$	$x_{j_{m-1}}$	x_{j_m}
$m-2$	$m+1$	1	2	3	$\cdots$	$m-3$	$m-1$
$m-2$	$m+1$	2	1	3	$\cdots$	$m-3$	$m-1$
$m-2$	$m+1$	1	3	2	$\cdots$	$m-3$	$m-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$m-2$	$m+1$	1	2	3	$\cdots$	$m-1$	$m-3$
m	$m+1$	1	2	3	$\cdots$	$m-3$	$m-2$
$m+1$	$m-1$	1	2	3	$\cdots$	$m-3$	$m-2$

(35)

The first $m-2$ rows are type 2 J-circuits, all of which satisfy (33) at equality. The last two rows are type 1 and type 3 J-circuits, respectively, chosen as above to satisfy (33) at equality. The nonsingular $(m-2)\times(m-2)$ submatrix in the upper right is obtained by swapping pairs of elements in the first row. After suitable row operations (35) becomes a matrix that is triangular after rearranging columns:

x_{m-1}	x_m	x_{j_3}	x_{j_4}	x_{j_5}	$\cdots$	$x_{j_{m-1}}$	x_{j_m}
$(m-2)/s$	$(m+1)/s$	1	0	0	$\cdots$	0	0
$(m-2)/s$	$(m+1)/s$	0	1	0	$\cdots$	0	0
$(m-2)/s$	$(m+1)/s$	0	0	1	$\cdots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$(m-2)/s$	$(m+1)/s$	0	0	0	$\cdots$	0	1
$2+(m-2)/s$	$(m+1)/s$	0	0	0	$\cdots$	0	0
$3+2(2s-3)/(m+1)$	0	0	0	0	$\cdots$	0	0

where $s=\frac{1}{2}(m-1)(m-2)+1$ is the sum of the elements in an arbitrary row of the $(m-2)\times(m-2)$ submatrix. Because each element on the diagonal is nonzero, the entire matrix is nonsingular, and the rows are affinely independent. When $m=3$, we use instead the affinely independent J-circuits $(3,4,1)$, $(1,4,2)$, and $(4,2,1)$, which again are of types 1, 2 and 3 and satisfy (30) at equality.

Affinely independent J-circuits of types 2, 4 and 5 can be similarly exhibited for (31), and circuits of types 2, 3 and 4 for (32). Affinely independent J-circuits for (29) are slightly different because only circuits of types 2 and 5 can satisfy (29) at equality. Here we are given that $m\geq 4$. We use the first $m-2$ circuits in (35) and the following two circuits of type 5:

x_{m-1}	x_m	x_{j_3}	x_{j_4}	x_{j_5}	$\cdots$	$x_{j_{m-1}}$	x_{j_m}
1	$m-1$	2	3	4	$\cdots$	$m-2$	m
2	$m-1$	1	3	4	$\cdots$	$m-2$	m

These satisfy (29) at equality because x_{m-1} has the same coefficient as x_1, x_2. An argument similar to the above shows that the J-circuits are affinely independent. $\square$

The above theorems provide a complete description of facets that appear for all $m\geq d+2$ on levels $d=0,1,2$. We can verify this by exhaustive enumeration of facets for $m = d + 2$ using Theorems 5 and 7. That is, for each d we use the greedy algorithm to generate all undominated J-circuits for $J = \{3, \ldots, d + 4\}$. We then consider the set I of all inequalities (10), up to a positive scalar multiple, that are satisfied at equality by an affinely independent subset of $d + 2$ undominated J-circuits. Finally, we list the inequalities in I that are satisfied by all the undominated J-circuits. This list contains all inequalities that are facet defining for $m = d + 2$, and all of them are described above. This method can, in principle, be used to identify families of facets on levels 3 and higher, although for each family one must prove that it is facet defining for all $m \geq d + 2$, as is done above.

9. Separation Algorithms

There are polynomial-time separation algorithms for all of the classes of facets described in the previous two sections. Each algorithm identifies a separating facet whenever one exists.

The separation problem is to identify a facet that separates a given solution value $\bar{x}$ of $x = (x_1, \ldots, x_n)$ from the hamiltonian circuit polytope; that is, to find a facet-defining inequality $ax \geq \alpha$ that is violated by $x = \bar{x}$. Consider first the family (16) of permutation facets. Let $j_1, \ldots, j_{n-2}$ be an ordering of the indices $3, \ldots, n$ such that $\bar{x}_{j_1} \leq \cdots \leq \bar{x}_{j_{n-2}}$. Then for $m = 1, \ldots, n$, check whether

$$\sum_{i=1}^{m} x_{j_i} \geq \tfrac{1}{2}m(m+1) \tag{36}$$

is violated by setting $(x_{j_1}, \ldots, x_{j_m}) = (\bar{x}_{j_1}, \ldots, \bar{x}_{j_m})$. Continue until (36) is violated, at which point a separating facet is discovered. The procedure has worst-case running time of $\mathcal{O}(n \log n)$, the time required to sort n values.

This procedure identifies a separating permutation facet in the family (16) if one exists. To see this, suppose $\sum_{j \in J'} x_j \geq \frac{1}{2}m(m+1)$ is a separating permutation facet, where $m = |J'|$ and $1, 2 \notin J'$. Then because $\bar{x}_{j_1}, \ldots, \bar{x}_{j_m}$ are the m smallest values among $\bar{x}_{j_1}, \ldots, \bar{x}_{j_{n-2}}$, we have

$$\sum_{i=1}^{m} \bar{x}_{j_i} \leq \sum_{j \in J'} \bar{x}_j < \tfrac{1}{2}m(m+1)$$

Thus (36) is also separating.

Separation requires only $\mathcal{O}(n)$ time for the two-term facets (18)–(21). A separating facet of the form (18) can be found, if one exists, by checking whether (18) is violated by setting $(x_i, x_j) = (\bar{x}_{j_1}, \bar{x}_{j_2})$, where $\bar{x}_{j_1}$ and $\bar{x}_{j_2}$ are the two smallest values among $\bar{x}_1, \ldots, \bar{x}_n$. If so, then (18) is separating with $(i, j) = (j_1, j_2)$. Facets (19)–(21) can be separated by enumerating at most n values of the index i.

Level 0 facets, level 1 facets of the form (24), and level 2 facets of the form (28) can be separated with the algorithms just described. A single initial sort of the values $\bar{x}_1, \ldots, \bar{x}_n$ provides the basis for separating all other facets on levels 1 and 2. For any fixed $m \geq 2$, we can find a separating level 1 facet of the form (25) as follows, if one exists. Let $\bar{x}_{j_2}, \ldots, \bar{x}_{j_m}$ be the $m - 1$ smallest values in $\{\bar{x}_{m+1}, \ldots, \bar{x}_n\}$. These values can be identified in $\mathcal{O}(n)$ time by looking through the sorted elements of $\{\bar{x}_1, \ldots, \bar{x}_n\}$ and selecting the first $m-1$ elements $\bar{x}_j$ with $j > m$. Now check whether (25) is violated by setting $(x_m, x_{j_1}, \ldots, x_{j_m})$ equal to $(\bar{x}_m, \bar{x}_{j_1}, \ldots, \bar{x}_{j_m})$. If so, then (25) is separating. It can be shown as above that this procedure finds a separating facet for any fixed m if one exists. We use a similar procedure for the level 2 facets (29)–(32). Thus for each m, we can identify a separating level 1 and level 2 facet of each type in $\mathcal{O}(n)$ time, if one exists. By enumerating $\mathcal{O}(n)$ values of m, we can execute the entire separation algorithm in time $\mathcal{O}(n \log n + n^2) = \mathcal{O}(n^2)$.

As an illustration, consider circuit$(x_1, \ldots, x_7)$ with each $D_j = \{1, \ldots, 7\}$. Suppose that $(\bar{x}_1, \ldots, \bar{x}_7) = (7, 2.6, 1, 6.25, 7, 2.2, 1.95)$. This point belongs to the affine hull described by (15), but it is infeasible if only because it does not consist of values in the domain. The following separating cuts are identified by the above algorithms:

$$x_3 + x_7 \geq 3 \tag{37}$$

$$x_2 + 2x_3 \geq 5 \tag{38}$$

$$x_3 + 2x_6 + 2x_7 \geq 10 \tag{39}$$

$$2x_3 + x_4 + 2x_6 + 2x_7 \geq 17 \tag{40}$$

$$2x_3 + x_4 + 4x_6 + 4x_7 \geq 25 \tag{41}$$

$$3x_2 + 2x_3 + 4x_7 \geq 19 \tag{42}$$

$$3x_2 + 2x_3 + 5x_7 \geq 21 \tag{43}$$

Here, (37) is a permutation facet as well as a 2-term facet, (38) is a level 1 facet as well as a 2-term facet, (39) is a level 1 facet, and (40)–(43) are level 2 facets of the form (29)–(32), respectively.

10. Conclusions and Future Research

We studied the structure of the hamiltonian circuit polytope by establishing its dimension, developing tools for the identification of facets, and using these tools to derive several families of facets. The tools include necessary and sufficient conditions for an inequality with at most $n-4$ variables to be facet defining, stated in terms of undominated circuits, and a greedy algorithm for generating undominated circuits, for which we proved completeness. We used a novel approach to identifying families of facet-defining inequalities, based on the structure of variable indices rather than on structured subgraphs. Finally, we described a hierarchy of facets of increasing combinatorial complexity and derived all facets on the first three levels. We also presented complete polynomial-time separation algorithms for all facets described here.

Appendix. Proof of Theorem 10

To prove Theorem 10, we first define for any given circuit $\bar{x}$ an *implied ordering* with respect to (J_+, J_-). The proof will show that if $\bar{x}$ is undominated with respect to (J_+, J_-), then a J-circuit that is greedily constructed according to the implied ordering is identical to $\bar{x}(J)$.

For a given J-circuit $\bar{x}(J)$, and partition (J_+, J_-), let $J_+ = \{i_1, \ldots, i_p\}$ where $\bar{x}_{i_1} < \cdots < \bar{x}_{i_p}$, and let $J_- = \{j_1, \ldots, j_q\}$ where $\bar{x}_{j_1} > \cdots > \bar{x}_{j_q}$.

The implied ordering will be $k_1, \ldots, k_m$. As we construct the ordering, we construct a J-circuit $y(J)$ that is greedy with respect to the ordering. The basic idea is that at each step ℓ of the procedure, we assign the greedy value to y_{i_r} for the next $i_r \in J_+$ (if any remain) and let $k_\ell = i_r$, provided this assigns y_{i_r} the same value as $\bar{x}_{i_r}$. Otherwise, we assign the greedy value to y_{j_s} for the next $j_s \in J_-$ and let $k_\ell = j_s$. If no indices j_s remain in J_-, we assign the greedy value to y_{i_r} regardless of whether it agrees with $\bar{x}_{i_r}$. The precise algorithm appears in Fig. .2.

As an example, suppose $\bar{x} = (v_2, v_3, v_4, v_7, v_6, v_1, v_5)$, $J_+ = \{1, 3, 6, 7\}$, and $J_- = \{4, 5\}$. Thus $\bar{x}(J) = (\bar{x}_1, \bar{x}_3, \bar{x}_4, \bar{x}_5, \bar{x}_6, \bar{x}_7) = (v_2, v_4, v_7, v_6, v_1, v_5)$. Based on the values in $\bar{x}(J)$, we order the contents of J_+ so that $J_+ = \{i_1, \ldots, i_4\} = \{6, 1, 3, 7\}$. Similarly, $J_- = \{j_1, j_2\} = \{4, 5\}$. The progress of the algorithm appears in Table .1. Note that when $\ell = 4$, we first consider assigning $v_{\min}$ to y_{i_r}. But this results in $y_7 = v_3$, which deviates from $\bar{x}$ because $\bar{x}_7 = v_5$. We therefore assign $v_{\max}$ to y_{j_s}, which yields $y_4 = v_7$. When $\ell = 5$, we again consider assigning $v_{\min}$ to y_{i_r}, but because $v_{\min}$ has

Let $V = \{v_1, \ldots, v_n\}$.
Let $J_+ = \{i_1, \ldots, i_p\}$ where $\bar{x}_{i_1} < \cdots < \bar{x}_{i_p}$.
Let $J_- = \{j_1, \ldots, j_q\}$ where $\bar{x}_{j_1} > \cdots > \bar{x}_{j_q}$.
Let $r = 1$ and $s = 1$.
For $\ell = 1, \ldots, m$:
    Let $v_{\min}$ be the smallest value in V such that setting $y_{i_r} = v_{\min}$
        creates no cycle with the elements of y assigned so far.
    Let $v_{\max}$ be the largest value in V such that setting $y_{j_s} = v_{\max}$
        creates no cycle with the elements of y assigned so far.
    If $r \leq p$ and ($\bar{x}_{i_r} = v_{\min}$ or $s > q$) then
        Let $k_\ell = i_r$, $y_{i_r} = v_{\min}$, and $r = r + 1$.
        Remove $v_{\min}$ from V.
    Else
        Let $k_\ell = j_s$, $y_{j_s} = v_{\max}$, and $s = s + 1$.
        Remove $v_{\max}$ from V.

Figure .2: Algorithm for generating an implied ordering $k_1, \ldots, k_m$ for J-circuit $\bar{x}(J)$ with respect to (J_+, J_-), where $m = |J|$. The resulting J-circuit $y(J)$ is greedily constructed with respect to the ordering $k_1, \ldots, k_m$ and (J_+, J_-). The algorithm is used to help prove Theorem 10, not to identify undominated J-circuits or construct facets.

changed, we now obtain an assignment $y_7 = v_5$ that agrees with $\bar{x}$. When $\ell = 6$, the indices in J_+ are exhausted, and we therefore assign $v_{\min}$ to y_{j_s}, so that $y_5 = v_6$. The resulting $y(J)$ is identical to $\bar{x}(J)$, and the implied ordering is $(k_1, \ldots, k_6) = (6, 1, 3, 4, 5, 7)$.

Proof of Theorem 10. Let $\bar{x}(J)$ be a J-circuit that is undominated with respect to (J_+, J_-). Let $J_+ = \{i_1, \ldots, i_p\}$ where $\bar{x}_{i_1} < \cdots < \bar{x}_{i_p}$, and let $J_- = \{j_1, \ldots, j_q\}$ where $\bar{x}_{j_1} > \cdots > \bar{x}_{j_q}$.

Let $k_1, \ldots, k_m$ be the implied ordering for $\bar{x}$ with respect to (J_+, J_-) as computed above, and let $(y_{k_1}, \ldots, y_{k_m})$ be the greedy solution with respect to this ordering. We claim that $\bar{x}_{k_\ell} = y_{k_\ell}$ for $\ell = 1, \ldots, m$, which suffices to prove the theorem. Supposing to the contrary, let $\bar{\ell}$ be the smallest index for which $\bar{x}_{k_{\bar{\ell}}} \neq y_{k_{\bar{\ell}}}$. Clearly $\bar{x}_{k_{\bar{\ell}}} \prec y_{k_{\bar{\ell}}}$ is inconsistent with the greedy choice, because $\bar{x}_{k_{\bar{\ell}}}$ is available when $y_{k_{\bar{\ell}}}$ is assigned a value. Thus we have $\bar{x}_{k_{\bar{\ell}}} \succ y_{k_{\bar{\ell}}}$

By hypothesis, $\bar{x}$ is undominated with respect to $(J_+ \cup J_-)$. We therefore

Table .1: Computation of the implied ordering for $\bar{x} = (v_2, v_3, v_4, v_7, v_6, v_1, v_5)$, where $J_+ = \{1, 3, 6, 7\}$ and $J_- = \{4, 5\}$ (indicated by the the signs above $\bar{x}$).

								$+$		$+$	$-$	$-$	$+$	$+$	
							$\bar{x} =$	v_2	v_3	v_4	v_7	v_6	v_1	v_5	
ℓ	r	s	i_r	j_s	$v_{\min}$	$v_{\max}$		y_1	y_2	y_3	y_4	y_5	y_6	y_7	k_ℓ
1	1	1	6	4	v_1	v_7							v_1		6
2	2	1	1	4	v_2	v_7		v_2					v_1		1
3	3	1	3	4	v_4	v_7		v_2		v_4			v_1		3
4	4	1	7	4	v_3	v_7		v_2		v_4	v_7		v_1		4
5	4	2	7	5	v_5	v_6		v_2		v_4	v_7		v_1	v_5	5
6	5	2		5		v_6		v_2		v_4	v_7	v_6	v_1	v_5	7

have $\bar{x}_{k_\ell} \prec y_{k_\ell}$ for some $\ell \in \{\bar{\ell}+1, \ldots, m\}$. Let $\hat{\ell}$ be the smallest such index. Then there are two cases: (1) $k_{\bar{\ell}}$ and $k_{\hat{\ell}}$ are both in J_+ or both in J_-, or (2) they are in different sets.

Case 1: $k_{\bar{\ell}}$ and $k_{\hat{\ell}}$ are both in J_+ or both in J_-. We will suppose that both are in J_+. The argument is similar if both are in J_-.

Let t be the index such that $i_t = k_{\bar{\ell}}$, and u the index such that $i_u = k_{\hat{\ell}}$. Then $\bar{x}_{i_t} > y_{i_t}$ because $\bar{x}_{i_t} \succ y_{i_t}$ and $i_t \in J_+$. Let t' be the largest index in $\{t, \ldots, u-1\}$ such that $\bar{x}_{i_{t'}} > y_{i_{t'}}$. We know that t' exists because $\bar{x}_{i_t} > y_{i_t}$. Thus we have two sequences of values related as follows:

$$\begin{array}{ccccccccccccccccc}
\bar{x}_{i_1} & < \cdots < & \bar{x}_{i_{t-1}} & < & \bar{x}_{i_t} & < \cdots < & \bar{x}_{i_{t'-1}} & < & \bar{x}_{i_{t'}} & < \cdots < & \bar{x}_{i_{u-1}} & < & \bar{x}_{i_u} \\
\| & & \| & & \vee & & \text{\rotatebox{90}{$\le$}}\!\!\!\;\geq & & \vee & & \geq & & \wedge \\
y_{i_1} & \cdots & y_{i_{t-1}} & & y_{i_t} & \cdots & y_{i_{t'-1}} & & y_{i_{t'}} & \cdots & y_{i_{u-1}} & & y_{i_u}
\end{array}$$

We first show that value $\bar{x}_{i_u}$ has not yet been assigned in the greedy algorithm when y_{i_u} is assigned a value. That is, we show that $\bar{x}_{i_u} \notin \{y_{i_1}, \ldots, y_{i_{u-1}}\}$ and $\bar{x}_{i_u} \notin \{y_{j_1}, \ldots, y_{j_{u'}}\}$. To see that $\bar{x}_{i_u} \notin \{y_{i_1}, \ldots, y_{i_{u-1}}\}$, suppose to the contrary that $\bar{x}_{i_u} = y_{i_w}$ for some $w \in \{1, \ldots, u-1\}$. This is impossible, because $\bar{x}_{i_u} > \bar{x}_{i_w} \geq y_{i_w}$. Also $\bar{x}_{i_u} \notin \{y_{j_1}, \ldots, y_{j_{u'}}\}$, because assigning value $\bar{x}_{i_u}$ to y_{j_w} for some $w \in \{1, \ldots, u'\}$ contradicts the greedy construction of y, due to the fact that value y_{i_u} was available at that time and is a superior choice.

We next show that value $\bar{x}_{i_{t'}}$ has not yet been assigned in the greedy algorithm when y_{i_u} is assigned a value. That is, we show that $\bar{x}_{i_{t'}} \notin \{y_{i_1}, \ldots, y_{i_{u-1}}\}$ and $\bar{x}_{i_{t'}} \notin \{y_{j_1}, \ldots, y_{j_{u'}}\}$. To begin with, we have that $\bar{x}_{i_{t'}} \notin \{y_{i_1}, \ldots, y_{i_{t'-1}}\}$, by virtue of the same reasoning just applied to $\bar{x}_{i_u}$. Also $\bar{x}_{i_{t'}} \neq y_{i_{t'}}$, since by hypothesis $\bar{x}_{i_{t'}} > y_{i_{t'}}$. To show that $\bar{x}_{i_{t'}} \notin \{y_{i_{t'+1}}, \ldots, y_{i_{u-1}}\}$, suppose to the contrary that $\bar{x}_{i_{t'}} = y_{i_w}$ for some $w \in \{t'+1, \ldots, u-1\}$. Then since $\bar{x}_{i_{t'}} < \bar{x}_{i_w}$, we must have $\bar{x}_{i_w} > y_{i_w}$. But this contradicts the definition of t' $(< w)$ as the largest index in $\{1, \ldots, u-1\}$ such that $\bar{x}_{i_{t'}} > y_{i_{t'}}$. Thus $\bar{x}_{i_{t'}} \neq y_{i_w}$. Finally, $\bar{x}_{i_{t'}} \notin \{y_{j_1}, \ldots, y_{j_{u'}}\}$ because assigning value $\bar{x}_{i_{t'}}$ to y_{j_w} for some $w \in \{1, \ldots, u'\}$ contradicts the greedy construction of y, due to the fact that y_{i_u} was available at the time and $y_{i_u} > \bar{x}_{i_u} > \bar{x}_{i_{t'}}$.

Because $\bar{x}_{i_u} < y_{i_u}$ and value $\bar{x}_{i_u}$ has not yet been assigned, setting $y_{i_u} = \bar{x}_{i_u}$ must create a cycle in y, because otherwise setting $y_{i_u} = \bar{x}_{i_u}$ would have been the greedy choice. Also, setting $y_{i_u} = \bar{x}_{i_{t'}}$ was not the greedy choice because $y_{i_u} > \bar{x}_{i_u} > \bar{x}_{i_{t'}}$. Thus setting $y_{i_u} = \bar{x}_{i_{t'}}$ must likewise create a cycle in y, because $\bar{x}_{i_{t'}}$ has not yet been assigned. Now define $G_{y(J)}$ as before and consider the maximal subchain in $G_{y(J)}$ that contains y_{i_u}. Let the segment of the subchain up to y_{i_u} be

$$v_z \to \cdots \to v_{i_u} \to y_{i_u}$$

Because setting $y_{i_u} = \bar{x}_{i_u}$ creates a cycle in y, we must have $\bar{x}_{i_u} = v_z$. Similarly, because setting $y_{i_u} = \bar{x}_{i_{t'}}$ creates a cycle in y, we must have $\bar{x}_{i_{t'}} = v_z$. This implies $\bar{x}_{i_u} = \bar{x}_{i_{t'}}$, which is impossible because $\bar{x}_{i_u} > \bar{x}_{i_{t'}}$.

Case 2: $k_{\bar{\ell}} \in J_+$ and $k_{\hat{\ell}} \in J_-$, or $k_{\bar{\ell}} \in J_-$ and $k_{\hat{\ell}} \in J_+$. We can rule out the latter subcase immediately, because $k_{\bar{\ell}}$ can be in J_- only if $r > p$ when $y_{k_{\bar{\ell}}}$ is assigned a value. This means $k_{\hat{\ell}}$ must be in J_- as well, because $y_{k_{\hat{\ell}}}$ is assigned a value after $y_{k_{\bar{\ell}}}$ is assigned a value, and the situation reverts to Case 1. We therefore suppose $k_{\bar{\ell}} \in J_+$ and $k_{\hat{\ell}} \in J_-$.

Let t be the index such that $i_t = k_{\bar{\ell}}$, and u the index such that $j_u = k_{\hat{\ell}}$. Again $\bar{x}_{i_t} > y_{i_t}$ because $\bar{x}_{i_t} \succ y_{i_t}$ and $j_t \in J_+$. Thus, at the time value y_{i_t} was assigned a value, we had $\bar{x}_{j_s} < v_{\max}$ for the current value of s. So we have two sequences of values related as follows:

$$\begin{array}{ccccccccc} \bar{x}_{j_1} > \cdots > & \bar{x}_{j_{s-1}} & > & \bar{x}_{j_s} & > \cdots & \bar{x}_{j_{u-1}} & > & \bar{x}_{j_u} \\ \parallel & \parallel & & \leq & & \leq & & \vee \\ y_{j_1} \quad \cdots & y_{j_{s-1}} & & y_{j_s} & \cdots & y_{j_{u-1}} & & y_{j_u} \end{array} \qquad (.1)$$

where $v_{\max} > \bar{x}_{j_s}$. Let t' be the largest index for which $y_{i_{t'}}$ has been assigned a value at the time y_{j_u} is assigned a value. We have two sequences of values related as follows:

$$\begin{array}{ccccccccc} \bar{x}_{i_1} & < \cdots < & \bar{x}_{i_{t-1}} & < & \bar{x}_{i_t} & < \cdots < & \bar{x}_{i_{t'}} \\ \| & & \| & & \vee & & \\ y_{i_1} & \cdots & y_{i_{t-1}} & & y_{i_t} & \cdots & y_{i_{t'}} \end{array}$$

We first show that a cycle must be created if value $\bar{x}_{j_u}$ is assigned to y_{j_u}. Because $y_{j_u} < \bar{x}_{j_u}$, it suffices to show that value $\bar{x}_{j_u}$ has not yet been assigned in the greedy algorithm when y_{j_u} is assigned a value. That is, we show that $\bar{x}_{j_u} \notin \{y_{j_1}, \ldots, y_{j_{u-1}}\}$ and $\bar{x}_{j_u} \notin \{y_{i_1}, \ldots, y_{i_{t'}}\}$. If $\bar{x}_{j_u} = y_{j_w}$ for some $w \in \{1, \ldots, u-1\}$, then $\bar{x}_{j_u} < \bar{x}_{j_w} \leq y_{j_w}$, which is impossible. Thus $\bar{x}_{j_u} \notin \{y_{j_1}, \ldots, y_{j_{u-1}}\}$. Also $\bar{x}_{j_u} \notin \{y_{i_1}, \ldots, y_{i_{t'}}\}$, because assigning value $\bar{x}_{j_u}$ to y_{i_w} for some $w \in \{1, \ldots, t'\}$ contradicts the greedy construction of y, due to the fact that value y_{j_u} was available at that time and is a superior choice.

We next show that a cycle must be created if value $v_{\max}$ is assigned to y_{j_u}. Note that $v_{\max} \notin \{y_{i_1}, \ldots, y_{i_{t'}}\}$, because assigning value $v_{\max}$ to y_{i_w} for some $w \in \{1, \ldots, t'\}$ contradicts the greedy construction of y, due to the fact that value y_{j_u} was available at that time and is a superior choice because $v_{\max} > \bar{x}_{j_s} > \bar{x}_{j_u}$. Now suppose, contrary to the claim, that assigning $v_{\max}$ to y_{j_u} does not create a cycle. Then since $v_{\max} > y_{j_u}$, the value $v_{\max}$ must have already been assigned in the greedy algorithm at the time y_{j_u} is assigned a value. This implies $v_{\max} \in \{y_{j_s}, \ldots, y_{j_{u-1}}\}$. But in this case we must have $y_{j_s} = v_{\max}$, because assigning $v_{\max}$ to y_{j_s} does not create a cycle and, by definition, is the most attractive choice at the time. Thus (.1) becomes

$$\begin{array}{ccccccccccccc} \bar{x}_{j_1} & > \cdots > & \bar{x}_{j_{s-1}} & > & \bar{x}_{j_s} & > \cdots > & \bar{x}_{j_{s'-1}} & > & \bar{x}_{j_{s'}} & > \cdots > & \bar{x}_{j_{u-1}} & > & \bar{x}_{j_u} \\ \| & & \| & & \wedge & & \leq & & \wedge & & \geq & & \wedge \\ y_{j_1} & \cdots & y_{j_{s-1}} & & y_{j_s} & \cdots & y_{j_{s'-1}} & & y_{j_{s'}} & \cdots & y_{j_{u-1}} & & y_{j_u} \end{array}$$

where $y_{j_s} = v_{\max}$ and where s' is the largest index in $\{s, \ldots, u-1\}$ such that $y_{j_{s'}} < \bar{x}_{j_{s'}}$. Now we can argue as in Case 1 that assigning $\bar{x}_{j_u}$ to y_{j_u} creates a cycle, and assigning $\bar{x}_{j_{s'}}$ to y_{j_u} creates a cycle, which implies $\bar{x}_{j_{s'}} = \bar{x}_{j_u}$, a contradiction because $\bar{x}_{j_{s'}} > \bar{x}_{j_u}$. We conclude that assigning $v_{\max}$ to y_{j_u} creates a cycle.

Having shown that assigning $\bar{x}_{j_u}$ to y_{j_u} creates a cycle, and assigning $v_{\max}$ to y_{j_u} creates a cycle, we derive as in Case 1 that $v_{\max} = \bar{x}_{j_u}$, a contradiction because $v_{\max} \geq \bar{x}_{j_s} > \bar{x}_{j_u}$. The theorem follows. □

References

[1] J.-L. Laurière, A language and a program for stating and solving combinatorial problems, Artificial Intelligence 10 (1978) 29–127.

[2] Y. Caseau, F. Laburthe, Solving small TSPs with constraints, in: L. Naish (Ed.), Proceedings, Fourteenth International Conference on Logic Programming (ICLP 1997), Vol. 2833, The MIT Press, 1997, pp. 316–330.

[3] J. A. Shufelt, H. J. Berliner, Generating hamiltonian circuits without backtracking, Theoretical Computer Science 132 (1994) 347–375.

[4] P. H. Schoute, Analytic treatment of the polytopes regularly derived from the regular polytopes, in: Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam, Vol. 11(3), 1911.

[5] L. Genç-Kaya, J. N. Hooker, A filter for the circuit constraint, in: F. Benhamou (Ed.), Principles and Practice of Constraint Programming (CP 2006), Vol. 4204 of Lecture Notes in Computer Science, Springer, 2006, pp. 706–710.

[6] D. Naddef, Polyhedral theory and branch-and-cut algorithms for the symmetric TSP, in: G. Gutin, A. P. Punnen (Eds.), The Traveling Salesman Problem and its Variations, Kluwer, Dordrecht, 2002, pp. 29–116.

[7] E. Balas, M. Fischetti, Polyhedral theory for the asymmetric traveling salesman problem, in: G. Gutin, A. P. Punnen (Eds.), The Traveling Salesman Problem and its Variations, Kluwer, Dordrecht, 2002, pp. 117–168.

[8] M. Jünger, G. Reinelt, G. Rinaldi, The traveling salesman problem, in: M. O. Ball, T. L. Magnanti, C. L. Monma, G. L. Nemhauser (Eds.), Network Models, Handbooks in Operations Research and Management Science, Elsevier, Amsterdam, 1995, pp. 225–330.

[9] J. N. Hooker, Logic-Based Methods for Optimization: Combining Optimization and Constraint Satisfaction, Wiley, New York, 2000.

[10] H. P. Williams, H. Yan, Representations of the all_different predicate of constraint satisfaction in integer programming, INFORMS Journal on Computing 13 (2001) 96–103.

[11] J. N. Hooker, Integrated Methods for Optimization, 2nd ed., Springer, 2012.