Scheduling Home Hospice Care with Logic-Based Benders Decomposition

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Outline

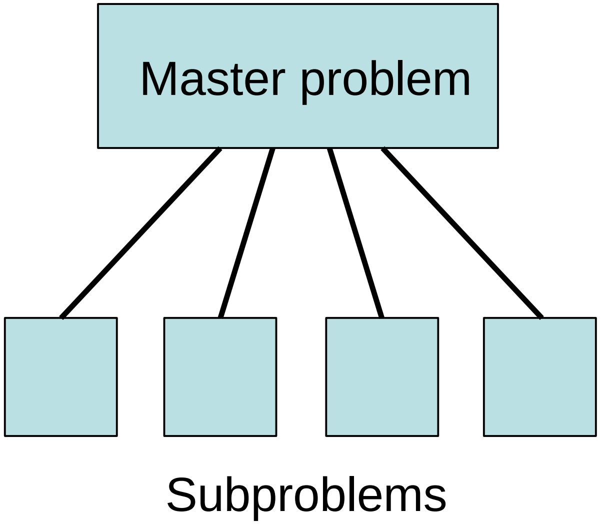
- Logic-based Benders tutorial
  - The algorithm
  - Inference duality
  - Machine scheduling
  - Other applications
  - Logical inference and SAT
- Home health care
  - The problem
  - Logic-based Benders model
  - Computational results
  - Alternate relaxations
  - LBBD References

Decomposition

- **Decomposition** breaks a large problem into subproblems that can be solved separately.
  - But with some kind of **communication** among the subproblems.
  - Decomposition is an **essential strategy** for solving today's ever larger and more interconnected models.

Benders Decomposition

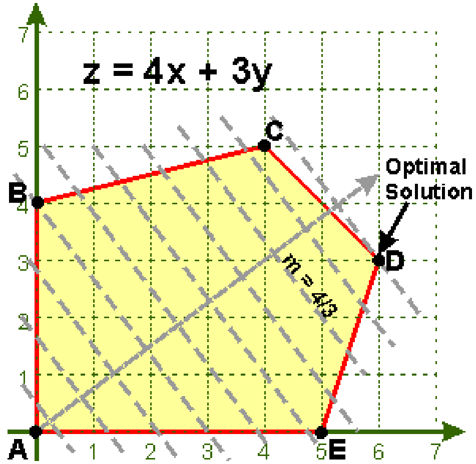
- **Benders decomposition** is a classical strategy that does not sacrifice overall optimality.
  - Separates the problem into a **master problem** and multiple **subproblems**.
    - Variables are partitioned between master and subproblems.
    - Exploits the fact that the problem may **radically simplify** when the master problem variables are fixed to a set of values.



Benders Decomposition

- But classical Benders decomposition has **a serious limitation.**
  - The subproblems must be **linear programming** problems.
    - Or continuous nonlinear programming problems.
    - The **linear programming dual** provides the Benders cuts.

Benders 1962

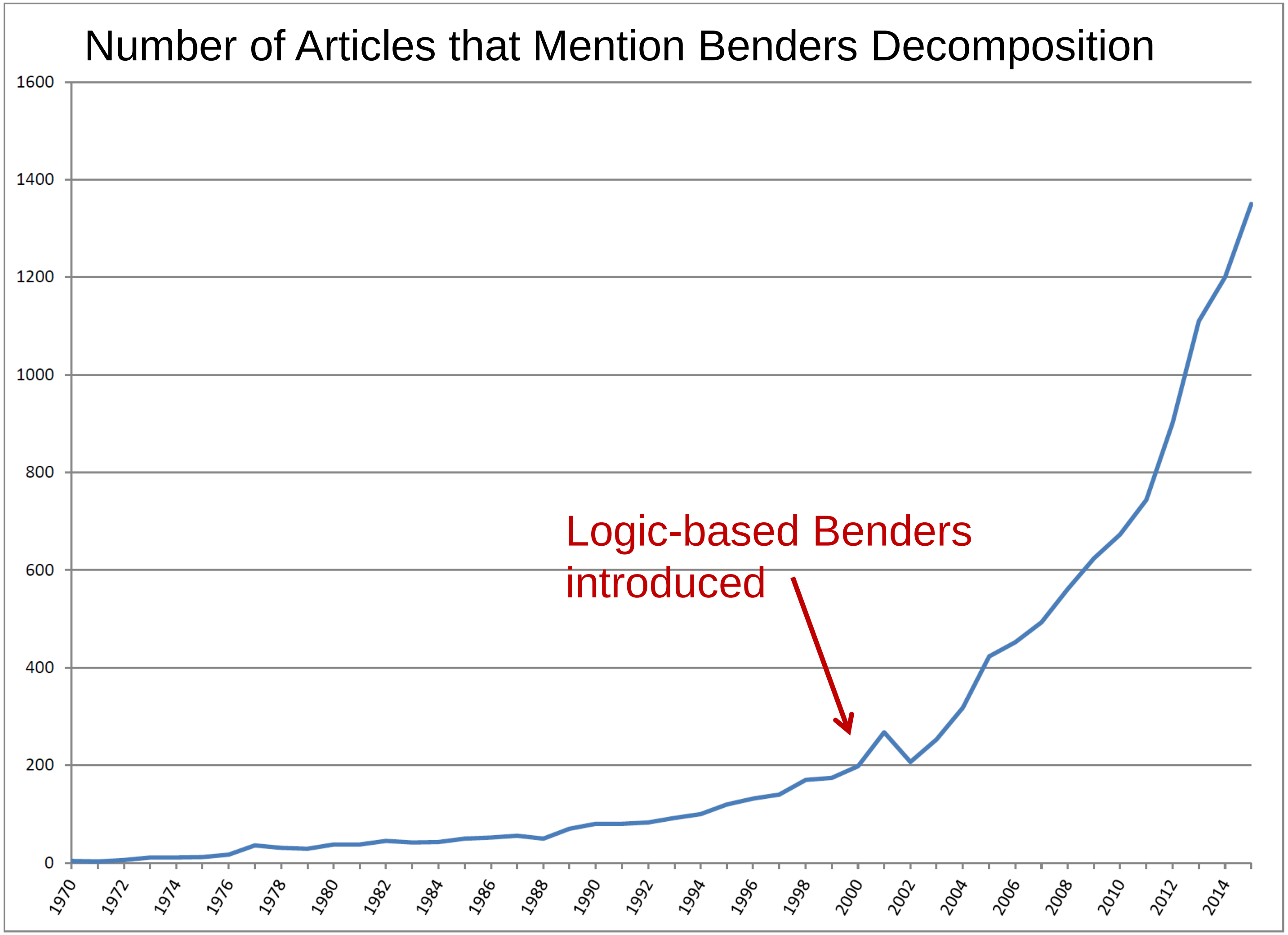


Logic-Based Benders

- **Logic-based Benders decomposition** attempts to overcome this limitation.
  - The subproblems can, in principle, be **any kind of optimization problem**.
    - The Benders cuts are obtained from an **inference dual**.
  - Speedup over state of the art can be **several orders of magnitude**.
  - Yet the Benders cuts must be designed specifically for every class of problems.

JH 1996, 2000
JH & Ottosson 2003

Logic-Based Benders



Source: Google Scholar

Logic-Based Benders

- Logic-based Benders decomposition solves a problem of the form

$$\min f(x,y)$$
$$(x,y) \in S$$
$$x \in D_x, y \in D_y$$

  – Where the problem simplifies when ***x* is fixed** to a specific value.

Logic-Based Benders

- Decompose problem into master and subproblem.
  - Subproblem is obtained by fixing x to solution value in master problem.

Master problem

$$\min z$$
$$z \geq g_k(x) \quad \text{(Benders cuts)}$$
$$x \in D_x$$

Minimize cost z subject to bounds given by Benders cuts, obtained from values of x attempted in previous iterations k.

→ Trial value $\bar{x}$ that solves master

← Benders cut $z \geq g_k(x)$

Subproblem

$$\min f(\bar{x}, y)$$
$$(\bar{x}, y) \in S$$

Obtain proof of optimality (solution of **inference dual**). Use same proof to deduce cost bounds for other assignments, yielding Benders cut.

Logic-Based Benders

- Iterate until master problem value equals best subproblem value so far.
  - This yields optimal solution.

Master problem

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$$x \in D_x$$

Minimize cost z subject to bounds given by Benders cuts, obtained from values of x attempted in previous iterations k.

Trial value $\bar{x}$ that solves master →

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Logic-Based Benders

- Fundamental concept: inference duality

Primal problem:
optimization

$$\min f(x)$$
$$x \in S$$

Find best feasible solution by searching over values of *x*.

Dual problem:
Inference

$$\max v$$
$$x \in S \overset{P}{\Rightarrow} f(x) \geq v$$
$$P \in \mathcal{P}$$

Find a proof of optimal value *v** by searching over proofs *P.*

In classical LP, the proof is a tuple of dual multipliers

Logic-Based Benders

- The proof that solves the dual in iteration k gives a bound $g_k(\bar{x})$ on the optimal value.
  - **The same proof** gives a bound $g_k(x)$ for other values of x.

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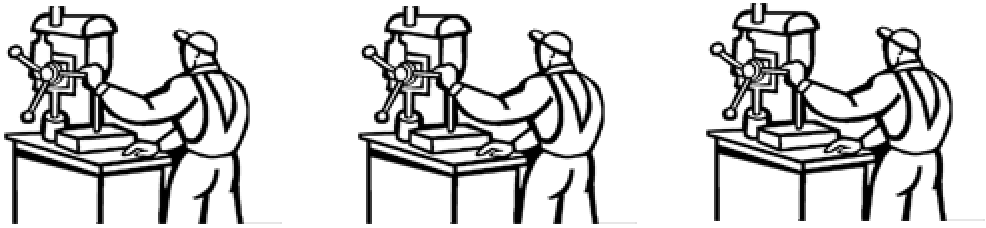
Logic-Based Benders

- Popular optimization duals are special cases of the inference dual.
  - Result from different choices of inference method.
  - For example....
    - Linear programming dual (gives **classical Benders cuts**)
    - Lagrangean dual
    - Surrogate dual
    - Subadditive dual

Machine Scheduling

- Assign tasks to machines.
- Then schedule tasks assigned to each machine.
  - Subject to time windows.
  - Cumulative scheduling: several tasks can run simultaneously, subject to resource limits.
  - Scheduling problem **decouples** into a separate problem for each machine.



Jain & Grossmann 2001

Machine Scheduling

- Assign tasks in master, schedule in subproblem.
  - Combine **mixed integer programming** and **constraint programming**

Master problem

Assign tasks to resources to minimize cost.

Solve by **mixed integer programming**.

Trial assignment $\bar{x}$ →

← Benders cut $z \geq g_k(x)$

Subproblem

Schedule jobs on each machine, subject to time windows.

Constraint programming obtains proof of optimality (dual solution).

Use **same proof** to deduce cost for some other assignments, yielding Benders cut.

Machine Scheduling

- Objective function
  - Cost is based on **task assignment only**.

$$\text{cost} = \sum_{ij} c_{ij} x_{ij}, \quad x_{ij} = 1 \text{ if task } j \text{ assigned to resource } i$$

  - So cost appears only in the **master problem**.
  - Scheduling subproblem is a **feasibility problem**.

Machine Scheduling

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    - So cost appears only in the **master problem**.
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- Benders cuts
  - They have the form $\sum_{j \in J_i} (1 - x_{ij}) \geq 1, \text{ all } i$
    - where J_i is a set of tasks that create infeasibility when assigned to resource i.

Machine Scheduling

- Resulting Benders decomposition:

Master problem

$$\min\ z$$
$$z = \sum_{ij} c_{ij} x_{ij}$$
Benders cuts

Trial assignment $\bar{x}$ →

← Benders cuts

$$\sum_{j \in J_i} (1 - x_{ij}) \geq 1,$$
for infeasible resources *i*

Subproblem

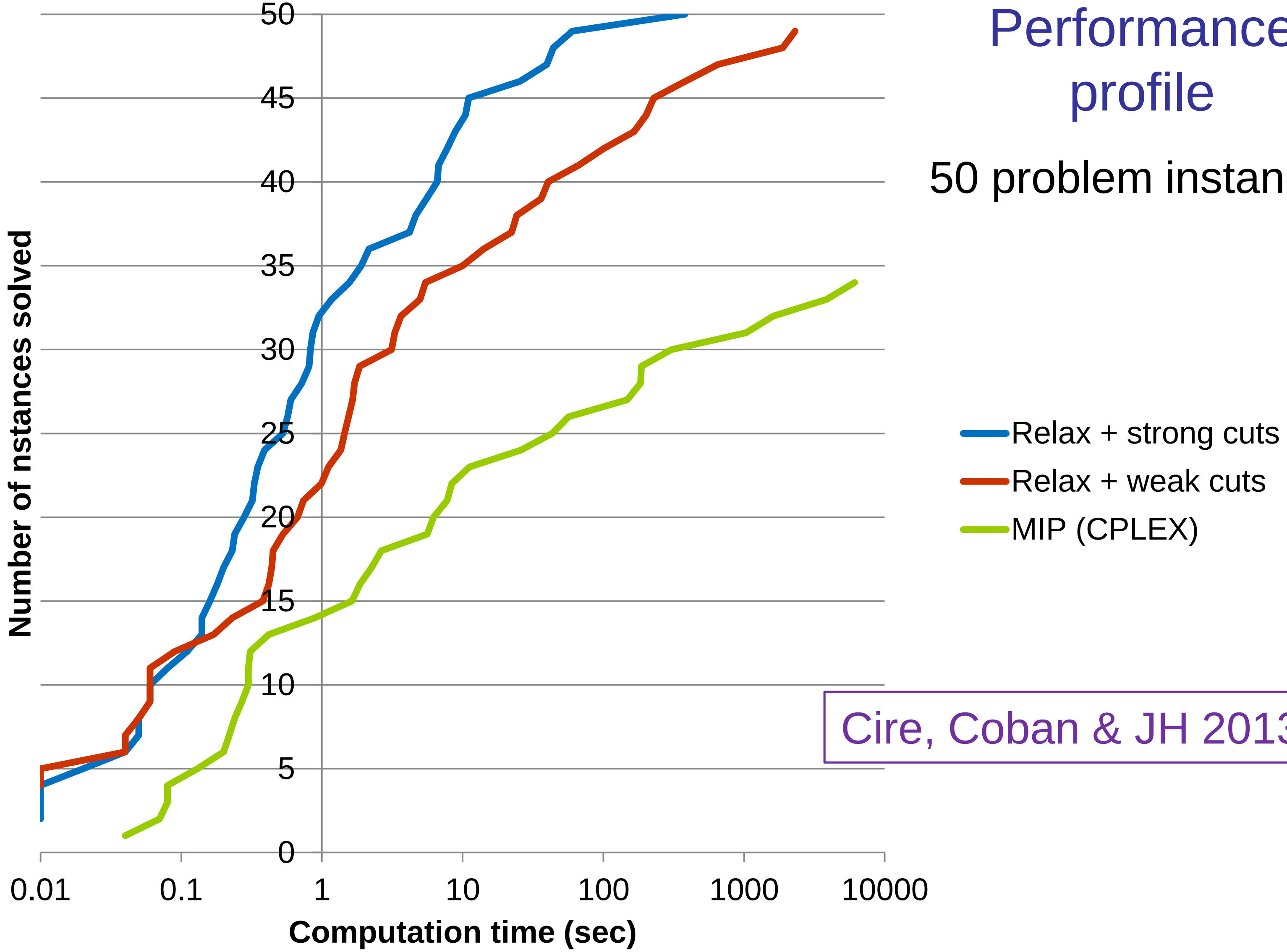
Schedule jobs on each resource.

Constraint programming may obtain proof of infeasibility on some resources (dual solution).

Use **same proof** to deduce infeasibility for some other assignments, yielding Benders cut.

Performance profile

50 problem instances



Cire, Coban & JH 2013

Logic-Based Benders Applications

- Planning and scheduling:
  - Machine allocation and scheduling
  - Steel production scheduling
  - Chemical batch processing (BASF, etc.)
  - Auto assembly line management (Peugeot-Citroën)
  - Allocation and scheduling of multicore processors (IBM, Toshiba, Sony)
  - Worker assignment in a queuing environment

Logic-Based Benders Applications

- Other scheduling
  - Lock scheduling
  - Shift scheduling
  - Permutation flow shop scheduling with time lags
  - Resource-constrained scheduling
  - Hospital scheduling
  - Optimal control of dynamical systems
  - Sports scheduling

Logic-Based Benders Applications

- Routing and scheduling
  - Vehicle routing
  - Home health care
  - Food distribution
  - Automated guided vehicles in flexible manufacturing
  - Traffic diversion around blocked routes
  - Concrete delivery

Logic-Based Benders Applications

- Location and Design
  - Wireless local area network design
  - Facility location-allocation
  - Stochastic facility location and fleet management
  - Capacity and distance-constrained plant location
  - Queuing design and control

Logic-Based Benders Applications

- Other
  - Logical inference
  - Logic circuit verification
  - Bicycle sharing
  - Service restoration in a network
  - Inventory management
  - Supply chain management
  - Space packing

Logical Inference

- A fundamental problem in the information age.
  - Can use **SAT solvers** or **logic-based Benders** to deduce facts from a knowledge base.
  - SAT solvers are a **special case** of Benders!

Logical Inference

- Draw inferences from a clause set
  - Infer everything we can about propositions x_1, x_2, x_3

We can deduce

$$x_1 \vee x_2$$

$$x_1 \vee x_3$$

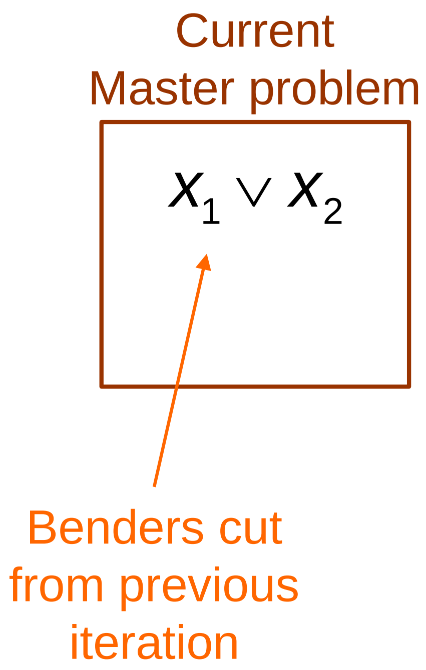
This is a **projection** onto x_1, x_2, x_3

$$
\begin{array}{llllll}
x_1 & & & \vee\, x_4 & \vee\, x_5 & \\
x_1 & & & \vee\, x_4 & \vee\, \bar{x}_5 & \\
x_1 & & & & \vee\, x_5 & \vee\, x_6 \\
x_1 & & & & \vee\, x_5 & \vee\, \bar{x}_6 \\
 & x_2 & & & \vee\, \bar{x}_5 & \vee\, x_6 \\
 & x_2 & & & \vee\, \bar{x}_5 & \vee\, \bar{x}_6 \\
 & & x_3 & \vee\, \bar{x}_4 & \vee\, x_5 & \\
 & & x_3 & \vee\, \bar{x}_4 & \vee\, \bar{x}_5 &
\end{array}
$$

JH 2015

Logical Inference

- Benders decomposition computes the **projection.**
  - Benders cuts describe projection onto master problem variables.



Logical Inference

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Current Master problem

$x_1 \vee x_2$

→ solution of master $(x_1,x_2,x_3) = (F,T,F)$ →

Resulting subproblem

$$x_4 \vee x_5$$
$$x_4 \vee \bar{x}_5$$
$$x_5 \vee x_6$$
$$x_5 \vee \bar{x}_6$$
$$\bar{x}_4 \vee x_5$$
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Subproblem is **infeasible**.
$(x_1,x_3)=(F,F)$ creates infeasibility

Logical Inference

- Benders decomposition computes the **projection**.
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Current Master problem

$$x_1 \vee x_2$$
$$x_1 \vee x_3$$

solution of master
$(x_1,x_2,x_3) = (F,T,F)$

Benders cut
(nogood)

Resulting subproblem

$$x_4 \vee x_5$$
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Current Master problem

$x_1 \vee x_2$

$x_1 \vee x_3$

→ solution of master $(x_1,x_2,x_3) = (F,T,T)$ →

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$x_4 \vee x_5$

$x_4 \vee \bar{x}_5$

$x_5 \vee x_6$

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Subproblem is **feasible**

Logical Inference

- Benders decomposition computes the **projection**.
  - Benders cuts describe projection onto master problem variables.

Current Master problem

$x_1 \vee x_2$

$x_1 \vee x_3$

$x_1 \vee \bar{x}_2 \vee \bar{x}_3$

solution of master (x_1,x_2,x_3) = (F,T,T)

Enumerative Benders cut

Resulting subproblem

$x_4 \vee x_5$

$x_4 \vee \bar{x}_5$

$x_5 \vee x_6$

$x_5 \vee \bar{x}_6$

Subproblem is **feasible**

Logical Inference

- Benders decomposition computes the **projection**.
  - Benders cuts describe projection onto master problem variables.

Current Master problem

$x_1 \vee x_2$

$x_1 \vee x_3$

$x_1 \vee \bar{x}_2 \vee \bar{x}_3$

→ solution of master $(x_1,x_2,x_3) = (F,T,T)$ →

Resulting subproblem

$x_4 \vee x_5$

$x_4 \vee \bar{x}_5$

$x_5 \vee x_6$

$x_5 \vee \bar{x}_6$

← Enumerative Benders cut ←

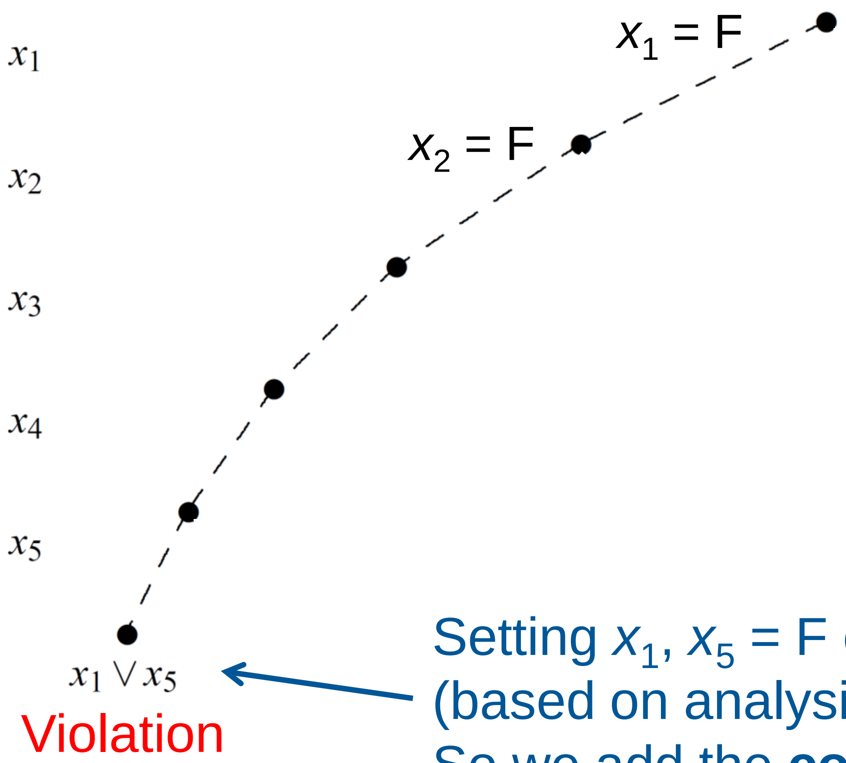
Continue until **master** is infeasible.
Black Benders cuts describe projection.

Logical Inference

- **Satisfiability methods** solve the problem by generating **Benders cuts!**
  - Conflict clauses = Benders cuts

Logical Inference

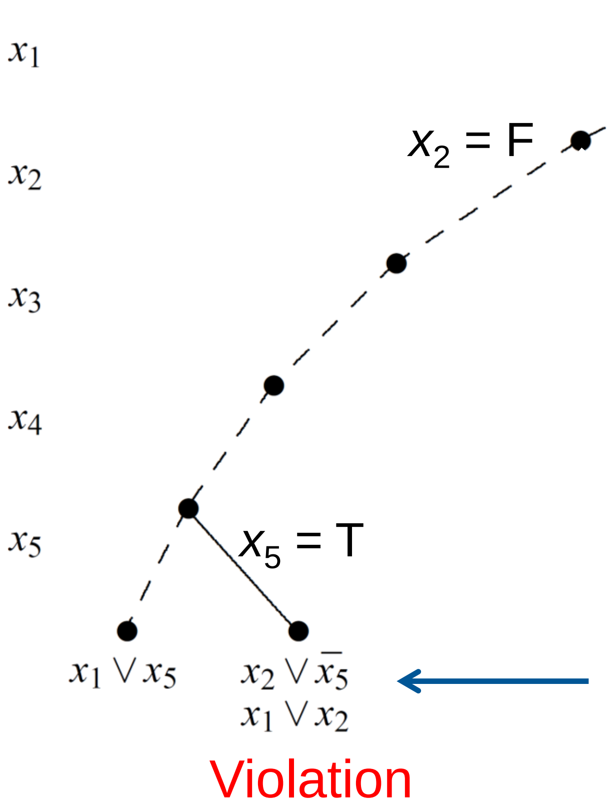
Start branching on variables in depth-first fashion.
At each node of the branching tree, check if a clause is violated



Setting x_1, x_5 = F causes the violation (based on analysis of an implication graph)
So we add the **conflict clause** $x_1 \vee x_5$ to avoid this assignment in the future

Logical Inference

Start branching on variables in depth-first fashion.
At each node of the branching tree, check if a clause is violated



Here we have a **conflict clause** $x_1 \vee \neg x_5$
Resolve with $x_1 \vee x_2$ to get $x_1 \vee x_2$

Logical Inference

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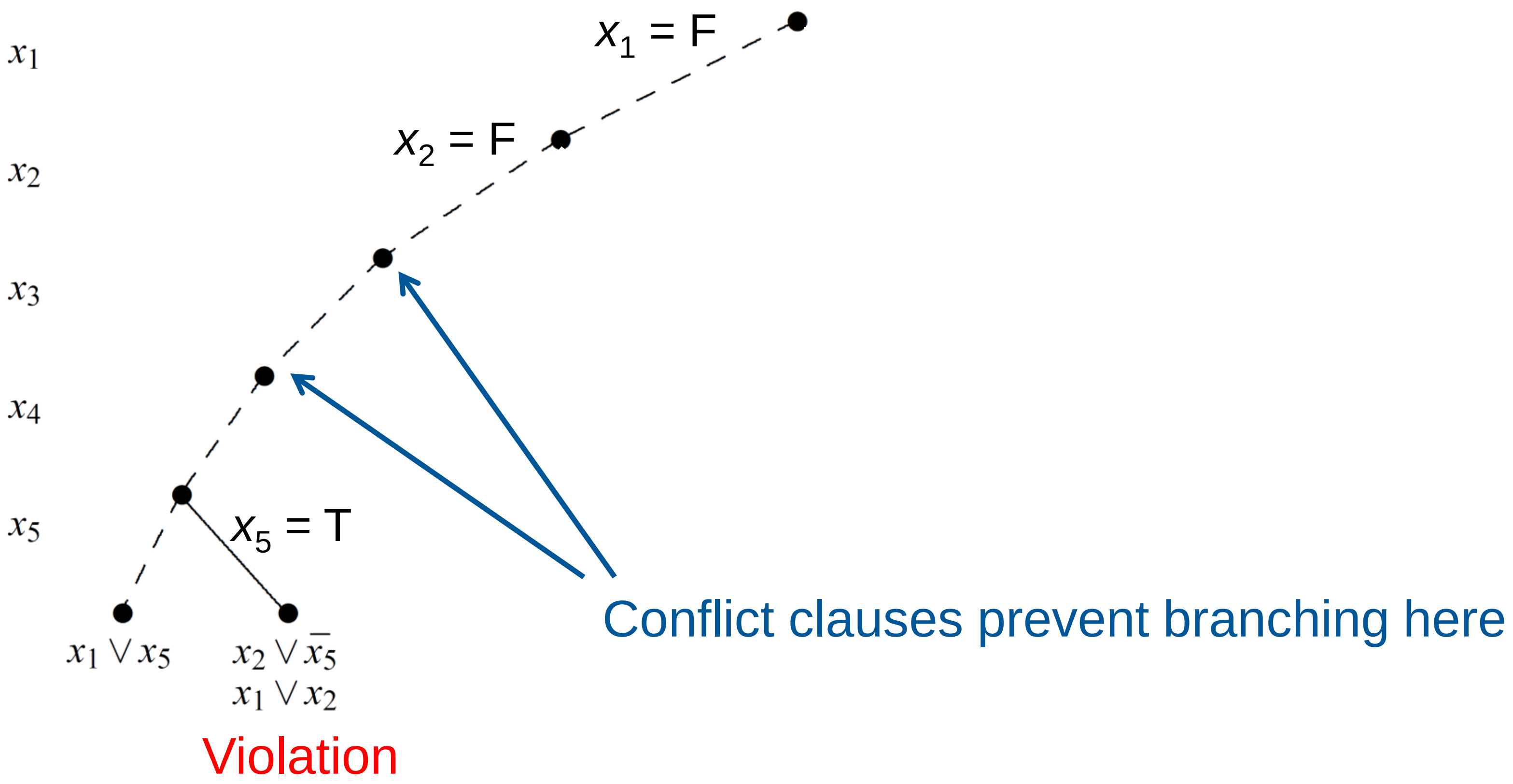

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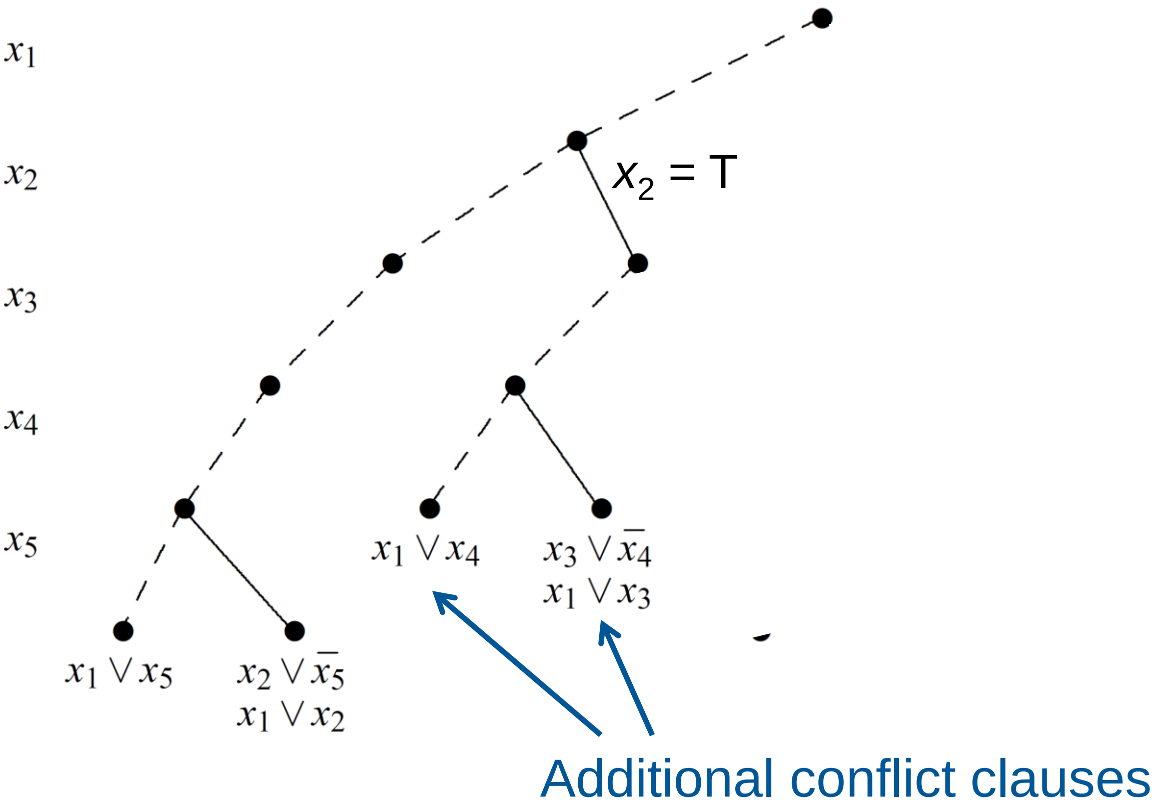
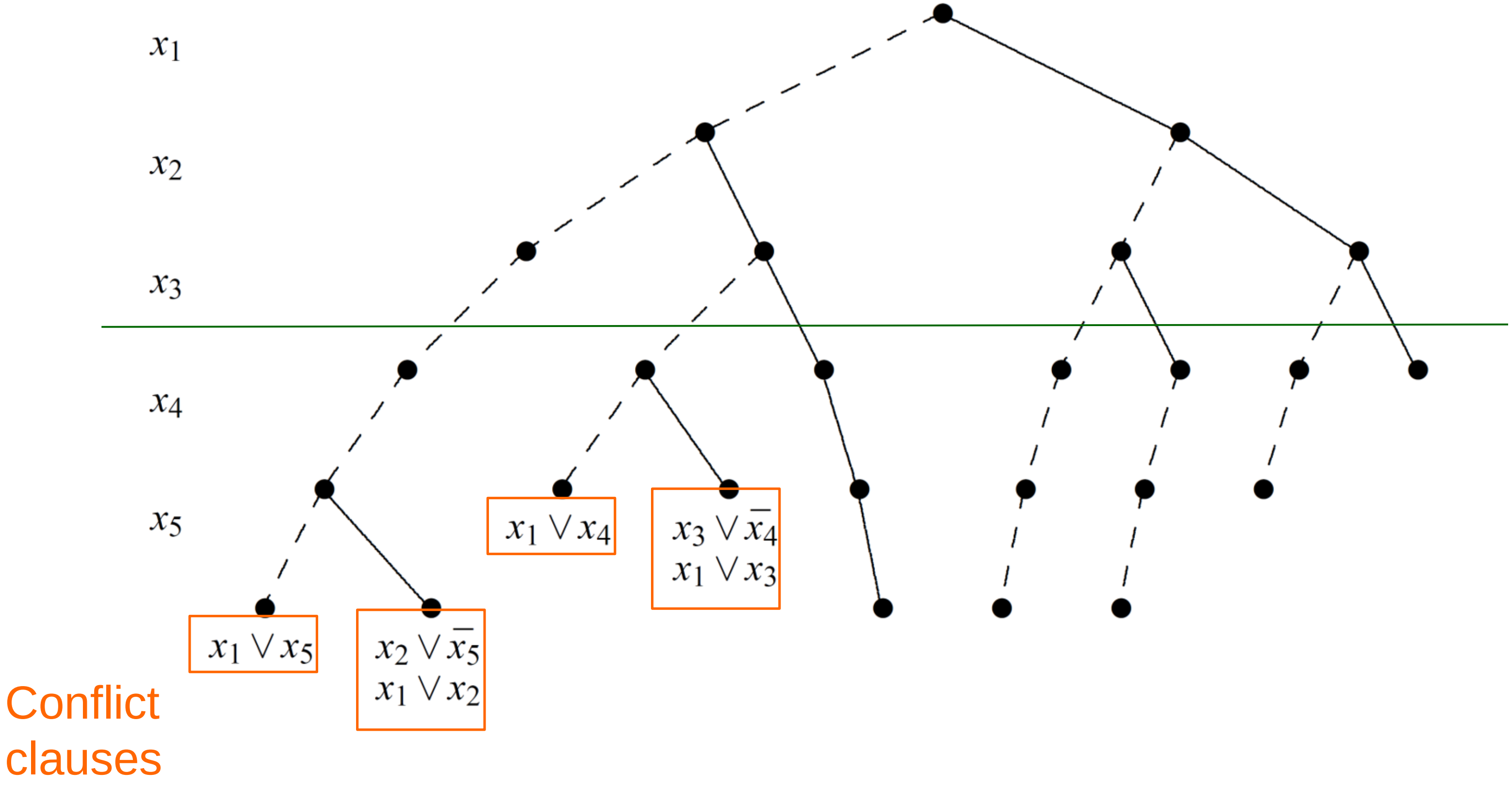


Logical Inference

- Benders cuts = conflict clauses in a SAT algorithm
  - Branch on x_1, x_2, x_3 first.



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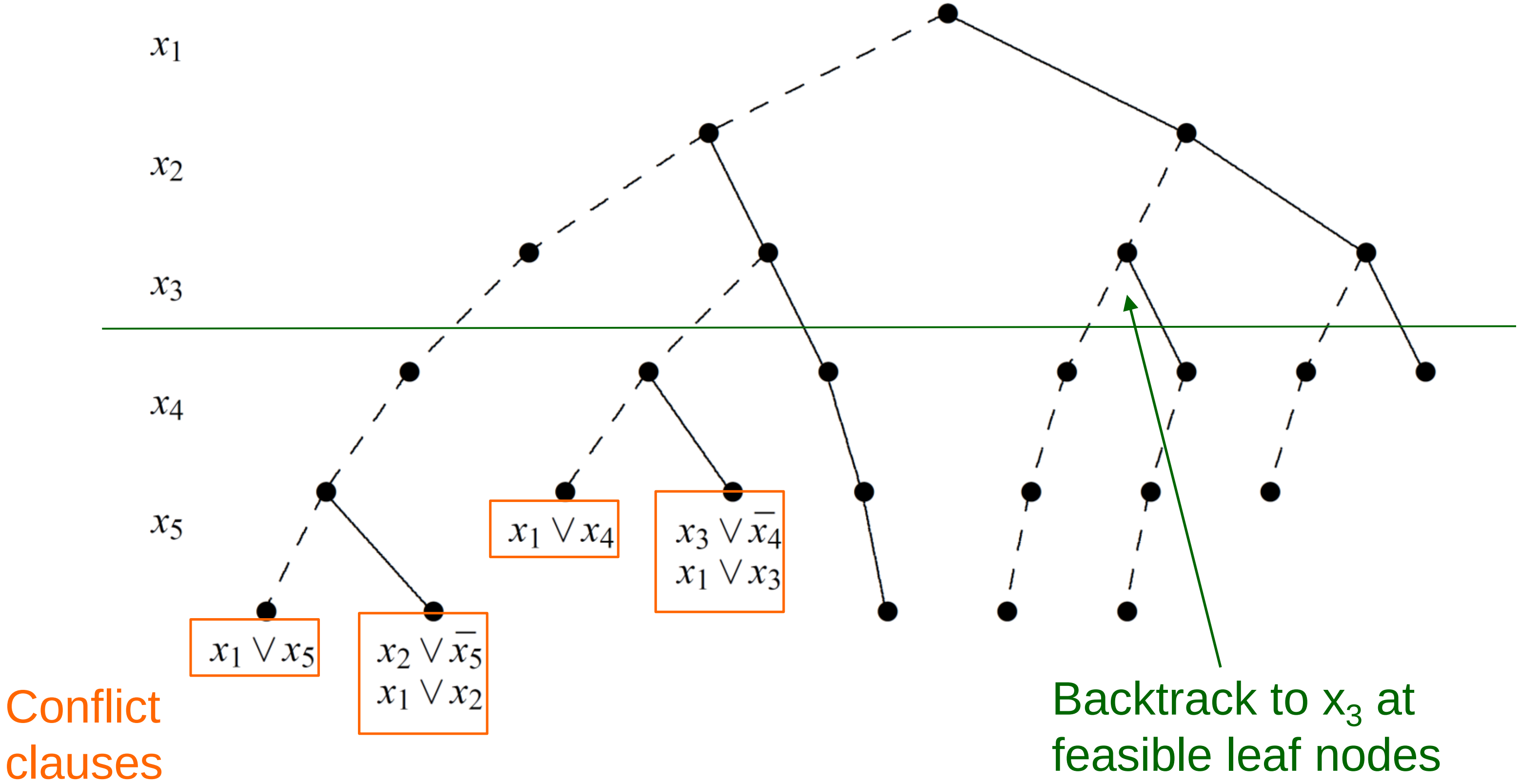

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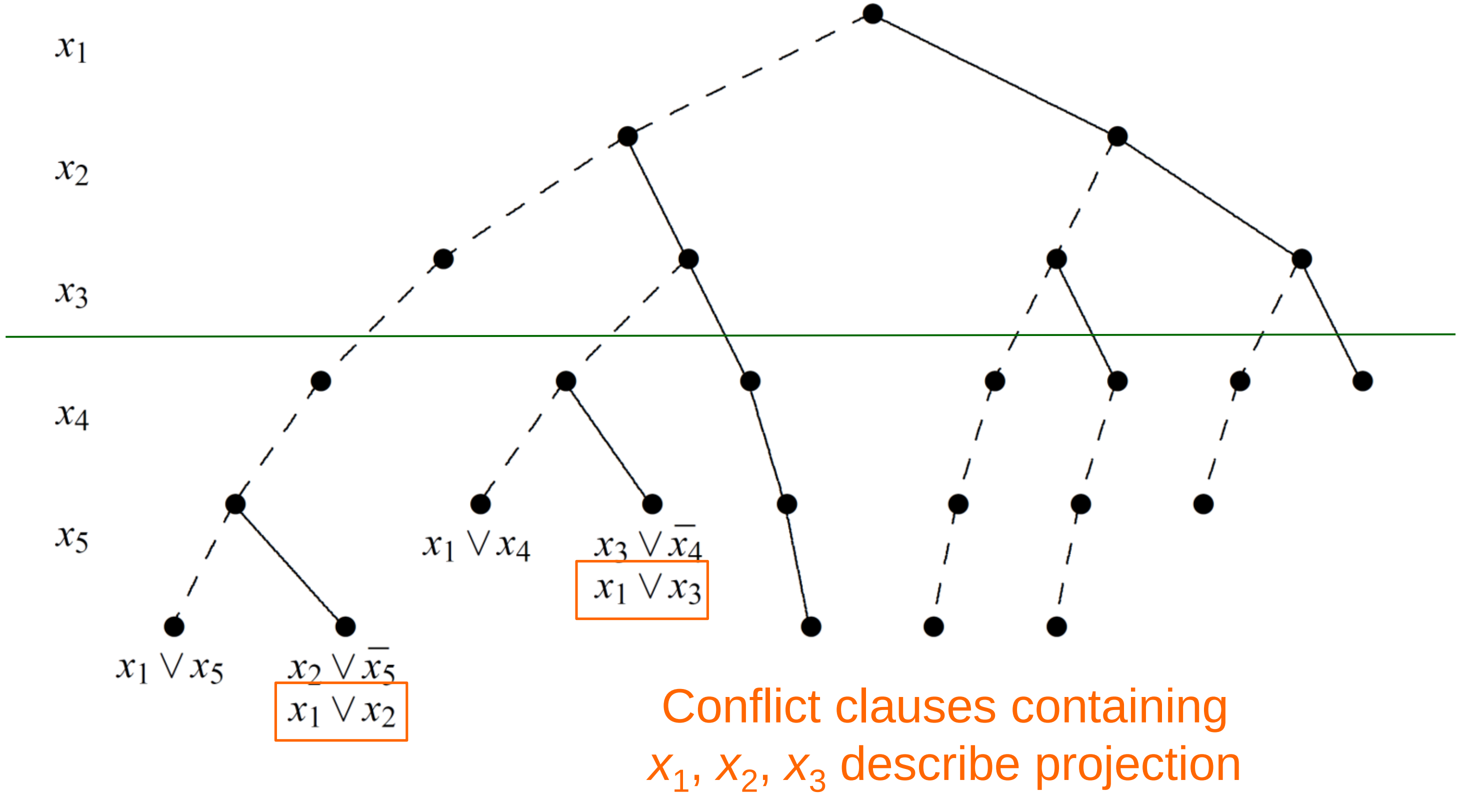


Home Health Care

- General home health care problem.
  - Assign **aides** to homebound **patients**.
    - …subject to constraints on aide qualifications and patent preferences.
    - One patient may require a team of aides.
  - **Route** each aide through assigned patients, observing **time windows**.
    - …subject to constraints on hours, breaks, etc.

Home Health Care

- A large industry, and **rapidly growing**.
  - Roughly as large as all courier and delivery services.

Projected Growth
of Home Health Care Industry

	2014	2018
U.S. revenues, $ billions	75	150
World revenues, $ billions	196	306

Increase in U.S. Employment, 2010-2020

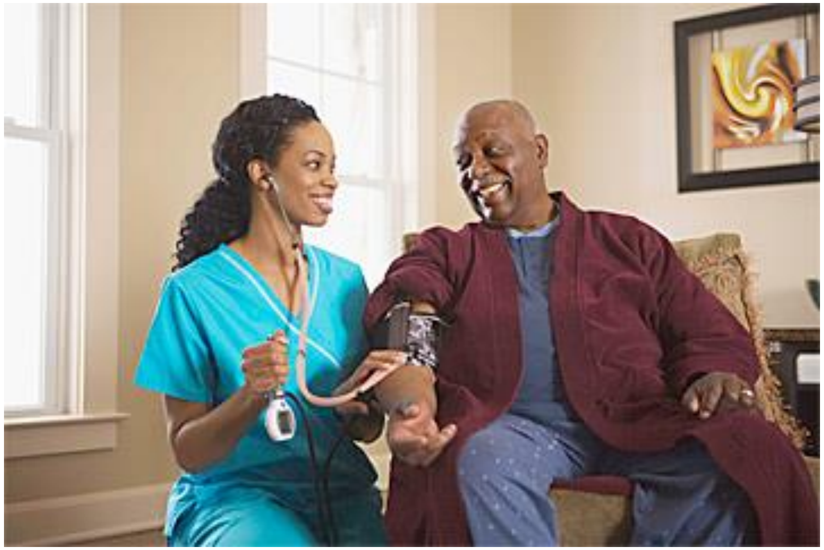
Home health care industry	70%
Entire economy	14%

Home Health Care

- Advantages of home health care
  - Lower cost
    - Hospital & nursing home care is very expensive.
  - No hospital-acquired infections
    - Less exposure to superbugs.
  - Preferred by patients
    - Comfortable, familiar surroundings of home.
    - Sense of control over one’s life.
  - Supported by new equipment & technology
    - IT integration with hospital systems.
    - Online consulting with specialists.

Home Health Care

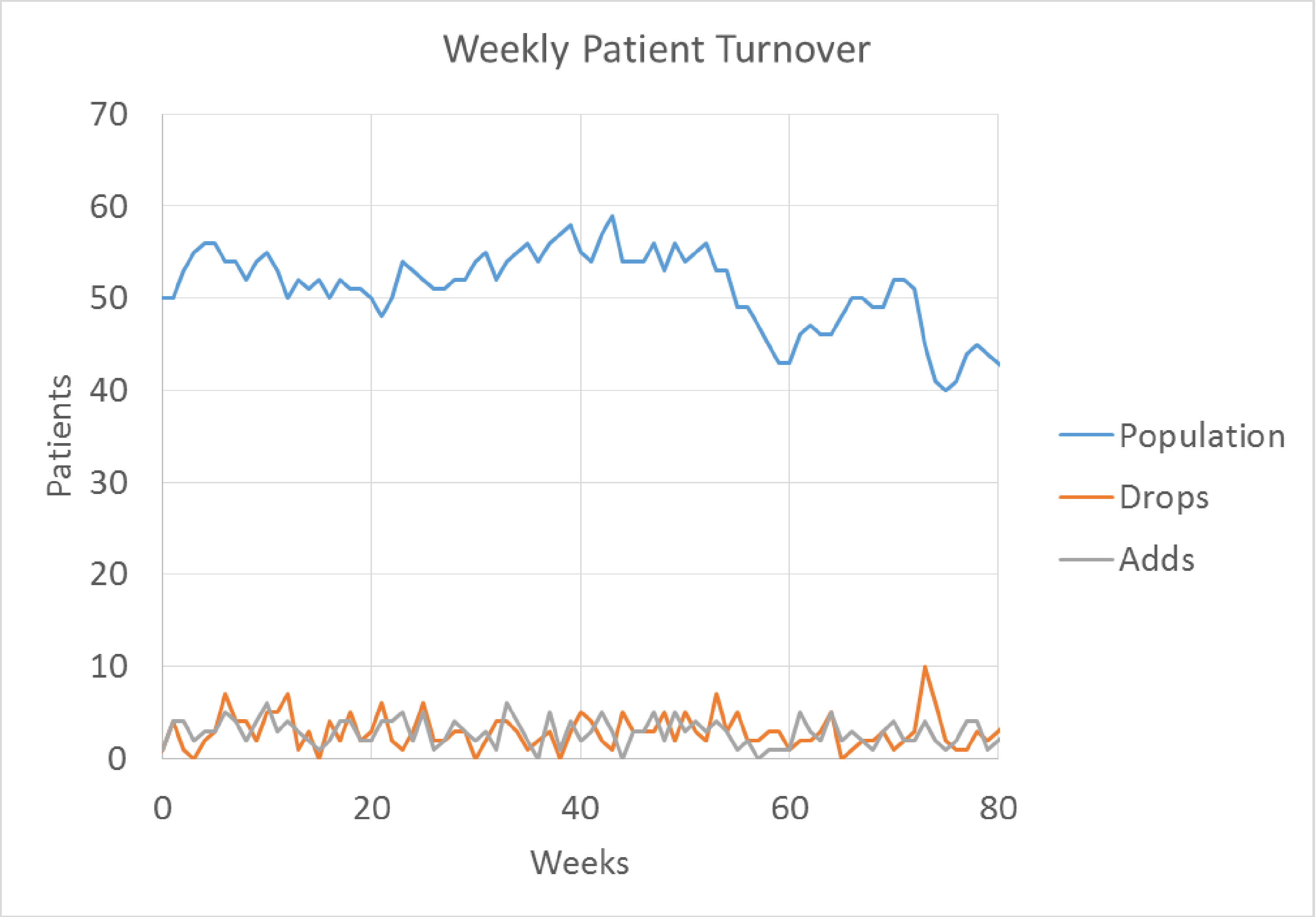
- Critical factor to realize cost savings:
  - Aides must be **efficiently** scheduled.
- This is our task.
  - Focus on home hospice care.



Home Hospice Care

- Distinguishing characteristics of hospice care
  - Personal & household services
  - Regular weekly schedule
    - For example, Mon-Wed-Fri at 9 am.
  - Same aide each visit
  - Long planning horizon
    - Several weeks
  - Rolling schedule
    - Update schedule as patient population evolves.

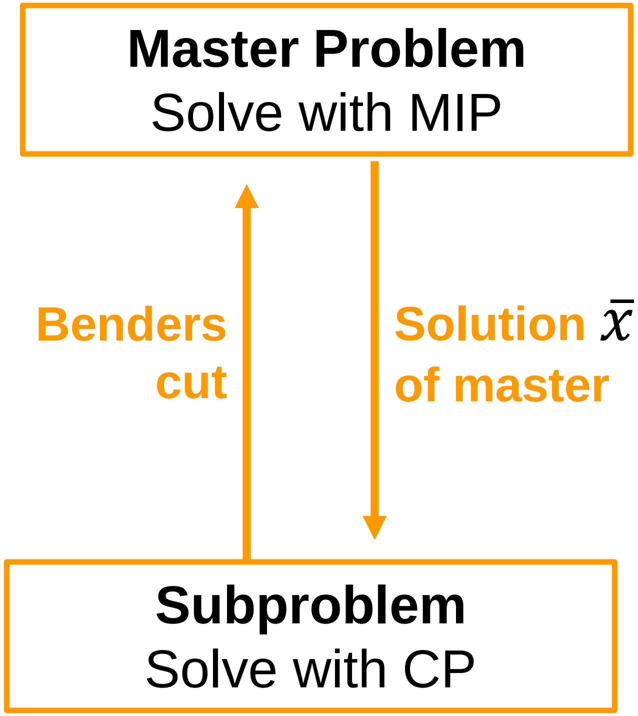
Home Hospice Care



5-8% weekly turnover

Home Hospice Care

- Solve with Benders decomposition.
  - **Assign aides to patients** in master problem.
    - Maximize number of patients served by a given set of aides.



Heching & JH 2016

Home Hospice Care

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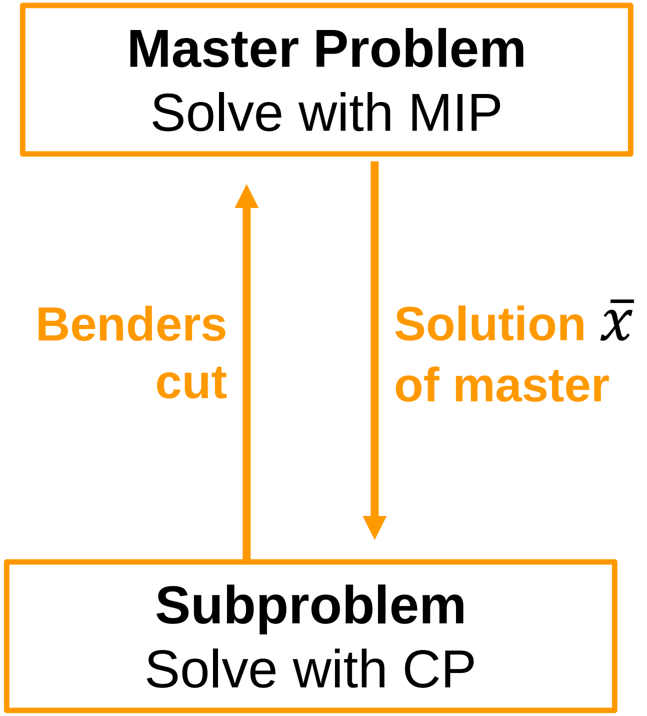
  - **Schedule home visits** in subproblem.
    - Cyclic weekly schedule.
    - No visits on weekends.



Home Hospice Care

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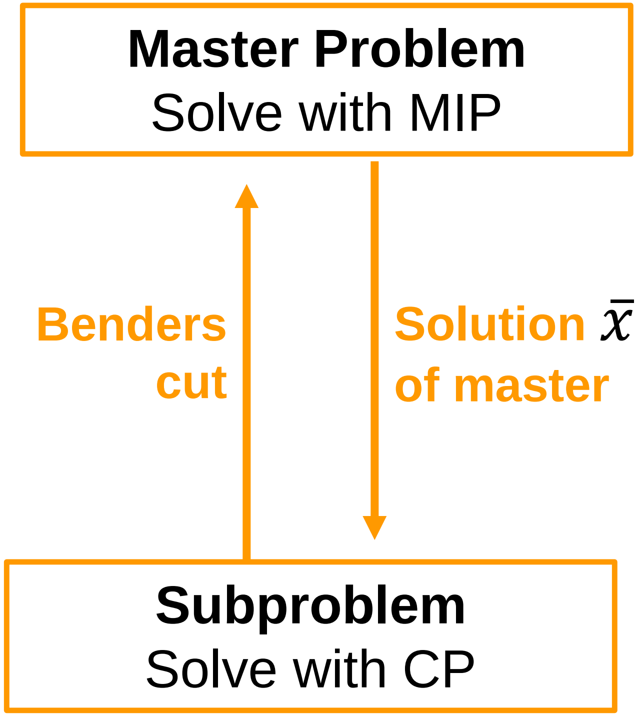
    - Maximize number of patients served by a given set of aides.
  - **Schedule home visits** in subproblem.
    - Cyclic weekly schedule.
    - No visits on weekends.
  - Subproblem **decouples** into a scheduling problem for each aide and each day of the week.



Master Problem

= 1 if patient *j* scheduled

= 1 if patient *j* assigned to aide *i*

$$\max \sum_j \delta_j$$

$$\sum_i x_{ij} = \delta_j, \quad \text{all } j$$

= 1 if patient *j* assigned to aide *i* on day *k*

Required number of visits per week

$$\sum_{i,k} y_{ijk} = v_j \delta_j, \quad \text{all } j$$

$$y_{ijk} \le x_{ij}, \quad \text{all } i, j, k$$

Spacing constraints on visit days

Benders cuts

Relaxation of subproblem

$$\delta_j, x_{ij}, y_{ijk} \in \{0, 1\}$$

Master Problem

- For a rolling schedule:
  - Schedule **new patients**, drop **departing patients** from schedule.
    - Provide continuity for remaining patients as follows:
  - Old patients served by **same aide** on **same days**.
    - Fix $y_{ijk} = 1$ for the relevant aides, patients, and days.

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  - Old patients served by **same aide** on **same days**.
    - Fix $y_{ijk} = 1$ for the relevant aides, patients, and days.
  - Alternative: Also served **at same time**.
    - Fix time windows to enforce their current schedule.
  - Alternative: served only by **same aide**.
    - Fix $x_{ij} = 1$ for the relevant aides, patients.

Subproblem

Scheduling problem for aide *i*, day *k*

*n*th patient in sequence

Set of patients assigned to aide *i*, day *k*

$$\text{alldiff}\{\pi_n \mid n = 1, \ldots, |P_{ik}|\}$$

$$[s_j, s_j + p_j] \subseteq [r_j, d_j], \quad \text{all } j \in P_{ik}$$

$$s_{\pi_n} + p_{\pi_n} + t_{\pi_n \pi_{n+1}} \leq s_{\pi_{n+1}}, \quad n = 1, \ldots, |P_{ik}| - 1$$

start time

Visit duration

Travel time

Modeled with interval variables in CP solver.

Benders Cuts

- Generate a cut for each infeasible scheduling problem.
  - Solution of subproblem inference dual is a **proof** of infeasibility.
    - The proof may show **other** patient assignments to be infeasible.
    - Generate **nogood cut** that rules out these assignments.

Benders Cuts

- Generate a cut for each infeasible scheduling problem.
  - Solution of subproblem inference dual is a **proof** of infeasibility.
    - The proof may show **other** patient assignments to be infeasible.
    - Generate **nogood cut** that rules out these assignments.
  - Unfortunately, we **don’t have access** to infeasibility proof in CP solver.

Benders Cuts

- So, strengthen the nogood cuts heuristically.
  - Find a smaller set of patients that create infeasibility…
    - …by re-solving the each infeasible scheduling problem repeatedly.

$$\sum_{j \in \boxed{\bar{P}_{ik}}} (1 - y_{ijk}) \geq 1$$

Reduced set of patients whose assignment to aide *i* on day *k* creates infeasibility

Benders Cuts

- Auxiliary cuts based on symmetries.
  - A cut for valid for aide *i*, day *k* is also valid for aide *i* on other days.
    - This gives rise to a large number of cuts.
  - The auxiliary cuts can be summed without sacrificing optimality.
    - Original cut ensures convergence to optimum.
    - This yields 2 cuts per aide:

$$\sum_{j \in \bar{P}_{ik}} (1 - y_{ijk}) \geq 1$$

$$\sum_{k \neq k} \sum_{j \in \bar{P}_{ik}} (1 - y_{ijk'}) \geq 4$$

Subproblem Relaxation

- Include relaxation of subproblem in the master problem.
  – Necessary for good performance.
  – Use **time window relaxation** for each scheduling problem.
  – Simplest relaxation for aide *i* and day *k*:

$$\sum_{j \in \boxed{J(a,b)}} p_j y_{ijk} \leq b - a$$

Set of patients whose time window fits in interval [*a*, *b*].

Can use several intervals.

Subproblem Relaxation

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  - This is **weak** unless most assignments are **fixed**.
    - As in rolling schedule.
  - We partition day into 2 intervals.
    - Morning and afternoon.
    - Simplifies handling of aide time windows and home bases.
    - All patient time windows are in morning or afternoon.

Subproblem Relaxation

Time window relaxation for aide *i*, day *k* using intervals [*a*,*b*], [*b*,*c*]

$$\sum_{j \in J(a,b)} p'_{ijk} y_{ijk} \le b - a$$

$$\sum_{j \in J(b,c)} p''_{ijk} y_{ijk} \le c - b$$

where

$[a, c]$ = time window for aide i

$$p'_{ijk} = p_j + \min\left\{t_{ij}, \min_{j' \in Q_{ik}} \{t_{j'j}\}\right\}$$

$$p''_{ijk} = p_j + \min\left\{\min_{j' \in Q_{ik}} \{t_{jj'}\}, c\right\}$$

and where Q_{ik} = {patients unassigned or assigned to aide *i*, day *k*}

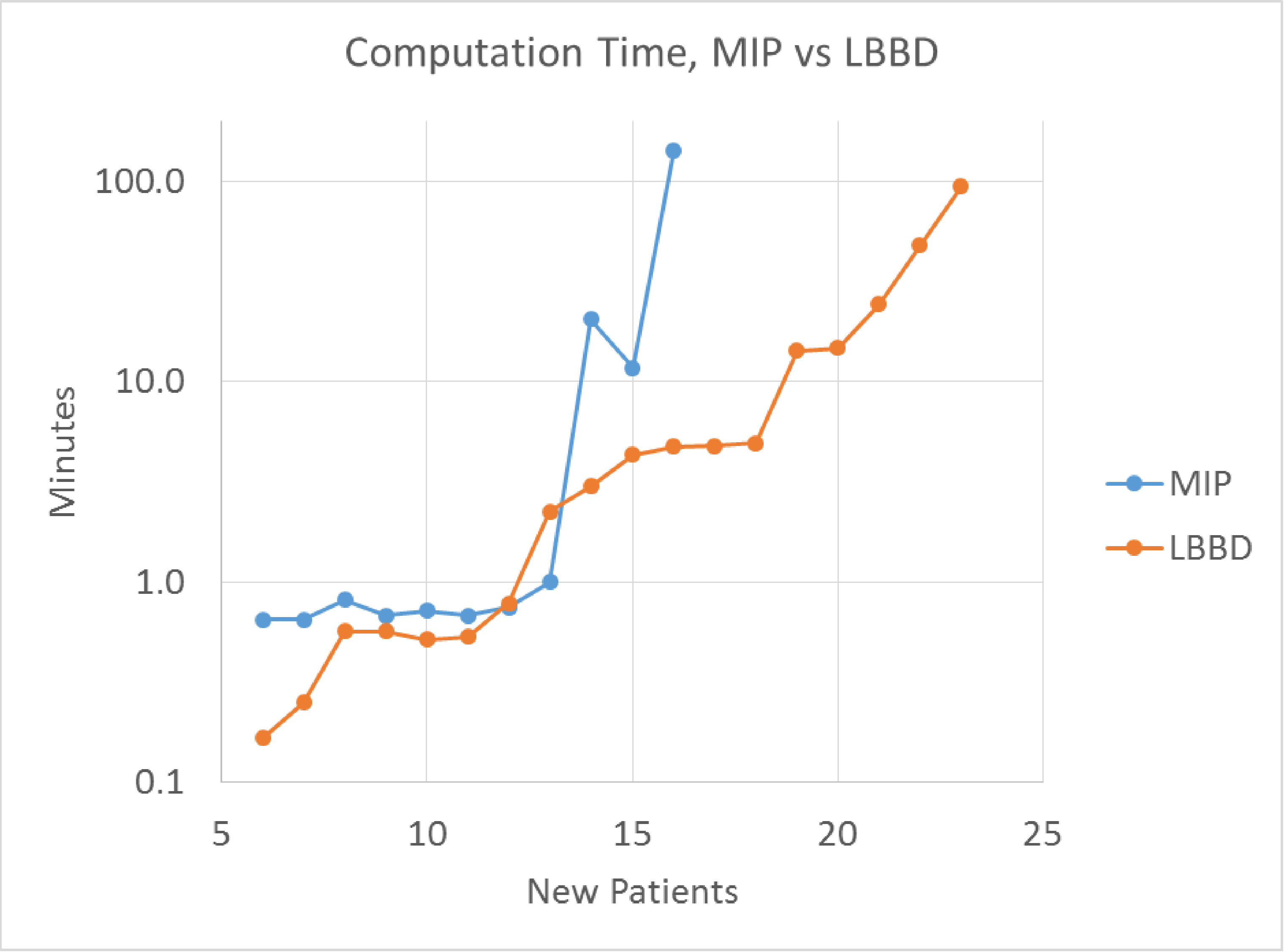
Computational Tests

- Dataset
  - 60 home hospice patients
    - 2, 3 or 5 visits per week (not on weekends)
  - 18 health care aides with time windows
  - Actual travel distances
- Solver
  - **LBBD:** IBM OPL Optimization Studio 12.6.2
    - CPLEX + CP Optimizer + user-supplied script
  - **MIP:** CPLEX in OPL Studio
    - Modified multicommodity flow model of VRPTW
- Computer
  - Laptop with Intel Core i7
    - 7.75 GB RAM

Computational Tests

- Instance generation
  - Start with (suboptimal) solution for the 60 patients
    - Fix this schedule for first *n* patients.
    - Schedule remaining 60 – *n* patients
  - Use 8 of the 18 aides to cover new patients
    - As well as the old patients they already cover.
    - This puts us near the phase transition.

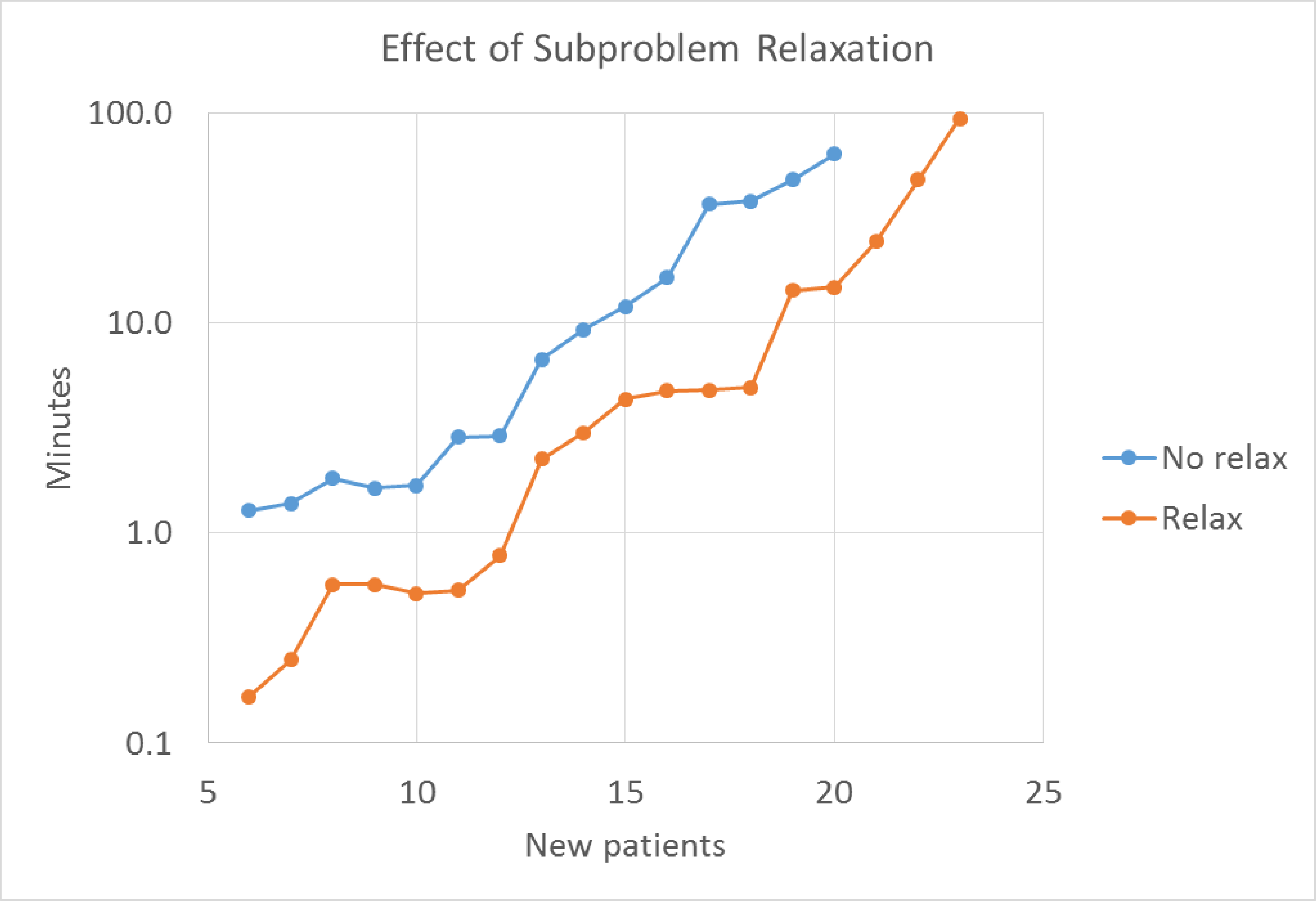
Computational Tests



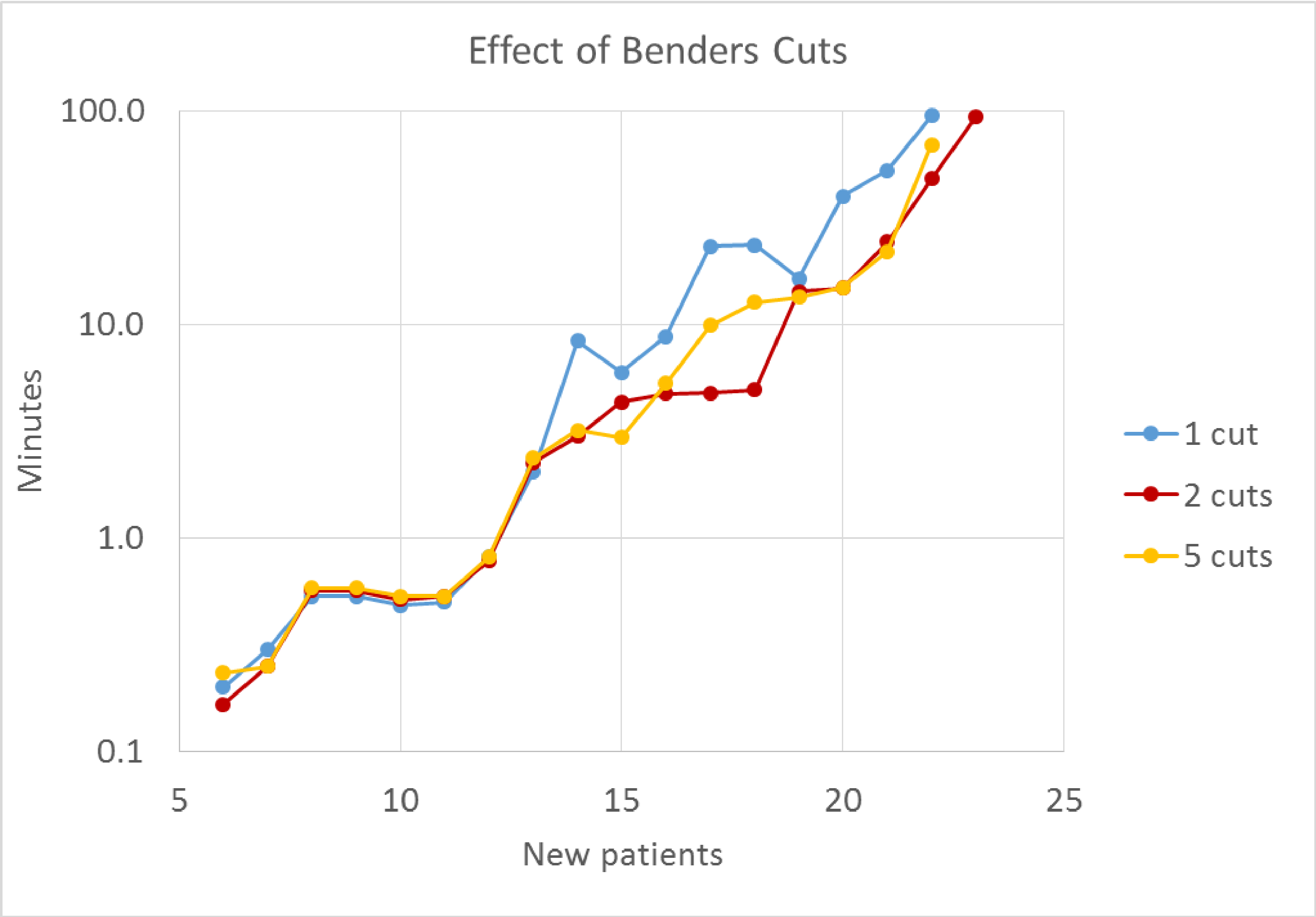
Computational Tests

- Practical implications
  - MIP or LBBD will work for smaller instances
  - LBBD **scales up** to realistic size
    - One month advance planning in 60 patient population
    - Assuming 5-8% weekly turnover
  - Advantage of **exact** solution method
    - We know **for sure** whether existing staff will cover projected demand.

Computational Tests



Computational Tests



Computational Tests

- Other relaxations
  - Multicommodity flow relaxation
    - Master problem too large, solves slowly
    - n^2 flow variables, where n = number of patients
    - Master must be re-solved in each iteration
    - Relaxation useless until many variables are fixed in B&B

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    - Master problem too large, solves slowly
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    - Master must be re-solved in each iteration
    - Relaxation useless until many variables are fixed in B&B
  - Assignment relaxation
    - Master problem still too large, solves slowly.
    - Relaxation very weak without separating TSP cuts.

Branch & Check

- Idea: use stronger relaxation with **branch & check**
  - Branch & check solves master problem **once** with search tree.
  - At feasible nodes, solve subproblem to obtain Benders cut.
  - **Not the same** as branch & bound.
- Large multicommodity or assignment relaxation is only solved **once**.

JH 2000
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- However, performance is **worse**…

JH 2000
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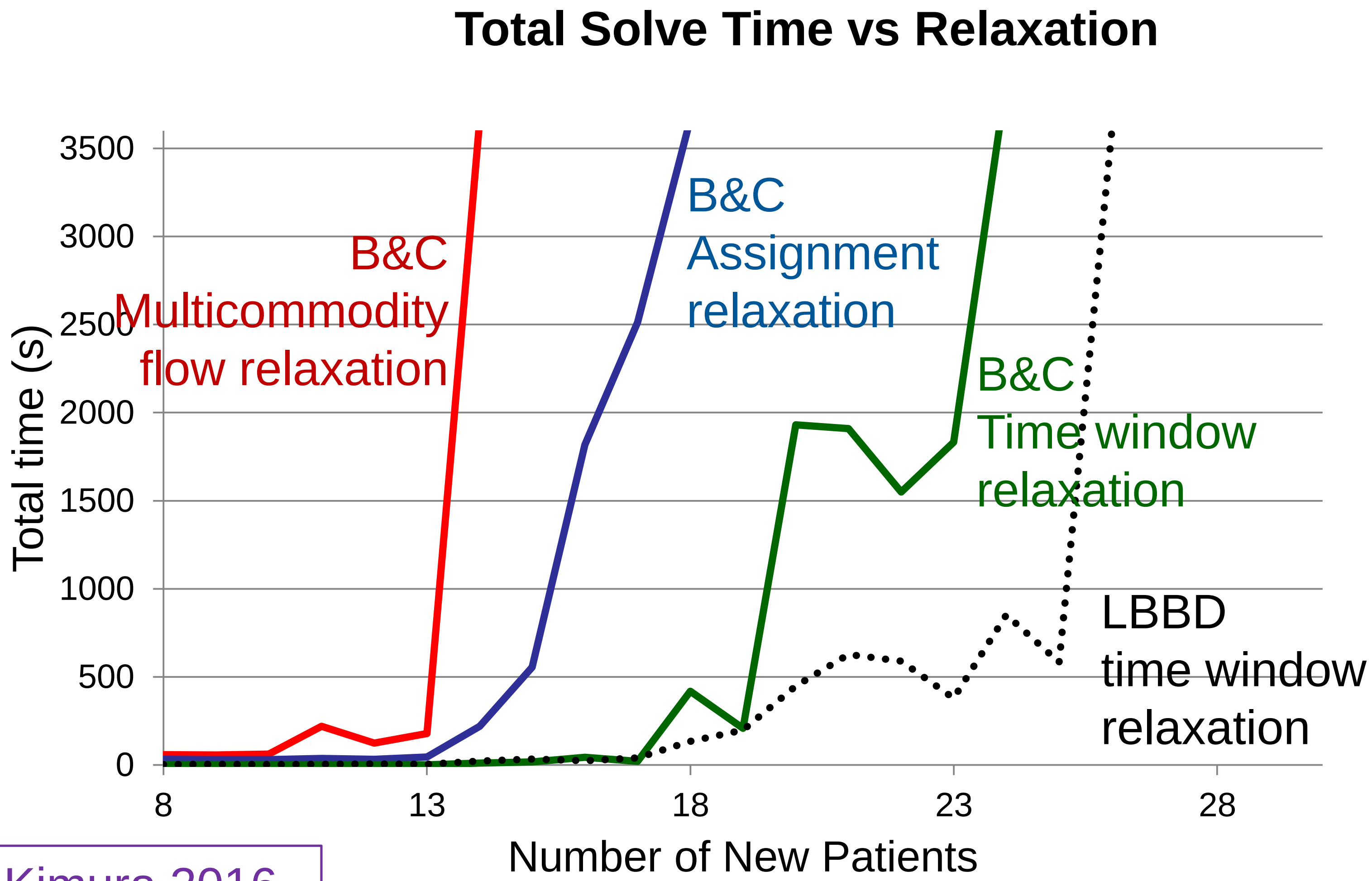
Branch & Check



Kimura 2016

Branch & check

- What is going on?
  - Because of superior relaxation, fewer feasible leaf nodes.
  - So fewer Benders cuts.
    - Less information obtained from subproblem.er

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- What is going on?
  - Because of superior relaxation, fewer feasible leaf nodes.
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    - Less information obtained from subproblem.er

- Good news…
  - This reimplementation of LBBD is substantially faster than OPL implementation.
    - Uses C++, SCIP, and Gecode.

Kimura 2016

Conclusions

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  - …when computing a **rolling** schedule
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    - Such as multicommodity flow and assignment relaxations
  - Relaxation must grow only linearly
    - Such as time window relaxation

Conclusions

- LBBD can scale up despite sequence-dependent costs…
  - …when computing a **rolling** schedule
    - Time window relaxation is tight enough
- Relaxation is key
  - Relaxation that grows quadratically is too large
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  - Relaxation must grow only linearly
    - Such as time window relaxation
- LBBD superior to branch & check

References
Applications of Logic-Based Benders Decomposition

Benders decomposition [7] was introduced in 1962 to solve applications that become linear programming (LP) problems when certain *search variables* are fixed. "Generalized" Benders decomposition, proposed by Geoffrion in 1972 [25], extended the method to nonlinear programming subproblems.

Logic-based Benders decomposition (LBBD) allows the subproblem to be any optimization problem. LBBD was introduced in [32], formally developed in 2000 [33], and tested computationally in [39]. *Branch and check* is introduced in [33] and tested computationally in [69]. *Combinatorial Benders cuts* for mixed integer programming are proposed in [18].

One of the first applications [43] was a planning and scheduling problem. Updated experiments [17] show that LBBD is orders of magnitude faster than state-of-the-art MIP, with the advantage over CP even greater). Similar results have been obtained for various planning and scheduling problems [15, 21, 30, 34, 35, 37, 71].

Other successful applications of LBBD include steel production scheduling [29], inventory management [74], concrete delivery [44], shop scheduling [3, 13, 27, 28, 59], hospital scheduling [57], batch scheduling in chemical plants [49, 70], computer processor scheduling [8, 9, 12, 22, 31, 46, 47, 48, 58, 62], logic circuit verification [40], shift scheduling [5, 60], lock scheduling [73], facility location [23, 66], space packing [20, 50], vehicle routing [19, 51, 53, 56, 61, 75], bicycle sharing [45], network design [24, 52, 63, 65], home health care [16], service restoration [26], supply chain management [68], food distribution [64], queuing design and control [67], optimal control of dynamical systems [11], propositional satisfiability [1], quadratic programming [2, 41, 42], chordal completion [10], and sports scheduling [14, 54, 55, 72]. LBBD is compared with branch and check in [6]. It is implemented in the general-purpose solver SIMPL [76].

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