Relaxation Based on Multivalued Decision Diagrams

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Relaxation in CP and MIP

- Relaxation in CP
  - Domain store (set of variable domains)
  - Fast and versatile
  - Suitable for propagation
  - But very weak
  - Imbalance of search and inference
  - Must process many nodes quickly
- Relaxation in MIP
  - Generally LP
  - Strong, rich structure
  - Can be strengthened with cutting planes
  - But requires linearization
  - May be weak for combinatorial or nonconvex constraints
  - Unsuitable for propagation

A New Relaxation

- **Multivalued decision diagram** (MDD).
  - Discrete structure
  - A compact representation of branching tree
  - Optimization = shortest path computation
  - Domain store is a special case
- MDD relaxation.
  - An approximate representation of constraints
  - Improve the relaxation with **node splitting , edge domain filtering**, and **shortest path reasoning**.
- Advantages.
  - Like domain store, suitable for propagation.
  - Like LP, provides strong relaxation
  - Rebalances search and inference
  - Natural for discrete problems

Objectives

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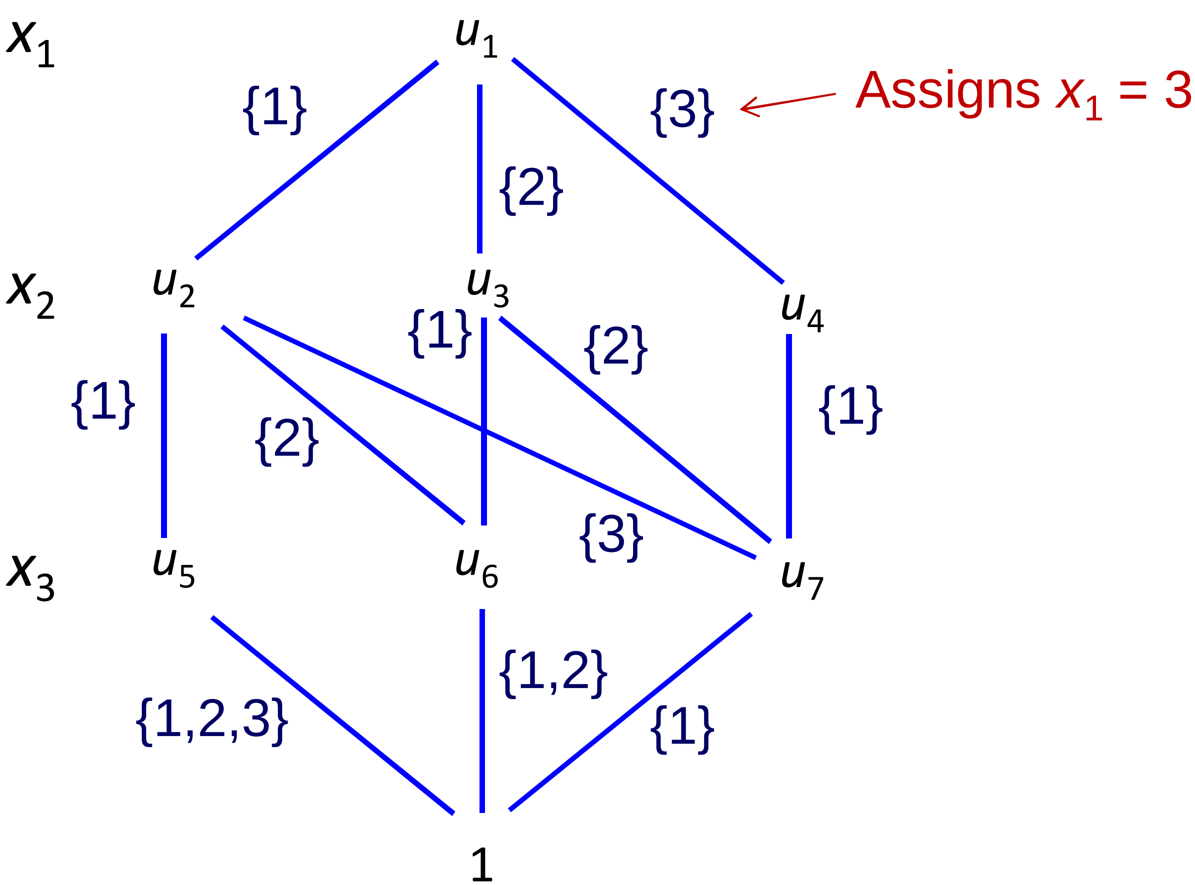
- Show that MDD-based propagation can follow a standard pattern.
  - And yet can be **specialized** to each type of constraint to exploit structure.
  - Implement a general **MDD-based constraint solver** based on this pattern.

- Test the MDD-based solver on a class of constraints.
  - Multiple **`among`** constraints.

MDD basics

MDD representing
$x_1 + x_2 + x_3 \leq 5$
$x_j \in \{1,2,3\}$



MDD basics

MDD representing

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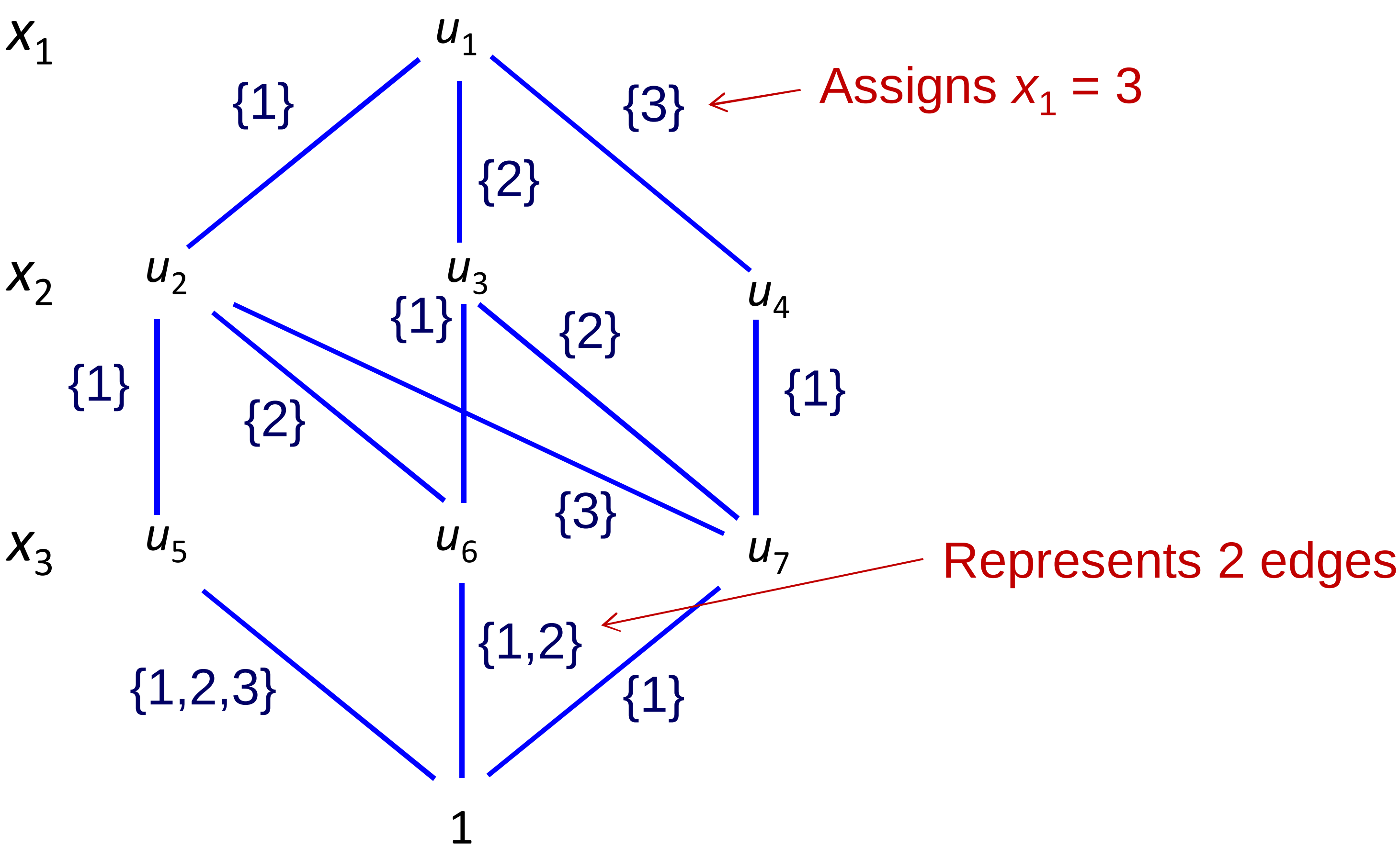

MDD basics

MDD representing
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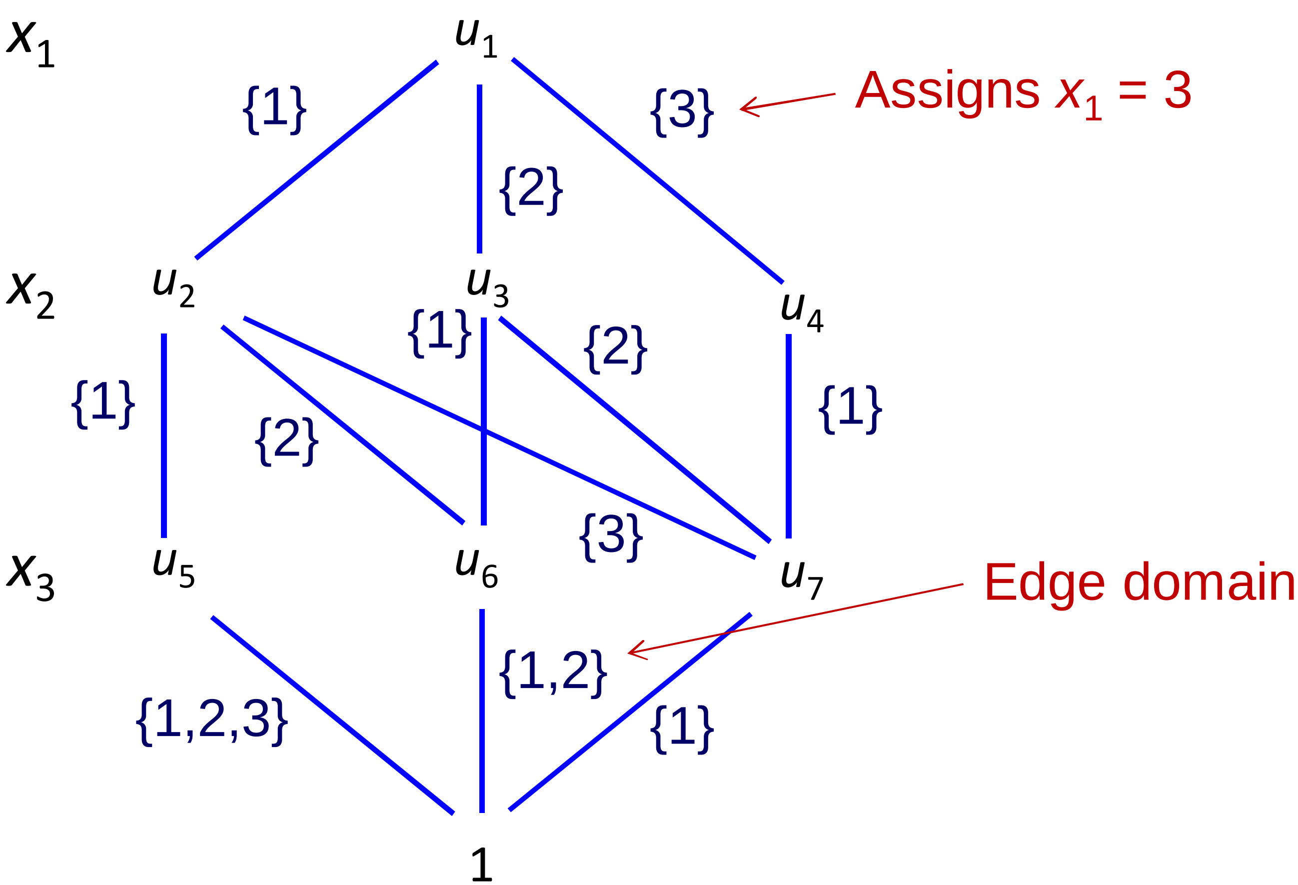

MDD basics

MDD representing
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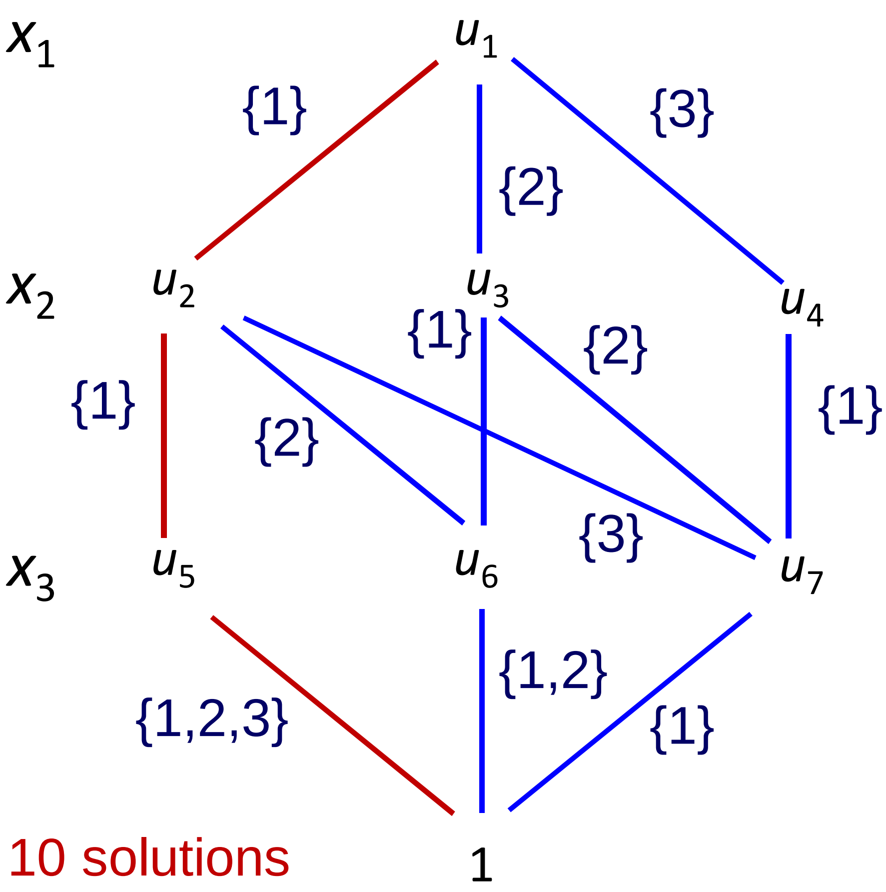


10 solutions total

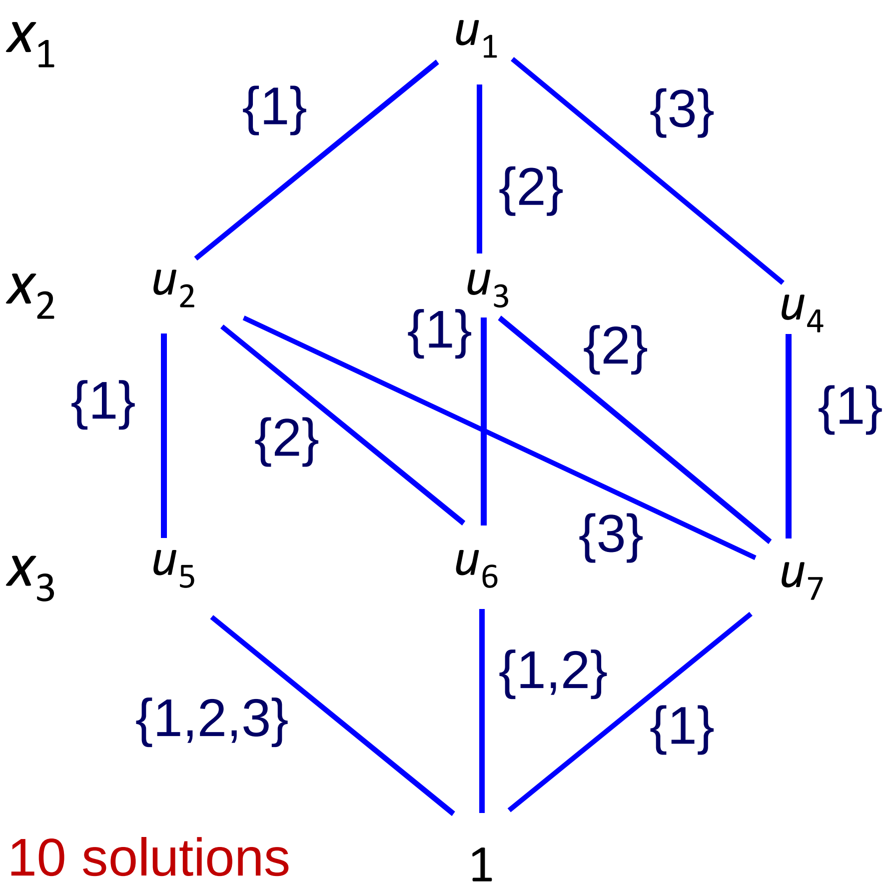
Each path from top to bottom represents a set of solutions

$\{1\}\times\{1\}\times\{1,2,3\} =$

$\{(1,1,1), (1,1,2), (1,1,3)\}$

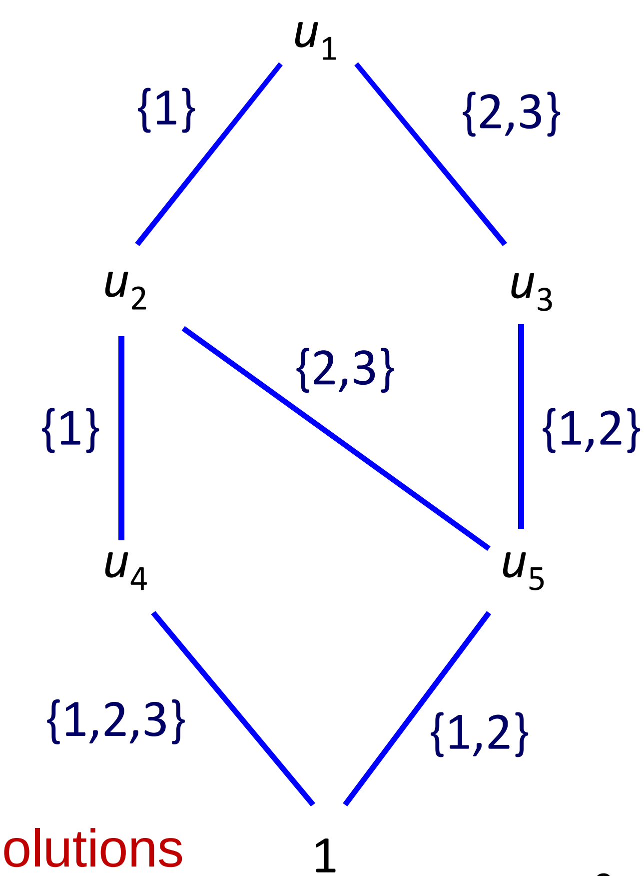
MDD relaxation

MDD representing
$x_1 + x_2 + x_3 \leq 5$
$x_j \in \{1,2,3\}$



10 solutions

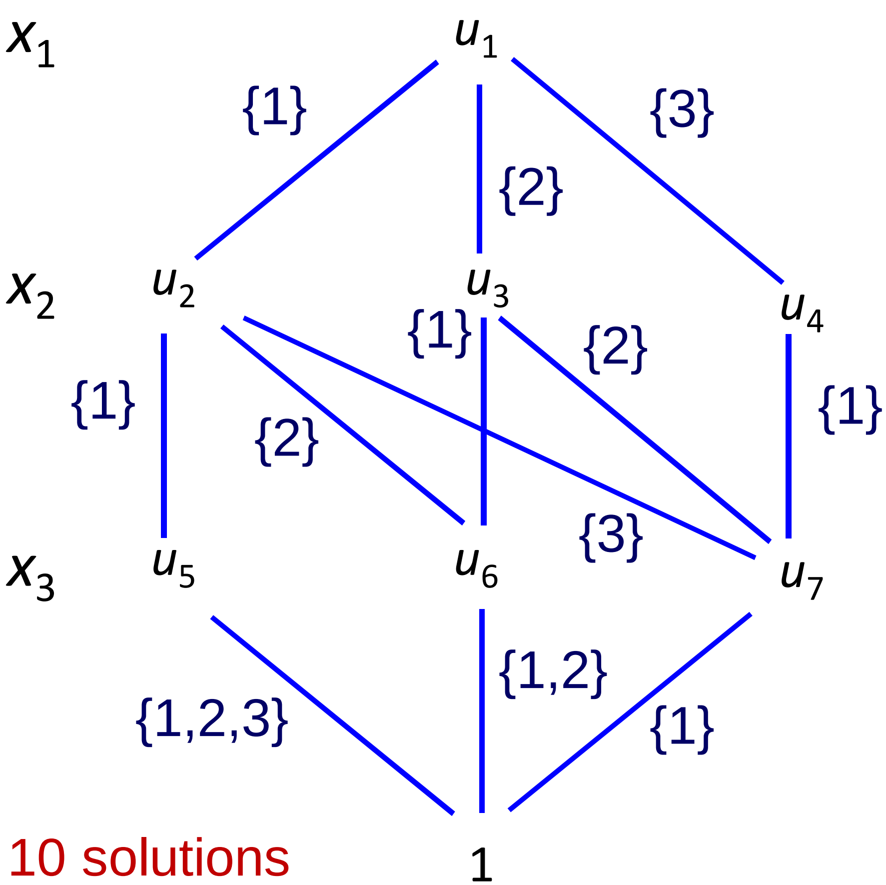
A **relaxed** MDD (width 2) representing a superset of the solutions



15 solutions

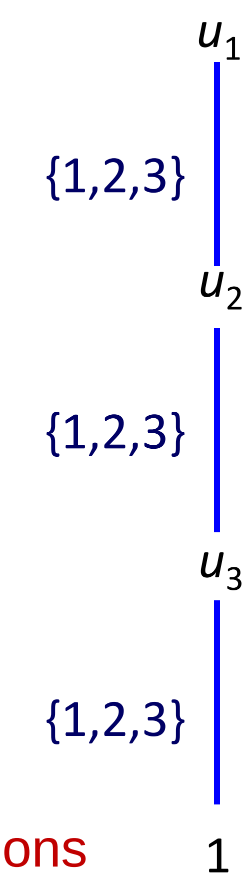
MDD relaxation

MDD representing
$x_1 + x_2 + x_3 \leq 5$
$x_j \in \{1,2,3\}$



10 solutions

Conventional **domain store**
(width 1 relaxation)



27 solutions

Among constraint

- We will use among constraint as a running example.
- At least L and at most U variables in set X take a value in set S:

$$\text{among}(X, S, L, U)$$

- Applications in sequencing and scheduling, etc.

Among constraint

- Example: Employee scheduling

x_t = shift assigned on day t.

S = {M,A,N} = {shifts}

Between 2 and 4 afternoon/night shifts in a week:

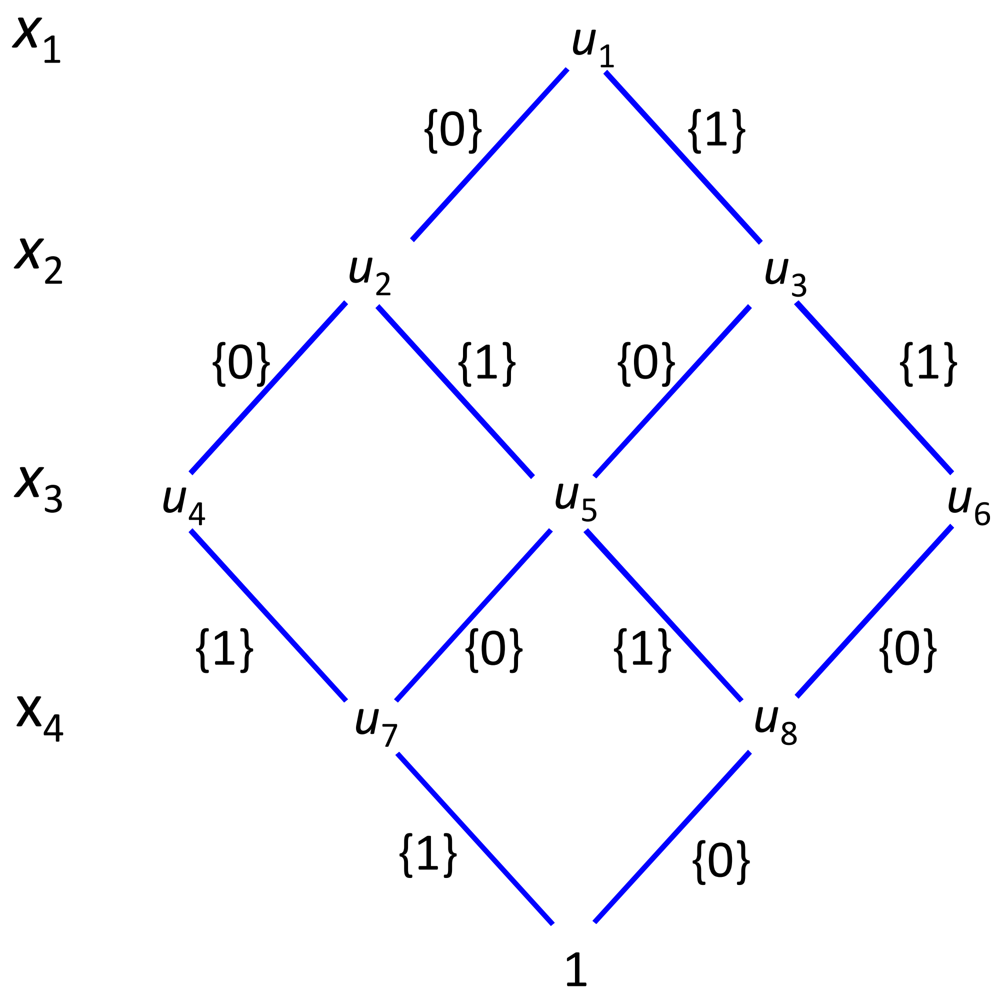
Among({x_1, …, x_7}, {A,N}, 2, 4)

t	1	2	3	4	5	6	7
x_t	M	**A**	**N**	M	M	**A**	M

WOLOG, suppose $x_t \in \{0,1\}$, $S = \{1\}$

t	1	2	3	4	5	6	7
x_t	0	**1**	**1**	0	0	**1**	0

Among constraint



Exact MDD for among($\{x_1,x_2,x_3,x_4\}$, {1}, 2, 2)

MDD-based constraint programming

- Maintain limited-width MDD
  - Serves as relaxation
  - Typically start with width 1 (domain store)
  - Update MDD by propagating constraints sequentially

- Constraint Propagation
  - Edge filtering: Remove provably inconsistent edges
  - Node refinement: Split nodes to separate edge information

- Search
  - As in classical CP, but may now be guided by MDD

- Easily extended to optimization
  - Shortest path in MDD

Example

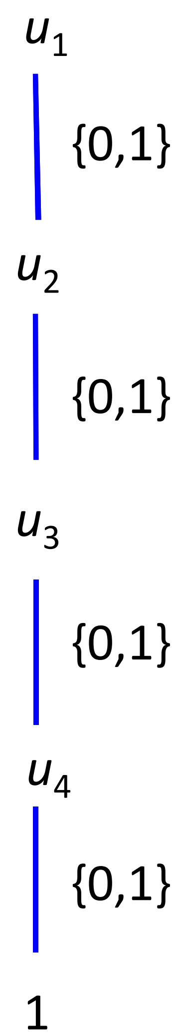
- We will propagate

$$\texttt{among}(\{x_1,x_2,x_3,x_4\}, \{1\}, 2\ ,2)$$

through a BDD of maximum width 3

  - All path lengths must be = 2.

Example



Initially, the MDD has width 1
(traditional domain store)

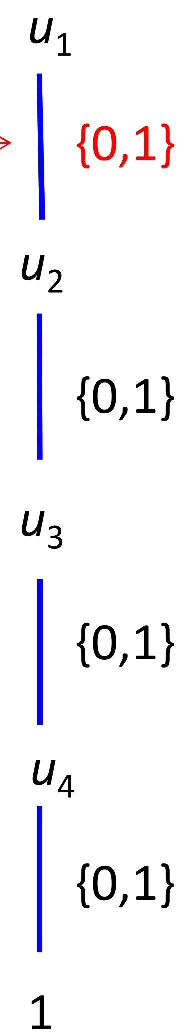
among($\{x_1,x_2,x_3,x_4\}$, {1}, 2 ,2)

Example

Try to filter edge domain

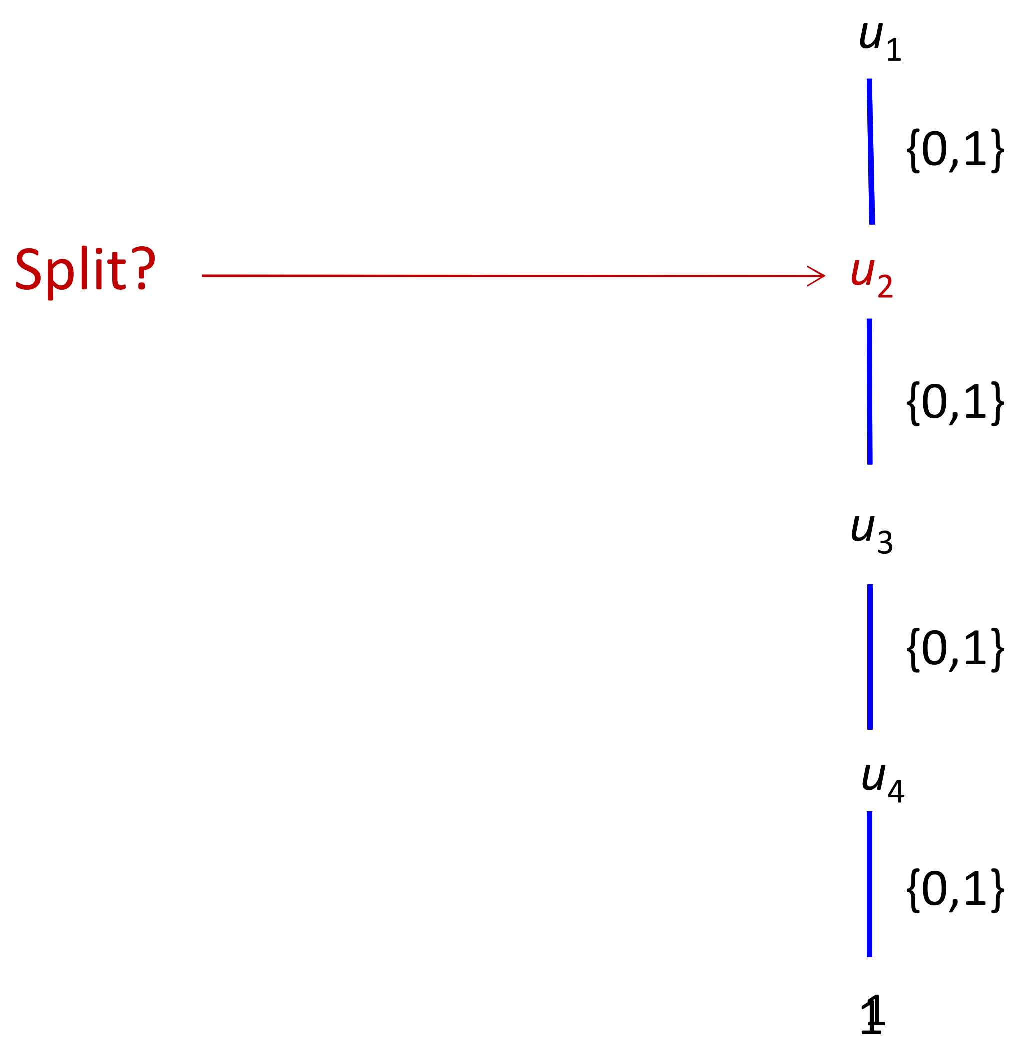
No filtering possible.

Initially, the MDD has width 1
(traditional domain store)



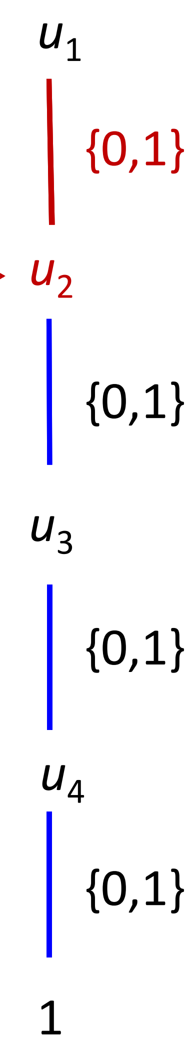
among($\{x_1,x_2,x_3,x_4\}$, {1}, 2 ,2)

Example



among({x_1,x_2,x_3,x_4}, {1}, 2 ,2)

Example



Split?

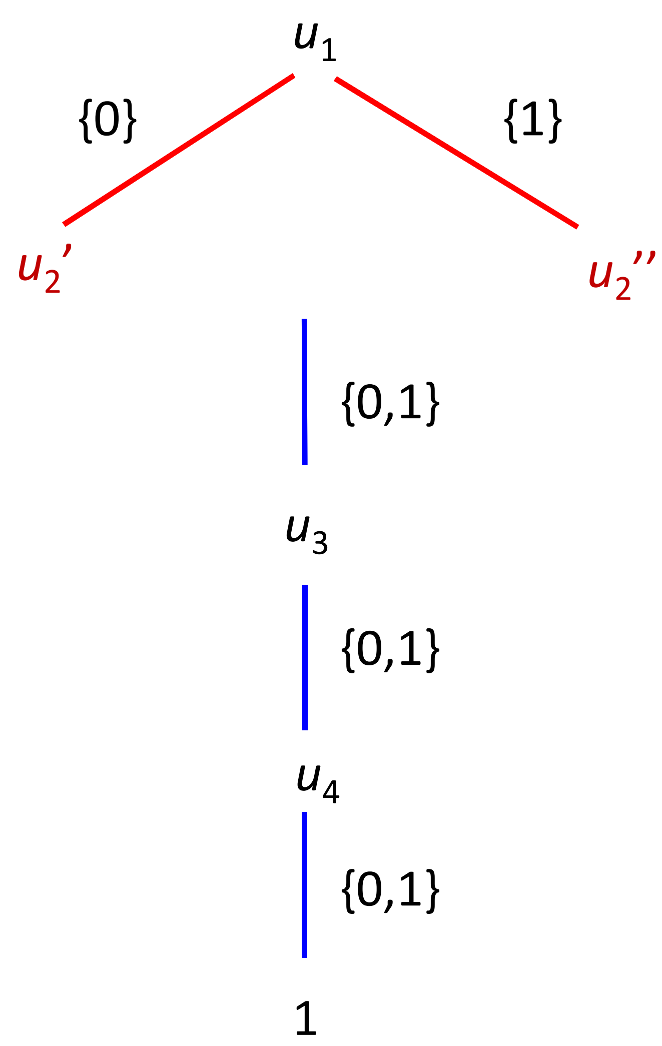
The 2 incoming edges (length 0 and 1) are not **equivalent**.

The partial assignments
$x_1 = 0$, $x_1 = 1$
don't have the same set of possible completions.

among($\{x_1,x_2,x_3,x_4\}$, {1}, 2 ,2)

Example

So we split.
This doesn't exceed max width of 3.

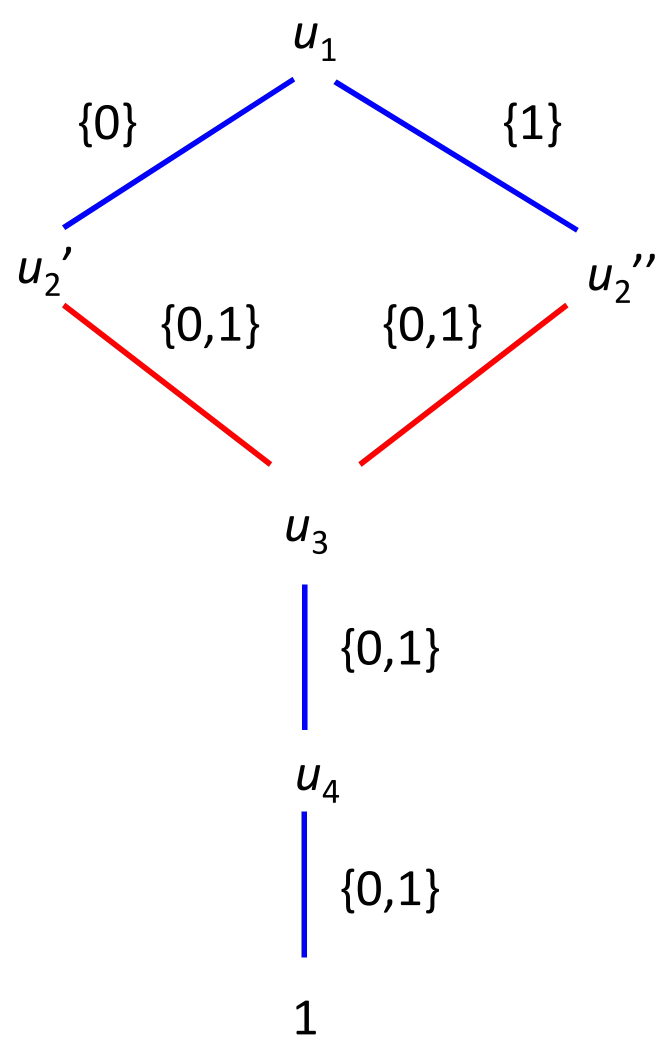


among({x_1,x_2,x_3,x_4}, {1}, 2 ,2)

Example

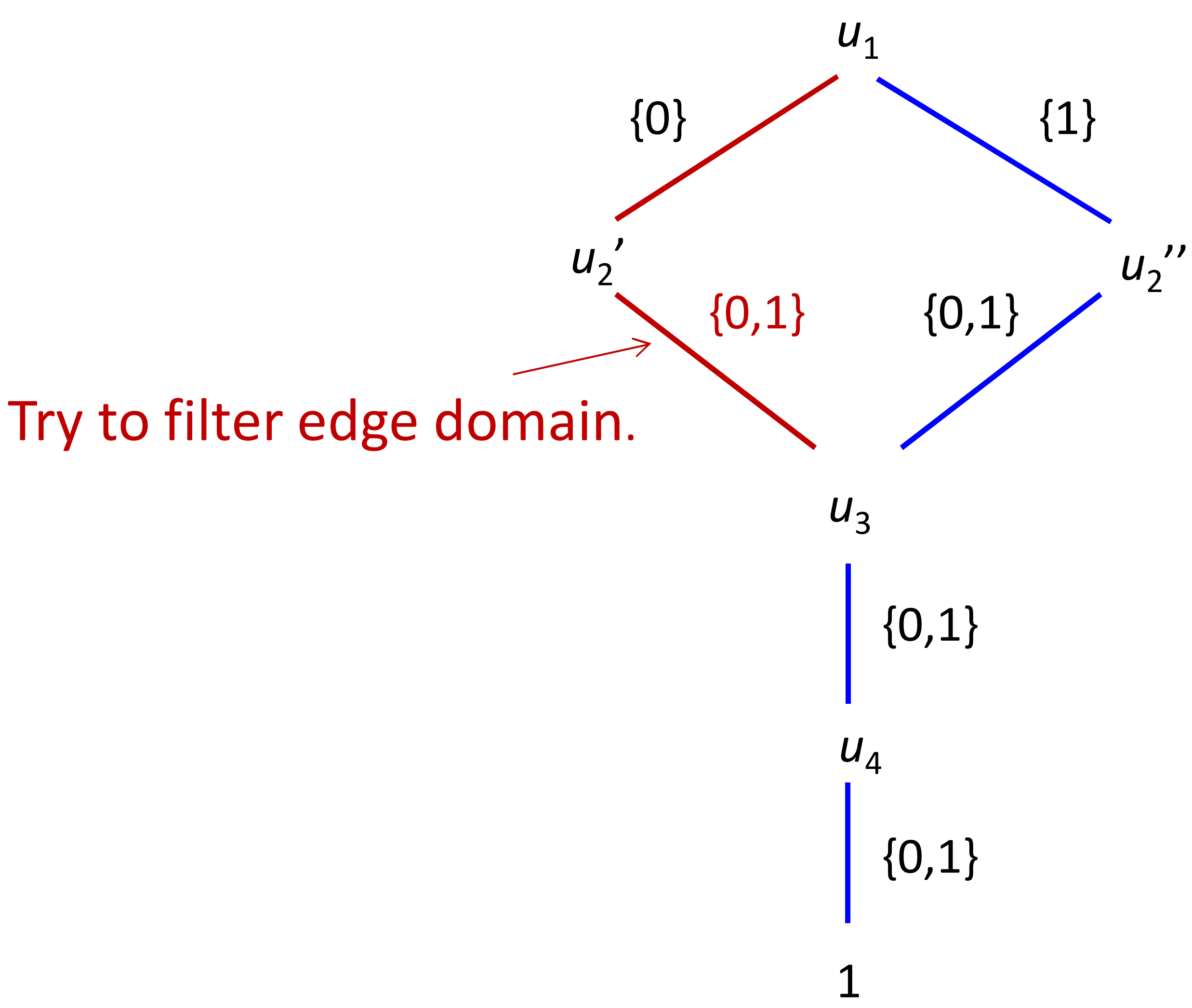
So we split.
This doesn't exceed max width of 3.

Duplicate outgoing Edges.



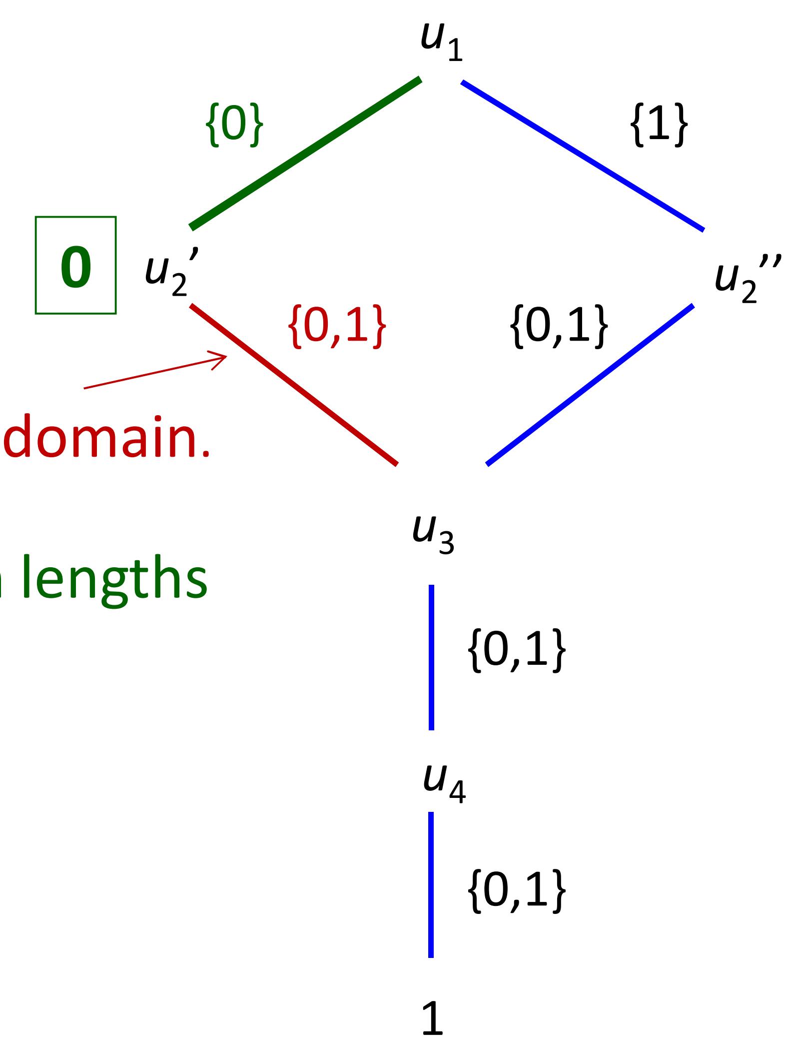
among($\{x_1,x_2,x_3,x_4\}$, {1}, 2 ,2)

Example



among($\{x_1,x_2,x_3,x_4\}$, {1}, 2 ,2)

Example

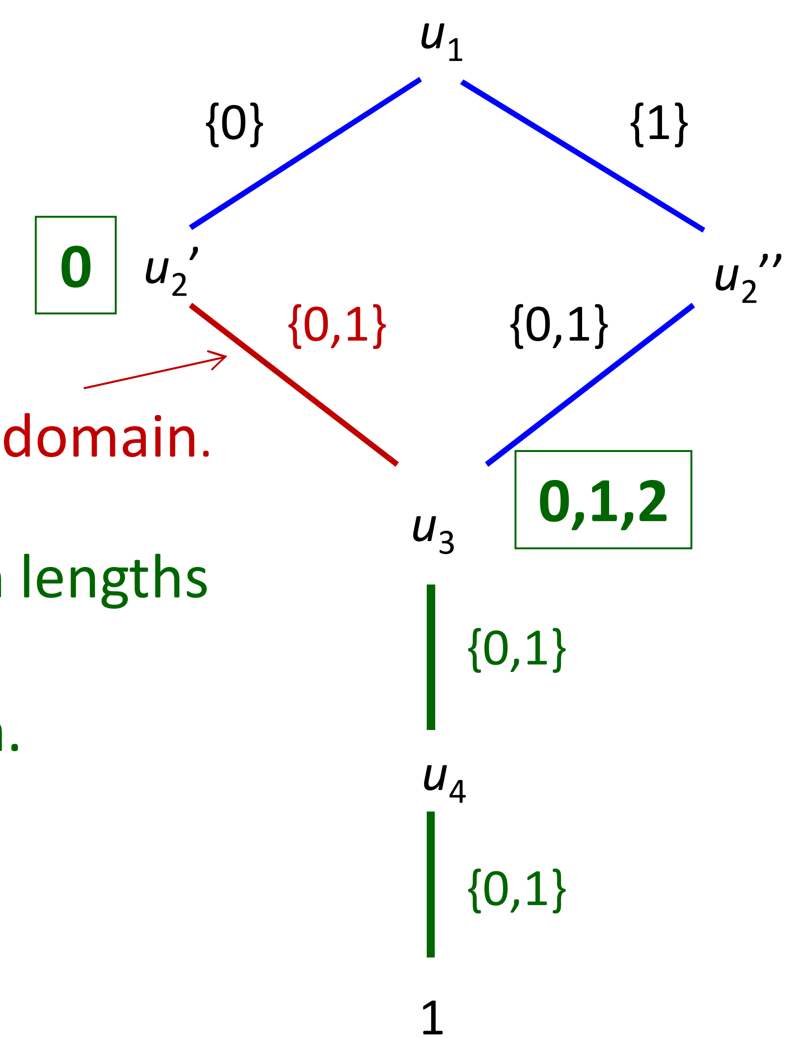


Try to filter edge domain.

Compute all path lengths from top…

among($\{x_1,x_2,x_3,x_4\}$, {1}, 2 ,2)

Example

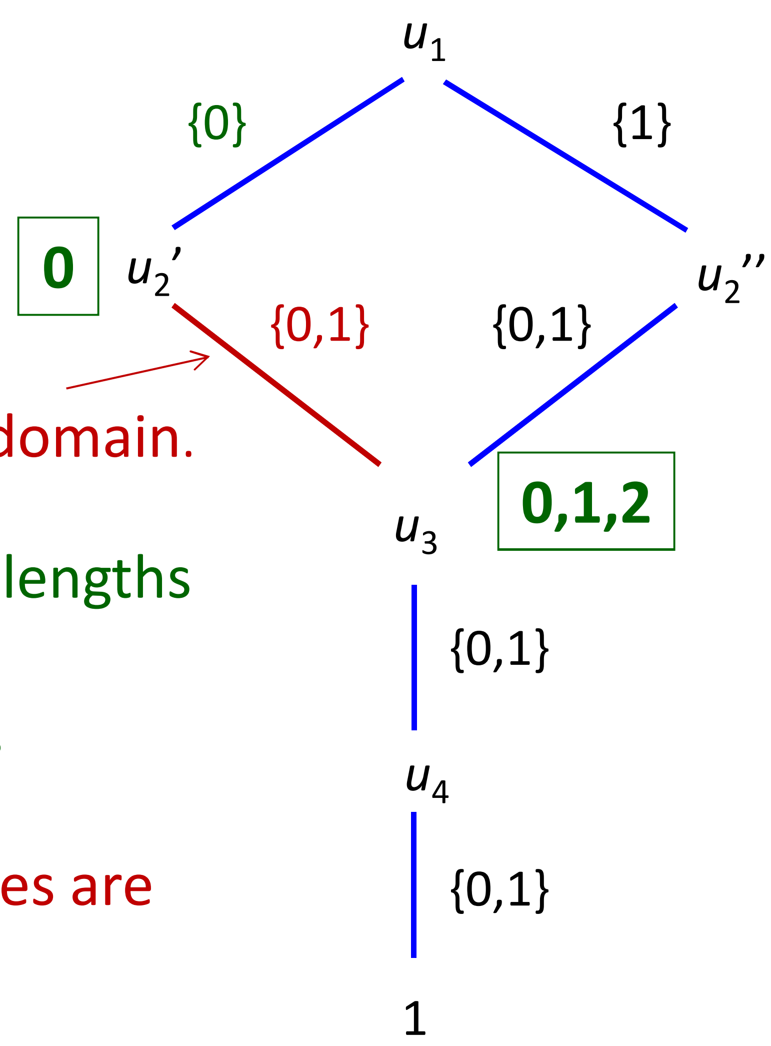


Try to filter edge domain.

Compute all path lengths
from top…
and from bottom.

among({x_1,x_2,x_3,x_4}, {1}, 2 ,2)

Example



Try to filter edge domain.

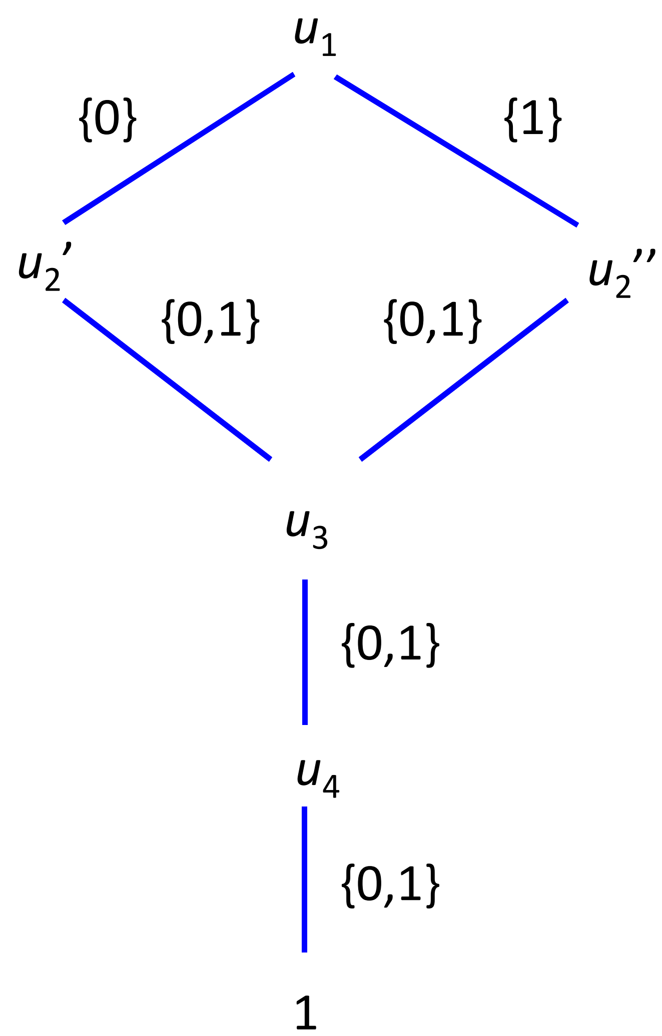
Compute all path lengths
from top…
and from bottom.

Both domain values are
consistent.

among($\{x_1,x_2,x_3,x_4\}$, {1}, 2 ,2)

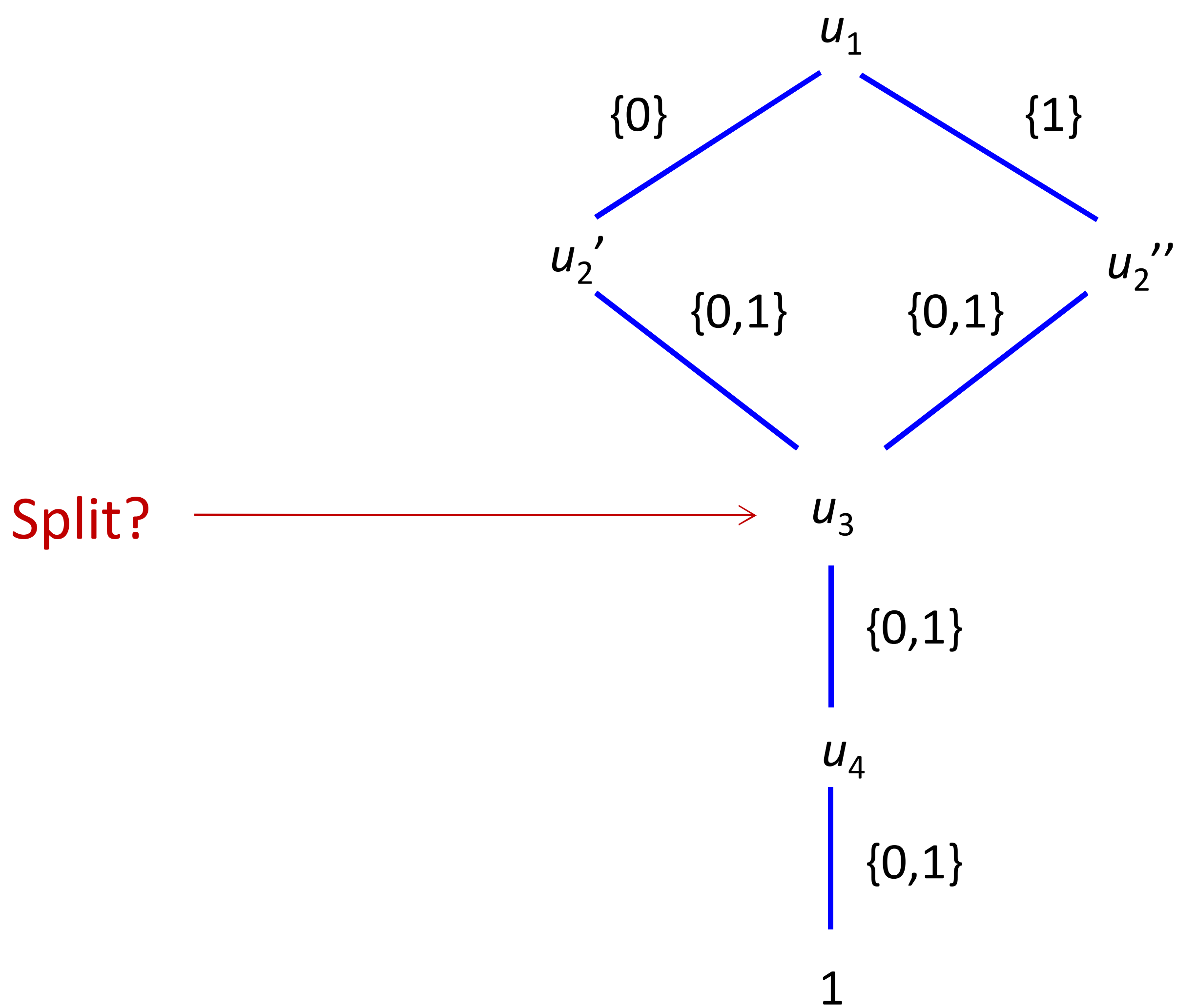
Example

No filtering possible for x_2.



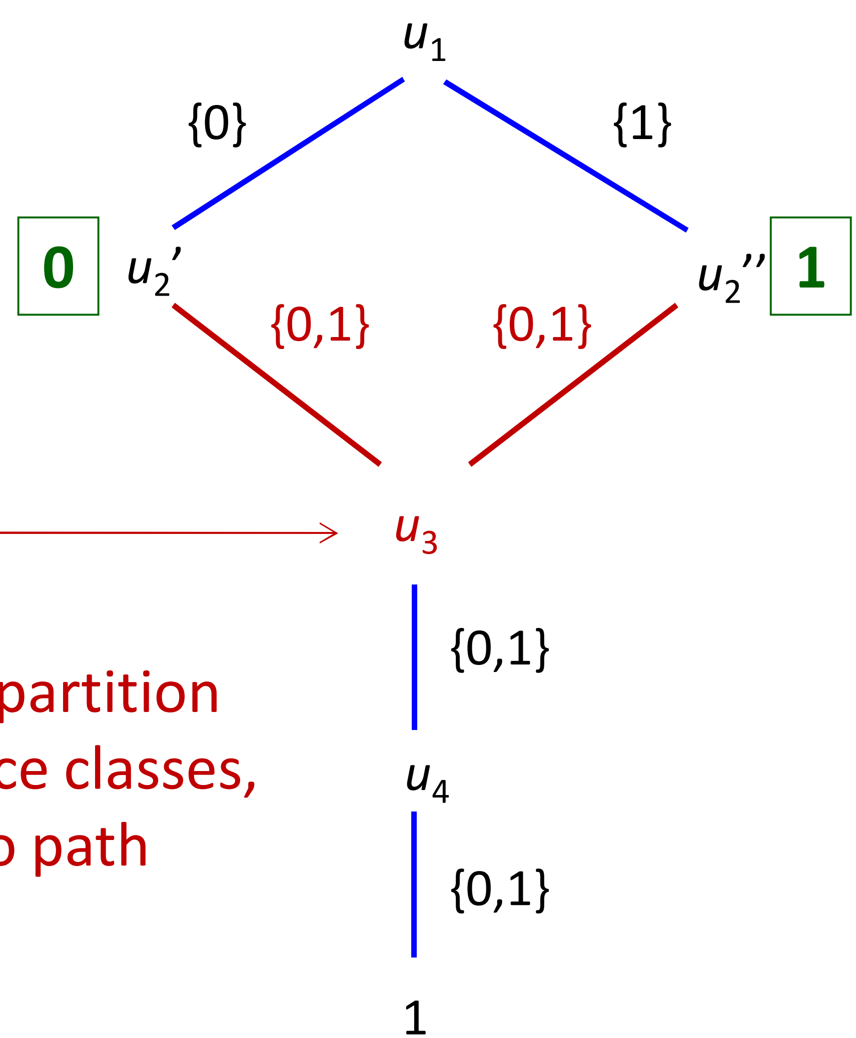
among($\{x_1,x_2,x_3,x_4\}$, {1}, 2 ,2)

Example



among($\{x_1,x_2,x_3,x_4\}$, {1}, 2 ,2)

Example

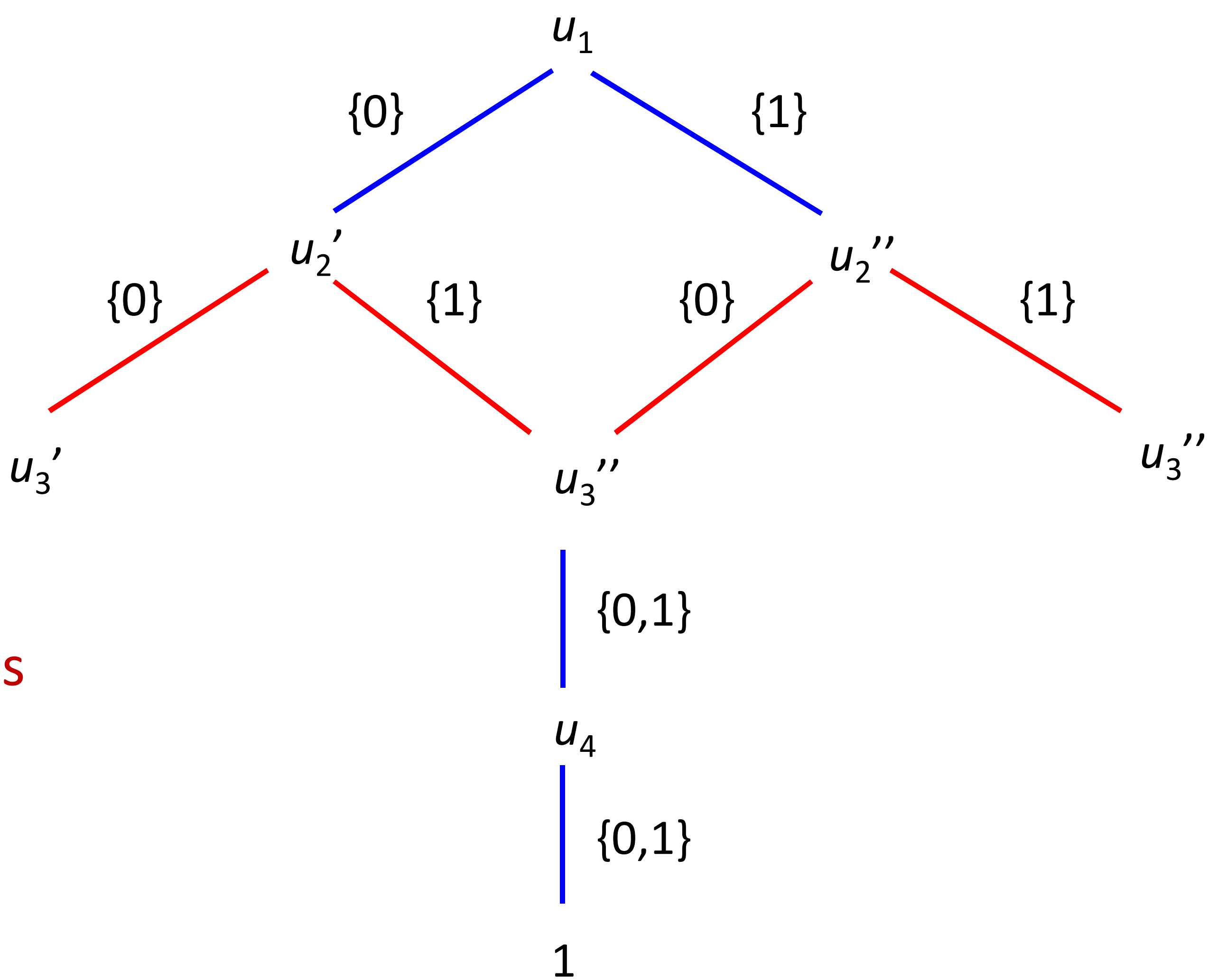


Split?

Incoming edges partition into 3 equivalence classes, corresponding to path lengths 0, 1, 2

among($\{x_1,x_2,x_3,x_4\}$, {1}, 2 ,2)

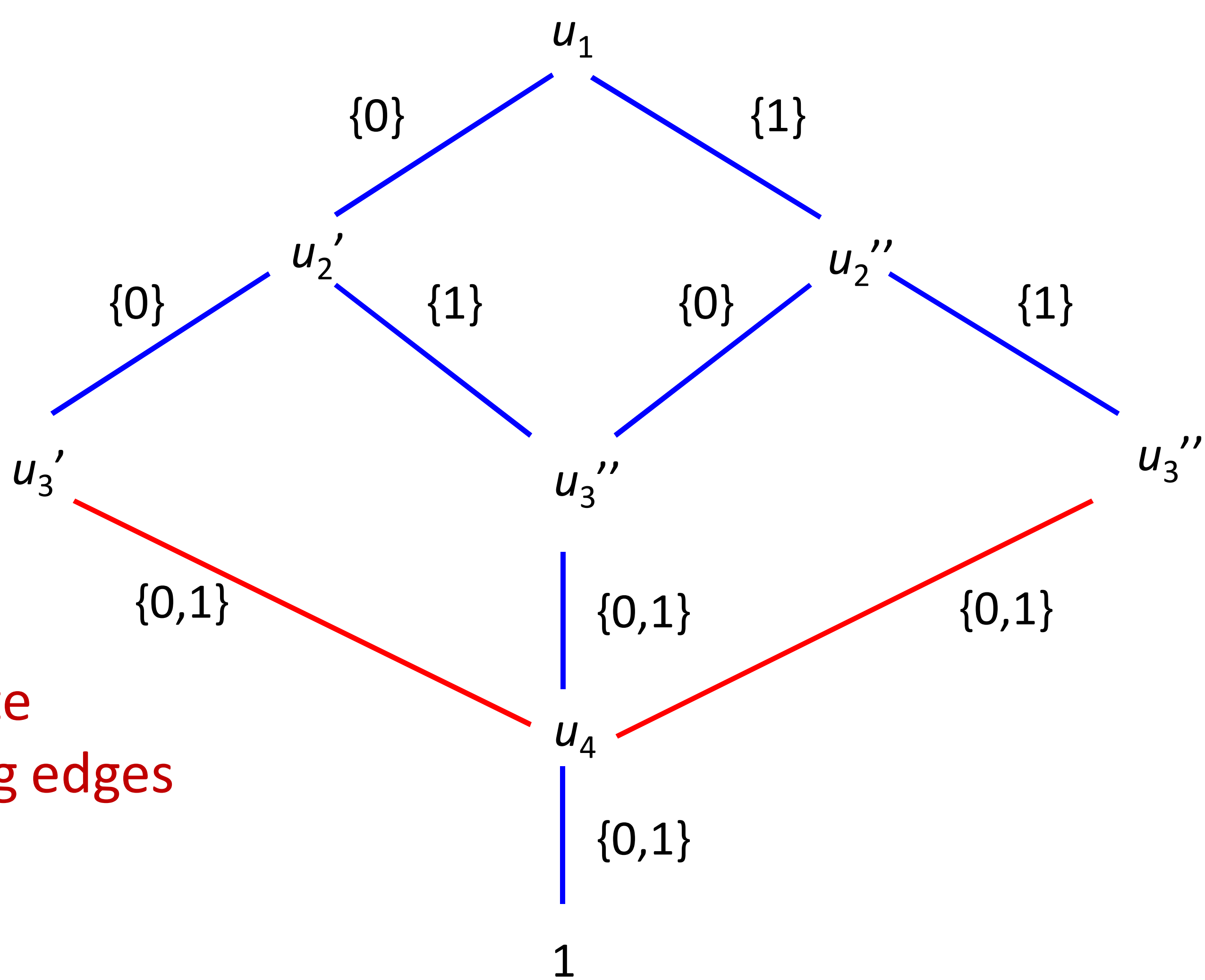
Example



So we split
into 3 nodes

among($\{x_1,x_2,x_3,x_4\}$, {1}, 2 ,2)

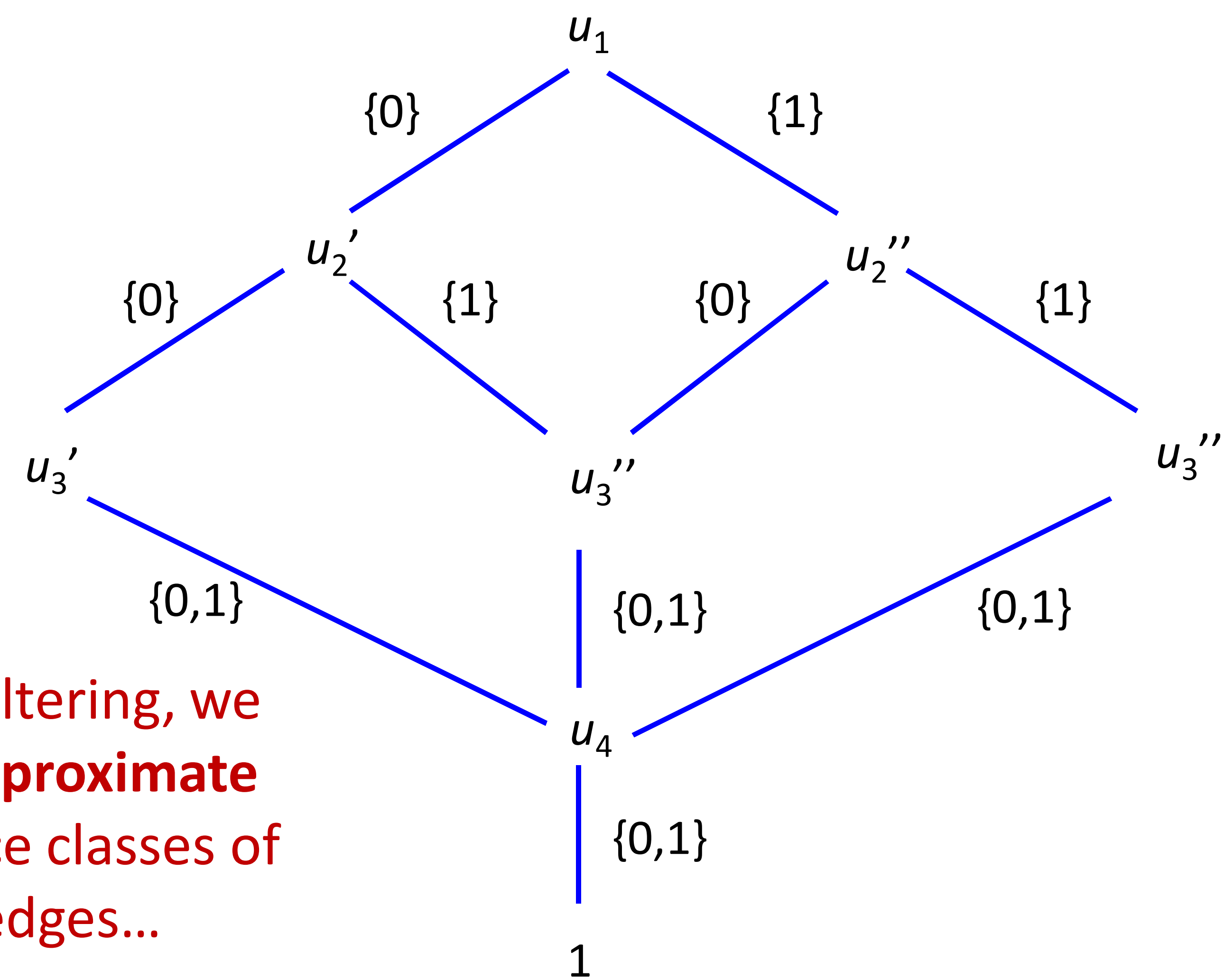
Example



Duplicate
outgoing edges

among($\{x_1,x_2,x_3,x_4\}$, {1}, 2 ,2)

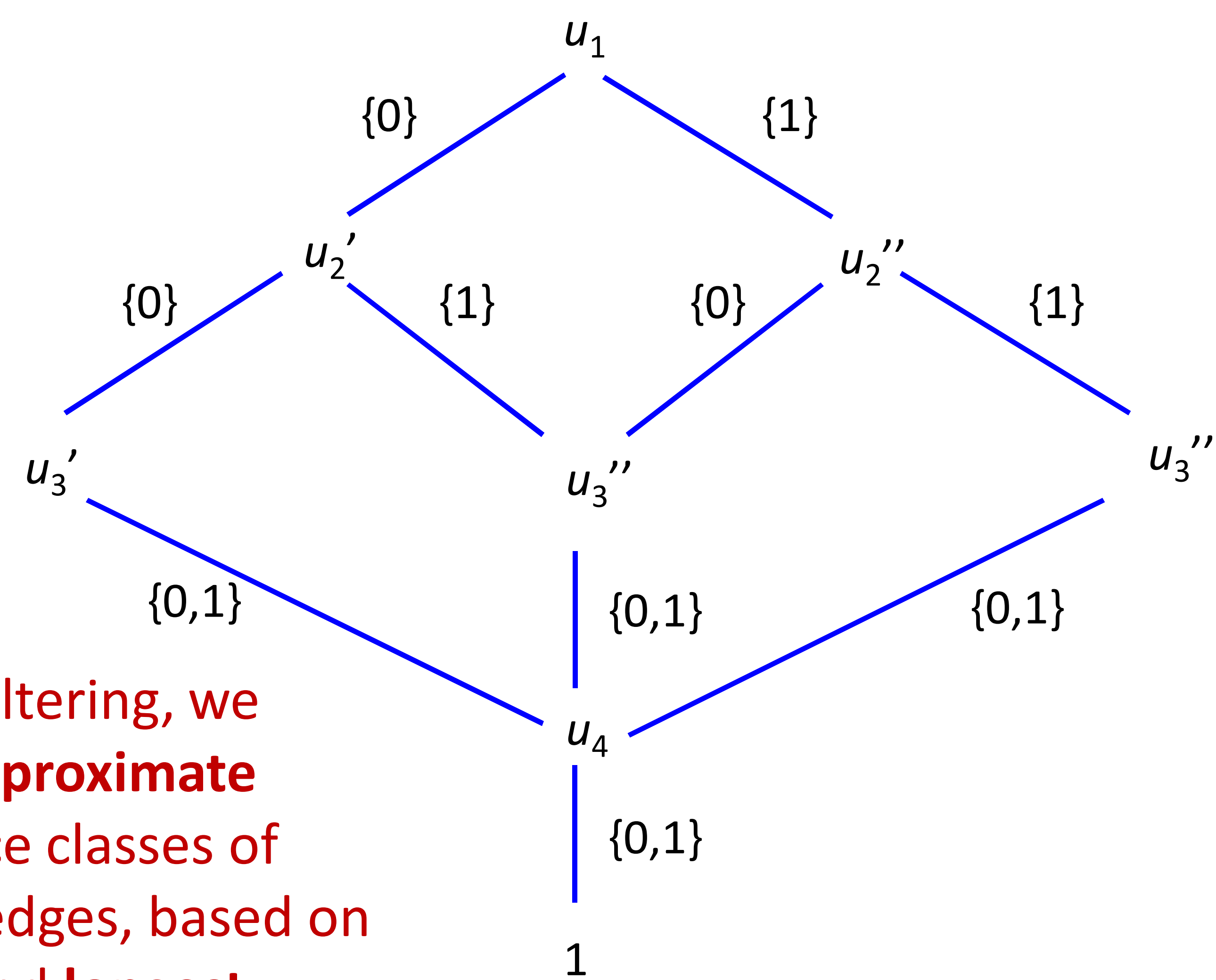
Example



To speed filtering, we identify **approximate** equivalence classes of incoming edges…

among($\{x_1, x_2, x_3, x_4\}$, {1}, 2 ,2)

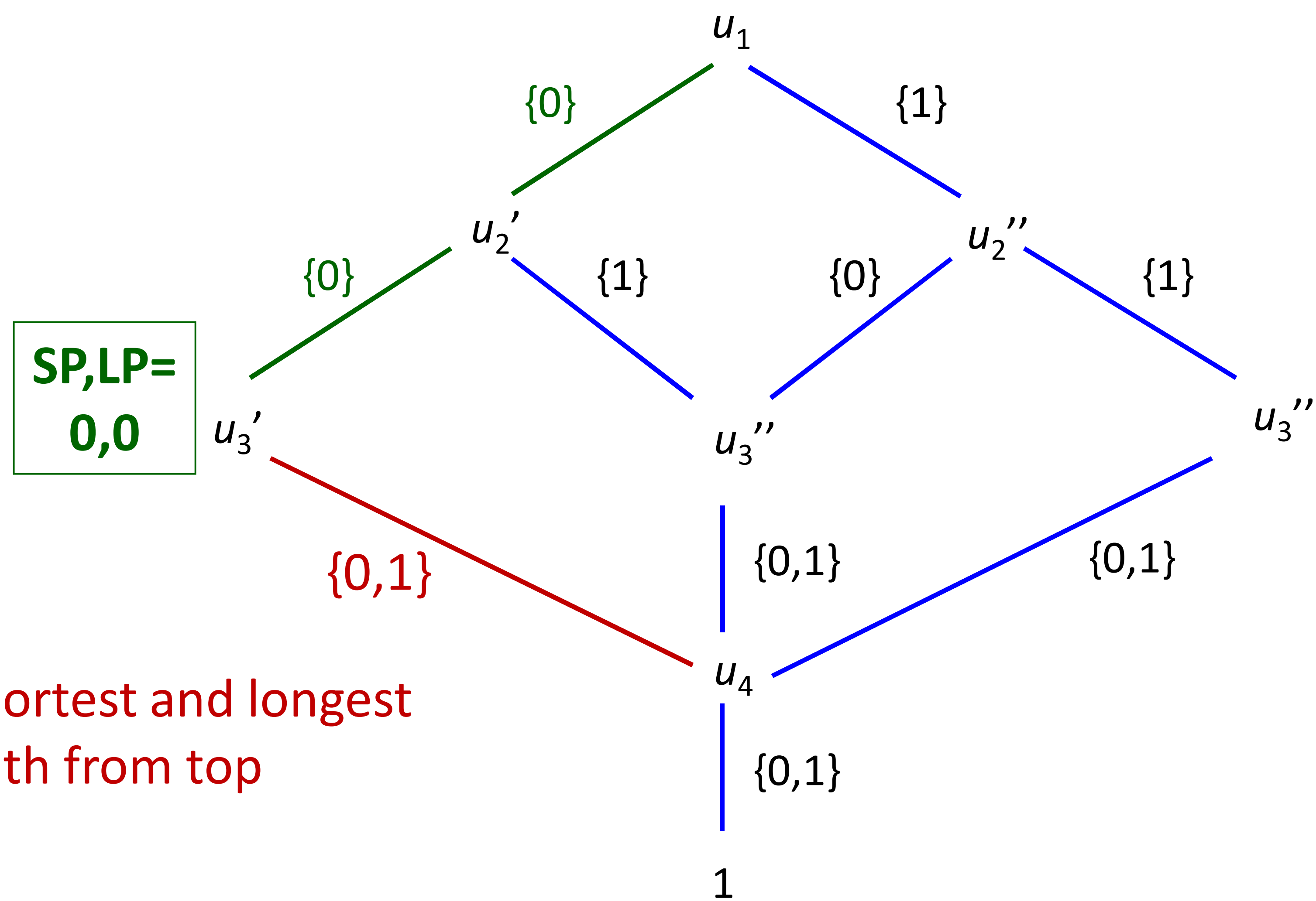
Example



To speed filtering, we identify **approximate** equivalence classes of incoming edges, based on **shortest** and **longest** **paths**.

among($\{x_1,x_2,x_3,x_4\}$, {1}, 2 ,2)

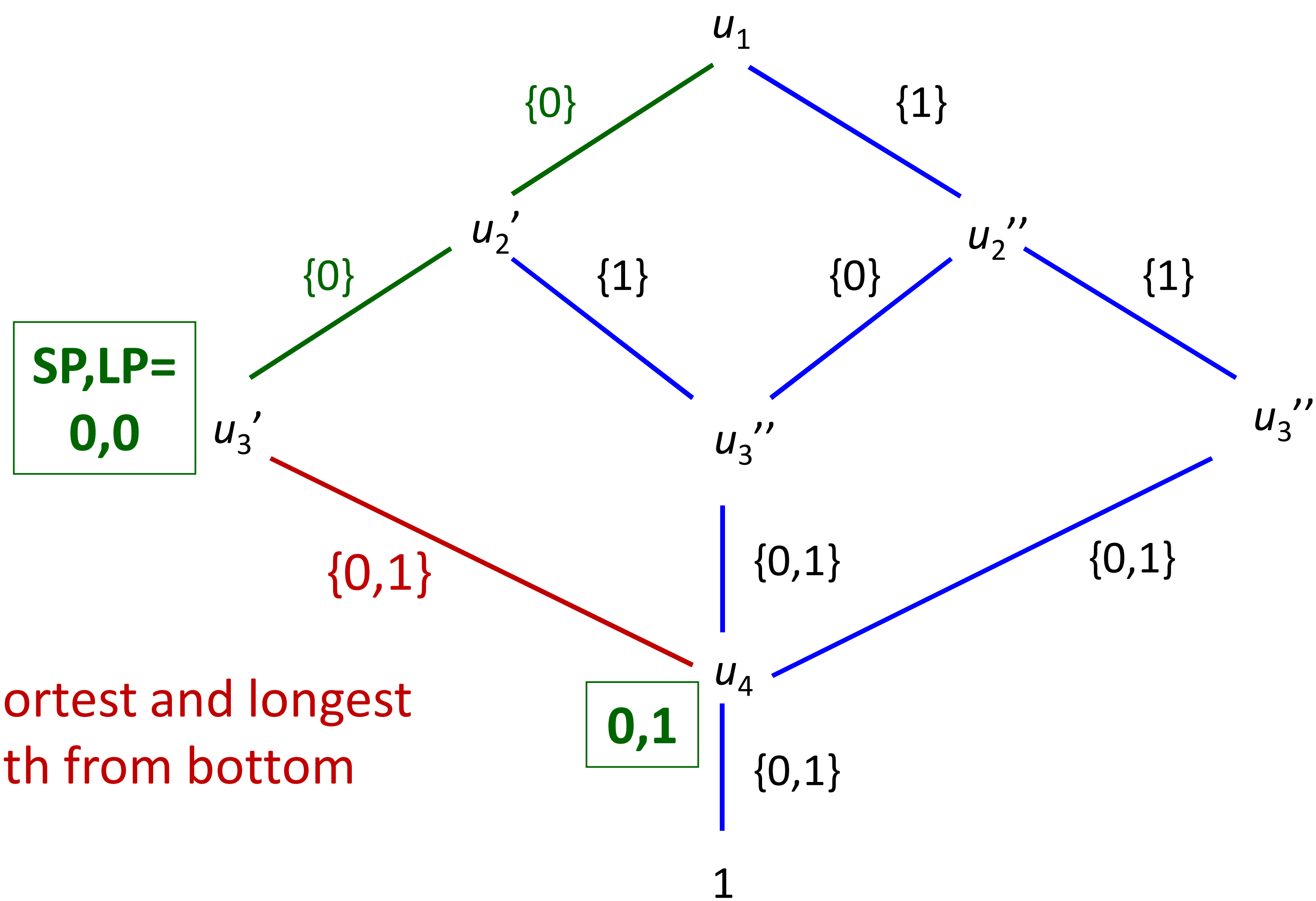
Example



Shortest and longest path from top

among($\{x_1,x_2,x_3,x_4\}$, {1}, 2 ,2)

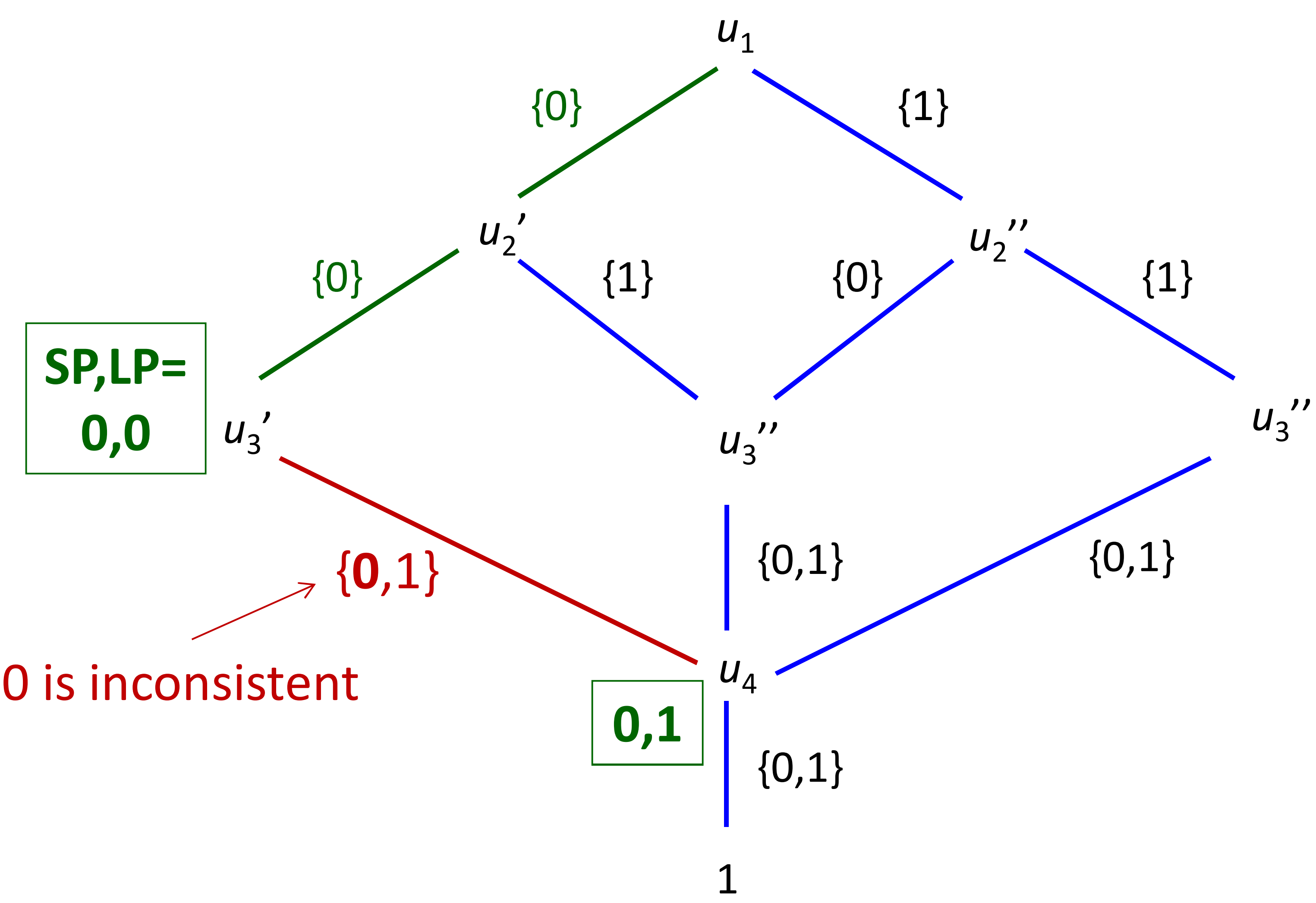
Example



Shortest and longest
path from bottom

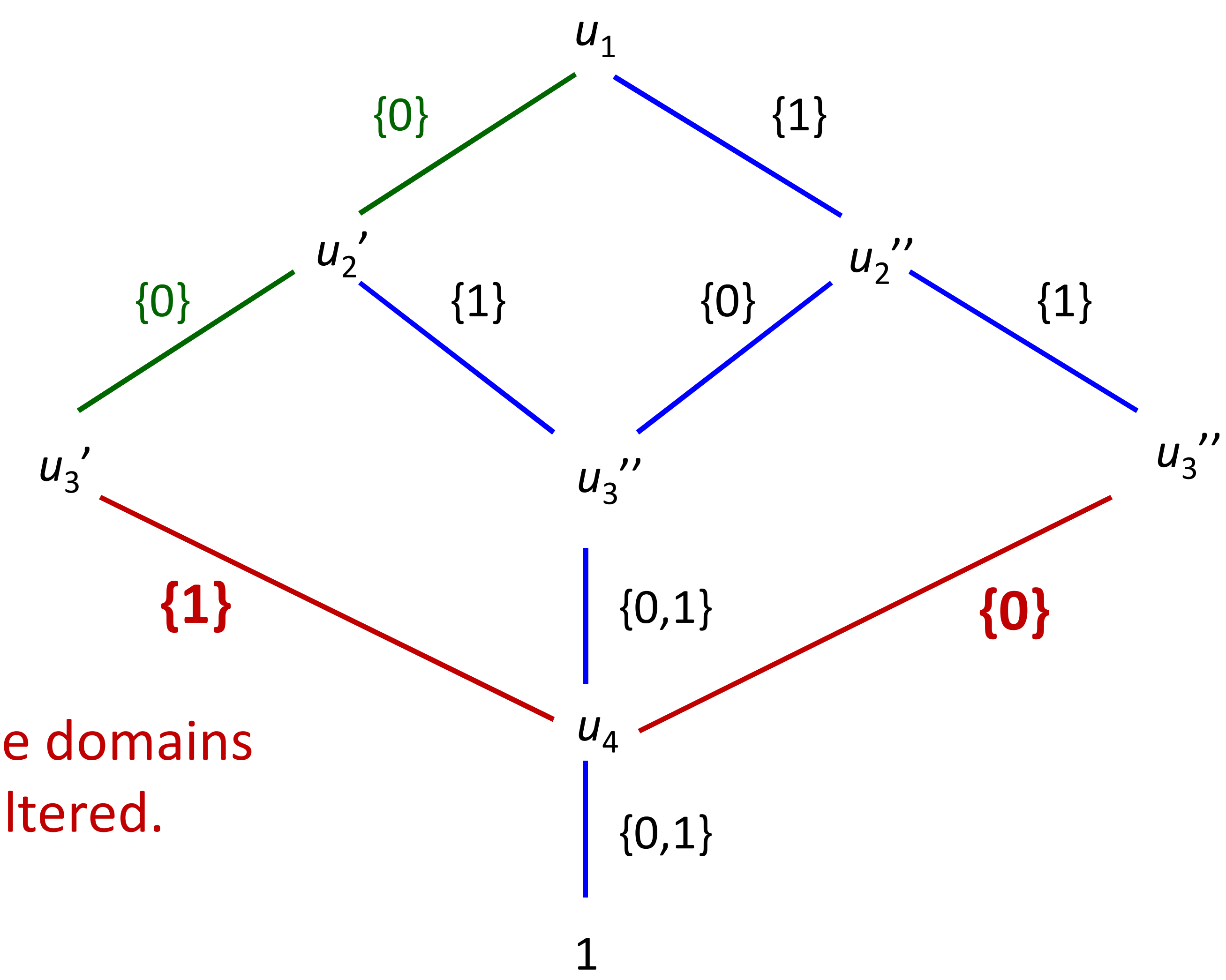
among($\{x_1,x_2,x_3,x_4\}$, {1}, 2 ,2)

Example



among({x_1,x_2,x_3,x_4}, {1}, 2 ,2)

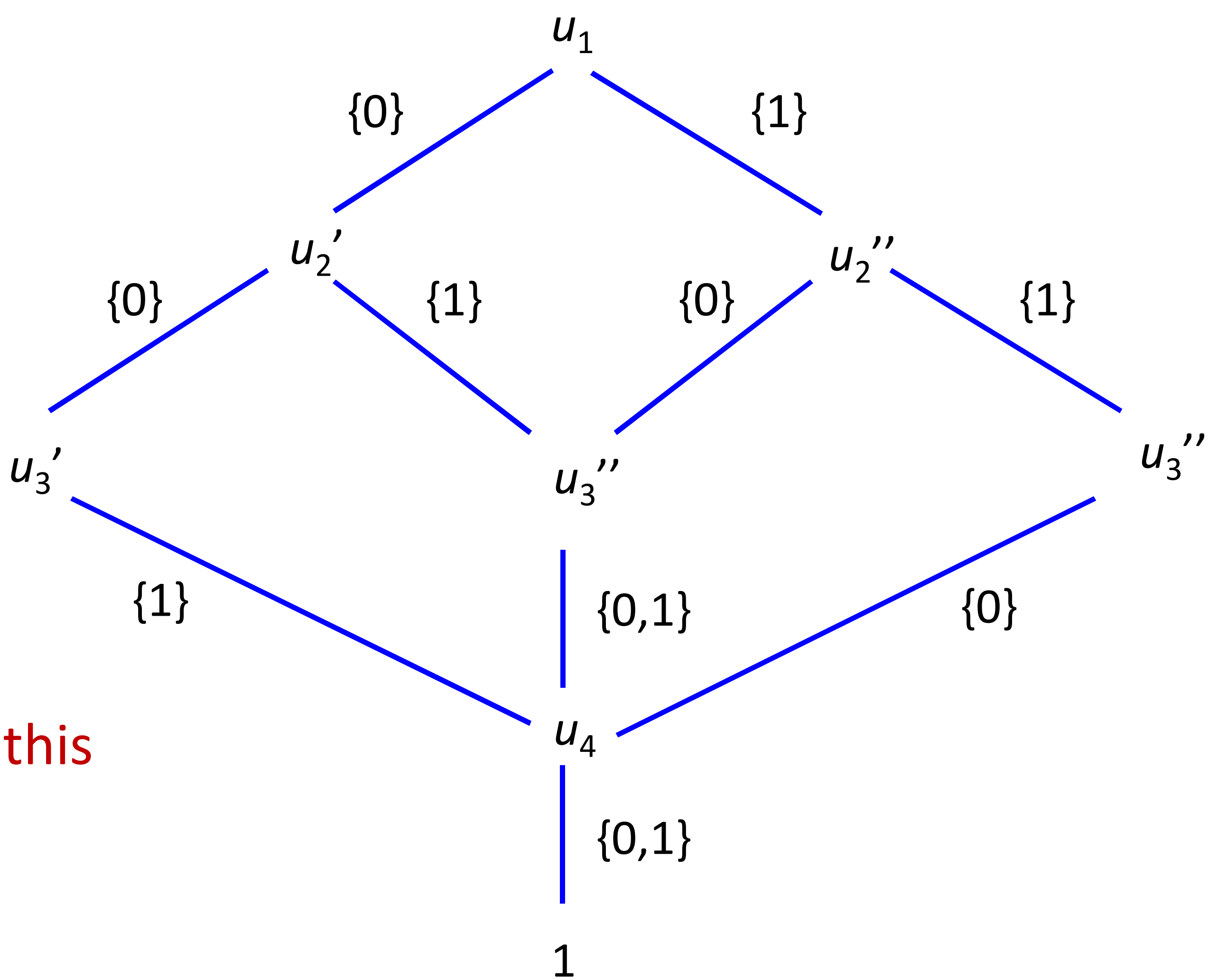
Example



Two edge domains can be filtered.

among($\{x_1,x_2,x_3,x_4\}$, {1}, 2 ,2)

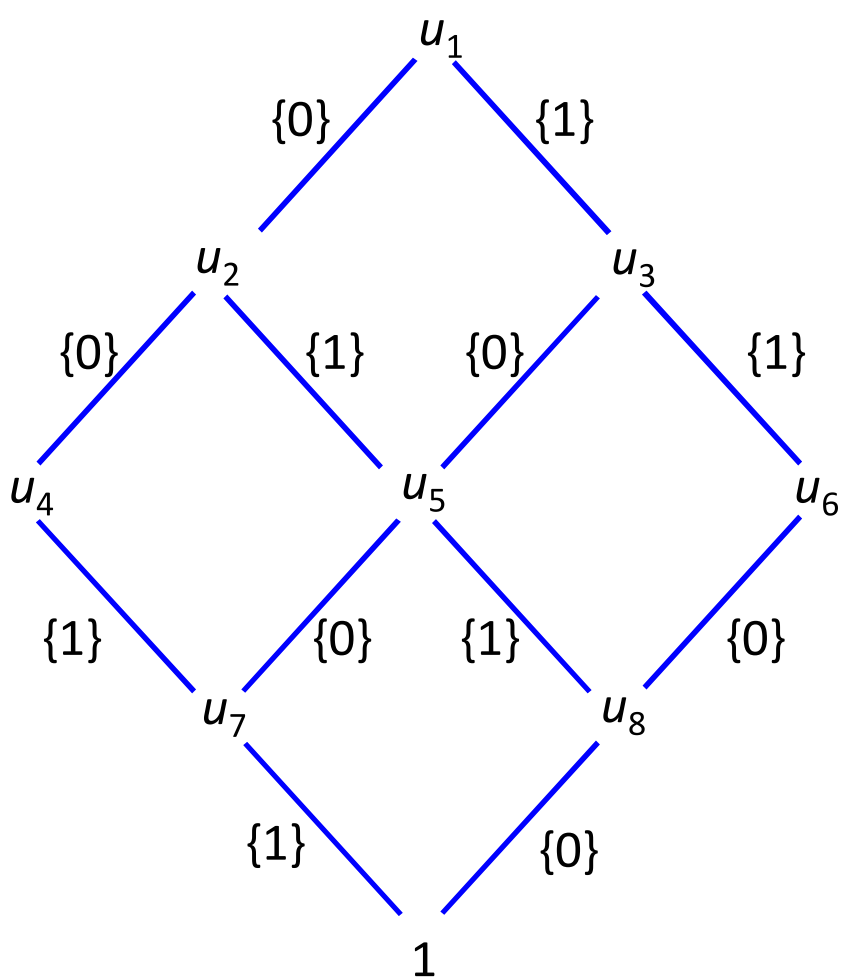
Example



Continue this process…

among($\{x_1,x_2,x_3,x_4\}$, {1}, 2 ,2)

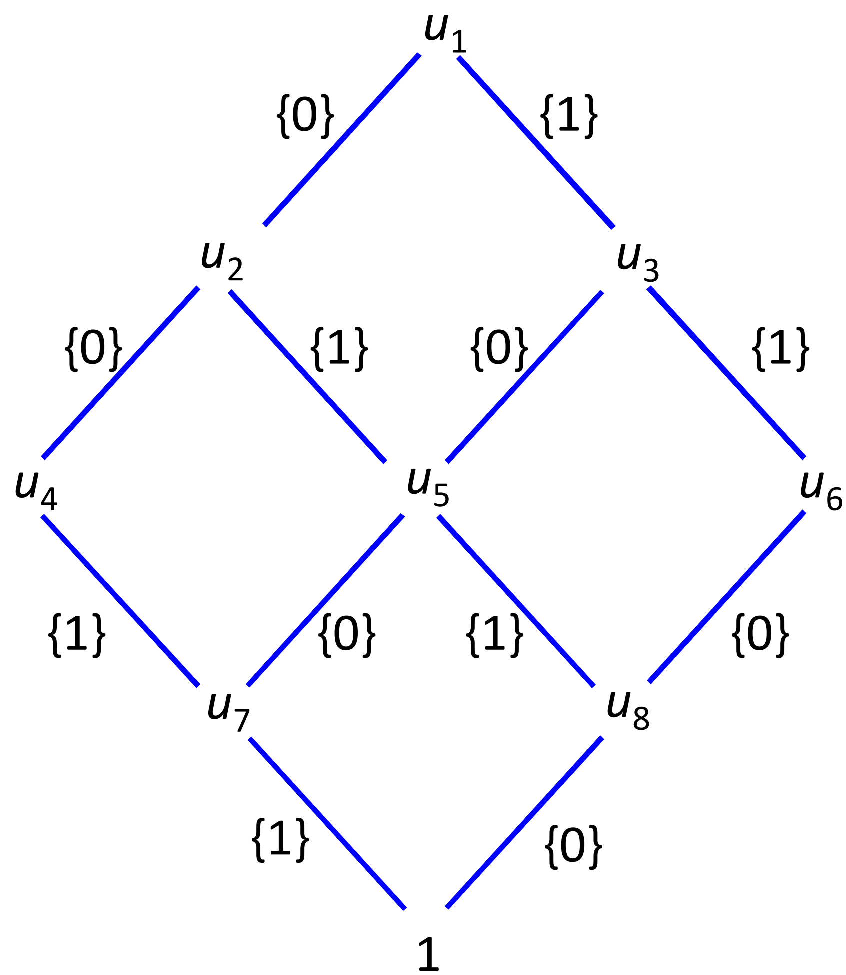
Example



...and obtain a relaxed MDD of width 3

among($\{x_1,x_2,x_3,x_4\}$, {1}, 2 ,2)

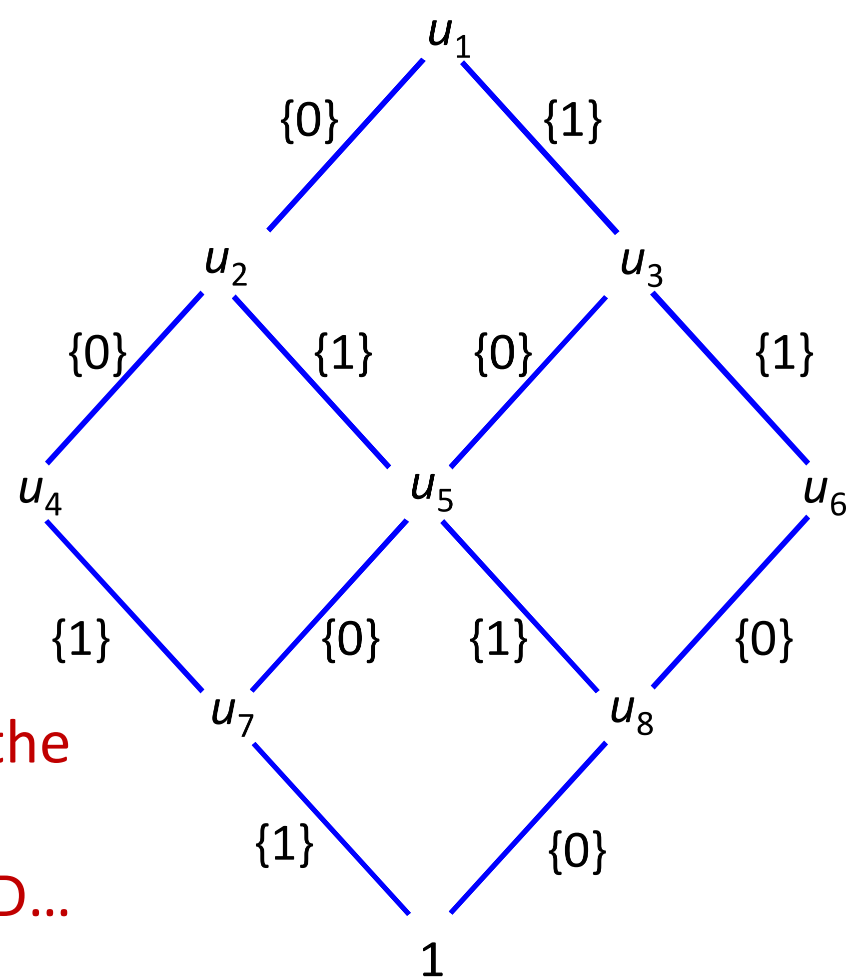
Example



In this case, the MDD is exact.

among($\{x_1,x_2,x_3,x_4\}$, {1}, 2 ,2)

Example



Now, propagate the next constraint through **this** MDD…

among($\{x_1,x_2,x_3,x_4\}$, {1}, 2 ,2)

General Scheme for Node Splitting

- Identify (approximate) equivalence classes of incoming edges.
- Create a copy of the node for each equivalence class.
  - But without exceeding the max width.
- Duplicate outgoing arcs.

- Method for identifying equivalence classes is **constraint-specific**.
  - This exploits **special structure**.
  - Can also be **dynamic**, e.g. depending on current width.

General Scheme for Filtering

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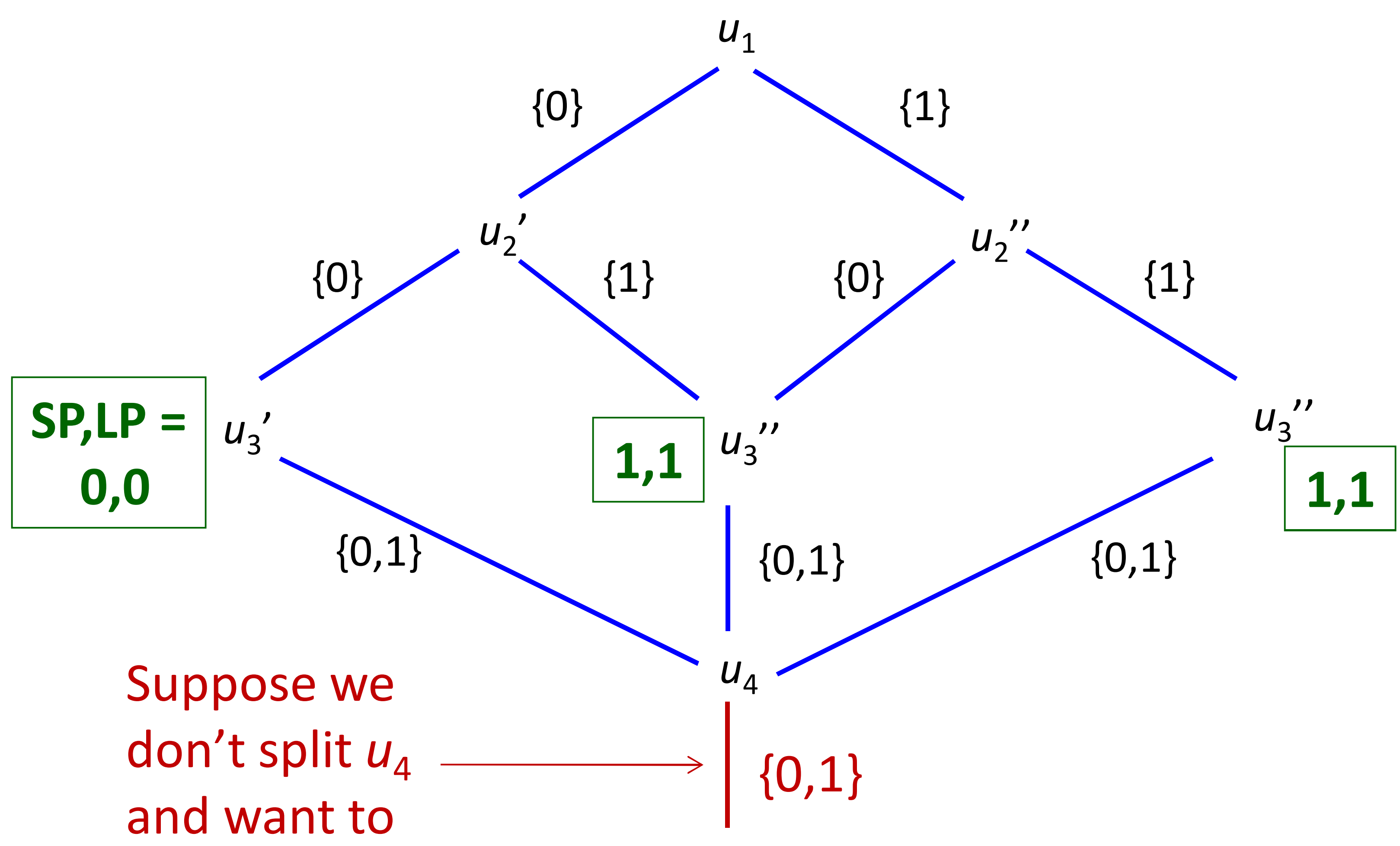
- Filtering on an edge is based on **information** at either end of the edge.

- Information is accumulated during **top-down** and **bottom-up** passes.
  - Based on **extend** ($\otimes$) and **merge** ($\oplus$) operations.

- For example…

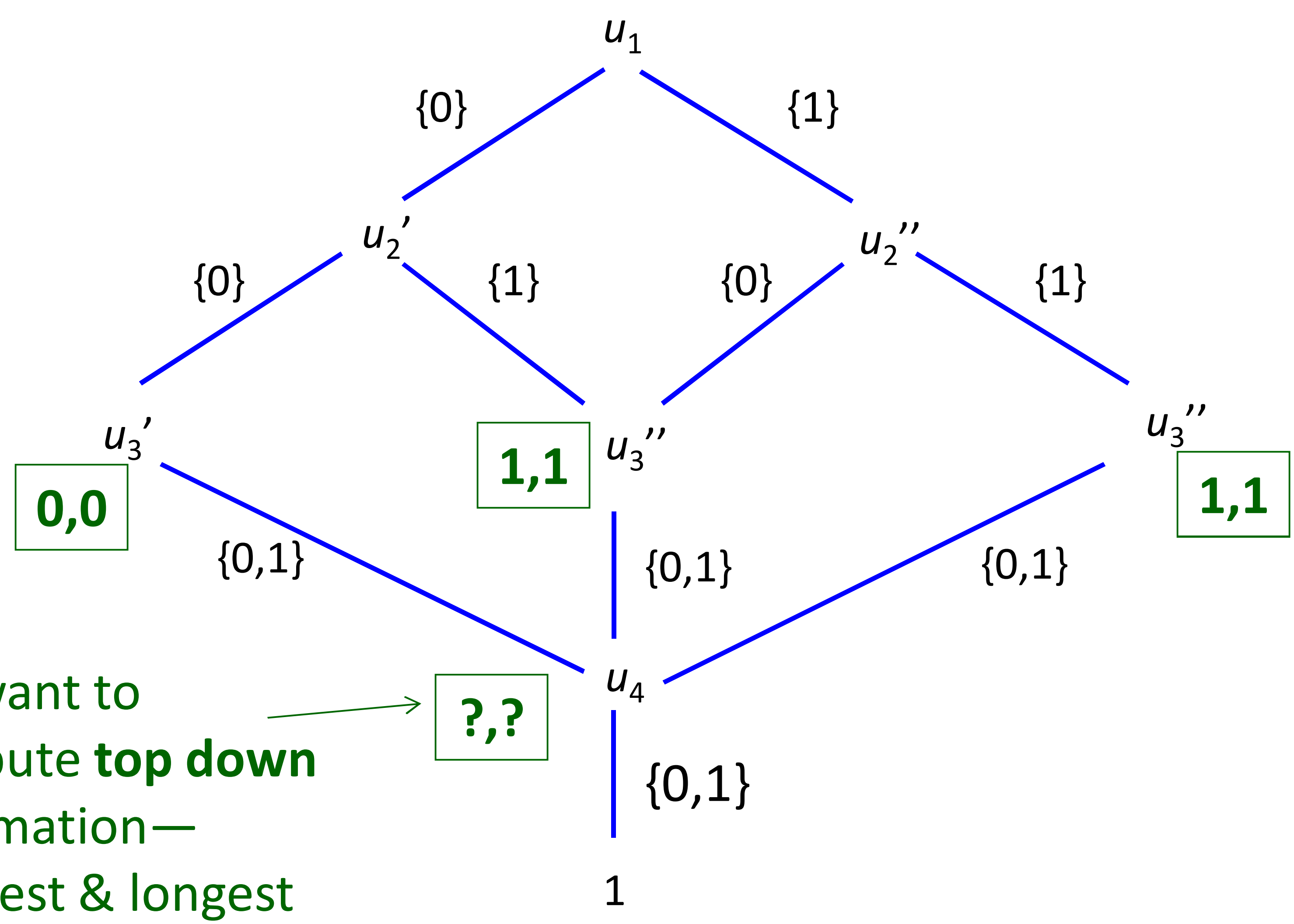
General Scheme for Filtering



Suppose we don't split u_4 and want to filter this edge domain

among($\{x_1, x_2, x_3, x_4\}$, {1}, 2 ,2)

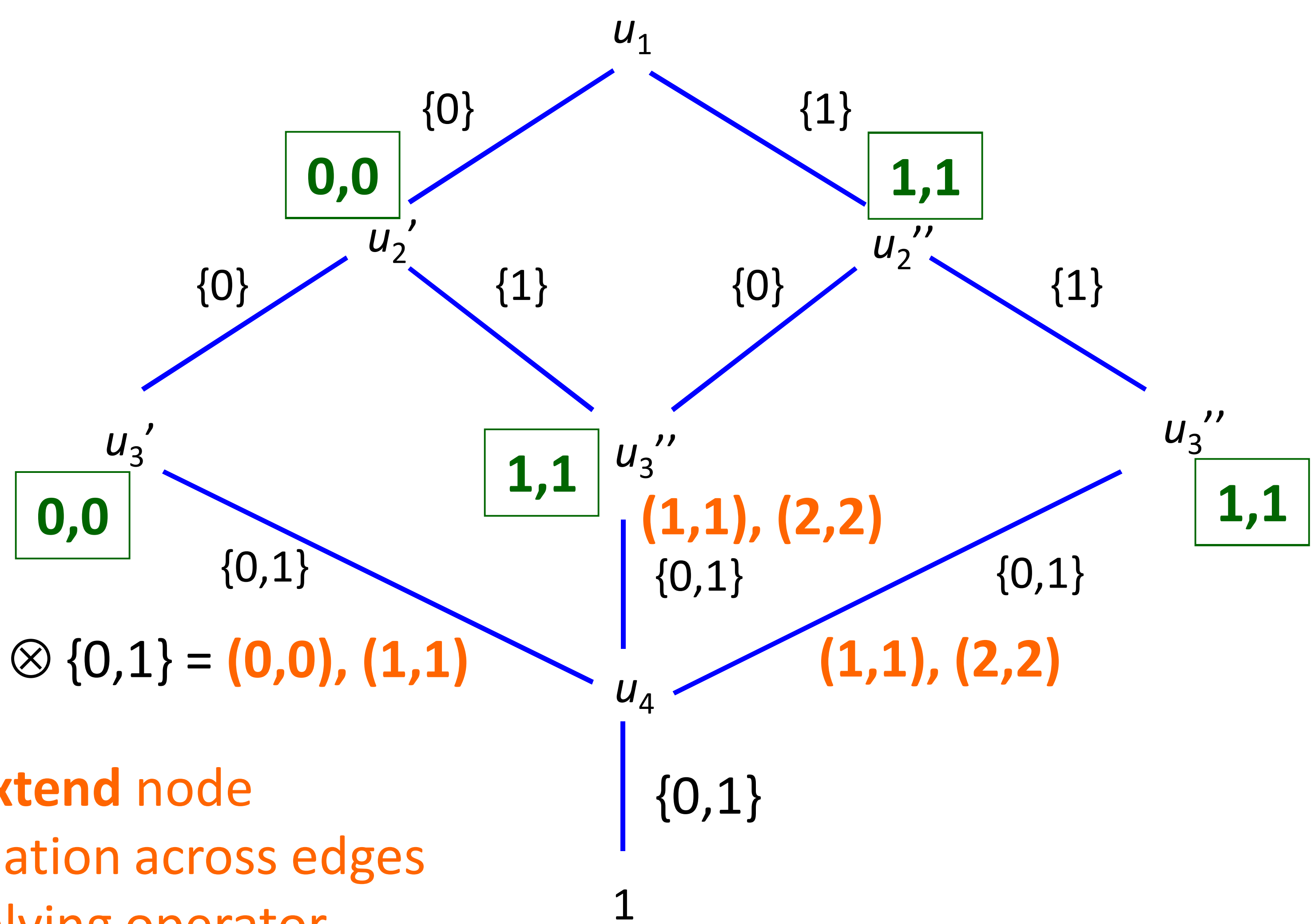
General Scheme for Filtering



We want to compute **top down** information—shortest & longest path length

among($\{x_1,x_2,x_3,x_4\}$, {1}, 2 ,2)

General Scheme for Filtering



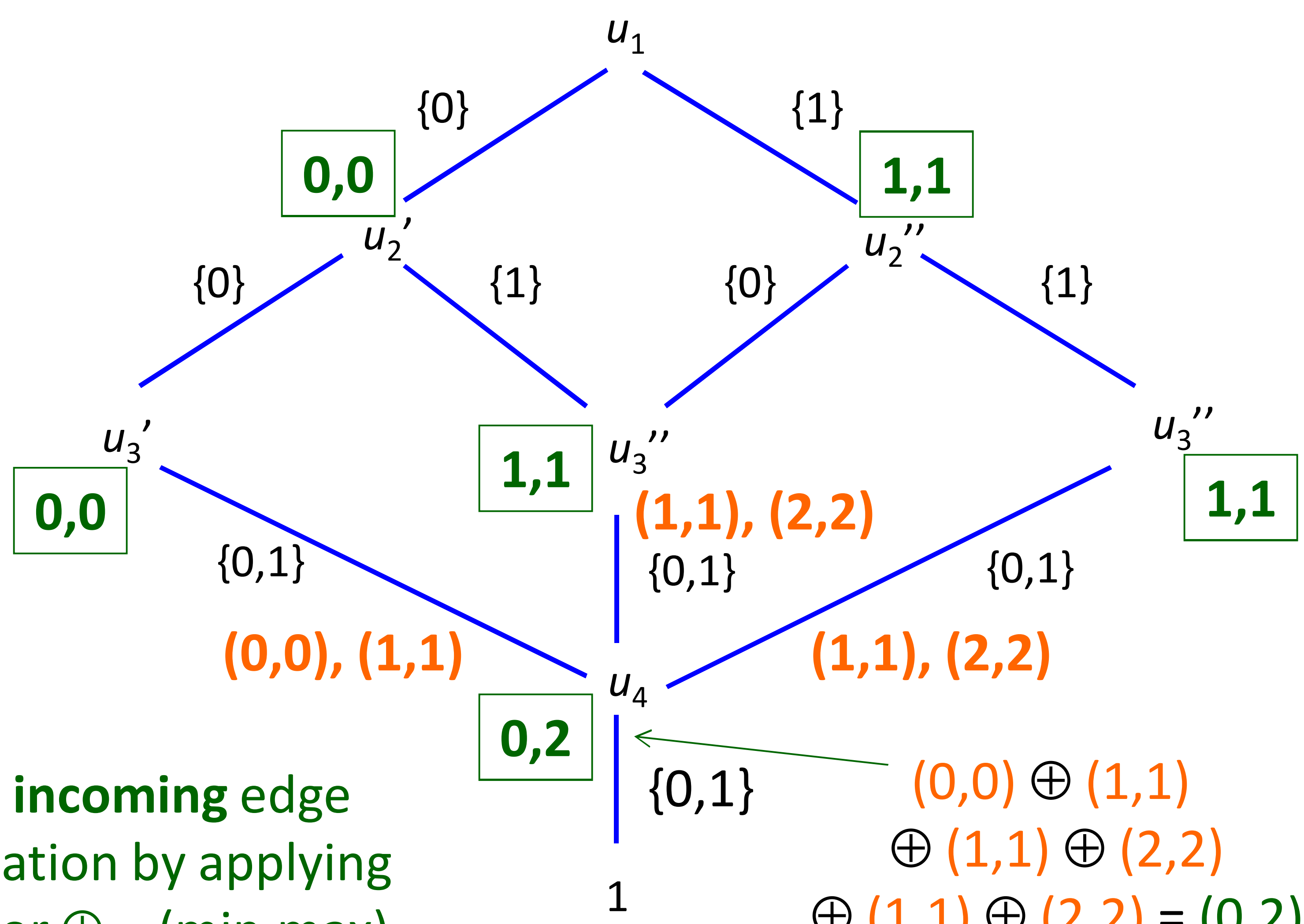
(0,0) ⊗ {0,1} = **(0,0), (1,1)**

First **extend** node information across edges by applying operator

⊗ = +

among({x_1,x_2,x_3,x_4}, {1}, 2 ,2)

General Scheme for Filtering

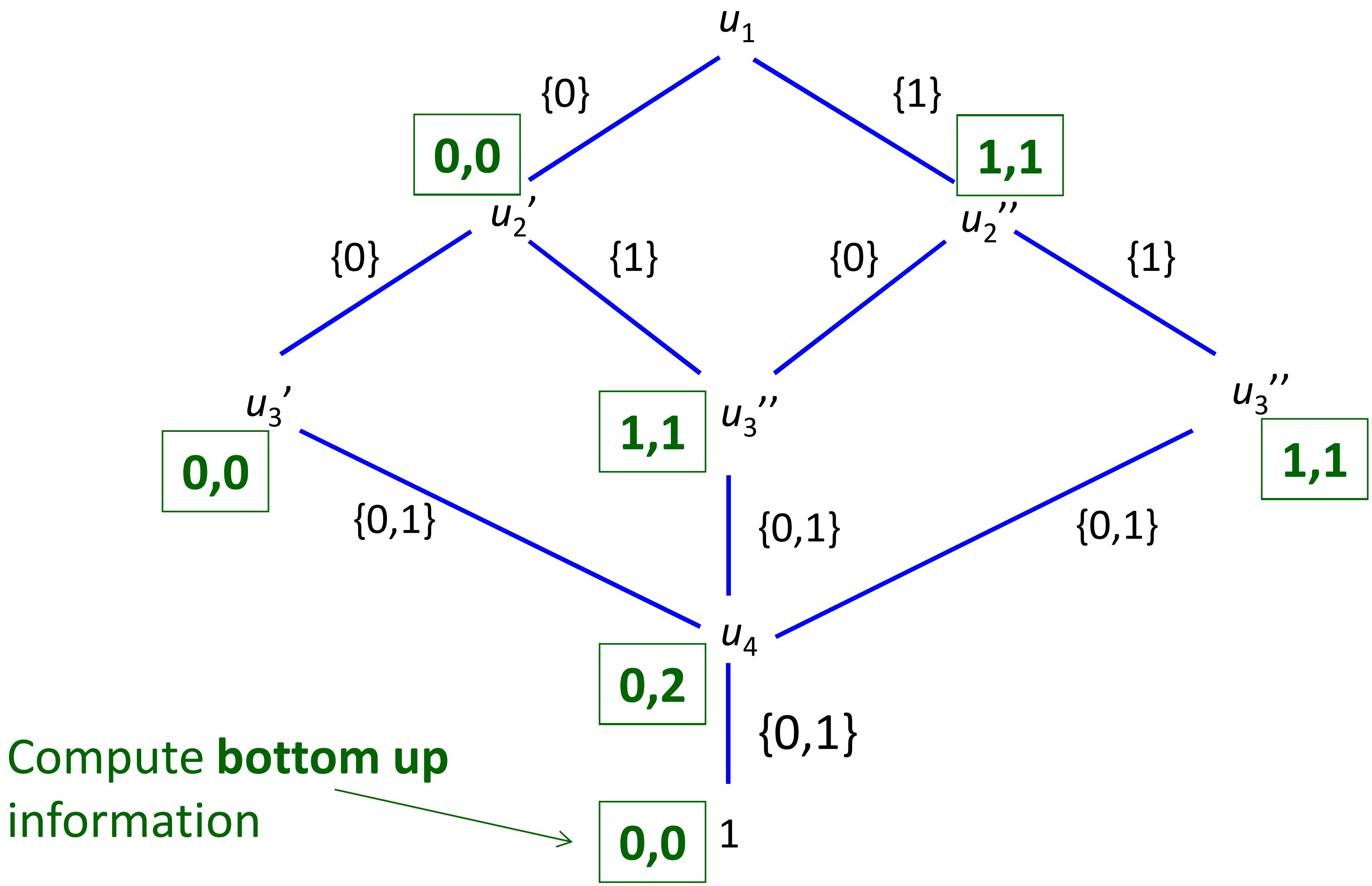


$(0,0) \oplus (1,1)$
$\oplus (1,1) \oplus (2,2)$
$\oplus (1,1) \oplus (2,2) = (0,2)$

Merge incoming edge information by applying operator $\oplus$ = (min,max)

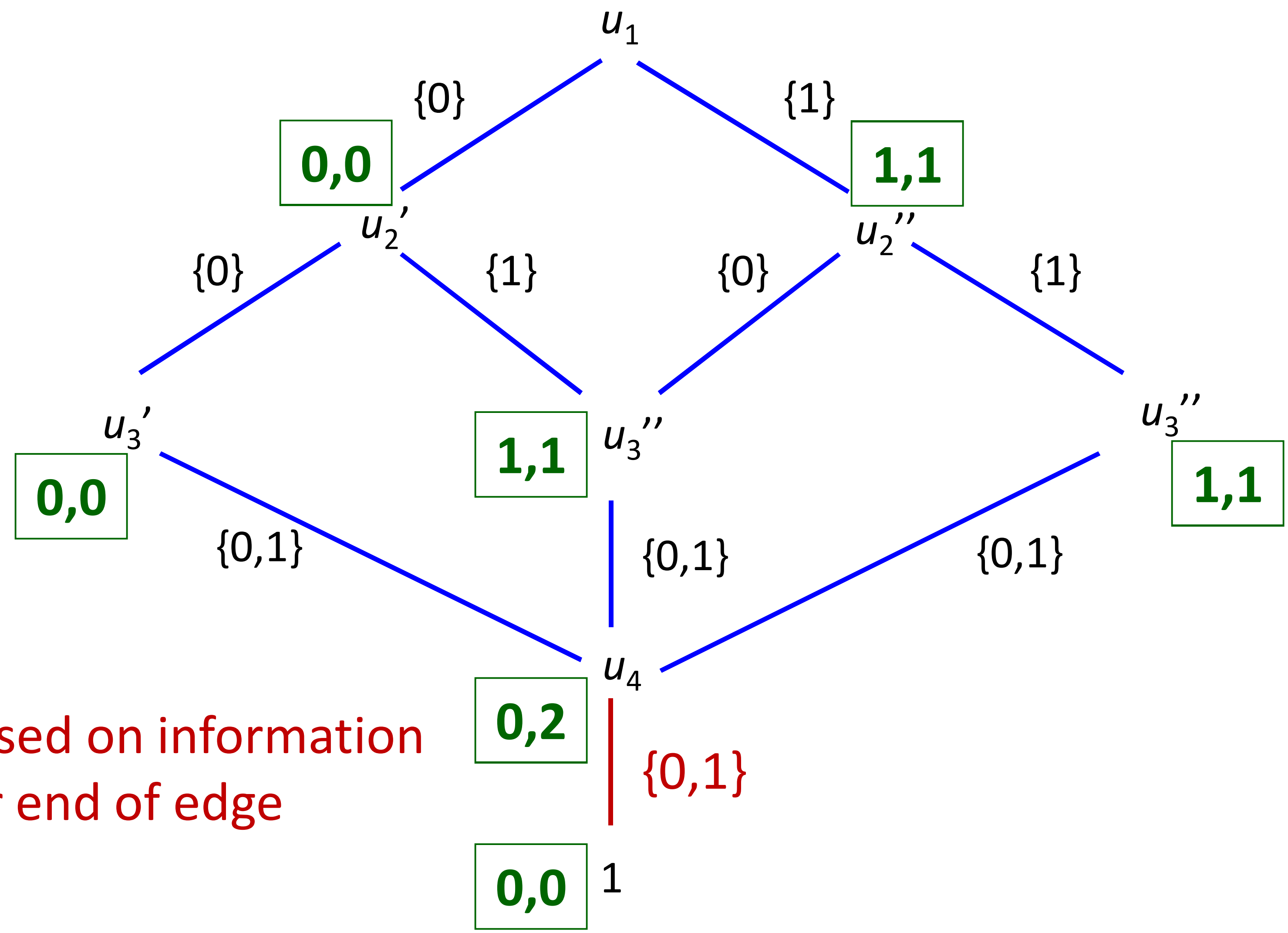
among($\{x_1,x_2,x_3,x_4\}$, {1}, 2 ,2)

General Scheme for Filtering



among($\{x_1,x_2,x_3,x_4\}$, {1}, 2 ,2)

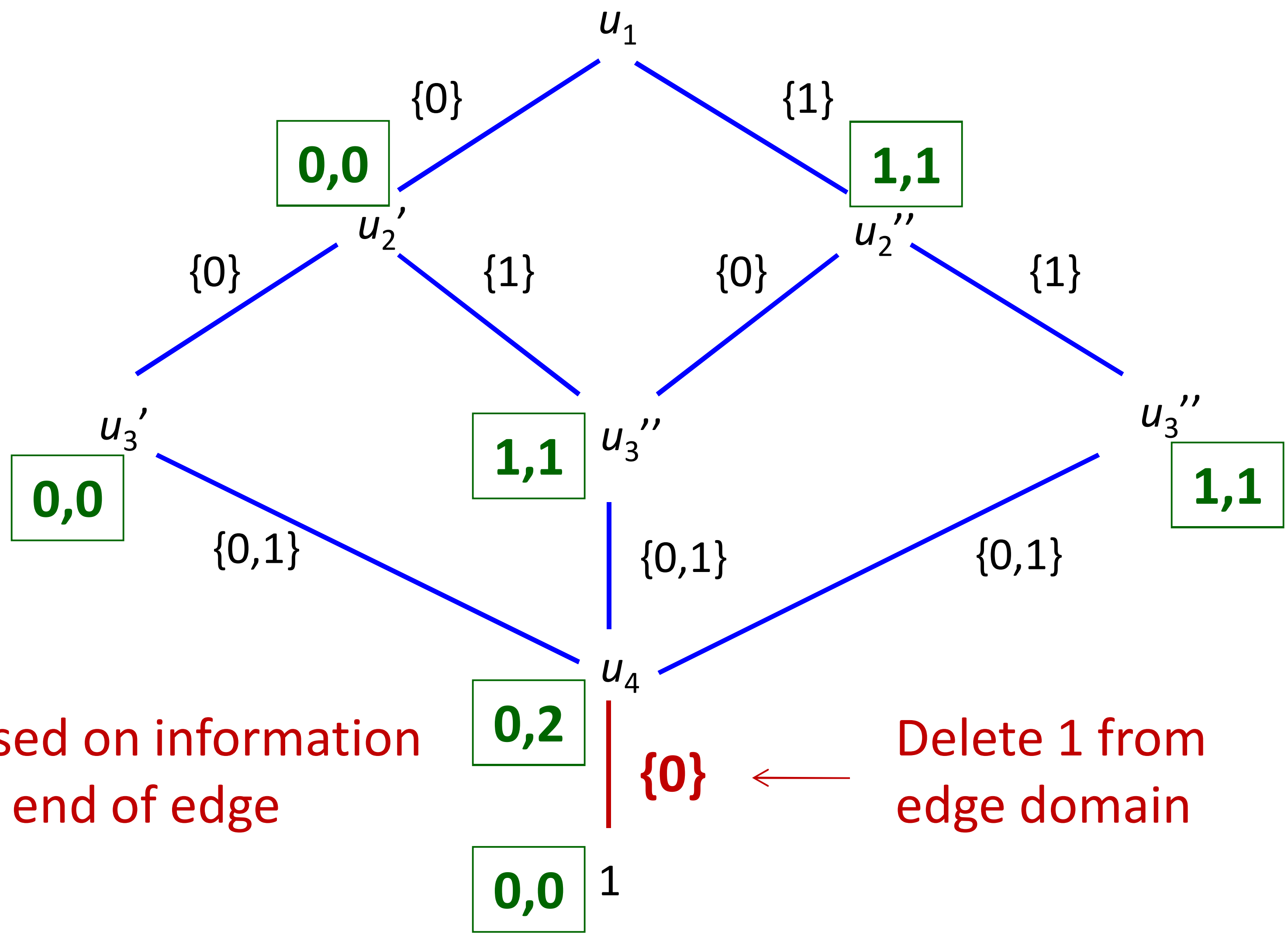
General Scheme for Filtering



Filter based on information at either end of edge

among($\{x_1,x_2,x_3,x_4\}$, {1}, 2 ,2)

General Scheme for Filtering



Filter based on information at either end of edge

Delete 1 from edge domain

among({x_1,x_2,x_3,x_4}, {1}, 2 ,2)

General Scheme for Filtering

among(X,S,L,U)

x_k u Top down information = (SP,LP)

Shortest, longest path to u

{e}

x_{k+1} u' Bottom up information = (SP,LP)

Shortest, longest path to u'

General Scheme for Filtering

among(X,S,L,U)

x_k u Top down information = (SP,LP)

$\{e\}$ Delete e if SP + e + SP′ > U or LP + e + LP′ < L

x_{k+1} u' Bottom up information = (SP′,LP′)

General Scheme for Filtering

among(*X*,*S*,*L*,*U*)

x_k u Top down information = (SP,LP)

$\{e\}$

Delete *e* if SP + *e* + SP′ > U
or LP + *e* + LP′ < L

x_{k+1} u' Bottom up information = (SP′,LP′)

$(SP,LP) \otimes \{e\} = (SP + e, LP + e)$

$(SP_1,LP_1) \oplus (SP_2,LP_2)$

$(\min\{SP_1,SP_2\}, \max\{LP_1,LP_2\})$

General Scheme for Filtering

Any binary constraint B with variables x_i, x_j

x_k u S = {all values assigned to x_i on some path to u}

{e}

x_{k+1} u' T = {all values assigned to x_j on some path to u'}

General Scheme for Filtering

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Any binary constraint B with variables x_i, x_j

x_k u

$\{e\}$

x_{k+1} u'

S = {all values assigned to x_i on some path to u}

Delete e if $k = j$ and $(x_i,x_j) = (v,e)$ satisfies B for no $v \in S$

or $\underline{k} = i$ and $(x_i,x_j) = (v,e)$ satisfies B for no $v \in T$

T = {all values assigned to x_j on some path to u'}

General Scheme for Filtering

Any binary constraint B with variables x_i, x_j

x_k u — S = {all values assigned to x_i on some path to u}

$\{e\}$ — Delete e if $k = j$ and $(x_i, x_j) = (v, e)$ satisfies B for no $v \in S$
or $\underline{k} = i$ and $(x_i, x_j) = (v, e)$ satisfies B for no $v \in T$

x_{k+1} u' — $\underline{T}$ = {all values assigned to x_j on some path to u'}

$S \otimes \{e\} = \{e\}$ if $k = i$, S otherwise

S_1 $\oplus$ S_2 → $S_1 \cup S_2$

General Scheme for Filtering

Any binary constraint B with variables x_i, x_j

Theorem. This scheme achieves MDD consistency for any binary constraint B in time that is polynomial in size of B and the MDD.

General Scheme for Filtering

Any binary constraint *B* with variables x_i, x_j

Theorem. This scheme achieves MDD consistency for any binary constraint *B* in time that is polynomial in size of *B* and the MDD.

Corollary. Polytime MDD consistency for:
equality, not-equal constraints, `element` constraint, etc.

General Scheme for Filtering

alldiff(X)

x_k $\quad u \quad (A,S)$

{all values that appear on **some** path to u}

{all values that appear on **all** paths to u}

$\{e\}$

x_{k+1} $\quad u' \quad (A',S')$

General Scheme for Filtering

alldiff(X)

x_k u (A,S)

$\{e\}$

Delete e if $e \in A \cup A'$, or:
$|S| = k - 1$ or $|S'| = n - k$
(variables above or below form a Hall set)

x_{k+1} u' (A',S')

General Scheme for Filtering

alldiff(X)

$x_k \quad u \quad (A,S)$

$\{e\}$

Delete e if $e \in A \cup A'$, or:

$|S| = k - 1$ or $|S'| = n - k$

(variables above or below form a Hall set)

$x_{k+1} \quad u' \quad (A',S')$

$(A,S) \otimes \{e\} = (A \cup \{e\}, S \cup \{e\})$

$(A_1,S_1) \quad \oplus \quad (A_2,S_2)$

$(A_1 \cap A_2, S_1 \cup S_2)$

General Scheme for Filtering

Unary resource constraint

x_i = activity in position *i* of sequence

r_i, d_i = release time, deadline

Enforce `alldiff`(X) plus:

x_k u EST

$\{e\}$

x_{k+1} u' LFT

Earliest start time of x_k, given previous activities in sequence

Latest finish time of x_{k+1}, given subsequent activities in sequence

General Scheme for Filtering

Unary resource constraint

x_i = activity in position *i* of sequence

r_i, d_i = release time, deadline

Enforce `alldiff`(X) plus:

x_k u EST

{e}

Delete e if $\max\{\text{EST}, r_e\} + p_e > \min\{\text{LFT}, d_e\}$

x_{k+1} u' LFT

General Scheme for Filtering

Unary resource constraint

x_i = activity in position i of sequence

r_i, d_i = release time, deadline

Enforce `alldiff`(X) plus:

x_k u EST

$\{e\}$ Delete e if $\max\{\text{EST}, r_e\} + p_e > \min\{\text{LFT}, d_e\}$

x_{k+1} u' LFT

$$\text{EST} \otimes \{e\} = \max\{\text{EST} + p_e,\ r_e + p_e\}$$

$$\text{LFT} \otimes \{e\} = \min\{\text{LFT} - p_e,\ d_e - p_e\}$$

$$\text{EST}_1 \oplus \text{EST}_2 = \min\{\text{EST}_1, \text{EST}_2\}$$

$$\text{LFT}_1 \oplus \text{LFT}_2 = \max\{\text{EST}_1, \text{EST}_2\}$$

Solver Architecture

- Our MDD-based solver is built around this general scheme for splitting and filtering.
- Plug in constraint-specific (approximate) equivalence test for splitting.
- Plug in constraint-specific $\otimes$, $\oplus$ for filtering.

Experiments – Previous work

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- Multiple `alldiff` constraints
  - Andersen et al., CP 2007
  - MDDs of width 5
  - Reduce search tree of 1 million+ nodes to 1 node.
  - Reduce time by factor of 30
  - Using old software

- Equality constraints $\sum_j g_j(x_j) = b$
  - Hadzic et al., CPAIOR 2008

- Certain configuration problems
  - Hadzic et al, CP 2008

Experiments

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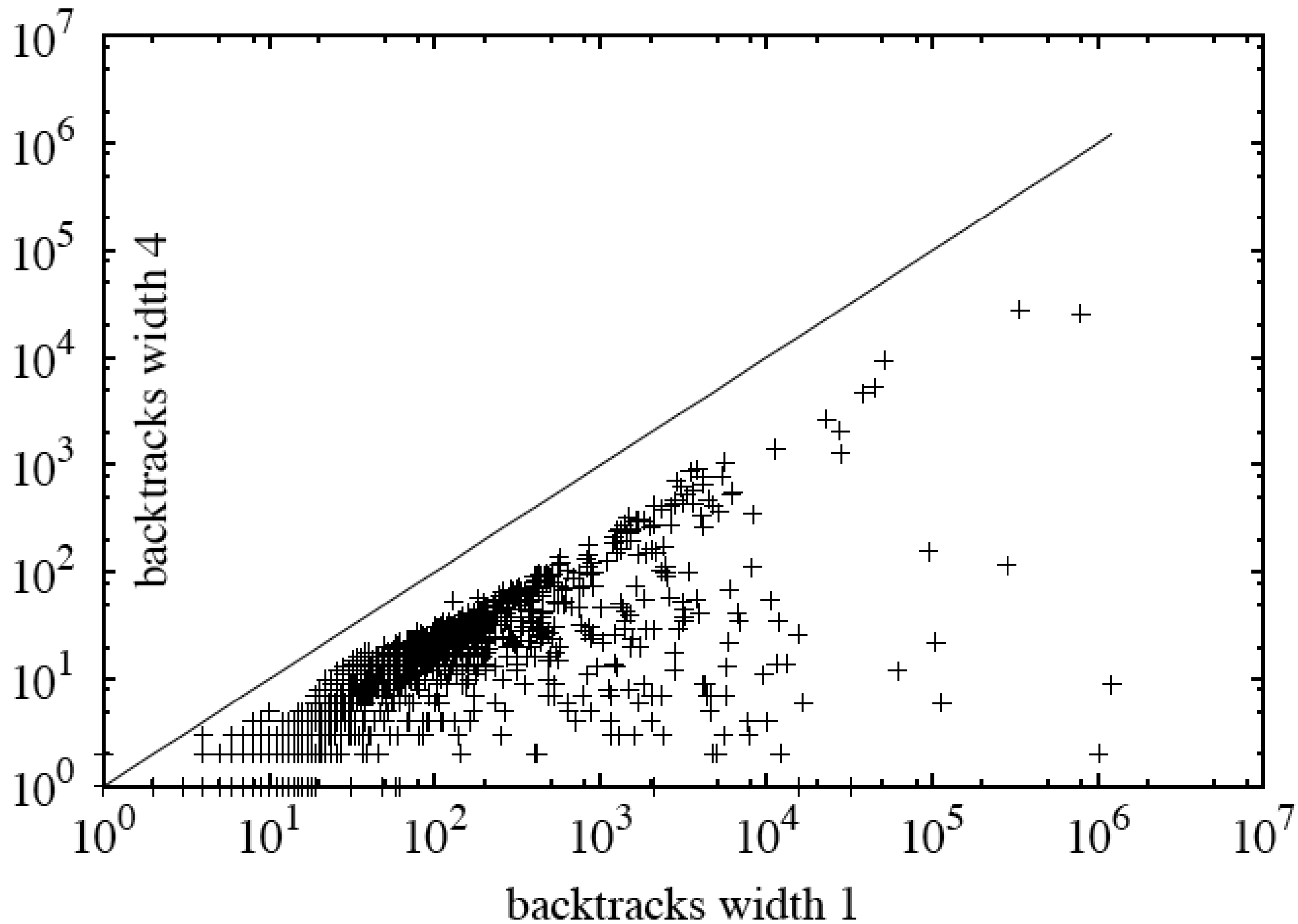
- Random multiple among constraints
- Nurse rostering instances
- Compare domain store (conventional filtering) with MDD store
  - Using MDDs of increasing widths.

Experiments

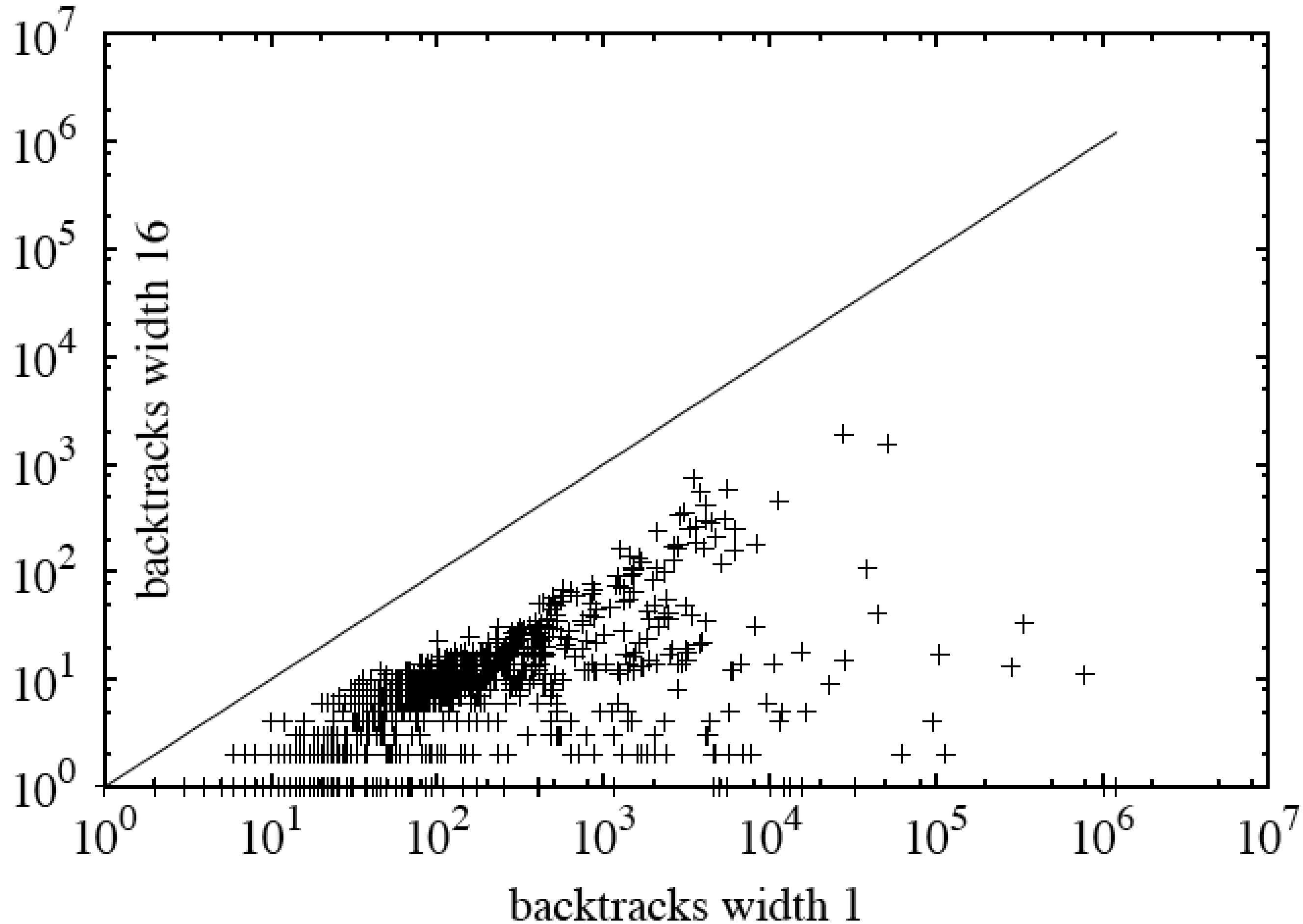
- Random multiple `among` constraints
  - 50 binary variables total
  - 5 variables per `among` constraint
  - Indices chosen from normal distribution with mean chosen from U[1..50] mod 50, and standard deviation 2.5
  - So, indices in a given `among` are near-consecutive, typical of applications.
  - $(L,U) = (2,3)$
  - Number of `among`s varies from 5 to 200 (in steps of 5), 100 instances for each number.
  - This range includes phase transition.

Multiple Amongs: Backtracks



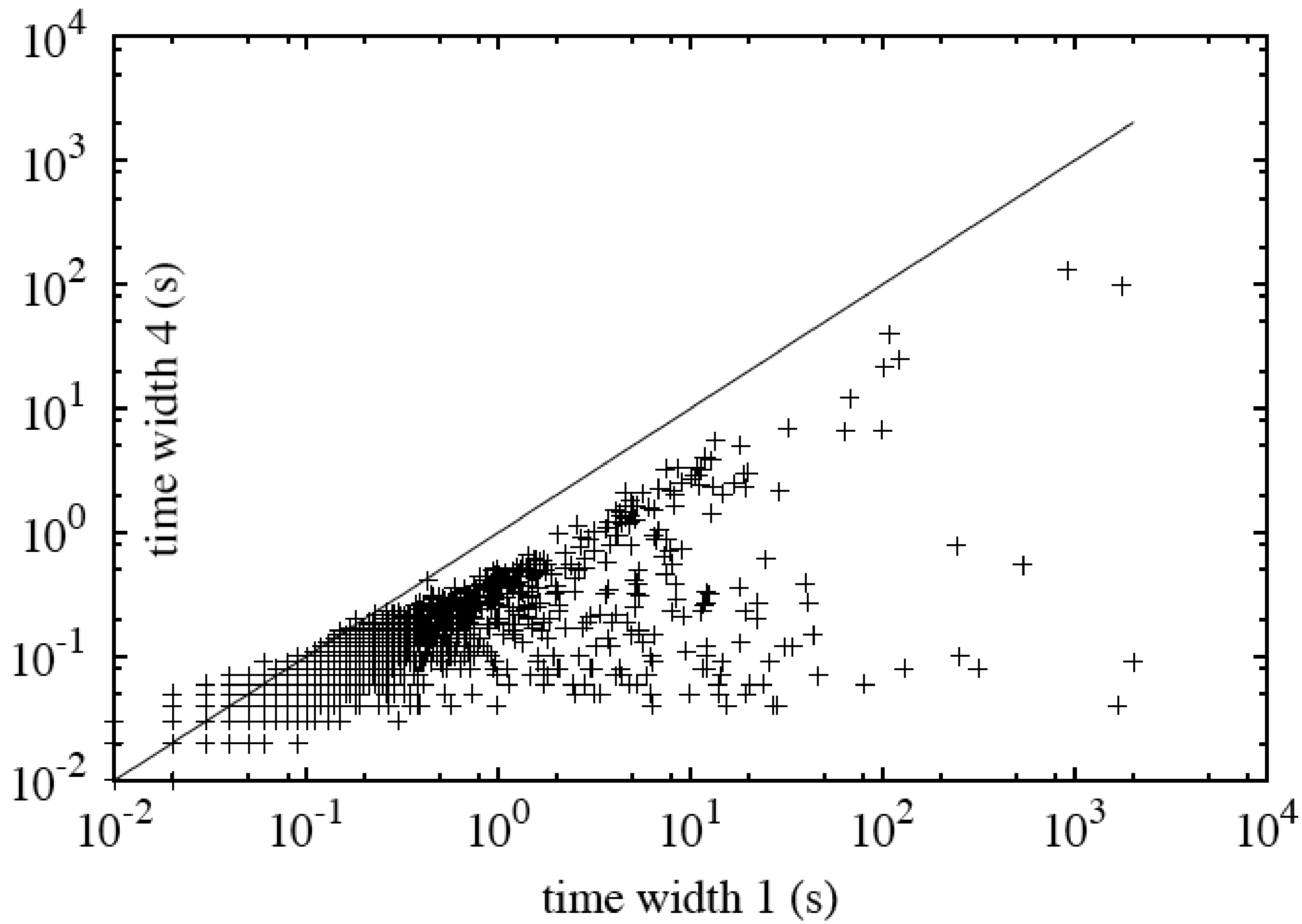
Domain store vs width **4** MDD

Multiple Amongs: Backtracks



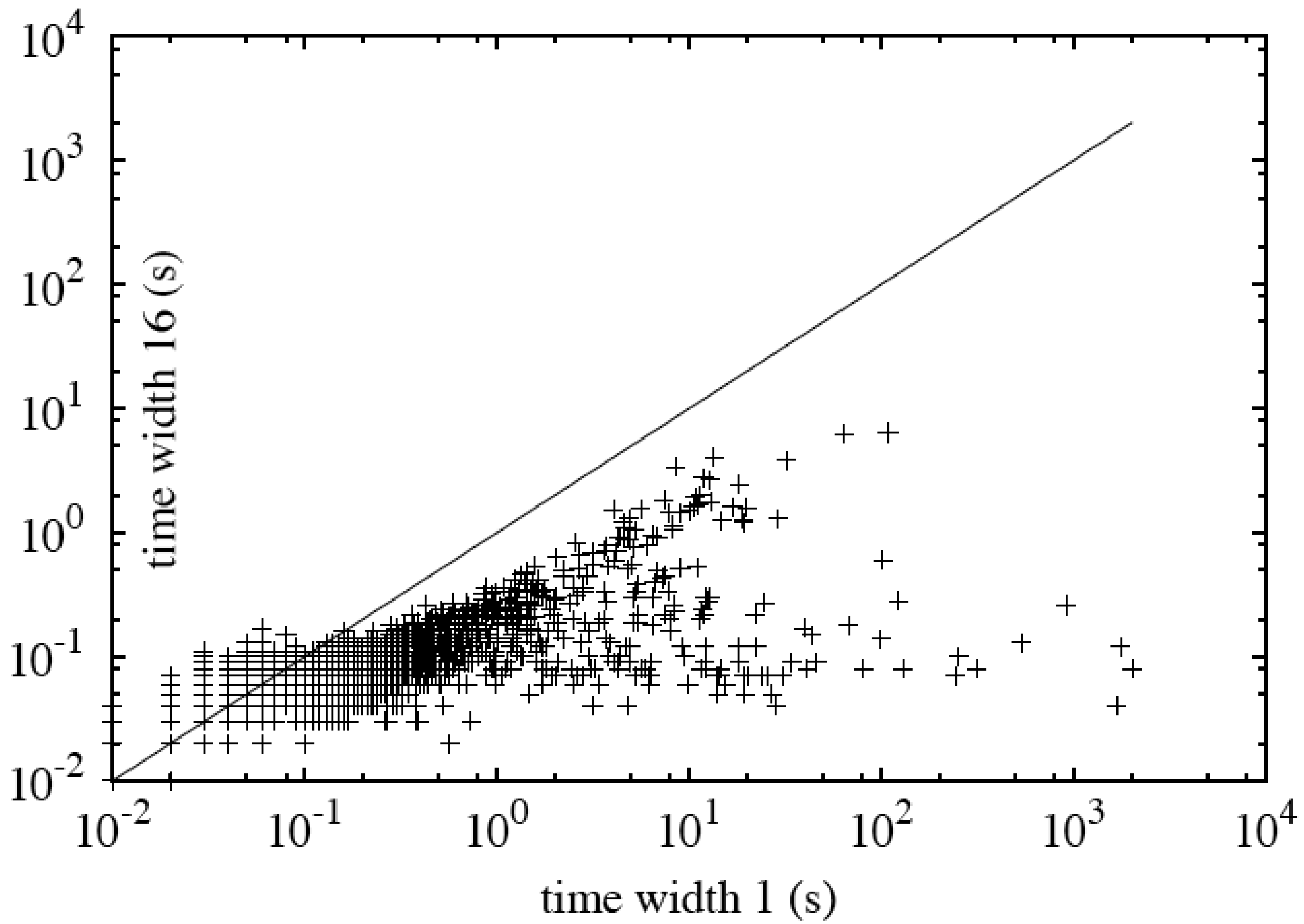
Domain store vs width **16** MDD

Multiple Amongs: Running Time



Domain store vs width **4** MDD

Multiple Amongs: Running Time



Domain store vs width **16** MDD

Experiments

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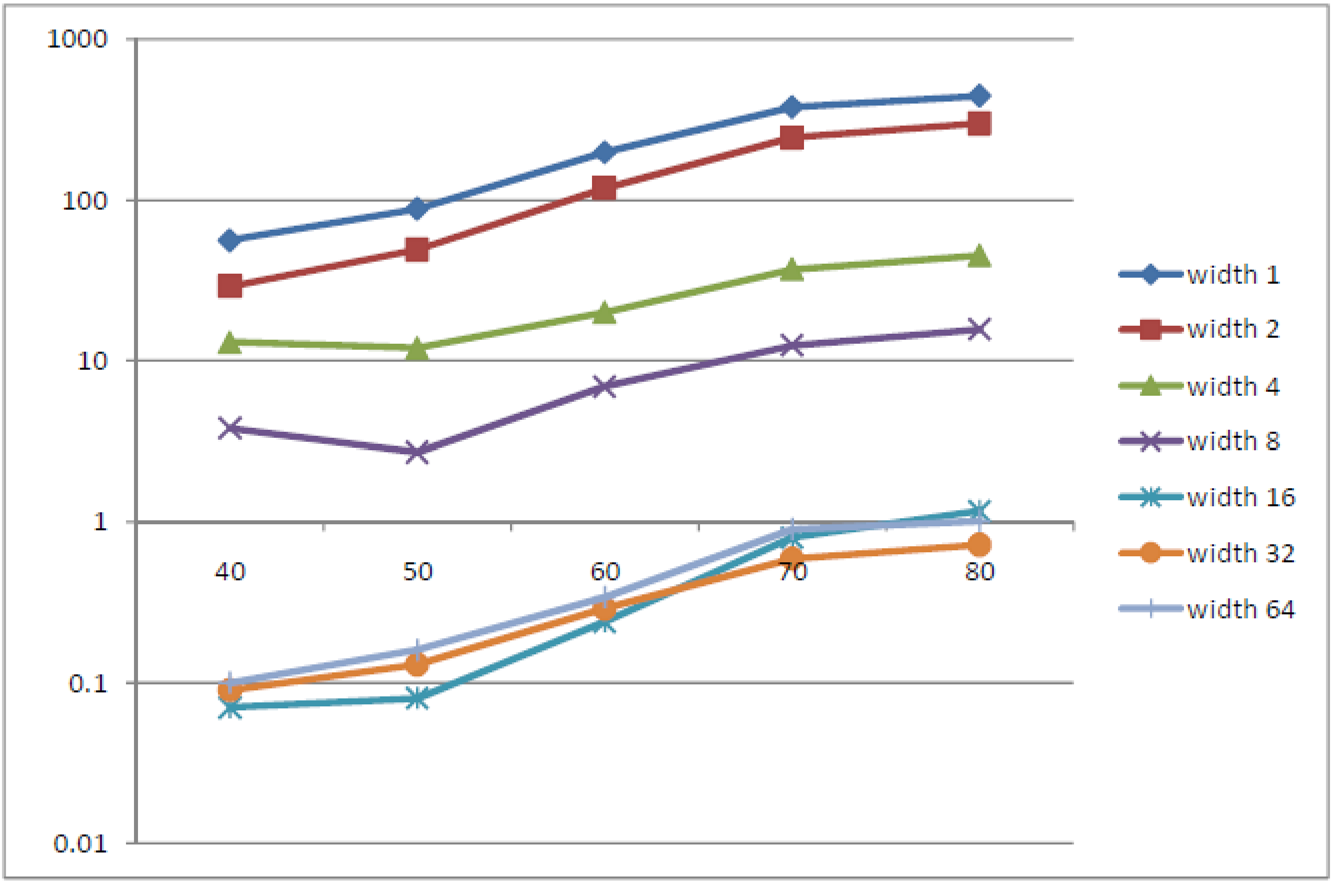
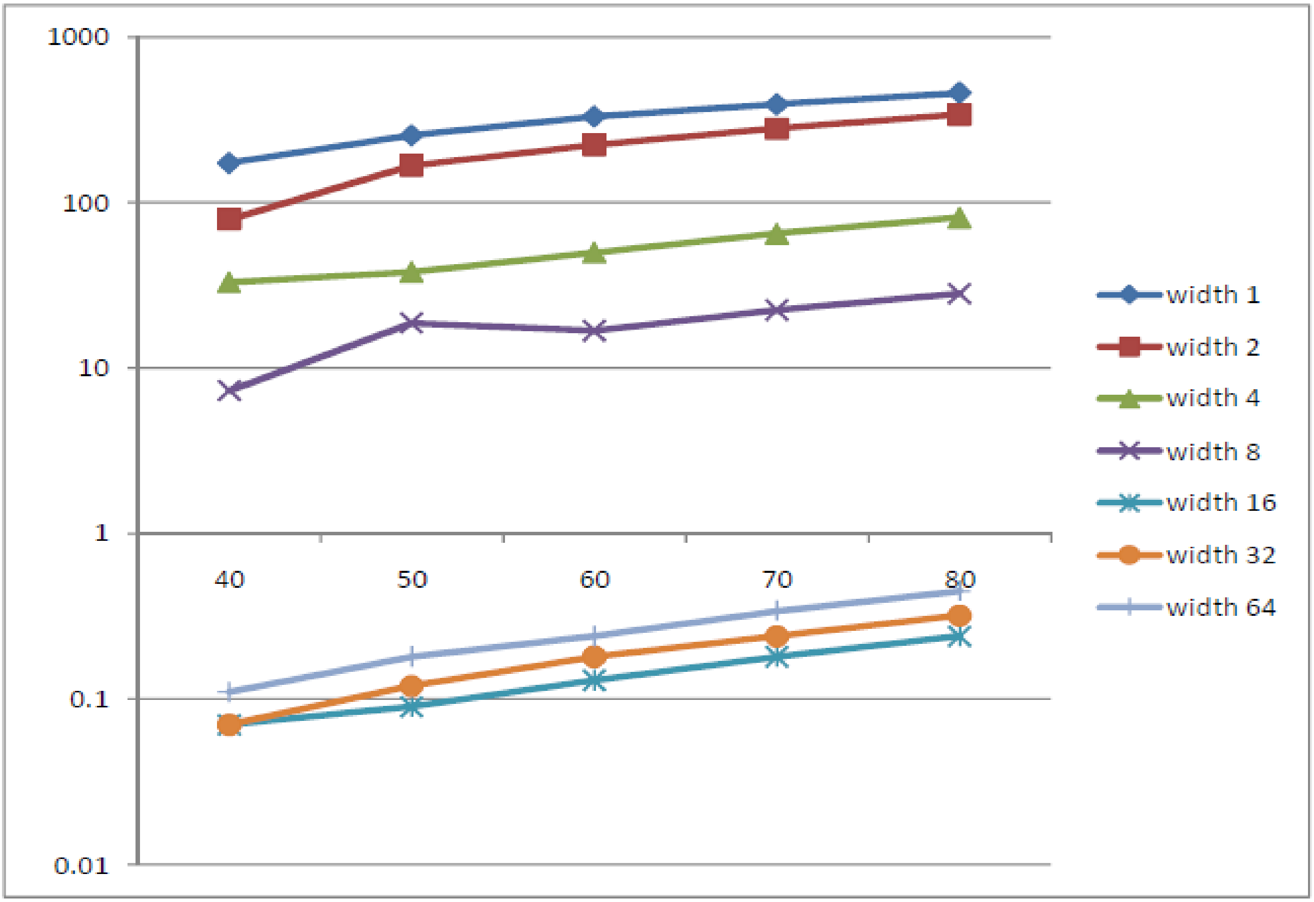
- Nurse rostering instances
  - Based on van Hoeve et al., *Constraints* 2009
  - Work 4-5 days per week
  - Max A days every B days (Max A/B)
  - Min C days every D days (Min A/B)
  - Time horizon ranges from 40 to 80 days
- Three problem classes
  - I: max 6/8, min 22/30
  - II: max 6/9, min 20/30
  - III: max7/9, min 22/30

Nurse rostering problems



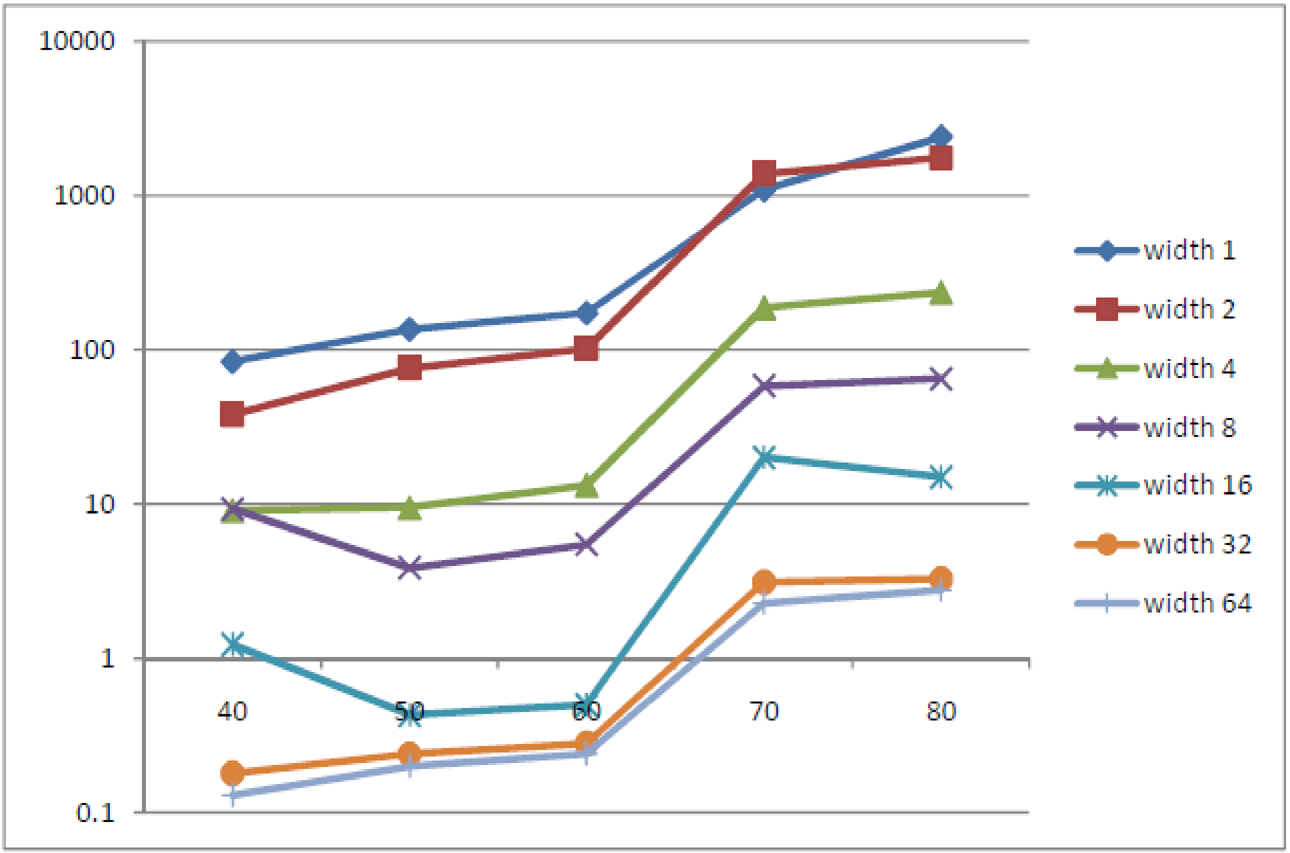
Computation time (sec) vs time horizon, Class I

Nurse rostering problems



Computation time (sec) vs time horizon, Class II

Nurse rostering problems



Computation time (sec) vs time horizon, Class III

Conclusions

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- MDD-based propagation can be carried out according to a general scheme.
  - Equivalence testing for node splitting
  - Information operators for filtering
  - Yet the scheme allows exploitation of constraint-specific structure

- MDD-based constraint solving yields substantial speedup over domain store for multiple among constraints.

- Intensive processing at search nodes can pay off when more information is communicated between constraints.