Transpn. Res.-A Vol. 22A, No. 3, pp. 183–201, 1988
Printed in Great Britain.

0191-2607/88 $3.00+.00

OPTIMAL DRIVING FOR SINGLE-VEHICLE FUEL ECONOMY

J. N. HOOKER
Graduate School of Industrial Administration, Carnegie-Mellon University, Pittsburgh, PA 15213, U.S.A.

(*Received* 11 *March* 1986; *in revised form* 15 *May* 1987)

Abstract—This study uses fuel consumption simulators for 15 late-model automobiles to determine how one ought to drive to maximize fuel economy. The simulation is based on extensive on-road and dynamometer testing of the 15 cars. Dynamic programming is used to determine the optimal way to accelerate from rest to cruising speed, to drive a block between stop signs, and to cruise on hilly terrain while maintaining a given average speed. The dependence of fuel economy on cruising speed is also characterized for various road grades. Findings include that optimal speeds are generally higher for larger cars and higher on downgrades than on upgrades, and that the relative fuel penalty for exceeding the speed limit is no worse for small cars than large cars. Optimal control for accelerate-and-cruise and for driving between stop signs varies considerably from car to car; in the latter case fuel economy is much improved by achieving a rather low peak speed. Optimal control on hills is consistent from car to car and can achieve fuel economy 7% to 30% better than that of constant-speed driving on 3% to 6% grades. Results that appear generalizable to other cars are reduced to advice for the fuel-conscious driver.

Driving style can have a significant bearing on fuel consumption, but it is often unclear how one should control the car to get the best possible fuel economy. The aim of the study described here is to generate some reliable advice in this matter. It does so by calculating optimal control of speed for fuel economy in several typical driving situations. The calculations are based on computer simulations of 15 late-model automobiles.

The simulated automobiles represent five manufacturers and range from a small 4-cylinder car to a large V-8 diesel pickup truck (Table 1). The simulation is based directly on extensive on-road and dynamometer testing of the 15 cars. To the writer's knowledge, vehicle-specific simulators for such a wide variety of cars have never before been available. They provide a unique opportunity to compute optimal control for a fairly representative cross-section of automobiles now on the road in the United States.

The motions of other vehicles and the timing of traffic signals can obviously have a bearing on optimal driving, but in this study we make no attempt to account for such interactions. Rather, we suppose that surrounding traffic does not prevent one from driving his car in an optimal fashion and that the timing of signals is not a factor.

Even so, optimal control depends on many varying details of automobile and road, and it would be impossible to compute optimal control for every possible situation. But optimal control calculated for a few typical situations can serve as an example for drivers in similar situations, since we will see that much of the benefit of optimal driving can be achieved by approximating it only roughly.

In this study, optimality and parametric analysis are carried out so as to answer four basic questions:

(i) What is the optimal way to cruise on a level road or on a constant grade, and how does fuel economy vary with speed?
(ii) What is the optimal way to accelerate from rest to a cruising speed typical of surburban or highway driving, and how does fuel economy depend on the rate of acceleration?
(iii) What is the optimal way to drive a block between two stop signs, and how does fuel economy depend on the duration of the trip?
(iv) What is the optimal way to drive over hills so as to achieve a given average speed, and how does the resulting fuel economy compare with that of constant-speed driving?

Most of the optimal control problems are solved for 8 of the 15 simulated vehicles, for a total of 52 solutions. The parametric analysis is carried out for all 15 cars. Since all the cars have automatic transmissions, optimal control of speed alone (not gearshift) is calculated directly.

Relatively little work has been done in the area of determining optimal driver control for fuel economy. Several investigators have measured the effect on fuel economy (and driving behavior) of following generalized instructions, such as "drive as economically as possible," "drive like a very cautious driver," "maintain the vacuum gauge in the green region," etc. (Everall, 1968; Chang *et al.*, 1976a, 1976b; Chang and Herman, 1976; Evans, 1979). Evans and Takasaki (1981) and Akcelik and Biggs (1987) studied how the rate of acceleration from rest

Table 1. The 15 simulated vehicles

Vehicle Make & Model	Engine Type	Engine Displacement (Liters)	Fuel Metering	Transmission Type†
1982 Datsun 210	4 cyl.	1.5	Carburetor with return fuel line	A3L
1983 Ford Escort‡	4 cyl.	1.6	Carburetor with return fuel line	A3
1982 Toyota Corolla‡	4 cyl.	1.8	Carburetor	A3
1982 Chevrolet Chevette	4 cyl. diesel	1.8	Injector	A3
1984 Chevrolet S-10 pickup‡	4 cyl.	2.0	Carburetor	A4
1982 Ford Fairmont	4 cyl.	2.3	Carburetor	A3
1982 Chevrolet Citation‡	4 cyl.	2.5	Throttle-body injection	A3
1983 Plymouth Reliant‡	4 cyl.	2.6	Carburetor	A3
1983 Pontiac Firebird‡	V-6	2.8	Carburetor	A4
1982 Ford Futura	6 cyl.	3.3	Carburetor	A3
1983 Chevrolet Monte Carlo‡	V-6	3.75	Carburetor	A3
1981 Buick Century‡	V-6	3.8	Carburetor	A3
1982 Chevrolet Caprice station wagon‡	V-8	5.0	Carburetor	A4L
1981 Chevrolet Caprice	V-8 diesel	5.7	Injector	A3
1983 Chevrolet Silverado pickup	V-8 diesel	6.2	Injector	A4

†A3 and A4 indicate 3-speed and 4-speed automatic transmissions, respectively. An L indicates that the torque converter has a lockup feature.

‡Most of the optimal control problems are solved for these eight cars.

to cruising speed affects "excess" acceleration fuel consumption, which is total fuel consumption during acceleration minus the fuel the car would have consumed while covering the same distance at the cruising speed (see also Biggs and Akcelik, 1985). But these investigators did not compute optimal control, and the minimization of excess fuel consumption is often not an appropriate objective, as we explain in Section 2.

Apparently, the first attempt to calculate optimal trajectories was that of Schwarzkopf and Leipnik (1977), who did so both for acceleration to cruising speed and for driving up a hill to a plateau. But their analytical solution technique (Pontriagin maximum principle) obliged them to model the car's fuel consumption rate with a highly simplified function. Hooker *et al.* (1983b) obtained greater accuracy by calculating optimal control with a fuel consumption simulator and a dynamic programming technique very much like those described here. But they did so for only one car, a 1979 Ford Fairmont station wagon. The present study extends this work to 15 automobiles and benefits from refinements in the simulation and dynamic programming techniques that have been achieved since the earlier paper. It is more completely described in a longer report (Hooker, 1985).

Below we first describe the simulation method, the mathematical formulation of the problem, the importance of convexity in the fuel-flow function, and the solution technique (Sections 1–4). Nontechnical readers may skip Sections 2–4 without serious loss of continuity. We then describe the results (Sections 5–8). Afterwards we draw some general conclusions (Section 9) and formulate some recommendations for efficient driving (Sections 10–13).

1. THE SIMULATION METHOD

The simulation technique is only briefly described here; see Hooker *et al.* (1983a), McGill *et al.* (1985) and Rose *et al.* (1982) for details.

The simulator relies on a statistical rather than an engineering model of the car's behavior. Its predictions are based not on how the car *should* behave, given its design, but on how the car *does* behave on a chassis dynamometer and a test track. This approach forgoes an understanding of the mechanisms that influence fuel economy, but it permits a single body of software to produce highly data-intensive simulations of 15 cars that have very different sets of performance characteristics.

The simulator is built in three stages. In the first stage the car is mounted on a chassis dynamometer, where a few thousand observations of steady-state fuel flow rates are made at various engine speeds and loads. Engine load is indicated by intake manifold vacuum in gasoline engines and throttle position in diesel engines. Regression and cubic spline interpolation techniques derive from the data points a single, smooth, piecewise cubic surface that represents the fuel flow rate as a function of engine speed and load. Dynamometer testing is necessary because it is difficult to measure instantaneous fuel flow rates on the road while the car is accelerating.

In the second stage, the car is driven on a level track so as to produce nearly all speeds and accelerations in its operating range. Several thousand measurements of engine speed and load are taken at various speeds and accelerations. The operating region of the car in each gear is inferred from the distribution of observations in the speed/acceleration plane.

The final stage relies on the fact that conditions in the carburetor or fuel injector during acceleration, as indicated by engine speed and load, can be very nearly duplicated on the dynamometer in steady state, where fuel flow rates can be accurately measured. Thus each engine speed and load measurement taken on the road is replaced with the corresponding fuel flow rate predicted in the first stage. The resulting data are used to build for each gear a piecewise cubic surface showing fuel flow rate as a function of vehicle speed and acceleration, using the same technique as in the first stage. Corrections for ambient temperature were derived, and all simulations done for this study assume a 20°C (68°F) ambient temperature.

To simulate a car's behavior on a grade, the acceleration is taken to be the "effective acceleration"; i.e., the algebraic sum of the vehicle's acceleration in the direction of motion and gravitational acceleration times the sine of the angle of road slope.

The simulation software and data sets are available to the public and may be obtained by contacting the author.

2. FORMULATION OF THE OPTIMAL CONTROL PROBLEM

The problem of optimal driver control can be formulated mathematically as follows. Let $f(v,a)$ be the automobile's rate of fuel consumption at speed v and effective acceleration a. Then the optimal control problem is,

$$\min_{v,T} \int_0^T f(v(t),a(t))\mathrm{d}t \qquad (1)$$

subject to

$$\begin{aligned} \dot{s}(t) &= v(t) \\ a(t) &= \dot{v}(t) + g \sin \theta(s(t)) \\ a(t) &\leq a_{\max}(v(t)) \\ v(0) &= v_0, \quad s(0) = 0 \\ v(T) &= v_1, \quad s(T) = s_1 \text{ (optional)} \\ T &= T_0 \text{ (optional).} \end{aligned}$$

The object is to find a speed trajectory $v(t)$ and duration T that minimize total fuel consumption, as measured by the integral in (1). T may be fixed at T_0 if desired. The system equation is $\dot{s}(t) = v(t)$. The angle of road slope $\theta(s)$ is given at any distance s from the start of the road. The maximum acceleration $a_{\max}(v)$ at any given speed v is dictated by maximum engine power. An initial speed v_0 and optionally a terminal speed v_1 are given as boundary conditions. Also one can set the distance s_1 to be covered so as to require the car to achieve an average speed s_1/T.

The following special cases of the optimal control problem are investigated in this study.

The crusing problem. The cruising problem asks how one can minimize fuel consumption on a level road while achieving a stated average speed s_1/T_0. We set $v_0 = v_1 = s_1/T_0$, $t = T_0$, and $\theta(s) = 0$ for all s.

Acceleration to cruising speed. The problem is to determine how to accelerate from rest to cruising speed most efficiently, where cruising speed is to be maintained for a long period. It may appear that one should formulate the problem simply by requiring the car optimally to achieve a given terminal speed v_1 while covering a given distance s_1. But this is an unsatisfactory formulation when the car's most efficient speed v^* (generally 30–75 km/h, or 20–45 mph) is less than v_1, as it generally is in this study. It is unsatisfactory because for any sufficiently large s_1, an optimally driven car simply accelerates to v^* and cruises at v^* until s_1 is nearly covered, when it accelerates to v_1. Thus the optimal trajectory depends on an arbitrary choice of s_1. The situation is equally unsatisfactory if one does not fix s_1 and minimizes the "excess" acceleration fuel consumption. The reason is that for any trajectory one can always produce a better one by cruising sufficiently long at speed v^* before accelerating to v_1; the "excess" fuel consumption in fact eventually becomes increasingly negative as one cruises longer and longer at v^*. Thus there is no finite optimal trajectory. (See Hooker, 1983b, pp. 155–157, for a more detailed discussion of this point.) One way to overcome these difficulties is to define the problem over a very long fixed time horizon T and very long fixed distance s_1, adjusted so that the average speed s_1/T is about the same as the desired cruising speed. The problem is then divided into two periods. There is a short period from time 0 to t_1 (where t_1 is fixed in advance, to one minute in most cases) in which all acceleration takes place. Following this there is a long period from time t_1 to T in which the speed $v(t)$ is required to be constant. The solution value of $v(t)$ for $t_1 \leq t \leq T$ is taken to be the cruising speed to which the car is accelerating.

Driving between stop signs. The objective is to minimize fuel consumption while covering a fixed distance s_1 on a level road, starting and ending at rest ($v_0 = v_1 = \theta = 0$). In this problem T is not fixed.

Driving over hills. This is a family of problems identical to the cruising problem except that the road grade is not identically zero. Thus we set $T = T_0$ and $v_0 = v_1 = s_1/T$.

It will be useful to apply Pontriagin's maximum principle (adapted to minimization) to problem (1). Let T, s_1, and v_1 be fixed. The engine power constraint can be effected by making the fuel cost $f(v,a)$ prohibitively high when a exceeds maximum acceleration at speed v. We can assume without practical consequences that f is differentiable. Acceleration $\dot{v}$ is the control, and the car's speed and position form the state vector (v,s). If adjoint variables λ and μ are associated with the speed and position states, respectively, the Hamiltonian is

$$H(v,s,\dot{v},t) = f(v, \dot{v} + g \sin \theta(s)) - \lambda \dot{v} - \mu v. \quad (2)$$

Let f_i be the partial derivative of f with respect to its ith argument. Then the maximum principle states that at any time t the optimal control $\dot{v}$ minimizes $H(v,s,\dot{v},t)$, where

$$\dot{\lambda} = -H_1(v,s,\dot{v},t) = -f_1(v, \dot{v} + g \sin \theta(s)) + \mu \quad (3)$$

$$\dot{\mu} = -H_2(v,s,\dot{v},t) = -f_2(v, \dot{v} + g \sin \theta(s))\, \theta'(s)\, g \cos \theta(s). \quad (4)$$

If the road grade is constant, $\theta'(s) = 0$, and by (4), $\dot{\mu} = 0$.

If the Hamiltonian is differentiable and convex in $(v,s,\dot{v})$, and if the adjoint variables are continuous in time, the above necessary conditions for optimality are also sufficient (Seierstad and Sydsaeler, 1977).

3. THE ROLE OF CONVEXITY

The character of optimal control depends critically on whether the fuel flow function f in (1) is convex. A function $g: R^n \rightarrow R$ is *convex* if $g((1 - \alpha)x + \alpha y) \leq (1 - \alpha)g(x) + \alpha g(y)$ for all $x, y \in R^n$ and all $\alpha \in [0,1]$.

The importance of convexity is evident in the cruising problem. Most drivers solve it simply by holding a constant speed equal to the average speed. We can show that this solution is optimal *if the fuel flow function is convex*. This holds for any constant road grade, level or otherwise.

To see why, we show that the constant speed solution satisfies the optimality conditions of the previous section when f is convex. Thus we set $v(t) = v_0 = v_1$ and $\dot{v}(t) = 0$ for all t. As we observed earlier, the constant grade implies that $\dot{\mu} = 0$. If we set $\mu = f_1(v_1,0)$ and suppose λ is constant, (3) and (4) are satisfied. Since f is convex and $\dot{v}$ and v are constant, there is a constant value of λ for which $\dot{v} = 0$ minimizes the Hamiltonian in (2) at all t. We set λ to this value and note that all of the conditions are satisfied. Finally, the convexity of f implies the convexity of the Hamiltonian in $(v,s,\dot{v})$, so that the satisfaction of these conditions is sufficient for optimality.

When the fuel flow function is not convex, the solution of a cruising problem is unpredictable. The car may be instructed alternately to speed up and slow down. This same oscillating behavior may occur in other problems, such as an acceleration problem.

The fuel flow functions for the 15 simulated cars are not convex primarily because of discontinuities at gearshift points. Thus, if a problem covers a range of speeds and accelerations that calls for gear shifts, one may see an oscillation between gears, and this was observed in three cars.

4. SOLUTION TECHNIQUE

The technique used to solve the optimal control problems is described in detail in Hooker *et al.* (1983b). Briefly, it is a forward dynamic programming technique with time stages. The state variables are the vehicle's speed, position, and gear, and speed is the control variable. For $k = 1, \ldots, K$ let $F_k(v,s,r)$ be the fuel used along an optimal trajectory from stage 0 to stage k (time $k\Delta t$), given that the car is at speed v, position s and gear r in stage k. Then optimal control is given by a recursive formula similar to,

$$F_{k+1}(v,s,r) = \min_{v'} \{f(v_{avg},a)\Delta t + F_k(v',s',r') \mid a \leq a_{max}(v_{avg})\}, \quad k = 1, \ldots, K,$$

where the speeds v, v' and positions s, s' range over discrete values. Also $v_{avg} = (1/2)(v + v')$, $a = (v' - v)/\Delta t + g \sin \theta(s')$, s' is the discrete value nearest $s - (1/2)(v + v')\Delta t$, and r' is the gear in which the car operates at speed v_{avg} and effective acceleration a. There is an optional prohibition of downshifting, $r \geq r'$. The fuel flow function f is evaluated by a subroutine call to the simulator. The boundary condition is,

$$F_1(v,s,r) = f(v_{avg},a)\Delta t,$$

where $v_{avg} = (1/2)(v + v_0)$, $a = (v - v_0)/\Delta t + g \sin \theta(0)$, s is the discrete position nearest $(1/2)(v + v_0)\Delta t$, and v_0 is the initial speed. The minimal fuel consumption for the entire trip is $F_K(v_1,s_1,r_1)$, where r_1 is the terminal gear.

To improve accuracy, the problem is solved in two iterations. The second iteration uses more closely spaced discrete speeds and positions centered about the optimal trajectory from the first iteration.

The accelerate and cruise problem is solved by gathering the time between t_1 and T into one long terminal stage, with $T - t_1 = 1{,}000$ seconds. The terminal cruising speed v_1 is then chosen so as to minimize $F_K(v_1,s_1r_1)$, where r_1 is high gear.

The problem of driving between stop signs is solved by first picking the stage k for which $F_k(0,s_1,r_1)$ is minimized in iteration 1, where r_1 is the highest gear below overdrive. This estimates the optimal duration to be $k\Delta t$. Then the problem is re-solved from scratch, through both iterations, with $T_1 = k\Delta t$.

Since the state space is rather large due to a large number of discrete distances s, solution often requires an hour or more of CPU time on a DEC-20 computer.

5. RESULTS: OPTIMAL CRUISING ON A LEVEL ROAD OR CONSTANT GRADE

The cruising problem on a level road, or on a constant grade, is the most basic fuel-economy prob-

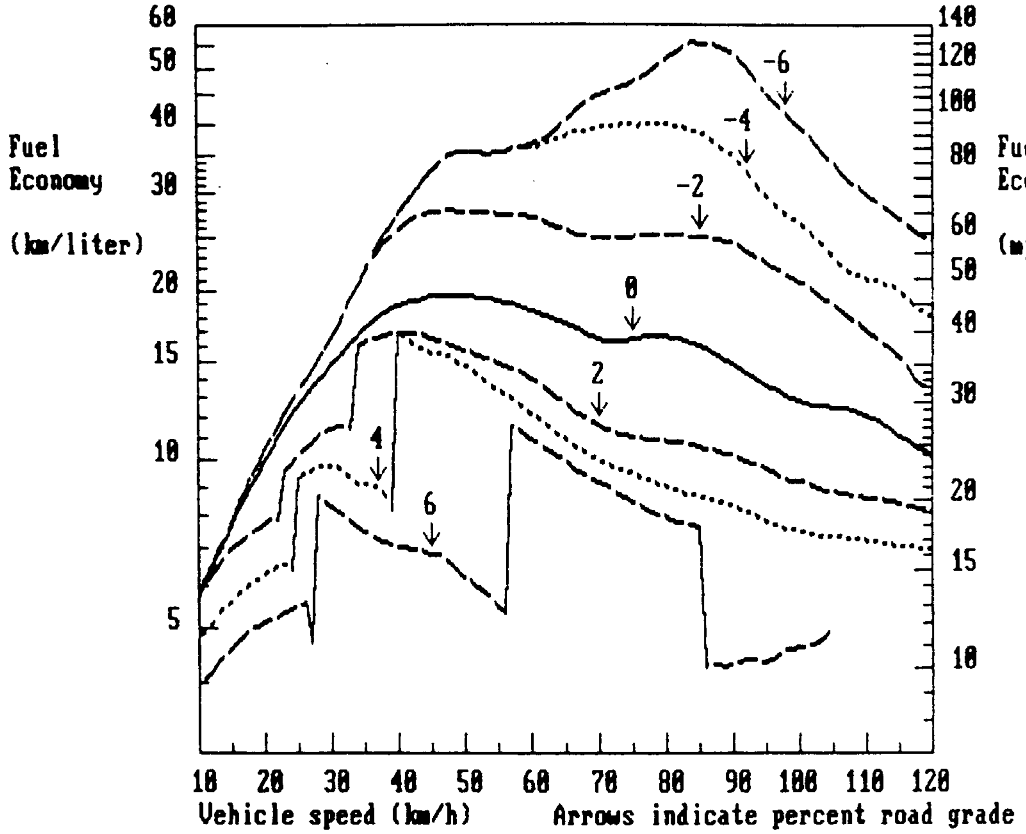

Fig. 1. Ford Escort: Fuel economy vs. speed and road grade.

lem. It asks how one should drive so as to achieve a given average speed if the initial and terminal speeds are set equal to the average speed. No attempt was made systematically to solve this optimal control problem for several cars, since drivers will in practice solve it in the simplest way: by cruising at constant speed. It remains only to determine how fuel economy depends on the cruising speed.

Figures 1 and 2 illustrate typical curves of fuel economy vs. speed on various grades. The discontinuous changes on the uphill grades correspond to gearshifts. Table 2 indicates each car's optimal speed on a level road, and Table 3 shows how fuel economy varies with speed on a level road.

6. RESULTS: OPTIMAL ACCELERATION TO CRUISING SPEED

Optimal control was computed for acceleration from rest to cruising speeds of 55 and 90 km/h (34 and 56 mph), representing city and highway driving, respectively. The car is allowed to accelerate only during an initial acceleration phase. After this point it is constrained to maintain a constant cruising speed, whatever it might be, for a total of 1,000 seconds. The average speed was set at either 55 or 90 km/h, and due to the length of the trip, the cruising speed attained is quite near 55 or 90 km/h.

Thus, the problem can be seen as optimizing the tradeoff between the disadvantage of fast acceleration, which is having to use more fuel to overcome inertia, and the disadvantage of slow acceleration, which is having to cruise slightly faster to make up for lost time. Thus the solution applies only to situations in which one intends to cruise at least for a short while after accelerating.

The acceleration phase is made long enough so that the speed of the car levels off at cruising speed before the end of the phase. This leveling off occurs within a minute or so for seven of the eight cars for which the problem was solved. But there is a theoretical possibility that a car's speed will fail to level off at the desired cruising speed within any reasonable time, and one car (the Toyota Corolla) exhibited this behavior. The Toyota instead levels off at its most efficient cruising speed (28 km/h or 17 mph) and holds it until near the end of the acceleration phase, when it is obliged to accelerate to the desired cruising speed.

Two optimal trajectories appear in Figs. 3 and 4. Here the dashed line shows the optimal speed at each time. The dotted line at the bottom is the profile of the road, which in this case is level. The solid line

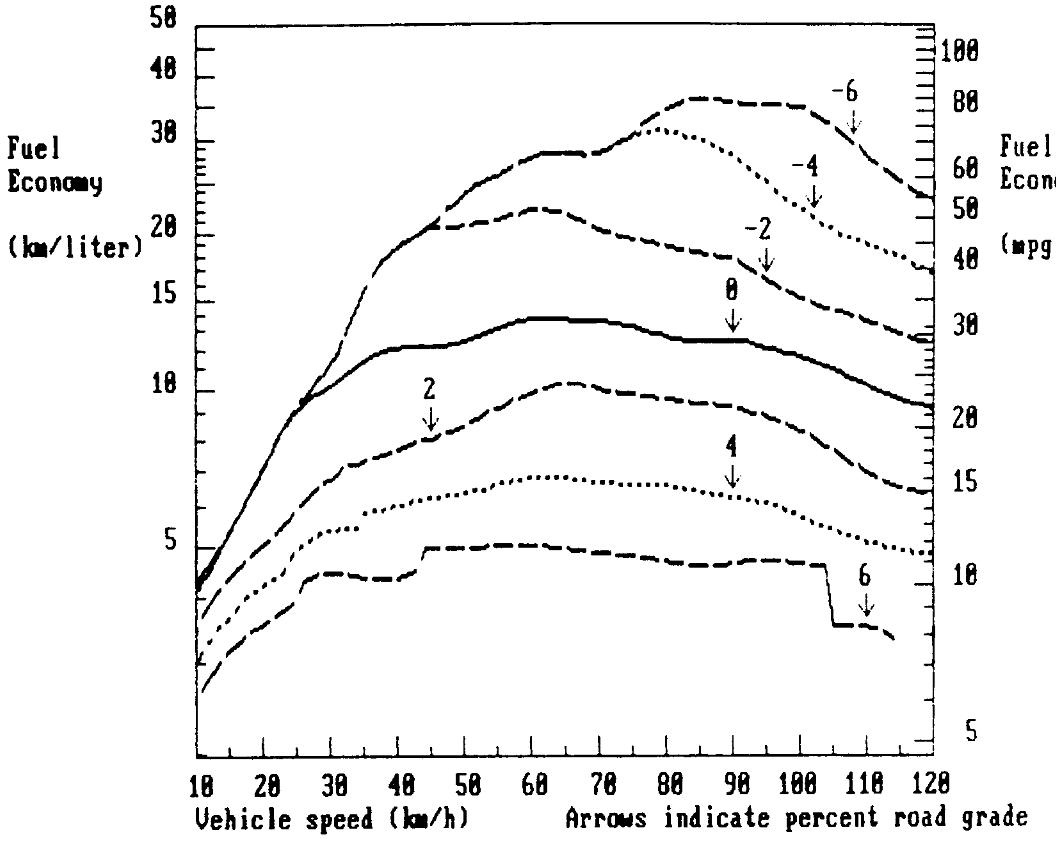

Fig. 2. Chevrolet Monte Carlo: Fuel economy vs. speed and road grade.

indicates what might be described as the percent of available engine power that the car is using. More precisely, it is

$$100 \cdot (a - a_{min})/(a_{max} - a_{min}),$$

where a is the effective acceleration, a_{min} is the coasting acceleration on a level road (a negative number, since the car slows as it coasts), and a_{max} is the maximum acceleration at the current speed on a level road. The solid line is at 100% when the throttle is wide open and at zero when the car is coasting in gear; it is below zero when brakes are applied. Intermediate positions indicate roughly the position of the gas pedal. Rapid fluctuations in the solid line do not indicate that an efficient driver must jiggle the

Table 2. Simulated fuel economy at optimal speed and at 90 km/h on a level road

Automobile	Optimal Speed (km/h)	Optimal Speed (mph)	Fuel Economy at Optimal Speed (km/liter)	Fuel Economy at Optimal Speed (mpg)	Fuel Economy at 90 km/h (56 mph) (km/liter)	Fuel Economy at 90 km/h (56 mph) (mpg)
Datsun 210	43	27	24.6	58	15.7	37
Ford Escort	48	30	20.0	47	15.0	35
Toyota Corolla	28	17	16.4	39	13.2	31
Chevette diesel	37	23	30.0	71	15.9	37
Chevrolet S-10	45	28	17.5	41	10.7	25
Ford Fairmont	65	41	14.2	33	11.6	27
Chevrolet Citation	78	49	16.0	38	13.8	32
Plymouth Reliant	50	31	14.2	33	11.8	28
Pontiac Firebird	51	32	17.7	42	14.6	34
Ford Futura	47	29	14.5	34	11.8	28
Monte Carlo	62	38	13.8	33	12.5	30
Buick Century	76	47	13.4	32	12.2	29
Caprice SW	74	46	13.0	31	9.5	22
Caprice diesel	70	44	15.4	36	13.8	32
Silverado diesel	60	37	13.8	32	10.1	24

Table 3. Simulated fuel economy relative to that at 90 km/h (56 mph) on level road

Automobile (km/h)	10	20	30	40	50	60	70	80	90	100	110	120
(mph)	6	12	19	25	31	37	44	50	56	62	68	75
Datsun 210	0.76	1.22	1.23	1.49	1.51	1.45	1.20	1.06	1.00	0.93	0.89	0.85†
Ford Escort	0.38	0.67	1.07	1.27	1.33	1.25	1.12	1.12	1.00	0.86	0.81	0.69
Toyota Corolla	0.60	1.10	1.24	1.16	1.14	1.21	1.11	1.02	1.00	0.95	0.85	0.76
Chevette diesel	0.75	1.24	1.79	1.84	1.49	1.54	1.30	1.13	1.00	0.81	0.64	0.56†
Chevrolet S-10	0.48	0.77	0.95	1.10	1.38	1.36	1.35	1.18	1.00	0.88	0.79	0.73†
Ford Fairmont	0.33	0.66	0.95	1.13	1.15	1.20	1.20	1.11	1.00	0.88	0.81	0.74
Chevy Citation	0.24	0.40	0.60	0.77	0.92	1.05	1.10	1.16	1.00	0.80	0.67	0.60
Plymouth Reliant	0.29	0.64	0.93	1.17	1.21	1.14	1.10	1.07	1.00	0.90	0.82	0.72
Pontiac Firebird	0.32	0.55	0.75	0.88	1.20	1.18	1.05	1.00	1.00	0.94	0.84	0.80
Ford Futura	0.36	0.74	1.04	1.21	1.22	1.21	1.15	1.09	1.00	0.90	0.78	0.67
Monte Carlo	0.21	0.57	0.82	0.97	0.99	1.10	1.09	1.02	1.00	0.93	0.82	0.73
Buick Century	0.43	0.71	0.86	1.00	1.02	1.03	1.05	1.05	1.00	0.96	0.85	0.73
Caprice SW	0.40	0.70	0.84	0.89	0.86	1.09	1.26	1.19	1.00	0.92	0.88	0.71
Caprice diesel	0.27	0.54	0.78	0.95	1.03	1.06	1.12	1.06	1.00	0.89	0.79	0.66
Silverado diesel	0.44	0.70	0.89	1.07	1.12	1.34	1.06	1.03	1.00	0.94	0.73	0.70

†Figure indicates fuel economy at 115 km/h (71 mph), since the simulation of these cars does not extend to 120 km/h.

gas pedal. They result partly from the discrete nature of dynamic programming, and it was found that smoothing the trajectory has little effect on simulated fuel economy.

Table 4 reveals that the optimal trajectories are quite different from car to car. Figures 3 and 4 should therefore not be taken as representative.

Many of the solutions differ from the findings of Evans and Takasaki (1981), which recommend very gradual acceleration, and from those of Akcelik and Biggs (1987), which recommend brisk acceleration at first. But as noted previously their findings are based on minimizing the "excess" acceleration fuel consumption, which is quite different from mini-

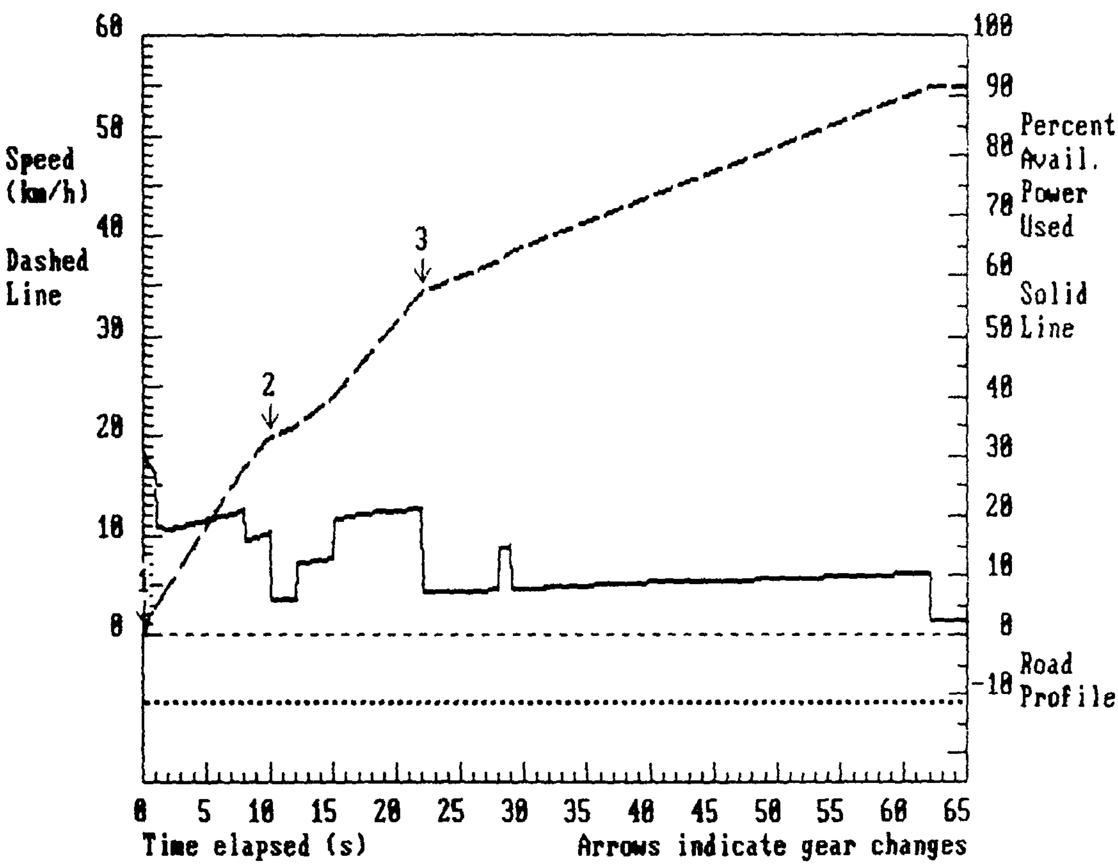

Fig. 3. Chevrolet Citation: Optimal acceleration to 55 km/h.

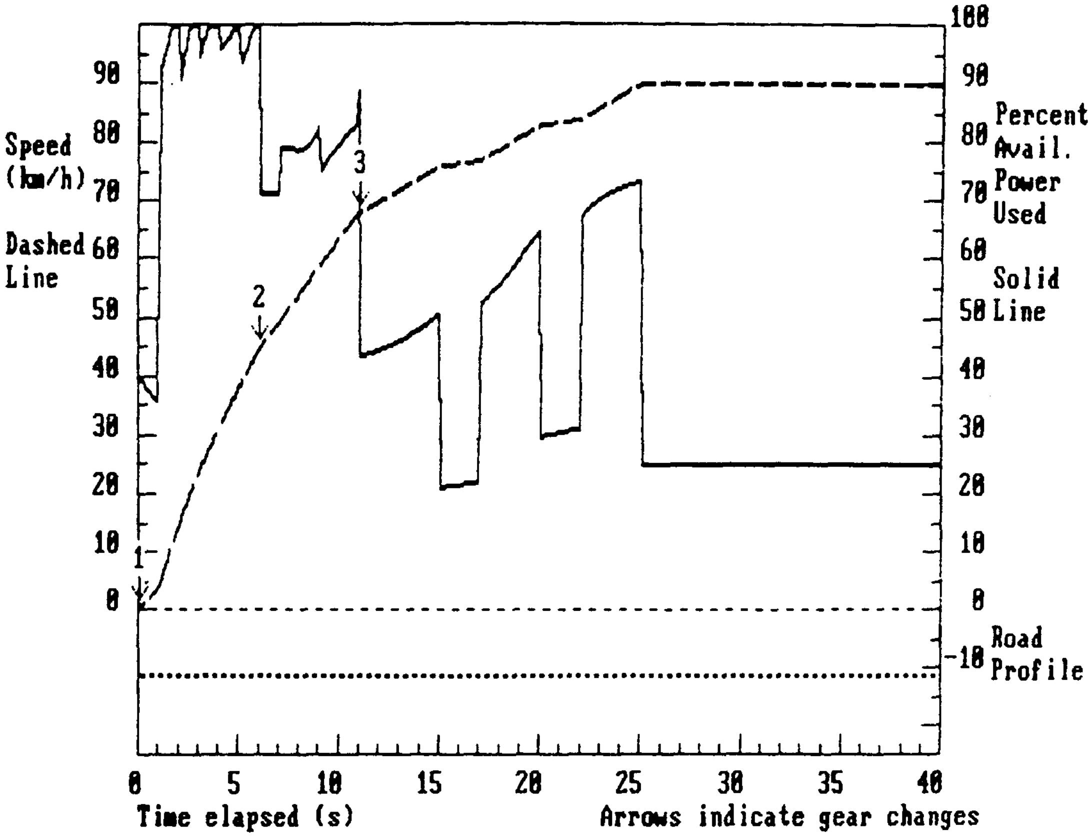

Fig. 4. Chevrolet Citation: Optimal acceleration to 90 km/h.

Table 4. Some points on the optimal trajectories for acceleration to cruising speed

Automobile	Desired Cruising Speed (km/h)	(mph)	Shift into Second Gear at time (sec)	speed (km/h)	(mph)	Shift into Third Gear at time (sec)	speed (km/h)	(mph)	Reach Cruising Speed (sec)
Ford Escort	55	34	5	23	14	10	33	21	32
	90	56	3	30	19	6	39	24	35
Chevrolet S-10	55	34	3	21	13	28†	47	29	40
	90	56	6	45	28	19	74	46	51
Chevy Citation	55	34	10	20	12	22	34	21	61
	90	56	6	45	28	11	68	42	25
Plymouth Reliant	55	34	4	34	21	6	40	25	36
	90	56	4	35	22	12	66	41	37
Pontiac Firebird	55	34	2	20	12	8‡	36	22	30
	90	56	3	32	20	6†	49	30	31
Buick Century	55	34	5	21	13	13	37	23	36
	90	56	4	26	16	26	90	56	26
Caprice SW	55	34	4	30	19	11†	55	34	11
	90	56	5	47	29	9§	60	37	40

†At this point the car shifts into third and then immediately into fourth.
‡The Firebird shifts into fourth at 19 sec, at speed 46 km/h (29 mph).
§The Caprice shifts into fourth at 11 sec, at speed 65 km/h (40 mph).

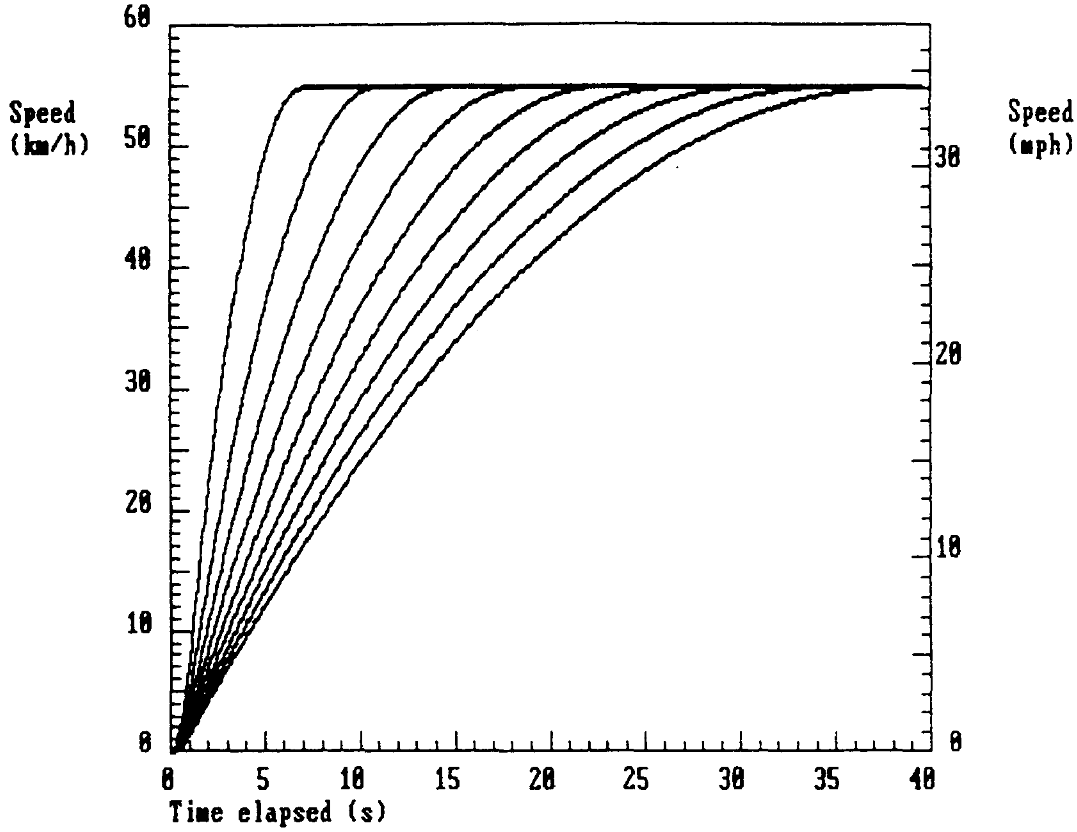

Fig. 5. Speed/time trajectories for acceleration to 55 km/h cruising speed.

mizing accelerate-and-cruise fuel consumption. One may ask why these investigators did not confirm our earlier statement that one can always reduce excess fuel consumption by cruising sufficiently long at the car's most efficient speed v^* before accelerating to the terminal speed. It is because they restrict their attention to a limited family of trajectories, none of which allow for a long cruise at speed v^*. Evans and Takasaki compare several trajectories that a driver achieved on a test track, and Akcelik and Biggs compare trajectories described by a certain class of polynomials. Each found the optimal trajectory within the class investigated and therefore arrived at different results.

Some indication of the sensitivity of fuel economy to the rate of acceleration was obtained by running all 15 cars over the acceleration paths depicted in Figs. 5 and 6. In each curve the acceleration rate increases linearly from zero to a peak during the first second and thereafter decreases linearly until reaching zero.

Table 5 shows that fuel economy varies relatively little over this very wide range of acceleration rates. Here, fuel economy is calculated over a 2-km test section that extends well beyond the acceleration phase, since the object is to maximize fuel economy over a combination accelerate-and-cruise trajectory. It would be inappropriate to check fuel economy in the acceleration phase alone, since the optimal solution may sacrifice economy in this phase so as to maximize overall economy. Thus, if the acceleration path roughly resembles one of those in Figs. 5 or 6, it should make little difference to fuel economy how rapidly one accelerates to cruising speed.

7. RESULTS: OPTIMAL DRIVING BETWEEN STOP SIGNS

Optimal control was computed for 10 cars on the condition that the cars start from rest and come to a stop after covering 300 m, a typical length for a suburban block. The aim is to determine efficient driving for streets where stop signs are prevalent. No speed limit was imposed, because it is never optimal to exceed 44 km/h (27 mph) in such a situation.

Optimal driving on a block depends on how much time one wishes to spend covering the block, but it is useful to know the *optimal* time one should spend. If it is too short, the car wastes fuel in rapid acceleration, and if it is too long, an idling engine consumes unnecessary fuel as the car creeps along. Thus the optimal time, as well as the optimal trajectory, was determined.

A car with unlimited braking power is instructed to drive right up to the stop sign and slam on the brakes. This may seem paradoxical due to the fact that braking is normally wasteful, but in this case it is quite reasonable. The car's kinetic energy must

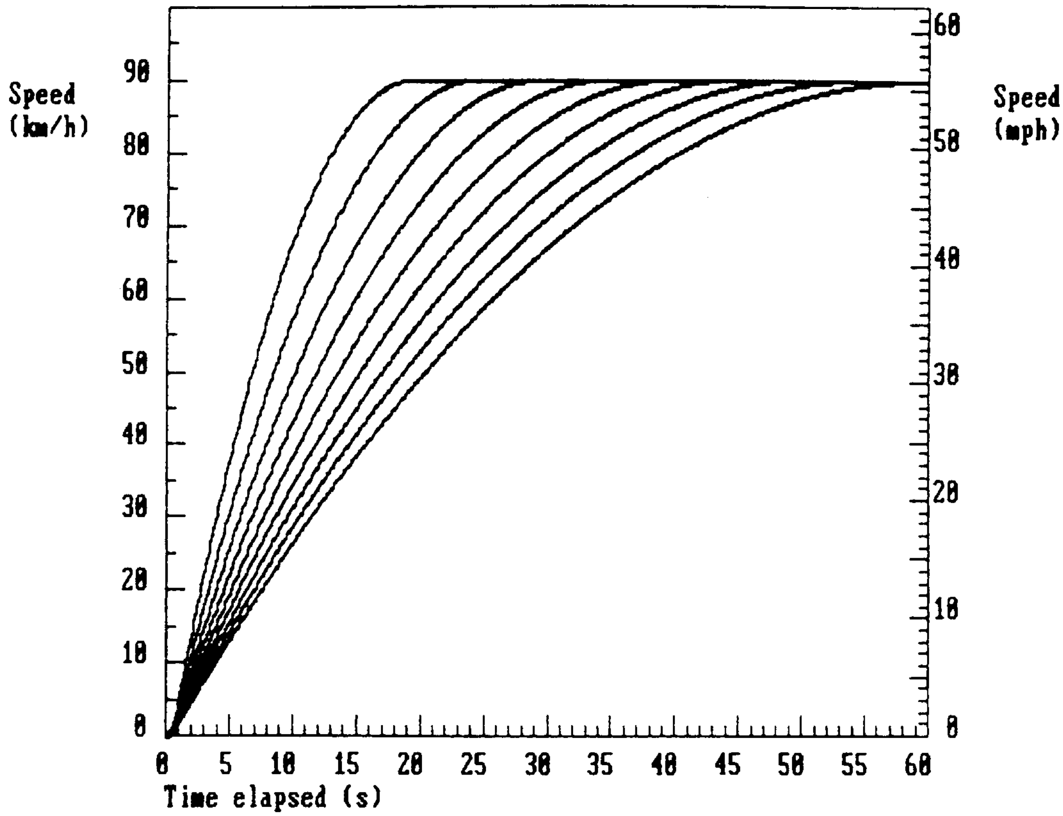

Fig. 6. Speed/time trajectories for acceleration to 90 km/h cruising speed.

be dissipated one way or another, and gradual braking dissipates it no less completely than hard braking. Furthermore, we can see why hard braking is more efficient. Consider any trajectory in which the braking is gradual, and let x be the last point on the road at which the accelerator is depressed. Then the fuel consumed to move from x to the stop sign is approximately tf, where t is the time required to cover the distance and f the idle fuel flow rate. Now consider another trajectory exactly like the first except that the car simply coasts from x to the stop sign, where it stops instantaneously. The fuel consumed to move from x to the stop sign is now approximately $t'f$, where t' is the time required. Since in general $t' < t$, we have $t'f < tf$, so that the second trajectory should in general require less fuel. Since the car cannot in practice stop suddenly, we impose a maximum braking deceleration of 0.3 g's.

Table 5. Variation of fuel economy over a family of trajectories for acceleration to cruising speed

	Range of Fuel Economies†	
Automobile	Accel. to 55 km/h (%)	Accel. to 90 km/h (%)
Datsun 210	5	6
Ford Escort	9	13
Toyota Corolla	2	7
Chevette diesel	6	12
Chevrolet S-10	2	3
Ford Fairmont	1	3
Chevy Citation	7	5
Plymouth Reliant	1	9
Pontiac Firebird	2	13
Ford Futura	3	13
Monte Carlo	3	7
Buick Century	2	3
Caprice SW	7	7
Caprice diesel	3	10
Silverado diesel	1	4

†Fuel economy is simulated over a distance of 2 km (1.24 miles). The highest fuel economy is the percentage base.

Two solutions appear in Figs. 7 and 8. Table 6 shows certain characteristics of all 10 solutions. Again there is substantial variation from car to car, without any apparent pattern to explain the variation. In particular, differences in vehicle weight and idle fuel flow rate, which should have an important bearing here, do not seem to explain the variation. The Toyota Corolla and Ford Escort, for instance, are rather similar cars but call for entirely different optimal control.

Sensitivity analysis was carried out by defining a family of comparison trajectories (Fig. 9), as for the acceleration problem. The acceleration phase is sim-

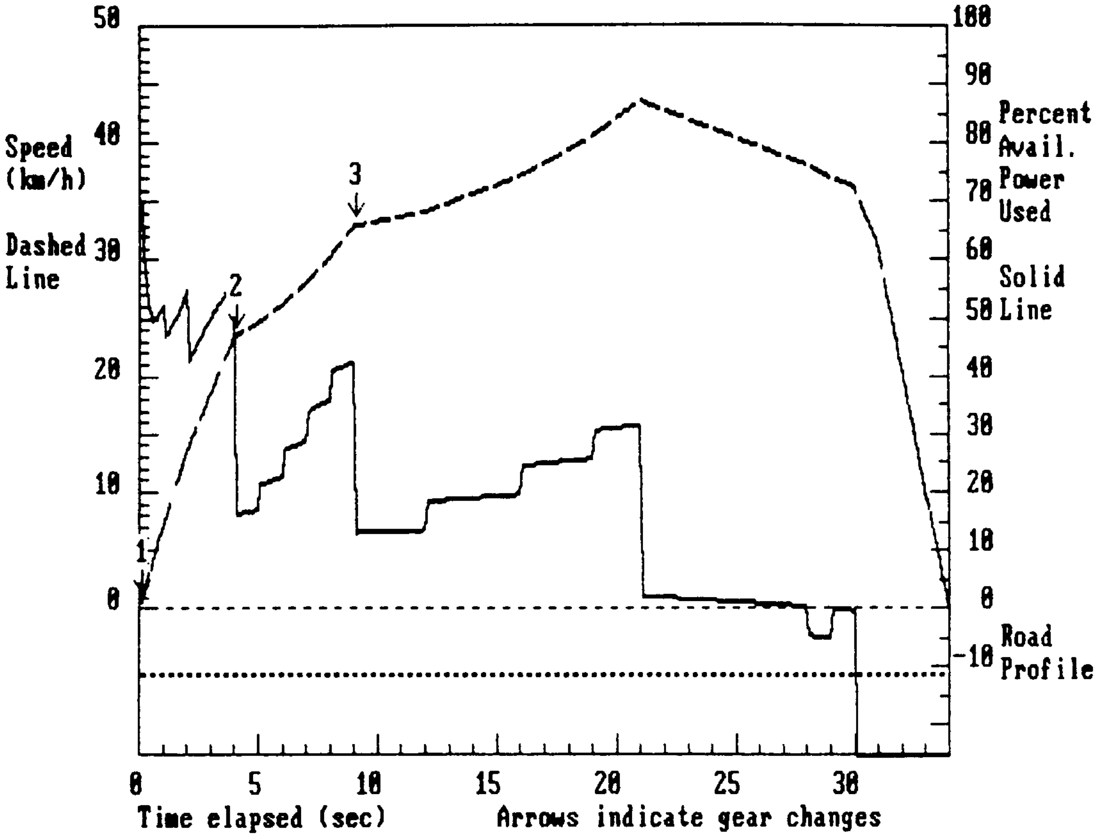

Fig. 7. Ford Escort: Optimal control on a 300 m block.

ilar to that in Figs. 5 and 6. The braking phase consists of deceleration increasing linearly from zero to 0.3 g's. The distance to be covered is fixed at 300 m. The results appear in Table 7. This time there is substantial variation in fuel economy over the family of trajectories. But one can approximate optimal fuel economy to within an average of 8% by using the best of the trajectories. Also, the cruising speed of the best trajectory is usually quite close to the cruising speed of the trajectory whose duration matches that of the optimal trajectory. These two facts suggest that fuel economy is sensitive to how quickly one covers the block, but less sensitive to the shape of the acceleration path, provided one covers the block in roughly the optimal time.

8. RESULTS: OPTIMAL DRIVING OVER HILLS

Optimal control problems were solved for two types of hills: an isolated hill with level road on either side, and rolling terrain. In every case the average, initial, and terminal speeds were fixed at 80 km/h (50 mph). Higher average speeds would entail violation of the U.S. 55 mph speed limit over part of the optimal trajectory. One could solve problems with a higher average speed and a 55 mph speed limit imposed, but there is little point in doing so, since the average and top speed would be so close as to leave little room for variations.

Isolated hill problems were solved for 3% and 6% grades. The 3% profile consists of level road for 200 m, then linearly increasing grade for 50 m, then 3% grade for 200 m, then linearly decreasing grade for 50 m, followed by a mirror image of this profile, for a total distance of 1 km. The 6% profile is the same but with all grades doubled.

Typical solutions appear in Figs. 10 and 11, where the vertical component of the road profile (dotted line) is exaggerated. Table 8 shows characteristics of eight solutions, all similar. The optimal fuel economy tends to be much better than that of cruising at constant speed (cruise-control driving).

Rolling terrain is represented by a single one-kilometer cycle, resembling one cycle of a sine wave, with road grade of 3% or 6%. The 3% profile consists of 3% grade for 200 m, linearly decreasing grade for 100 m, −3% grade for 400 m, linearly increasing grade for 100 m, and 3% grade for 200 m, for a total distance of 1 km. The 6% profile is analogous.

Typical solutions appear in Figs. 12 and 13, and Table 9 describes solutions for four cars, again similar. There is again a substantial advantage over cruise-control driving.

To get some notion of how closely one must ap-

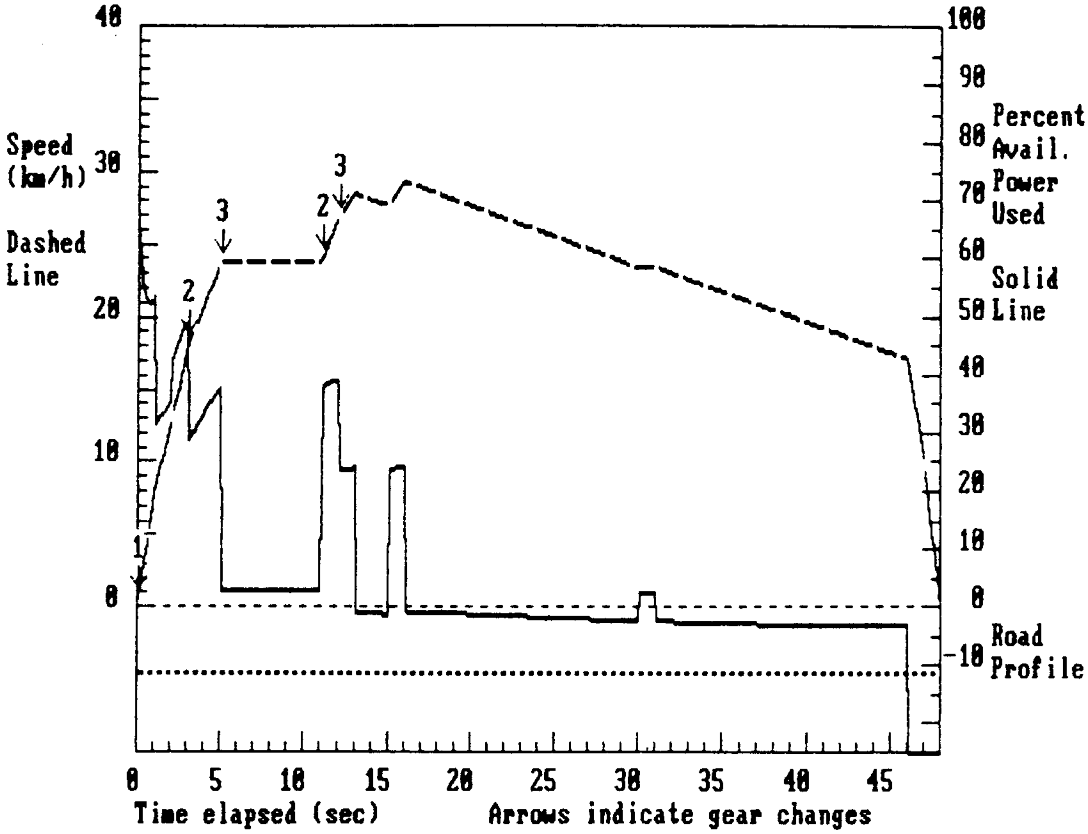

Fig. 8. Toyota Corolla: Optimal control on a 300 m block.

Table 6. Some characteristics of the optimal trajectories for driving a block between stop signs

	Shift into Third			Reach Top Speed			Cruising	Total
	at Time (sec)	Speed (km/h)	Speed (mph)	at Time (sec)	Speed (km/h)	Speed (mph)	Time† (sec)	Time (sec)
Ford Escort	9	33	21	21	44	27	9‡	34
Toyota Corolla	12§	27	17	16	29	18	30‡	48
Chevette diesel	13	27	17	18	29	18	16‖	44
Chevrolet S-10	17¶	36	22	17	36	22	15	40
Chevy Citation	14	37	23	29	44	27	0	34
Plymouth Reliant	9	33	21	23	38	24	7#	34
Pontiac Firebird	13	30	19	18	33	21	29††	40
Ford Futura	12	29	18	12	29	18	39††	44
Buick Century	6	25	16	12	27	17	29††	44
Caprice SW	5‡‡	26	16	11	28	17	28††	42

†Cruising time is the time between the point at which the car reaches top speed and the point at which the brakes are applied. It generally involves cruising at constant speed, coasting, or some combination of the two. Braking is usually firm (0.3 g's) at the end of the block.

‡These cars coast throughout the cruising time.

§The Toyota shifts into third at 5 sec, back into second at 11 sec, and into third at 12 sec.

‖The Chevette maintains constant speed for 12 sec, reduces throttle slightly for 4 sec, and coasts for 7 sec.

¶The S-10 shifts into fourth at this point. It then coasts 3 sec and shifts down to third, at which point a slightly open throttle is used for the remaining 12 sec of the coasting period. Brake pressure is very light for the first 5 sec of braking.

#Cruising is more or less at constant speed.

††These cars cruise at constant speed and then coast; the Firebird cruises at constant speed 8 sec, the Futura 16 sec, the Century 20 sec, and the Caprice station wagon 12 sec.

‡‡The Caprice downshifts momentarily at 10 sec to achieve 28 km/h at 11 sec.

Table 7. Variation of fuel economy over a family of trajectories for driving a block between stop signs

	Range of Fuel Economies† (%)	Best Trajectory: Cruising Speed (km/h)	Best Trajectory: Cruising Speed (mph)	Best Trajectory: Deviation from Optimal Fuel Economy (%)	Cruising Speed Corresponding to the Optimal Trajectory‡ (km/h)	Cruising Speed Corresponding to the Optimal Trajectory‡ (mph)
Datsun 210	25	20	12	§	§	§
Ford Escort	26	32	20	12	39	23
Toyota Corolla	32	25	16	8	25	15
Chevette diesel	23	32	20	11	28	17
Chevrolet S-10	8	30	19	10	31	18
Ford Fairmont	18	32	20	§	§	§
Chevy Citation	25	32	20	§	39	23
Plymouth Reliant	18	35	22	7	39	23
Pontiac Firebird	16	31	19	5	31	18
Ford Futura	18	29	18	5	28	17
Monte Carlo	20	33	21	§	§	§
Buick Century	25	24	15	5	28	17
Caprice SW	22	25	16	8	29	17
Caprice diesel	20	33	21	§	§	§
Silverado diesel	9	41	25	§	§	§

†The highest fuel economy is the percentage base.
‡This is the cruising speed of the comparison trajectory having the same duration as the optimal trajectory.
§Optimal control was not computed for these cars.

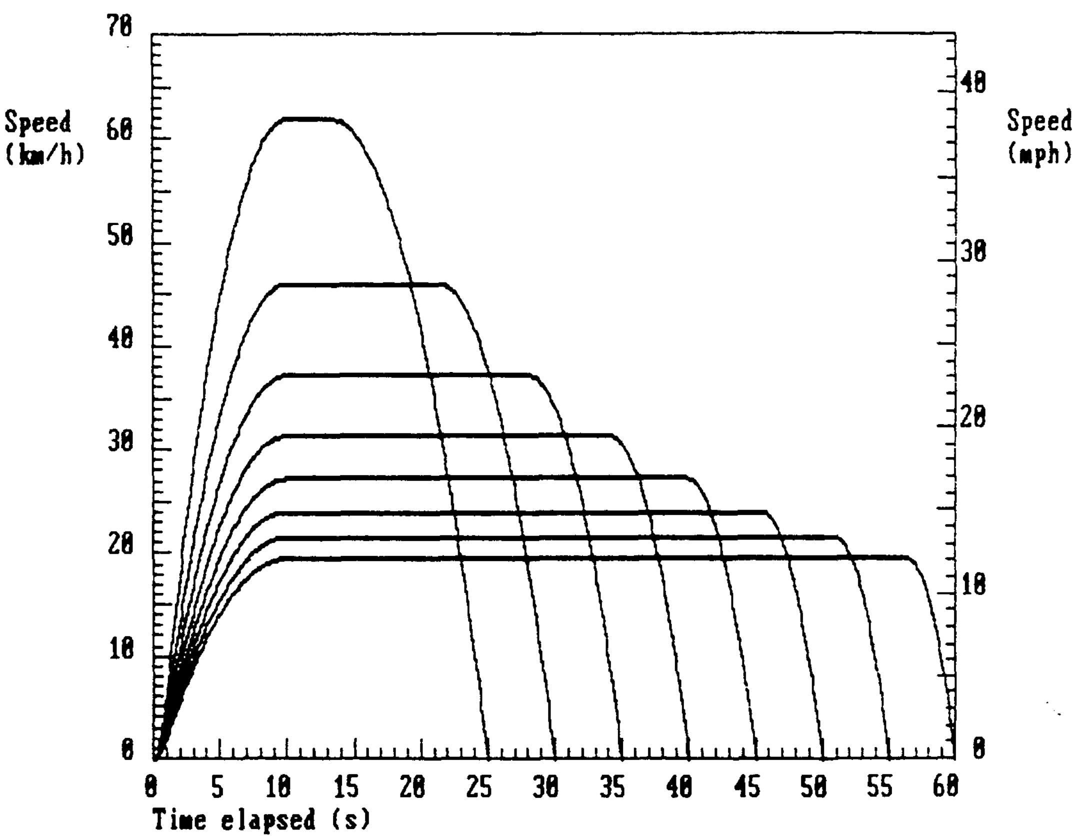

Fig. 9. Speed/time trajectories for a 300 m block.

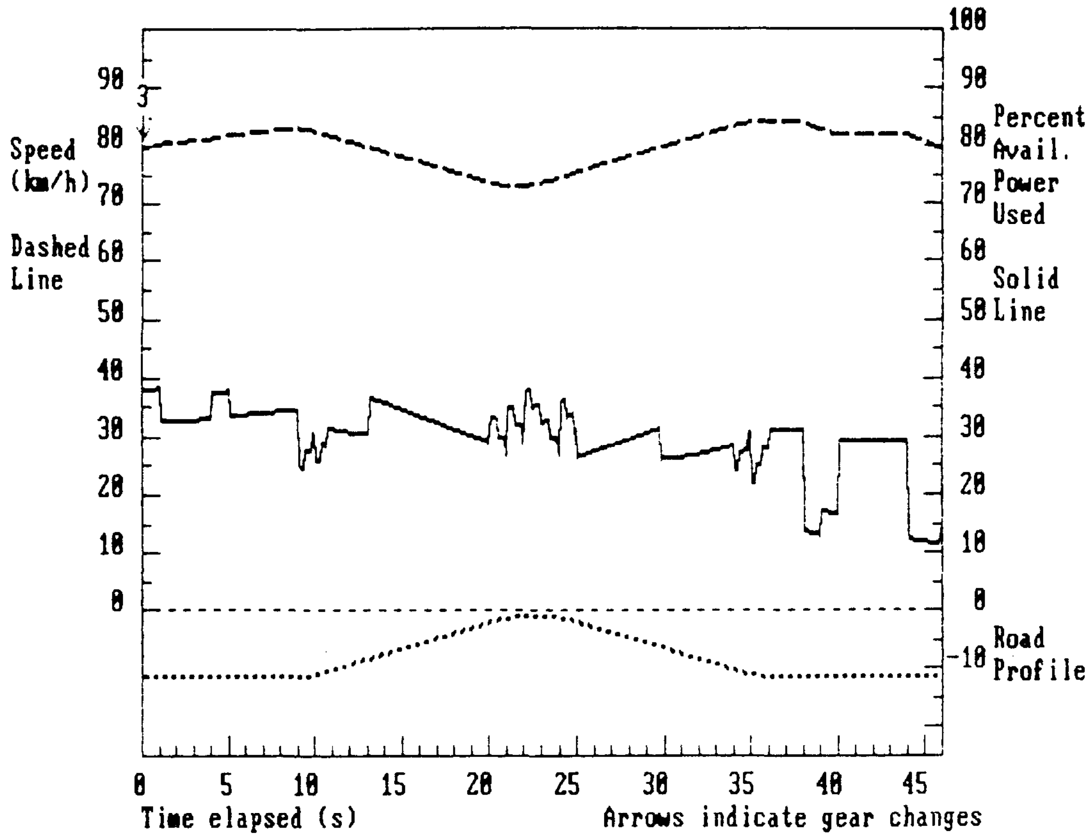

Fig. 10. Plymouth Reliant: Optimal control on a hill (average speed 80 km/h, 3% grade).

proximate the optimal trajectory to achieve near-optimal fuel economy, all of the cars for which the hill problems were solved were run over the Plymouth Reliant's optimal trajectory. Tables 8 and 9 indicate that the degradation in fuel economy is slight. This suggests that one generalized optimal trajectory (e.g., the Reliant's) is adequate for any car, and that one need not follow his car's optimal trajectory very closely to enjoy most of the advantage of optimal control.

9. GENERAL CONCLUSIONS

The most striking fact about optimal control of automobiles is that it can be very different for different cars when acceleration is an important factor, as when one is accelerating to cruising speed or driving between stop signs. On the other hand, optimal control is fairly consistent from car to car when acceleration is a minor factor, as when one is cruising on hills.

When the object is to accelerate from rest to cruising speed and then to cruise for some distance while achieving a fixed overall average speed, fuel economy is not very sensitive to the rate of acceleration. But fuel economy is quite sensitive to the way one drives between stop signs or over hills. Thus the potential fuel savings of optimal control are substantial in the latter two cases.

Despite the disparate results it is possible to draw several conclusions that should be generalizable to other cars. These are detailed in the following three sections.

10. RECOMMENDATIONS FOR CRUISING

The optimal cruising speed on level road tends to increase with the size class of the car. It averages 44 km/h (27 mph) for seven 4-cylinder cars, 59 km/h (37 mph) for four 6-cylinder cars, and 68 km/h (42 mph) for three 8-cylinder cars. Left out of this accounting is the only simulated car that has gasoline injection, the Chevrolet Citation, which has a rather high optimal speed of 79 km/h (49 mph).

If the car has an overdrive gear, one should drive just above the speed at which the car shifts into overdrive. If the torque converter has a lockup feature, one should drive just far enough for the lockup to engage.

On the average, one can deviate ±8 km/h (±5

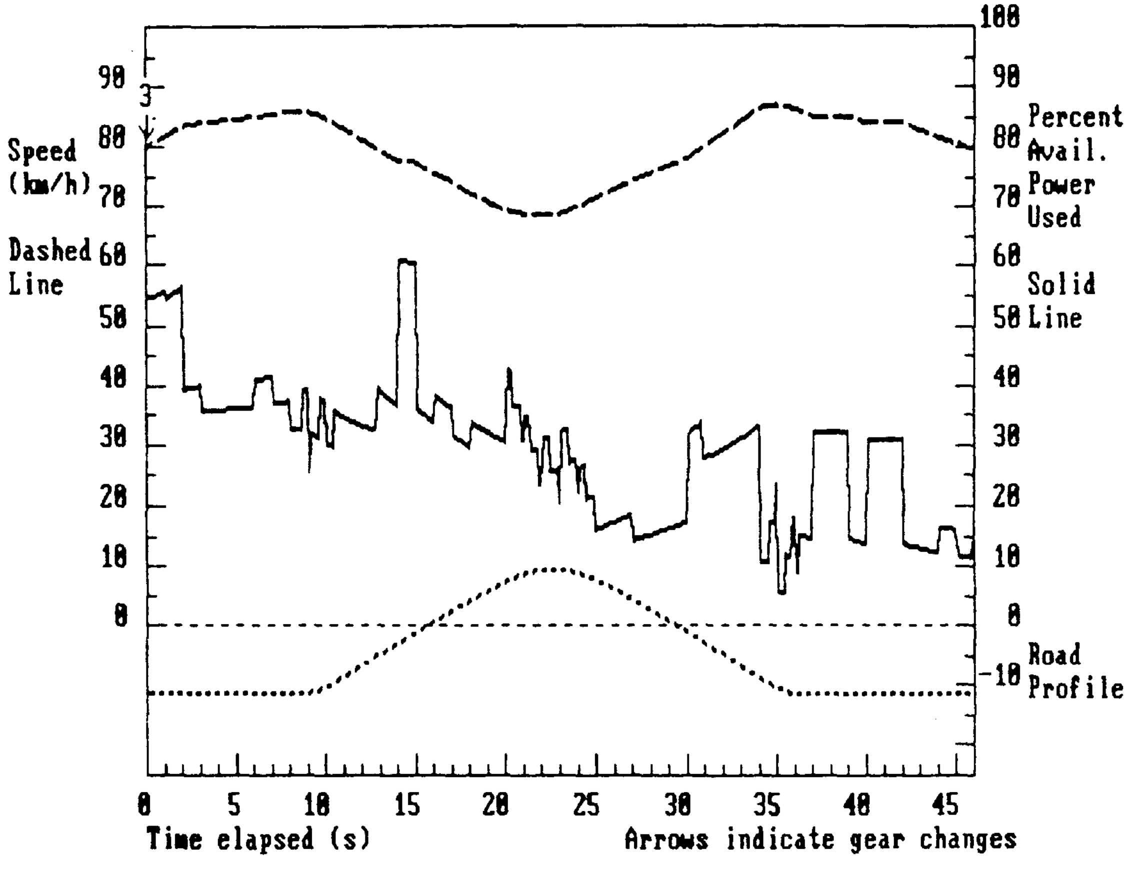

Fig. 11. Plymouth Reliant: Optimal control on a hill (average speed 80 km/h, 6% grade).

Table 8. Some characteristics of the optimal trajectories for driving over an isolated hill

		Peak Speeds						Fuel Penalty	
		Before Hill		After Hill		Minimum Speed		C.S.†	Reliant‡
Automobile	Grade (%)	(km/h)	(mph)	(km/h)	(mph)	(km/h)	(mph)	(%)	(%)
Ford Escort	3	81	50	85	53	69	43	2	0
	6	86	53	88	55	70	44	4	0
Toyota Corolla	3	85	53	85	53	70	44	4	2
	6	86	53	88	55	69	43	11	2
Chevrolet S-10	3	83	52	84	52	73	45	7	2
	6	84	52	89	55	68	42	16	4
Chevy Citation	3	83	52	84	52	73	45	17	2
	6	87	54	86	53	68	42	27	3
Plymouth Reliant	3	83	52	83	52	73	45	2	0
	6	86	53	87	54	68	42	17	0
Pontiac Firebird	3	82	51	85	53	73	45	4	3
	6	87	54	86	53	64	40	24	0
Buick Century	3	81	50	85	53	73	45	12	1
	6	89	55	90	56	66	41	22	7
Caprice SW	3	83	52	85	53	72	45	9	2
	6	88	55	87	54	68	42	23	1

†This is the fuel economy penalty for driving at a constant speed of 80 km/h rather than in the optimal way.
‡This is the fuel economy penalty for driving according to the Plymouth Reliant's optimal trajectory.

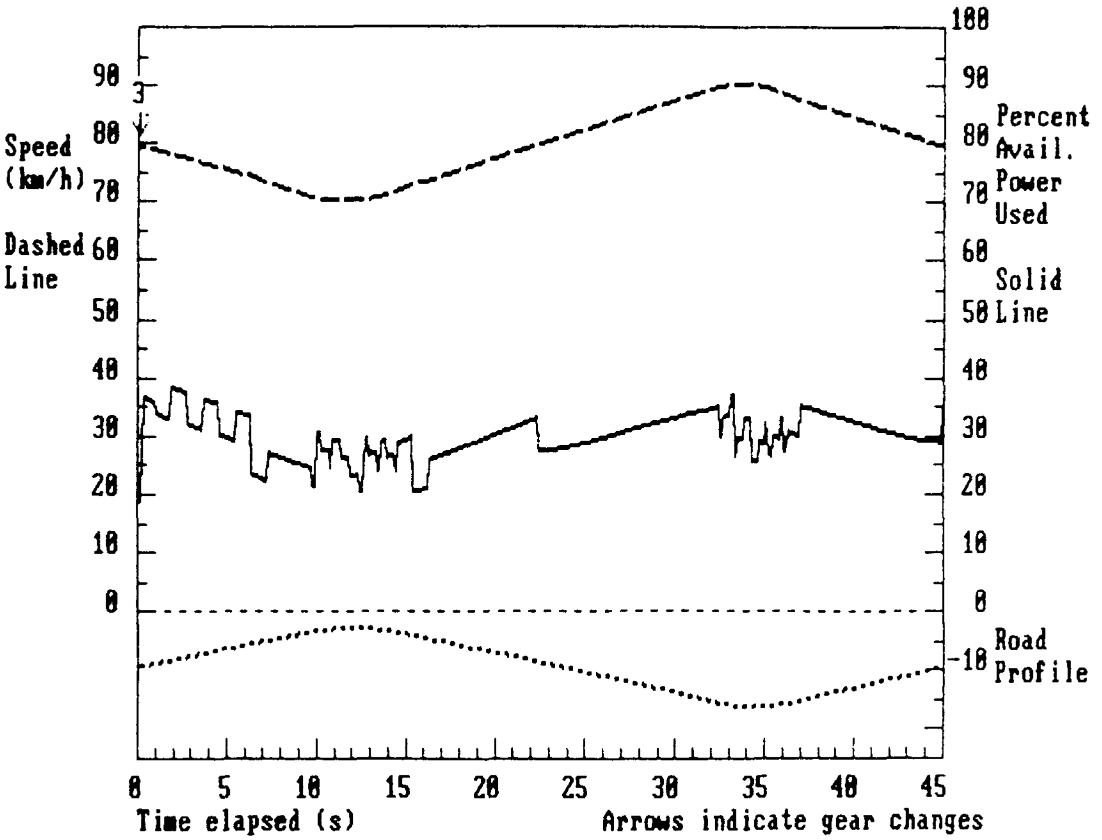

Fig. 12. Plymouth Reliant: Optimal control on a 1 km hill cycle (avg. 80 km/h, 3% grade).

mph) in cruise speed from the optimal cruise speed without sacrificing more than 5% of the optimal fuel economy, and ±19 km/h (±12 mph) without sacrificing more than 15%. The fuel economy penalty for driving at 55 mph (88 km/h) rather than the optimal speed ranges from 8% to 53%, with an average of 24%. The penalty for driving at 75 mph (121 km/h) rather than 55 mph ranges from 21% to 43%, with an average of 30%.

Although fuel economy is generally thought to be more sensitive to high speed in small cars, the figures do not confirm this. The penalty for driving at 75 rather than 55 mph averages 31% for seven 4-cylinder cars, 28% for four 6-cylinder cars, and 33% for three 8-cylinder cars. These figures omit the 4-cylinder Chevette diesel, whose fuel economy is much more sensitive to speed than that of the other cars tested.

The optimal cruising speed tends to increase with slope on downhill grades, roughly at the rate of 5–10 km/h (3–6 mph) for each percent increase in downhill grade. On uphill grades fuel economy tends to be fairly insensitive to speed while operating in a given gear, especially on steeper grades. Thus the major objective on uphill grades is simply to choose a speed that puts one in as high a gear as possible. Typically, the transmission chooses a lower gear when the car is moving uphill rather fast or rather slow and a higher gear when the speed is somewhere in between.

11. RECOMMENDATIONS FOR ACCELERATING TO CRUISING SPEED

The optimal way to accelerate to cruising speed differs greatly from one car to another, and it is unclear how these differences are to be explained. The time one should require to reach cruising speed varies from 11 to 62 seconds. It averages 35 seconds, whether one is accelerating to 55 or 90 km/h.

Although the optimal acceleration trajectories differ widely from car to car, it does not make a great deal of difference to fuel economy how one accelerates to cruising speed, provided he accelerates more rapidly at first and more gradually as cruising speed is approached. With this proviso, one can accelerate the average car at almost any rate within reason without affecting fuel economy over the first 2 km more than 4% when accelerating to 55 km/h, or more than 8% when accelerating to 90 km/h.

Although the rate of acceleration is not very important when a long cruise follows acceleration, it may yet be important in other contexts, as when one stops very soon after accelerating.

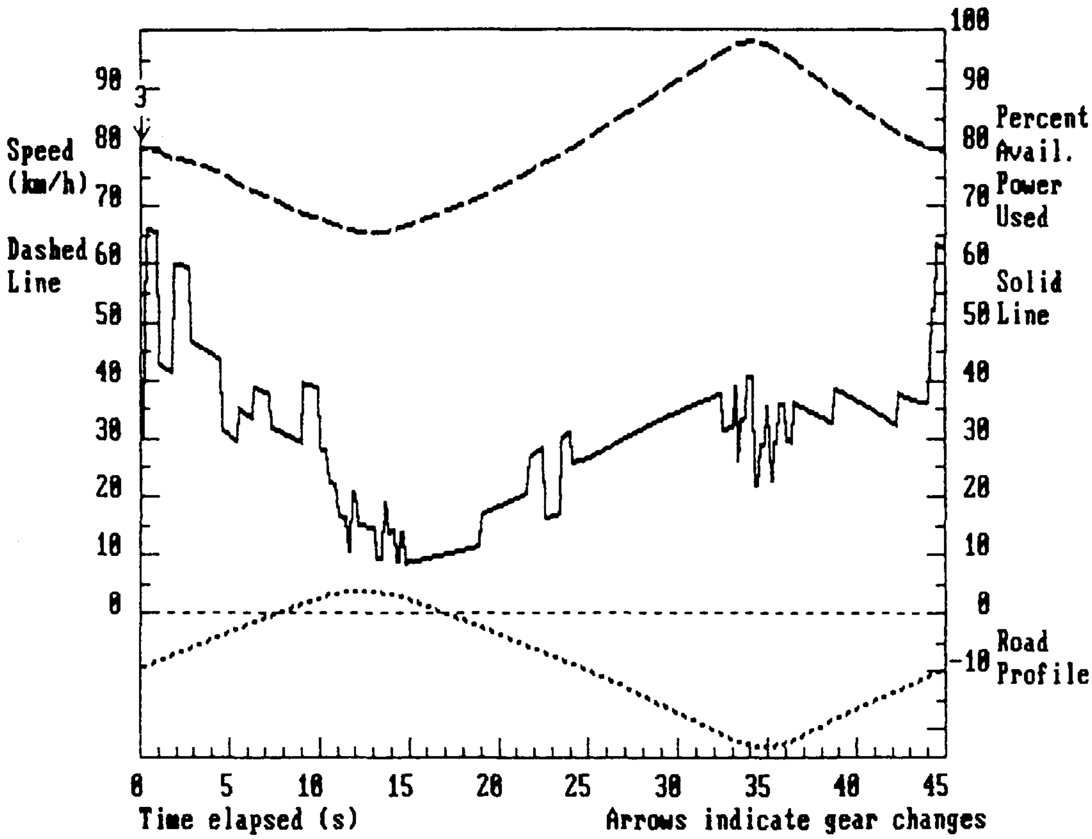

Fig. 13. Plymouth Reliant: Optimal control on a 1 km hill cycle (avg. 80 km/h, 6% grade).

12. RECOMMENDATIONS FOR DRIVING BETWEEN STOP SIGNS

Optimal control on a 300-m block with stop signs at either end differs substantially for different cars. The car should shift into third gear within 10 seconds on the average, but ranging from 5 to 17 seconds among the ten cars for which this problem was solved. Optimal peak speeds range from 27 to 43 km/h (17 to 27 mph), but *all* of these speeds are less than the speed most drivers achieve on a typical block. Thus most drivers can save fuel simply by driving more slowly between closely spaced stops.

One can approximate optimal fuel economy by doing all of his acceleration near the beginning of the block, cruising at constant speed for several seconds, and braking moderately at the end. The acceleration should be brisker at first and become more

Table 9. Some characteristics of the optimal trajectories for driving over a hill cycle

		Speed Extremes				Fuel Penalty	
		Minimum		Maximum			
Automobile	Grade (%)	(km/h)	(mph)	(km/h)	(mph)	Const. Sp.† (%)	Reliant‡ (%)
Ford Escort	3	65	40	90	56	4	0
	6	65	40	97	60	11	1
Plymouth Reliant	3	70	44	90	56	5	0
	6	66	41	98	61	30	0
Pontiac Firebird	3	72	45	88	55	4	2
	6	65	40	96	60	39	0
Buick Century	3	70	44	91	57	29	2
	6	63	39	98	61	42	6

†This is the fuel economy penalty for driving at a constant speed of 80 km/h rather than in the optimal way.

‡This is the fuel economy penalty for driving according to the Plymouth Reliant's optimal trajectory.

gradual as cruising speed is approached. If one drives is this manner and attains the cruising speed that is best for his car, he can probably get within 8% of optimal fuel economy on the average. Unfortunately, the optimal cruising speeds for this type of trajectory vary widely from 25 to 39 km/h (16 to 24 mph) for the 15 cars tested, with an average of 32 km/h (20 mph). Again, however, all of the cruising speeds are below what drivers normally achieve.

13. RECOMMENDATIONS FOR DRIVING ON HILLS

Optimal control on hills was computed with the average speed fixed (at 80 km/h). Unlike optimal control in the other situations studied, it is more or less the same from car to car.

Suppose one is driving on level road at 80 km/h (50 mph) and approaches a hill that climbs at a 3% grade and descends to the original elevation on the other side. The driver should pick up a bit of speed (2–4 km/h, or 1–3 mph) as he approaches the foot of the hill. As he ascends the hill his speed should drop at the rate of about 1 km/h every second (about 2 mph every 3 seconds) until he reaches the top, at which point his speed should start to rise at the same rate. This will require that he ease up on the accelerator shortly before reaching the crest. As he reaches the far side of the hill his speed will slightly exceed his original cruising speed, and he should not return the accelerator to its cruising position until he has slowed to cruising speed. On a typical hill (600 m from end to end) his speed should drop about 8 km/h (5 mph) below cruising speed by the time he reaches the crest. He should control the throttle so that the transmission never shifts down; one effect of the initial acceleration before reaching the hill is to avoid a downshift.

Optimal control is similar on a 6% grade, but the speed variations are greater. The driver should gain about 8 km/h (5 mph) before reaching the foot of the hill and let his speed drop at the rate of about 3 km/h every 2 seconds (about 1 mph per second) as he climbs the hill. It should rise at about the same rate on the downslope. On a typical hill (600 m) his speed should fall about 13 km/h (8 mph) below cruising speed by the time he reaches the crest. Again he should never let the transmission shift down, even if it is in overdrive. In some cars this can be effected by easing up on the accelerator slightly as the car climbs the hill.

The savings of optimal control vs. constant-speed driving can be substantial. Cruise-control driving reduces fuel economy an average of 7% on a 3% grade and 18% on a 6% grade. The penalty is usually greater for larger cars. (These figures reflect fuel economy for a stretch of road extending from 200 m in advance of the hill to 200 m beyond the hill.)

Suppose now that one is driving on a series of hills, up and down in a fairly regular fashion. He should consistently pick up speed as he descends and lose speed as he ascends, so that his speed about halfway down or up each hill is close to his average speed. On a 3% downgrade he should gain speed at the rate of about 1 km/h every second (2 mph every 3 seconds) and lose speed at the same rate on an upgrade. On a typical hill cycle (1 km or 45 seconds from crest to crest) his speed should vary over a range 20 km/h (12 mph) wide. He should never allow the transmission to shift down.

If the grade is 6%, the optimal speed variations may be too wide for safety if other traffic is present. The car should pick up speed at the rate of 3 km/h every 2 seconds (about 1 mph per second) as it descends and lose speed at the same rate as it ascends. On a 1-km cycle, the car's speed varies over a range 35 km/h (22 mph) wide. Again, the driver should never allow the transmission to shift down.

The penalty for cruise-control driving is again substantial, averaging 10% on 3% grade cycles and 30% on 6% grade cycles for four representative cars. It is clear that even if one realizes only half the advantage of optimal driving, the savings are worthwhile, especially on moderate to steep grades.

Acknowledgments—The calculation of optimal control and the original development of the automobile simulation technique were funded as part of the Driver Energy Conservation Awareness Training Program of the U.S. Department of Energy thorugh Oak Ridge National Laboratory (ORNL), operated by Union Carbide Corporation and more recently Martin Marietta Energy Systems for the Energy Department. The development of the 15 automobile simulators was carried out at ORNL with the support of the Office of Safety and Traffic Operations Research and Development of the Federal Highway Administration, U.S. Department of Transportation.

REFERENCES

Akcelik R. and Biggs D. C. (1987) Acceleration profile models for vehicles in road traffic. *Transpn. Sci.* **21**, 36–54.

Biggs D. C. and Akcelik R. (1985) Further work on modelling car fuel consumption. Australian Road Research Board Internal Report AIR 390-10, Vermont South, Victoria.

Chang, M.-F., Evans L., Herman R. and Wasielewski P. (1976a) The influence of vehicle characteristics, driver behavior, and ambient temperature on gasoline consumption in urban traffic. General Motors Research Laboratories, Research Publication GMR-1950.

Chang M.-F., Evans L., Herman R. and Wasielewski P. (1976b) Gasoline consumption in urban traffic. *Transpn. Res. Rec.* **599**, 23–30. Shortened version of previous reference.

Chang M.-F. and Herman R. (1976) Driver response to different driving instructions: Effect on speed, acceleration and fuel consumption. *Traffic Eng. Control* **21**, 545–550.

Evans L. (1979) Driver behavior effects on fuel consumption in urban driving. *Human Factors* **21**, 389–398.

Evans L. and Takasaki G. M. (1981) Fuel used to accelerate vehicles from rest to cruising speeds. SAE Paper No. 810781, presented at Passenger Car Meeting, Dearborn, Michigan.

Everall P. F. (1968) The effect of road and traffic conditions on fuel consumption. Road Research Laboratory, RRL Report LR 226, Crowthorne, England.

Hooker J. N. (1985) Fuel efficient driving strategies. Prepared for U.S. Dept. of Energy. Graduate School of Industrial Administration, Carnegie-Mellon University, Pittsburgh, PA.

Hooker J. N., Rose A. B. and Roberts G. F. (1983a) A holistic approach to vehicle simulation. *Math. Computers in Simulation* **25**, 259–267.

Hooker J. N., Rose A. B. and Roberts G. F. (1983b) Optimal control of automobiles for fuel economy. *Transpn. Sci.* **17**, 146–167.

McGill R. N., Hooker J. N. and Hodgson J. W. (1985) Fuel consumption and emission values for traffic models. Federal Highway Administration, U.S. Dept. of Transportation, Report No. FHWA/RD-85/053.

Rose A. B., Hooker J. N. and Roberts G. F. (1982) A data-based simulator for predicting vehicle fuel consumption. SAE Technical Paper No. 820302, Warrendale, Pennsylvania.

Schwarzkopf A. B. and Leipnik R. B. (1977) Control of highway vehicles for minimum fuel consumption over varying terrain. *Transpn. Res.* **11**, 279–286.

Seierstad A. and Sydsaeler K. (1977) Sufficient conditions in optimal control theory. *Int. Econ. Rev.* **18**, 367–391.