Annals of Operations Research 65(1996)1–20 

# Determining lower and upper bounds on probabilities of atomic propositions in sets of logical formulas represented by digraphs

K.A. Andersen
*Matematisk Institut, Aarhus Universitet, DK-8000 Århus C, Denmark*
E-mail: kima@mi.aau.dk

J.N. Hooker
*Graduate School of Industrial Administration, Carnegie Mellon University, Pittsburgh, PA 15213, USA*
E-mail: jh38@andrew.cmu.edu

In this paper we consider the problem of determining lower and upper bounds on probabilities of atomic propositions in sets of logical formulas represented by digraphs. We establish a sharp upper bound, as well as a lower bound that is not in general sharp. We show further that under a certain condition the lower bound is sharp. In that case, we obtain a closed form solution for the possible probabilities of the atomic propositions.

## 1 Introduction

In Andersen [1], a class of logical formulas representable by digraphs were described. Suppose a digraph has $n$ nodes. Then an atomic proposition is assigned to every node and it is possible to construct a set of $n$ logical formulas using those $n$ atomic propositions. This construction will be considered in more detail in section 3. Andersen considers in [1] the problem of characterizing the probability assignments that can consistently be made to a given set of formulas. He provided a closed-form characterization that is valid when the formulas in question are represented by a digraph with a perfect or a $t$-perfect underlying undirected graph. However, no effort was made to determine lower and upper bounds on the probabilities of the atomic propositions that are associated with the nodes in the digraph. This is the theme of the present paper.

*The second author is partially supported by ONR grant N00014-92-J-1028 and AFOSR grant 91-0287.

The results of the present paper are as follows:

- A sharp upper bound on the probability of the atomic propositions associated with the nodes in the digraph is given.
- A lower bound on the probability of the atomic propositions associated with the nodes in the digraph is given when the digraph is acyclic. The lower bound is not always sharp.
- When the digraph is acyclic and the sum of the probabilities of the logical formulas represented by the digraph is sufficiently high, then the lower bound is sharp. So in this particular case an exact probability interval for the possible probabilities of the atomic propositions can be given (both the lower and upper bound are attained).

The outline of the paper is as follows. In section 2 a short introduction to probabilistic logic is given. In Chandru and Hooker [3] a more thorough treatment is given. In this section we just define the inference problem in probabilistic logic and illustrate it by using a small example.

In section 3 the class of logical formulas representable by digraphs is defined. In fact, every digraph gives rise to a set of logical formulas. The definitions are illustrated by small examples. Furthermore, the main results of Andersen [1] are given.

It should be emphasized that the digraphs described in this paper represent sets of logical formulas and *do not* represent probabilistic dependence or independence relations among logical propositions. The latter is the case in Bayesian networks (also called Belief networks). Let us suppose we have a Bayesian network with three nodes and two arcs. One arc goes from node 1 to node 3 and the other arc goes from node 2 to node 3. Then this means that node 3 *depends probabilistically* on nodes 1 and 2, and, furthermore, the network shows that nodes 1 and 2 are *probabilistically independent.* A thorough treatment of Bayesian networks is given in Pearl [7]. It is also possible to extend probabilistic logic so as to reflect dependence and independence relations among propositions, where the relations are described using a Bayesian network. This approach is described in detail in Andersen and Hooker [2] and is called Bayesian logic.

Finally, in section 4 the main results of the present paper are described. The upper and lower bounds on probabilities are proven and are illustrated using small examples. Furthermore, examples are given to show that when the conditions in the lemmas and theorems are not fulfilled, the results are not necessarily true.

## 2 An introduction to probabilistic logic

As is well known, ordinary propositional logic is two-valued, because a formula can have only one of two possible truth values. In probabilistic logic, reintroduced by Nilsson [6], formulas can be assigned infinitely many truth values. The truth value assigned to a formula is the probability of the formula being true.

A possible world is an assignment of truth values, true or false, to every atomic proposition. For example, $(x_1, x_2, x_3) = (1, 0, 1)$ is a possible world ($x_1$, $x_2$ and $x_3$ are atomic propositions, 1 denotes true and 0 denotes false). In propositional logic, a model is simply a possible world. In probabilistic logic, a model is an assignment of probabilities to every possible world. The probabilities are nonnegative and sum to one, and the probability of a given formula is the sum of the probabilities of those possible worlds in which the formula is true.

**Example 1**

Suppose we have a tiny knowledge base consisting of three formulas

$$x_1, \tag{1}$$

$$x_2, \tag{2}$$

$$(x_1 \wedge x_2) \rightarrow x_3, \tag{3}$$

where $x_1$, $x_2$ and $x_3$ are atomic propositions, "$\wedge$" is the logical connective "and" and "$\rightarrow$" is the logical connective "implies". For example, $x_1$ and $x_2$ might represent two distinct symptoms and $x_3$ might represent a particular disease. The third formula then reads that if a person has the two distinct symptoms, then this implies that the person has the particular disease, too.

A model is given in table 1.

Table 1

A model.

| Possible world $(x_1, x_2, x_3)$ | *Pr.* | $x_1$ | $x_2$ | $(x_1 \wedge x_2) \rightarrow x_3$ |
|---|---|---|---|---|
| (0, 0, 0) | 0.2 | 0 | 0 | 1 |
| (0, 0, 1) | 0.2 | 0 | 0 | 1 |
| (0, 1, 0) | 0.1 | 0 | 1 | 1 |
| (0, 1, 1) | 0.05 | 0 | 1 | 1 |
| (1, 0, 0) | 0.15 | 1 | 0 | 1 |
| (1, 0, 1) | 0.2 | 1 | 0 | 1 |
| (1, 1, 0) | 0.05 | 1 | 1 | 0 |
| (1, 1, 1) | 0.05 | 1 | 1 | 1 |

Let $p = (p_1, \ldots, p_8)$ denote the probabilities of the eight possible worlds and let $\pi = (\pi_1, \pi_2, \pi_3)$ denote the probabilities of the formulas (1)–(3). Then there are the following connections between $p$'s and $\pi$'s:

$$\begin{aligned}
&& p_5 + p_6 + p_7 + p_8 &= \pi_1, \\
& p_3 + p_4 && + p_7 + p_8 = \pi_2, \\
& p_1 + p_2 + p_3 + p_4 + p_5 + p_6 && + p_8 = \pi_3, \\
& p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 &&= 1.0, \\
& p_i \geq 0, \quad \forall i.
\end{aligned} \tag{4}$$

The first equation in (4) says that the probability of the formula $x_1$ should be equal to $\pi_1$ because $x_1$ is true in the last four possible worlds. Similarly, the second and the third equations in (4) say that the probabilities of $x_2$ and $(x_1 \wedge x_2) \to x_3$ should be equal to $\pi_2$ and $\pi_3$, respectively. The last two equations in (4) express the fact that the $p$'s should constitute a probability distribution. In this particular example, $\pi = (0.45, 0.25, 0.95)$ is a valid assignment of probabilities to the formulas in the database.

Now suppose we want to determine the probability of formula $x_3$. Clearly, the probability of $x_3$ is equal to $p_2 + p_4 + p_6 + p_8$. So one possibility for the probability of $x_3$ is 0.5. In general, the following linear programming problems determine the range of possible probabilities for $x_3$:

$$\begin{aligned}
&\text{min / max} \quad p_2 + p_4 + p_6 + p_8 \\
&\text{subject to} \quad (4).
\end{aligned}$$

It turns out that the three formulas (1)–(3) in the database can be represented by a digraph (this is described in detail in section 3), namely the one in figure 1.

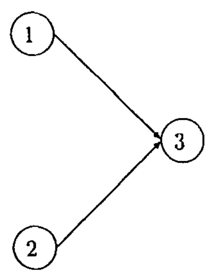


Figure 1.

From results in Andersen [1] (and reviewed as theorem 2 in section 3 in this paper), it follows that (4) is satisfiable iff $\pi_1 + \pi_3 \geq 1$ and $\pi_2 + \pi_3 \geq 1$. Furthermore, results in this paper show that the minimal probability of $x_3$ is equal to $\max\{0, \pi_1 + \pi_2 + \pi_3 - 2\}$ and that the maximal probability of $x_3$ is equal to $\pi_3$.

Notice that the digraph in figure 1 represents the logical formulas (1)–(3) and does *not* say anything about dependence or independence. *If* figure 1 were a Bayesian network *then* it would mean that the *probability* of node 3 *depends probabilistically* on the probabilities of nodes 1 and 2 and, furthermore, that nodes 1 and 2 are *probabilistically independent*. □

In general, suppose there are $n$ atomic propositions and $m$ formulas. Then there are $N = 2^n$ possible worlds. Let $p = (p_1,\ldots,p_N)^t$ be the distribution of probabilities over possible worlds (the exponent $t$ means transpose). Then the probability of a given formula is $a \cdot p$, where $a$ is a binary vector indicating in which possible worlds the formula is true. For formula $x_1$ in example 1, we have $a = (0, 0, 0, 0, 1, 1, 1, 1)$. Now let the matrix $A$ consist of rows $a_i$, $i = 1,\ldots,m$. The $i$th row indicates in which possible worlds the $i$th formula is true. Denote $\pi = (\pi_1,\ldots,\pi_m)^t$, where $\pi_i$ is the probability of the $i$th formula in the model. Then there is the following connection between $A$, $p$ and $\pi$:

$$Ap = \pi, \quad e^t p = 1, \quad p \geq 0. \tag{5}$$

In (5), the connection $Ap = \pi$ expresses the fact that the probability of a given formula should be equal to the sum of the probabilities of those possible worlds in which the formula is true. The last two statements in (5) mean that the vector $p = (p_1,\ldots,p_N)^t$ should be nonnegative and the sum of the coordinates should be equal to one ($e$ is the $N$-vector with all entries equal to 1). So what (5) really says is that $\pi$ makes the system satisfiable iff $\pi$ is a convex combination of the columns in $A$.

In example 1, we had

$$A = \begin{Bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \end{Bmatrix}.$$

The inference problem in probabilistic logic is the following one:

$$\begin{aligned} &\min / \max \quad a \cdot p \\ &\text{subject to} \quad (5), \end{aligned}$$

where $a$ is the binary vector indicating in which possible worlds the formula for which we want to determine its upper and lower probability is true. In the example we have $a = (0, 1, 0, 1, 0, 1, 0, 1)$.

## 3 Logical formulas associated with digraphs

A directed graph $\mathcal{D} = (\mathcal{V}, \mathcal{A})$ consists of a finite set $\mathcal{V} = \{1, 2,\ldots, n\}$ and a set $\mathcal{A} = \{e_1,\ldots,e_m\}$. $\mathcal{V}$ is called the nodeset of the digraph $\mathcal{D}$. The elements of $\mathcal{A}$ are ordered subsets of size 2 of $\mathcal{V}$. These elements are called arcs. The underlying graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is the graph on the same nodeset $\mathcal{V}$ and with edgeset $\mathcal{E}$. $\mathcal{E}$ is equal to $\mathcal{A}$ except that the subsets of $\mathcal{V}$ are not ordered.

In the following, we shall with every directed graph associate a set of logical formulas. We say that the digraph represents the set of formulas. The set of logical formulas that can be represented by digraphs is a subclass of the widely used Horn formulas.

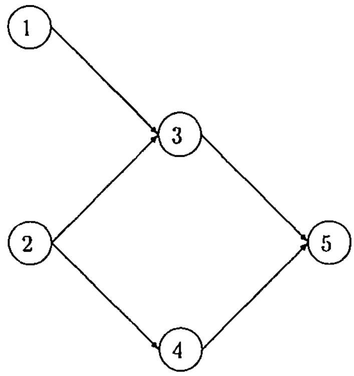


Figure 2.

**Example 2**
We consider the directed graph of figure 2, which represents the following set of logical formulas:

$$\begin{array}{ll} \text{Node 1:} & x_1 \\ \text{Node 2:} & x_2 \\ \text{Node 3:} & (x_1 \wedge x_2) \rightarrow x_3 \\ \text{Node 4:} & x_2 \rightarrow x_4 \\ \text{Node 5:} & (x_3 \wedge x_4) \rightarrow x_5 \end{array}$$

□

Given a digraph it represents the following set of logical formulas:

- If a node, say node $k$, has no incoming arcs, then it represents the formula $x_k$.
- If a node $k$ has several incoming arcs, say from nodes $i_1, i_2, \ldots, i_r$, then node $k$ represents the formula $(x_{i_1} \wedge x_{i_2} \ldots \wedge x_{i_r}) \rightarrow x_k$.

It should be noticed that the underlying graph of a digraph in principle represents several sets of logical formulas, namely one for each different orientation of the arcs (two digraphs are different if they are not isomorphic). For example, figure 3 is obtained from figure 2 by altering some of the directions of the arcs. The digraph in figure 3 represents the following set of logical formulas:

$$\begin{array}{ll} \text{Node 1:} & x_3 \rightarrow x_1 \\ \text{Node 2:} & (x_3 \wedge x_4) \rightarrow x_2 \\ \text{Node 3:} & x_5 \rightarrow x_3 \\ \text{Node 4:} & x_5 \rightarrow x_4 \\ \text{Node 5:} & x_5 \end{array}$$

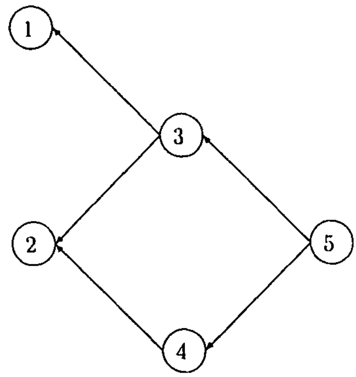


Figure 3.

Assume that we have been given a set of logical formulas represented by a digraph. It would be interesting to somehow characterize $A$ in the system (5). In fact, this has been done in Andersen [1]. Here, a characterization of the set of distinct columns of $A$ is given as the $0-1$ solutions to a certain linear system; see theorem 1 below. It turns out that if the underlying graph is either perfect or $t$-perfect, that result can be used to give a closed form solution set to the consistency problem in probabilistic logic for the set of formulas represented by the digraph. These results are given in theorems 2 and 3, respectively. The proofs of the theorems are somewhat technical and will not be reviewed in this paper. Before giving the results, a small illustrative example is given.

Let the nodes in the digraph be denoted by $\{1, 2, \ldots, n\}$. Let $x_i$, $i = 1, \ldots, n$, denote atomic propositions, one for each node in the digraph. Finally, let $y_i$, $i = 1, \ldots, n$, denote the truth values of the logical formulas associated with each node. The truth value of a particular formula $F$ is denoted by $v(F)$.

**Example 2 (continued)**

Let us consider the logical formulas associated with figure 2. We have five atomic propositions and five logical formulas. Now assign to each of the five nodes in the digraph a $y$-variable. The $y$'s should be the truth value of the formulas corresponding to the nodes. In this particular example, we have

$$\begin{aligned}
&\text{Node 1:} \quad y_1 = v(x_1) \\
&\text{Node 2:} \quad y_2 = v(x_2) \\
&\text{Node 3:} \quad y_3 = v((x_1 \wedge x_2) \rightarrow x_3) \\
&\text{Node 4:} \quad y_4 = v(x_2 \rightarrow x_4) \\
&\text{Node 5:} \quad y_5 = v((x_3 \wedge x_4) \rightarrow x_5)
\end{aligned}$$

It turns out that the set of distinct columns in $A$ are exactly the set of $0-1$ solutions to the following set of equations:

$$\begin{aligned} y_1 \quad + y_3 \qquad\qquad &\geq 1,\\ y_2 + y_3 \qquad\qquad &\geq 1,\\ y_2 \qquad + y_4 \qquad &\geq 1,\\ y_3 \qquad + y_5 &\geq 1,\\ y_4 + y_5 &\geq 1. \end{aligned}$$

In fact, the above equations can be rewritten as: $\{y \in B^n \mid Dy \geq e\}$, where $D$ is the arc-node incidence matrix for the underlying graph of the digraph, $B$ is the set of $\{0, 1\}$ vectors and $n$ is the number of nodes (or formulas). □

In example 2, we noticed that the set of distinct columns of $A$ could be characterized using the arc-node incidence matrix for the underlying graph of the digraph. This is true in general, as theorem 1 below shows.

**Theorem 1** [1]
Suppose we have a set of $n$ logical formulas represented by a connected digraph on $n$ nodes. The $i$th formula is the one corresponding to node $i$ in the digraph. Then the set of distinct columns of $A$ is exactly the set of solutions to the system $\{y \in B^n \mid Dy \geq e\}$, where $D$ is the arc-node incidence matrix for the underlying graph of the digraph. □

In fact, theorem 1 can be used to solve the satisfiability problem for some classes of logical formulas represented by digraphs, among them formulas represented by digraphs where the underlying undirected graph is perfect (theorem 2) or $t$-perfect (theorem 3). If $J \subseteq \{1,\ldots,n\}$, then $|J|$ denotes the number of elements in $J$.

**Theorem 2** [1]
Suppose we have a set of logical formulas represented by a connected digraph on $n$ nodes. Suppose that the underlying graph $\mathcal{G}$ is perfect. Then $\{Ap = \pi, e^t p = 1, p \geq 0\}$ is satisfiable if and only if $\pi \in \{x \in [0, 1]^n \mid Dx \geq e, \sum_{j \in \mathcal{Q}} x_j \geq |\mathcal{Q}| - 1$, for all maximal cliques $\mathcal{Q}$ in $\mathcal{G}\}$, where $D$ is the arc-node incidence matrix for $\mathcal{G}$. □

**Theorem 3** [1]
Suppose we have a set of logical formulas represented by a connected digraph on $n$ nodes. Suppose that the underlying graph $\mathcal{G}$ is $t$-perfect. Then $\{Ap = \pi, e^t p = 1, p \geq 0\}$ is satisfiable if and only if $\pi \in \{x \in [0, 1]^n \mid Dx \geq e, \sum_{j \in C} x_j \geq (|C| + 1)/2$, for all odd circuits $C$ in $\mathcal{G}\}$, where $D$ is the arc-node incidence matrix for $\mathcal{G}$. □

We will not discuss the properties of perfect or $t$-perfect graphs here. These two classes of graphs are well discussed in Grötschel et al. [4]. The book describes among other things several well-known classes of graphs which are perfect. It also describes how to recognize these graphs. In particular, if one can tell if a given graph is perfect or $t$-perfect, then theorem 2 or theorem 3 can be used to determine if a given assignment of probabilities to the logical formulas represented by the graph is consistent (meaning that (5) is feasible).

## 4 Probability bounds for atomic propositions associated with digraphs

Before proving our theorems, we shall illustrate the results in the following small example.

**Example 3**
We again consider the directed graph in figure 2. Now to each of the five formulas represented by the digraph we give a probability number, denoted by $\pi = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5)$. We will assume that the resulting probabilistic logical problem (5) is feasible. In fact, the graph in figure 2 is perfect and it follows from theorem 2 in section 3 that (5) is feasible iff:

$$\pi_1 + \pi_3 \geq 1, \;\; \pi_2 + \pi_3 \geq 1, \;\; \pi_2 + \pi_4 \geq 1, \;\; \pi_3 + \pi_5 \geq 1 \;\text{ and }\; \pi_4 + \pi_5 \geq 1,$$

where

$$\begin{array}{lrl} \text{(Node 1:)} & Pr(x_1) & = \pi_1 \\ \text{(Node 2:)} & Pr(x_2) & = \pi_2 \\ \text{(Node 3:)} & Pr((x_1 \wedge x_2) \rightarrow x_3) & = \pi_3 \\ \text{(Node 4:)} & Pr(x_2 \rightarrow x_4) & = \pi_4 \\ \text{(Node 5:)} & Pr((x_3 \wedge x_4) \rightarrow x_5) & = \pi_5 \end{array}$$

We want to determine the probabilities of formulas $x_1, \ldots, x_5$. The first two probabilities are given with the digraph. In this particular example, it will be demonstrated that

$$\begin{aligned} \max Pr(x_3) &= \pi_3 \\ \max Pr(x_4) &= \pi_4 \\ \max Pr(x_5) &= \pi_5 \end{aligned}$$

and that

$$\begin{aligned} \min Pr(x_3) &\geq \max\{0, \pi_1 + \pi_2 + \pi_3 - 2, 1 - \pi_5\} \\ \min Pr(x_4) &\geq \max\{0, \pi_2 + \pi_4 - 1, 1 - \pi_5\} \\ \min Pr(x_5) &\geq \max\{0, \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 - 4\} \end{aligned}$$

Table 2 shows the set of possible worlds and the associated probabilities for this example. The probabilities of the possible worlds are denoted by $p_1, p_2, \ldots, p_{32}$. How-

Table 2

Possible worlds for example 3.

| Possible world $(x_1, x_2, x_3, x_4, x_5)$ | *Pr.* | $x_1$ | $x_2$ | $(x_1 \wedge x_2) \rightarrow x_3$ | $x_2 \rightarrow x_4$ | $(x_3 \wedge x_4) \rightarrow x_5$ |
|---|---|---|---|---|---|---|
| (0, 0, 0, 0, 0) | $p_1$ | 0 | 0 | 1 | 1 | 1 |
| (0, 0, 0, 0, 1) | $p_2$ | 0 | 0 | 1 | 1 | 1 |
| (0, 0, 0, 1, 0) | $p_3$ | 0 | 0 | 1 | 1 | 1 |
| (0, 0, 0, 1, 1) | $p_4$ | 0 | 0 | 1 | 1 | 1 |
| (0, 0, 1, 0, 0) | $p_5$ | 0 | 0 | 1 | 1 | 1 |
| (0, 0, 1, 0, 1) | $p_6$ | 0 | 0 | 1 | 1 | 1 |
| (0, 0, 1, 1, 0) | $p_7$ | 0 | 0 | 1 | 1 | 0 |
| (0, 0, 1, 1, 1) | $p_8$ | 0 | 0 | 1 | 1 | 1 |
| (0, 1, 0, 0, 0) | $p_9$ | 0 | 1 | 1 | 0 | 1 |
| (0, 1, 0, 0, 1) | $p_{10}$ | 0 | 1 | 1 | 0 | 1 |
| (0, 1, 0, 1, 0) | $p_{11}$ | 0 | 1 | 1 | 1 | 1 |
| (0, 1, 0, 1, 1) | $p_{12}$ | 0 | 1 | 1 | 1 | 1 |
| (0, 1, 1, 0, 0) | $p_{13}$ | 0 | 1 | 1 | 0 | 1 |
| (0, 1, 1, 0, 1) | $p_{14}$ | 0 | 1 | 1 | 0 | 1 |
| (0, 1, 1, 1, 0) | $p_{15}$ | 0 | 1 | 1 | 1 | 0 |
| (0, 1, 1, 1, 1) | $p_{16}$ | 0 | 1 | 1 | 1 | 1 |
| (1, 0, 0, 0, 0) | $p_{17}$ | 1 | 0 | 1 | 1 | 1 |
| (1, 0, 0, 0, 1) | $p_{18}$ | 1 | 0 | 1 | 1 | 1 |
| (1, 0, 0, 1, 0) | $p_{19}$ | 1 | 0 | 1 | 1 | 1 |
| (1, 0, 0, 1, 1) | $p_{20}$ | 1 | 0 | 1 | 1 | 1 |
| (1, 0, 1, 0, 0) | $p_{21}$ | 1 | 0 | 1 | 1 | 1 |
| (1, 0, 1, 0, 1) | $p_{22}$ | 1 | 0 | 1 | 1 | 1 |
| (1, 0, 1, 1, 0) | $p_{23}$ | 1 | 0 | 1 | 1 | 0 |
| (1, 0, 1, 1, 1) | $p_{24}$ | 1 | 0 | 1 | 1 | 1 |
| (1, 1, 0, 0, 0) | $p_{25}$ | 1 | 1 | 0 | 0 | 1 |
| (1, 1, 0, 0, 1) | $p_{26}$ | 1 | 1 | 0 | 0 | 1 |
| (1, 1, 0, 1, 0) | $p_{27}$ | 1 | 1 | 0 | 1 | 1 |
| (1, 1, 0, 1, 1) | $p_{28}$ | 1 | 1 | 0 | 1 | 1 |
| (1, 1, 1, 0, 0) | $p_{29}$ | 1 | 1 | 1 | 0 | 1 |
| (1, 1, 1, 0, 1) | $p_{30}$ | 1 | 1 | 1 | 0 | 1 |
| (1, 1, 1, 1, 0) | $p_{31}$ | 1 | 1 | 1 | 1 | 0 |
| (1, 1, 1, 1, 1) | $p_{32}$ | 1 | 1 | 1 | 1 | 1 |

ever, in determining the probability bounds for the propositions $x_3$, $x_4$ and $x_5$, it is not necessary to consider all 32 possible worlds. For example, we only need 16 possible worlds to determine the probability bounds for $x_3$. For convenience we denote the probabilities of these 16 possible worlds by $p_1, p_2, \ldots, p_{16}$ even though it is not the first 16 possible worlds in table 2. The same thing has been done when establishing the probability bounds of $x_4$ and $x_5$.

*Probability bounds for $x_3$*
The linear programming problem is

$$\begin{aligned} \min / \max \quad & cp \\ \text{subject to} \quad & Ap = \pi, \\ & \sum_{i=1}^{16} p_i = 1, \\ & p_i \geq 0, \quad i = 1, \ldots, 16, \end{aligned} \tag{6}$$

where $c$ and $A$ are given as

$$c = \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1\},$$

$$A = \begin{Bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{Bmatrix}.$$

Notice that duplicate columns have been removed from the program (two columns are duplicate if they occur with the same coefficients in both the objective function and in the constraints). The above 16 columns correspond to the possible worlds 1, 9, 11, 17, 25, 27, 5, 7, 13, 15, 16, 21, 23, 29, 31 and 32, respectively.

Obviously the probability of $x_3$ is at least 0. From the fifth row in the program, we see that

$$\begin{aligned} & p_1 + p_2 + p_3 + p_4 + p_5 + p_6 \leq \pi_5 \\ & \Downarrow \\ & \pi_5 + Pr(x_3) \geq \textstyle\sum_{j=1}^{6} p_j + \sum_{j=7}^{16} p_j = 1 \\ & \Downarrow \\ & Pr(x_3) \geq 1 - \pi_5 \end{aligned}$$

(The reason the fifth row was used here is that if $v(x_3) = 0$, then $v((x_3 \wedge x_4) \rightarrow x_5) = 1$, which in turn implies that $Pr(x_3) + Pr((x_3 \wedge x_4) \rightarrow x_5) \geq 1$.)

We now show that the lower bound probability for $x_3$ is at least $\pi_1 + \pi_2 + \pi_3 - 2$.

Summing the first three equations in the program results in

$$p_1 + 2p_2 + 2p_3 + 2p_4 + 2p_5 + 2p_6 + p_7 + p_8 + 2p_9 + 2p_{10} + 2p_{11} + 2p_{12} + 2p_{13} + 3p_{14} + 3p_{15} + 3p_{16} = \pi_1 + \pi_2 + \pi_3$$

$$\Downarrow$$

$$2\sum_{j=1}^{16} p_j + p_{14} + p_{15} + p_{16} \geq \pi_1 + \pi_2 + \pi_3$$

$$\Downarrow$$

$$p_{14} + p_{15} + p_{16} \geq \pi_1 + \pi_2 + \pi_3 - 2$$

$$\Downarrow$$

$$Pr(x_3) = p_7 + \ldots + p_{16} \geq \pi_1 + \pi_2 + \pi_3 - 2$$

We next want to show that the upper probability bound for $x_3$ is given by $\pi_3$ and that this maximal probability is attained. Obviously, $\pi_3$ is an upper bound (if $v(x_3) = 1$, then also $v((x_1 \wedge x_2) \to x_3) = 1$). Now notice that if $(x_1 \wedge x_2) \to x_3$ is true in the $i$th column in $A$ with the $i$th objective function coefficient equal to zero, then there is another column in $A$, say the $j$th, such that the $i$th and the $j$th column are equal and the $j$th objective function coefficient is equal to 1 (this is true in general, as proposition 1 in this section shows). Therefore, if $p_i$ is positive, then we can construct another feasible $p$-vector by just adding the value of $p_i$ to the value of $p_j$ and then setting the new value of $p_i$ equal to zero. In this way we obtain a solution, such that positive $p$'s occur only in the third row (the $(x_1 \wedge x_2) \to x_3$ row) when there is also a 1 in the corresponding $c$-coefficient. But this gives a feasible solution with value $\pi_3$.

*Probability bounds for $x_4$*

The linear programming problem is

$$\begin{aligned} \text{min / max} \quad & cp \\ \text{subject to} \quad & Ap = \pi, \\ & \sum_{i=1}^{14} p_i = 1, \\ & p_i \geq 0, \quad i = 1, \ldots, 14, \end{aligned} \tag{7}$$

where $c$ and $A$ are given as

$$c = \{0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1\},$$

$$A = \left\{\begin{matrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{matrix}\right\}.$$

Obviously, the probability of $x_4$ is at least 0. From the fifth row in the program, we see that

$$\begin{aligned} & p_1 + p_2 + p_3 + p_4 + p_5 \le \pi_5 \\ & \Downarrow \\ & \pi_5 + Pr(x_4) \ge \textstyle\sum_{j=1}^{5} p_j + \sum_{j=6}^{14} p_j = 1 \\ & \Downarrow \\ & Pr(x_4) \ge 1 - \pi_5 \end{aligned}$$

We now show that the lower bound probability for $x_4$ is at least $\pi_2 + \pi_4 - 1$.

Summing the second and the fourth equations in the program results in

$$\begin{aligned} & p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + 2p_8 + 2p_9 + p_{10} + p_{11} \\ & \qquad + 2p_{12} + 2p_{13} + 2p_{14} = \pi_2 + \pi_4 \\ & \Downarrow \\ & p_8 + p_9 + p_{12} + p_{13} + p_{14} = \pi_2 + \pi_4 - 1 \\ & \Downarrow \\ & Pr(x_4) = p_6 + \ldots + p_{14} \ge \pi_2 + \pi_4 - 1 \end{aligned}$$

We also need to show that the upper bound on the probability of $x_4$ is equal to $\pi_4$. However, exactly the same argument that we used in the case of $x_3$ also applies in this case.

*Probability bounds for $x_5$*

The linear programming problem is

$$\begin{aligned} \min / \max \quad & cp \\ \text{subject to} \quad & Ap = \pi, \\ & \sum_{i=1}^{19} p_i = 1, \\ & p_i \ge 0, \quad i = 1, \ldots, 19, \end{aligned} \tag{8}$$

where $c$ and $A$ are given as

$$c = \{0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 1 \; 1 \; 1 \; 1 \; 1 \; 1 \; 1 \; 1\},$$

$$A = \left\{ \begin{matrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{matrix} \right\}.$$

Obviously, the minimal probability of $x_5$ is at least 0. So let us show that it is also at least $\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 - 4$.

Summing the five equations in the program results in

$$3p_1 + 2p_2 + 3p_3 + 4p_4 + 3p_5 + 4p_6 + 3p_7 + 3p_8 + 4p_9 + 4p_{10} + 4p_{11} + 3p_{12} + 3p_{13} + 4p_{14} + 4p_{15} + 3p_{16} + 4p_{17} + 4p_{18} + 5p_{19} = \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5$$

From this we obtain

$$4(p_1 + \ldots + p_{18}) + 5p_{19} \geq \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5$$
$$\Downarrow$$
$$4(p_1 + \ldots + p_{19}) + p_{19} \geq \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5$$
$$\Downarrow$$
$$p_{19} \geq \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 - 4(p_1 + \ldots + p_{19})$$

This shows that $p_{19} \geq \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 - 4$.

Noticing that $Pr(x_5) = p_{12} + \ldots + p_{19}$, we see that a lower bound for $Pr(x_5)$ is given by $\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 - 4$.

We need to show that the upper bound of the probability of $x_5$ is equal to $\pi_5$. Also in this case, the same argument that was used for $x_3$ applies. □

We are now in a position to prove the results illustrated in example 3. We start by finding the upper bound probability of the atomic propositions associated with a digraph (theorem 4). To do that we need the following simple proposition.

**Proposition 1**

Assume that there is a column $i$ in $A$ with the property that $v((x_{i_1} \wedge \ldots \wedge x_{i_r}) \rightarrow x_k) = 1$ and $v(x_k) = 0$. Then there is another column $j$ in $A$ with the property that column $j$ is equal to column $i$ and $v(x_k) = -1$.

*Proof*

Assume that $v((x_{i_1} \wedge \ldots \wedge x_{i_r}) \rightarrow x_k) = 1$ and $v(x_k) = 0$.

It has been shown in Andersen [1] that if $y$ is a column in $A$ then it can be generated using the possible world $x$ defined by $v(x_j) = y_j$, $j = 1,\ldots,n$. Now let $y$ be the $i$th column in $A$. Then the possible world $x$ defined by $v(x_j) = y_j$, $j = 1,\ldots,n$, will generate a column equal to the $i$th column in $A$ but with the property that $v(x_k) = 1$. This proves the result. □

In theorem 4, lemma 1 and lemma 2, the following assumption is used.

**Assumption**

A set of $n$ logical formulas represented by a connected digraph on $n$ nodes is given. The $i$th formula is the one corresponding to node $i$ in the digraph. It is assumed that the probabilistic logical problem is feasible, that is, $\pi$ is chosen such that the system $\{Ap = \pi, e^t p = 1, p \geq 0\}$ is non-empty.

If the graph is perfect ($t$-perfect), then theorem 2 (theorem 3) presented in section 3 can be used to determine whether the assumption is satisfied.

**Theorem 4**

Assume that the $k$th formula is given by: $(x_{i_1} \wedge \ldots \wedge x_{i_r}) \rightarrow x_k$. Then the maximal probability of $x_k$ is equal to $\pi_k$.

*Proof*

We first observe that $Pr(x_k) \leq Pr((x_{i_1} \wedge \ldots \wedge x_{i_r}) \rightarrow x_k)$. This follows from the fact that if $v(x_k) = 1$, then also $v((x_{i_1} \wedge \ldots \wedge x_{i_r}) \rightarrow x_k) = 1$. We therefore need to establish that some feasible solution $p$ to the system (5) has the property that $Pr(x_k) = Pr((x_{i_1} \wedge \ldots \wedge x_{i_r}) \rightarrow x_k)$. From this the result will follow.

Assume that $p$ is a feasible solution to the system. Then $p$ has the property that $Pr((x_{i_1} \wedge \ldots \wedge x_{i_r}) \rightarrow x_k) = \pi_k$. Now assume that some $p_i > 0$, where $i$ is some column in $A$ with the property that $v((x_{i_1} \wedge \ldots \wedge x_{i_r}) \rightarrow x_k) = 1$ and $v(x_k) = 0$. Using proposition 1, we notice that there exists some other column $j$ in $A$ with the property that column $i$ is equal to column $j$ and such that $v(x_k) = 1$. Therefore, we just construct another $p$-vector, say $\bar{p}$, defined as follows:

$$\bar{p}_l = \begin{cases} p_j + p_i & \text{if } l = j, \\ 0 & \text{if } l = i, \\ p_l & \text{otherwise.} \end{cases}$$

Clearly, this procedure produces a sequence of feasible $p$-vectors. The procedure ends with a feasible $p$-vector with the property that $Pr(x_k) = Pr(((x_{i_1} \wedge \ldots \wedge x_{i_r}) \rightarrow x_k)$. This completes the proof. □

We next want to determine the lower bound probabilities on the atomic propositions associated with a digraph (lemma 1). It turns out that it is necessary to assume that the digraph is acyclic (in this case, the nodes of the digraph can be labeled such that each arc goes from one node to another node with a higher number). Also, the lower bound probability is not necessarily attained. However, a condition on the graph can be imposed such that the lower bound is actually attained (lemma 2).

**Lemma 1**

Consider some particular node, say node $k$. Furthermore, consider all directed paths from nodes in the acyclic digraph to node $k$. Assume that there are directed paths from nodes $i_1, \ldots, i_r$ to node $k$. Then a lower bound on $Pr(x_k)$ is given by

$$\max\left\{0, \max\left\{1 - \pi_j \,|\, (k, j) \in \mathcal{A}\right\}, \sum_{j=1}^{r} \pi_{i_j} + \pi_k - r\right\}.$$

*Proof*
Clearly, $Pr(x_k) \geq 0$.

Next assume that $(k, j) \in \mathcal{A}$ and assume that node $j$ represents the formula

$$(x_k \wedge x_{j_1} \wedge \ldots \wedge x_{j_s}) \rightarrow x_j,$$

where $(j_1, j) \in \mathcal{A}, \ldots, (j_s, j) \in \mathcal{A}$. If $v(x_k) = 0$, then $v((x_k \wedge x_{j_1} \wedge \ldots \wedge x_{j_s}) \rightarrow x_j) = 1$. But then $Pr(x_k) + Pr((x_k \wedge x_{j_1} \wedge \ldots \wedge x_{j_s}) \rightarrow x_j) \geq 1$. This shows that $Pr(x_k) \geq 1 - \pi_j$, $(k, j) \in \mathcal{A}$.

$A$ has the following structure:

$$A = \begin{Bmatrix} B & \vdots & C \\ \cdots & & \cdots \\ D & \vdots & E \end{Bmatrix}.$$

$A$ has been partioned into four matrices, namely $B$, $C$, $D$ and $E$. We assume that the first $(r+1)$ rows of $A$ correspond to formulas for nodes $i_1, \ldots, i_r, k$. The remaining rows are for the formulas corresponding to nodes from which there are no directed path to node $k$. Assume that the number of columns in $A$ in which the first $(r+1)$ elements is 1 is given by $a_1$ and that these $a_1$ columns are the first columns in $A$. Then the dimension of $B$ is $(r+1) \times a_1$, the dimension of $C$ is $(r+1) \times (N - a_1)$, the dimension of $D$ is $(n - r - 1) \times a_1$, and finally the dimension of $E$ is $(n - r - 1) \times (N - a_1)$. Furthermore, all entries in $B$ are equal to 1.

We need to show that $Pr(x_k) \geq \sum_{j=1}^{r} \pi_{i_j} + \pi_k - r$ for every feasible solution $p$.

We start by noticing that if the formulas corresponding to nodes $i_1, \ldots, i_r, k$ are all true, then $v(x_k) = 1$. This follows easily from the fact that the digraph is acyclic.

Now summing rows 1 through $(r+1)$ of $A$ yields $(N = 2^n)$

$$(r+1) \sum_{j=1}^{a_1} p_j + \sum_{j=a_1+1}^{N} c_j p_j = \sum_{j=1}^{r} \pi_{i_j} + \pi_k.$$

Here, all $c_j$'s are numbers between 0 and $r$. Therefore, we get

$$(r+1) \sum_{j=1}^{a_1} p_j + r \sum_{j=a_1+1}^{N} p_j \geq \sum_{j=1}^{r} \pi_{i_j} + \pi_k.$$

But this means

$$\sum_{j=1}^{a_1} p_j + r \sum_{j=1}^{N} p_j \geq \sum_{j=1}^{r} \pi_{i_j} + \pi_k.$$

Therefore,

$$\sum_{j=1}^{a_1} p_j \geq \sum_{j=1}^{r} \pi_{i_j} + \pi_k - r,$$

which in turn implies

$$Pr(x_k) \geq \sum_{j=1}^{a_1} p_j \geq \sum_{j=1}^{r} \pi_{i_j} + \pi_k - r.$$

This proves the lemma. □

We now prove that the lower bound probability of the atomic propositions associated with the nodes in a digraph can actually be attained (lemma 2). Under the condition in lemma 2, we can therefore give the exact probability interval of the atomic propositions (corollary 1).

**Lemma 2**

Assume that $\sum_{j=1}^{n} \pi_j \geq (n-1)$. Let $k \in \{1, 2, \ldots, n\}$ and assume there are directed paths from nodes $i_1, \ldots, i_r$ to node $k$ in the acyclic digraph. Then the minimal probability of formula $x_k$ is equal to $\sum_{j=1}^{r} \pi_{i_j} + \pi_k - r$.

*Proof*

If $\sum_{j=1}^{n} \pi_j \geq (n-1)$, then also $\sum_{j \in J} \pi_j \geq (|J|-1)$, where $J \subseteq \{1, \ldots, n\}$. In particular, $\sum_{j=1}^{r} \pi_{i_j} + \pi_k \geq r$. It follows from lemma 1 that we only need to show that there is a feasible vector $p$ to (5) such that $Pr(x_k) = \sum_{j=1}^{r} \pi_{i_j} + \pi_k - r$ (the condition also implies that if $(k, j) \in \mathcal{A}$, then $\sum_{j=1}^{r} \pi_{i_j} + \pi_k - r \geq (1 - \pi_j)$).

Consider the following set of $(n+1)$ columns in $A$:

$$\left\{\begin{matrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 1 \end{matrix}\right\} \begin{matrix} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{matrix}$$

It follows from theorem 1 that these columns are in fact present in $A$.

To these columns we distribute the probabilities:

$$\left(1 - \pi_1, 1 - \pi_2, \ldots, 1 - \pi_n, \sum_{j=1}^{n} \pi_j - (n-1)\right).$$

All other columns in $A$ are given the probability 0. It is easily seen that this assignment of probabilities satisfies (5).

We now find $(n+1)$ possible worlds that generate the above $(n+1)$ columns in $A$.

- Column $k$.

  This column is generated by the possible world

$$v(x_j) = \begin{cases} 0 & \text{if } j = k, \\ 1 & \text{otherwise.} \end{cases}$$

In particular, $v(x_k) = 0$.

- Column $i$, where $i \in \{1,\ldots,n\} \setminus \{i_1,\ldots,i_r, k\}$.

  Then the $i$th column is generated by the possible world

$$v(x_j) = \begin{cases} 0 & \text{if } j = i, \\ 1 & \text{otherwise.} \end{cases}$$

  In particular, $v(x_k) = 1$.

- Column $i$, where $i \in \{i_1,\ldots,i_r\}$.

  Assume that there is a directed path from node $i$ to node $k$ using nodes $s_1,\ldots,s_l$ in that order (so $s_1,\ldots,s_l \in \{i_1,\ldots,i_r\}$), i.e. the path is $i \to s_1 \to \ldots \to s_l \to k$. Then the $i$th column is generated by the possible world (using the fact that the digraph is acyclic)

$$v(x_j) = \begin{cases} 0 & \text{if } j \in \{i, s_1, \ldots, s_l, k\}, \\ 1 & \text{otherwise.} \end{cases}$$

  In particular, $v(x_k) = 0$.

- Column $(n + 1)$.

  This column is generated by the possible world $(x_1,\ldots,x_n) = (1,\ldots,1)$.
  In particular, $v(x_k) = 1$.

We have now found $(n + 1)$ possible worlds which generate the chosen $(n + 1)$ columns in $A$. We have also found the corresponding truth value of $x_k$.

With the chosen assignment of probabilities to the above-mentioned possible worlds, the corresponding probability of $x_k$ can be calculated:

$$\begin{aligned} Pr(x_k) &= \sum_{j \in \{1,\ldots,n\} \setminus \{i_1,\ldots,i_r,k\}} (1 - \pi_j) + \left( \sum_{j=1}^{n} \pi_j - (n-1) \right) \\ &= (n - r - 1) - \sum_{j \in \{1,\ldots,n\} \setminus \{i_1,\ldots,i_r,k\}} \pi_j + \sum_{j=1}^{n} \pi_j - (n-1) \\ &= \sum_{j=1}^{r} \pi_{i_j} + \pi_k - r. \end{aligned}$$

This proves the lemma. □

This lemma gives rise to the following corollary.

**Corollary 1**

Suppose the assumptions in lemma 2 are fulfilled. Then: $Pr(x_k) \in [\sum_{j=1}^{r} \pi_{i_j} + \pi_k - r, \pi_k]$.

*Proof*
This follows directly from theorem 4 and lemma 2. □

It is interesting to note that the lower bound probability of $x_k$ in corollary 1 depends only on the sum of the probabilities of the logical formulas and not on that of any particular formula, as long as the sum of the probabilities is at least $(n-1)$.

The following example illustrates that the result in lemma 1 is not necessarily true if the digraph is not acyclic. It also shows that the lower bound probabilities given in lemma 1 are not sharp even if the digraph is acyclic.

**Example 4**

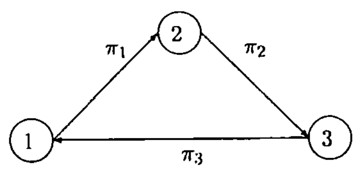


Figure 4.

The digraph in figure 4 corresponds to the formulas

$$x_1 \to x_2$$
$$x_2 \to x_3$$
$$x_3 \to x_1$$

Suppose that the probabilities $(\pi_1, \pi_2, \pi_3) = (0.8, 0.8, 0.8)$. In this case, the minimum probability of $x_1$ is 0.2, and $\pi_1 + \pi_2 + \pi_3 - 2 = 0.4$ (lemma 1 does not work here).

As another example, consider the directed graph in figure 2.

Assume that $(\pi_1, \pi_2, \pi_3, \pi_4, \pi_5) = (0.5, 0.55, 0.6, 0.65, 0.8)$. The minimum probability of $x_4$ in this particular case is 0.25. Furthermore, $\pi_2 + \pi_4 - 1 = 0.2$, and $1 - \pi_5 = 0.2$. We see that we can only give a lower bound on the probability of $x_4$ and not the exact lower probability number.

One could imagine that lemma 2 would also be true if for a particular end-node $k$ it is true that $\sum_{j=1}^{r} \pi_{i_j} + \pi_k - r \geq 0$, where $i_1, \ldots, i_r$ is the set of nodes from which there exists a directed path to node $k$. However, this is not true, as illustrated below. Consider the digraph in figure 5. This digraph corresponds to the formulas

$$x_1$$
$$x_1 \to x_2$$
$$x_2 \to x_3$$
$$x_2 \to x_4$$

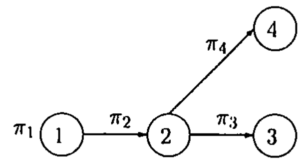


Figure 5.

Suppose that the probabilities $(\pi_1, \pi_2, \pi_3, \pi_4) = (0.7, 0.8, 0.7, 0.4)$. If one could use lemma 2, in this case the minimum probability of $x_3$ should be equal to $\pi_1 + \pi_2 + \pi_3 - 2 = 0.7 + 0.8 + 0.7 - 2 = 0.2$. However, the correct minimal probability is 0.3. □

It can be explained why lemma 1 (and lemma 2 and corollary 1) does not work for a digraph like the one in figure 4. Notice that in figure 4 there is a directed path from node 1 to node 1 passing through all other nodes in the digraph. In the proof of lemma 1 we used the fact that if all the formulas corresponding to nodes on a directed path to node $k$ were true for a possible world, then also $v(x_k) = 1$ (in the example, $k = 1$). However, this is not true in the example because the possible world $(x_1, x_2, x_3) = (0, 0, 0)$ makes all three logical formulas in figure 4 true (but $v(x_1) = 0$). The same reasoning applies, of course, for any digraph which contains a directed cycle (of length at least 2). This is the main reason why we have to require that the digraphs are acyclic in order to prove results like those in lemma 1, lemma 2 and corollary 1.

## 5 Conclusion

In this paper it has been demonstrated how to find probability bounds for atomic propositions associated with digraphs. It has been shown that the upper bound can be calculated exactly, whereas in most cases it is only possible to find lower bounds which are not necessarily attained. An exception occurs when the probabilities of the logical formulas in question are sufficiently high. In that case, an exact probability interval for the possible probabilities of the atomic propositions can be given.

## References

[1] K.A. Andersen, Characterizing consistency in probabilistic logic for a class of Horn clauses, Mathematical Programming 66, 1994, 257–271.

[2] K.A. Andersen and J. N. Hooker, Bayesian logic, Decision Support Systems 11, 1994, 191–210.

[3] V. Chandru and J. N. Hooker, *Optimization Methods for Logical Inference*, Wiley, to appear.

[4] M. Grötschel, L. Lovász and A. Schrijver, *Geometric Algorithms and Combinatorial Optimization*, Springer, Berlin, 1988.

[5] T. Hailperin, *Boole's Logic and Probability*, 2nd ed., Studies in Logic and the Foundations of Mathematics vol. 85, North-Holland, 1986.

[6] N.J. Nilsson, Probabilistic logic, Artificial Intelligence 28, 1986, 71–87.

[7] J. Pearl, *Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference*, Morgan Kaufmann, San Mateo, CA, 1988.