MDD-based Postoptimality Analysis for Mixed-integer Programs

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Two Perspectives on Optimization

Traditional

Nontransparent data structure

$$Ax \geq b$$

Solver $\Rightarrow$

Optimal solution, or list of (near-optimal) solutions

This wastes a wealth of information collected for the model, perhaps at great expense

Two Perspectives on Optimization

Traditional

Nontransparent data structure

$$Ax \geq b$$

Solver

Optimal solution, or list of (near-optimal) solutions

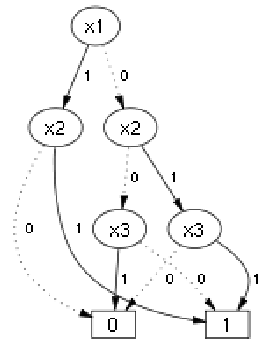
Proposed

Nontransparent data structure

$$Ax \geq b$$

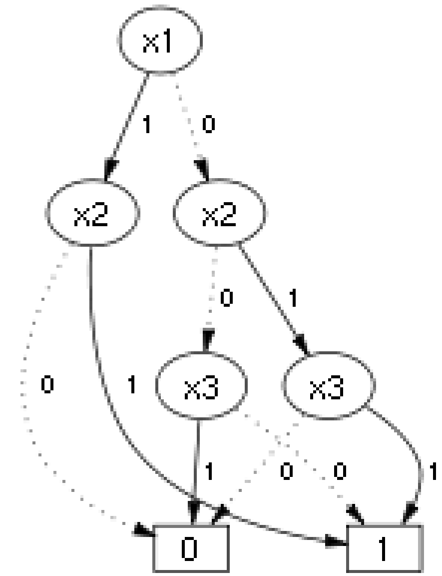
Solver

Transparent data structure



Postoptimality Analysis

- **Decision diagrams** provide a transparent data structure
  - Can compactly represent **all near-optimal solutions** (within Δ of optimum).
  - Open the door to more comprehensive postoptimality analysis
  - Can be **efficiently queried** with what-if questions.



Outline

- **Basic concepts**
- **Pure integer programming**
  - Sound diagrams for IP
  - Postoptimality analysis using sound DDs
  - Sound reduction
  - Computational results
- **Mixed-integer programming**
  - Sound diagrams for MILP
  - Identifying equivalent states
    - By computation of equivalency ranges
    - Arc deletion and contraction
    - Separable constraints
  - Introducing spurious solutions
    - By constraint dualization
    - By sound reduction

Near-optimal Solutions

- Let z^* = optimal cost.
- A solution is **Δ-optimal** if it is feasible and its cost is $\leq z^* + \Delta$
  - We wish to represent all Δ-optimal solutions in a decision diagram.
  - The diagram is generated once for multiple queries.
    - In general, the user will be interested in δ-optimal solutions for $\delta \leq \Delta$.

Hadžić and JH (2006)

Sound Decision Diagrams

- **Sound** DDs can store near-optimal solutions more compactly.
  - Sound = all Δ-**optimal solutions** are included…
    - …along with some **spurious** solutions (feasible and **infeasible**) that are **worse than Δ-optimal**
      - That is, cost > z^* + Δ.

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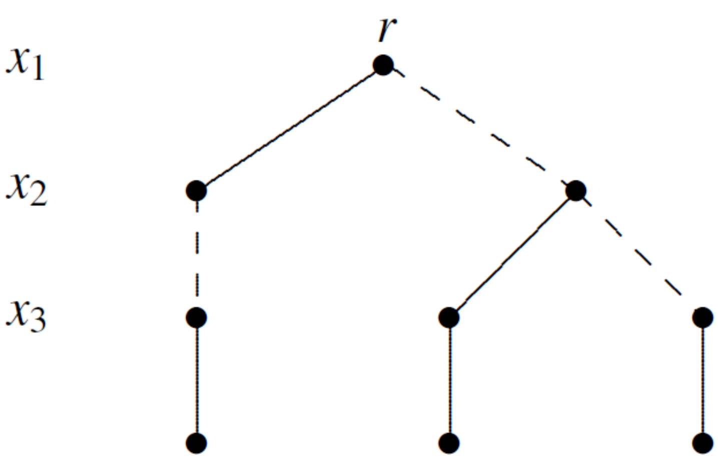
  - Sound = all Δ-**optimal solutions** are included…
    - …along with some **spurious** solutions (feasible and **infeasible**) that are **worse than Δ-optimal**
      - That is, cost > z^* + Δ.
    - These solutions are easily screened out.
    - No effect whatever on most queries.
  - Paradoxically, this can result in a **smaller** DD.

Hadžić and JH (2006)

Sound DDs for IP

$$\begin{array}{ll} \text{minimize} & 4x_1 + 3x_2 + 2x_3 \\ \text{subject to} & x_1 + x_3 \geq 1, \;\; x_2 + x_3 \geq 1, \;\; x_1 + x_2 + x_3 \leq 2 \\ & x_1, x_2, x_3 \in \{0, 1\} \end{array}$$

Branching tree



Optimal value = 2

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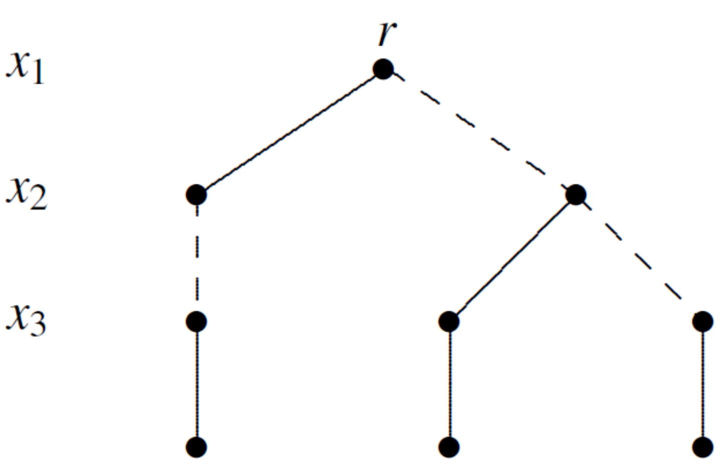
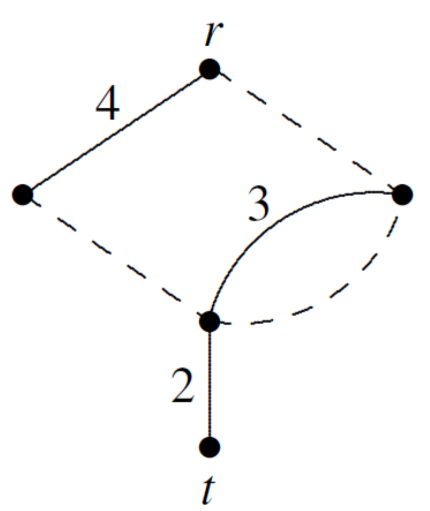


Reduced weighted DD



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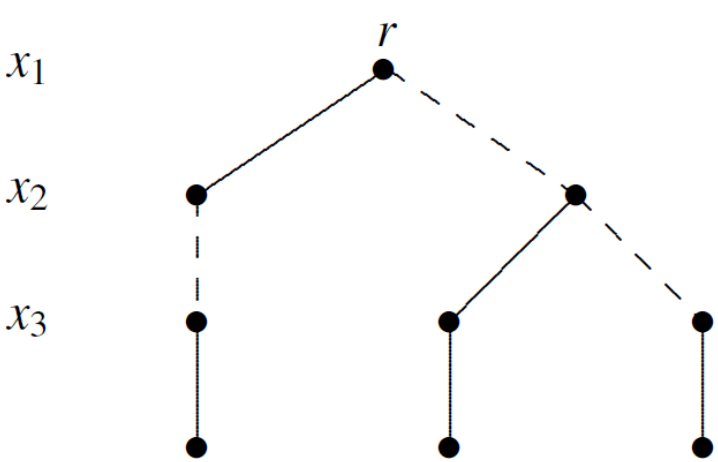

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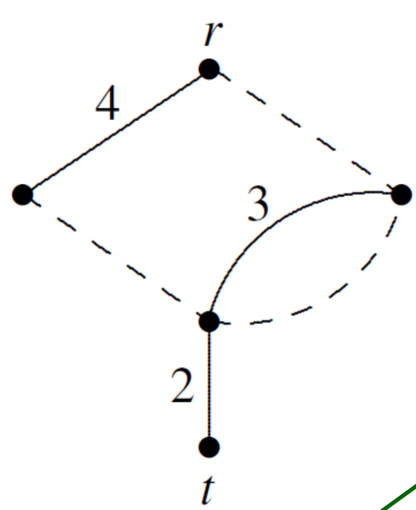
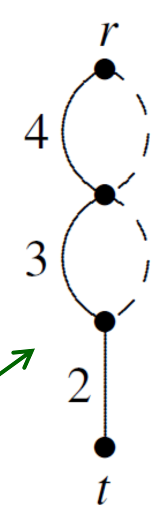


Sound DD for Δ = 6



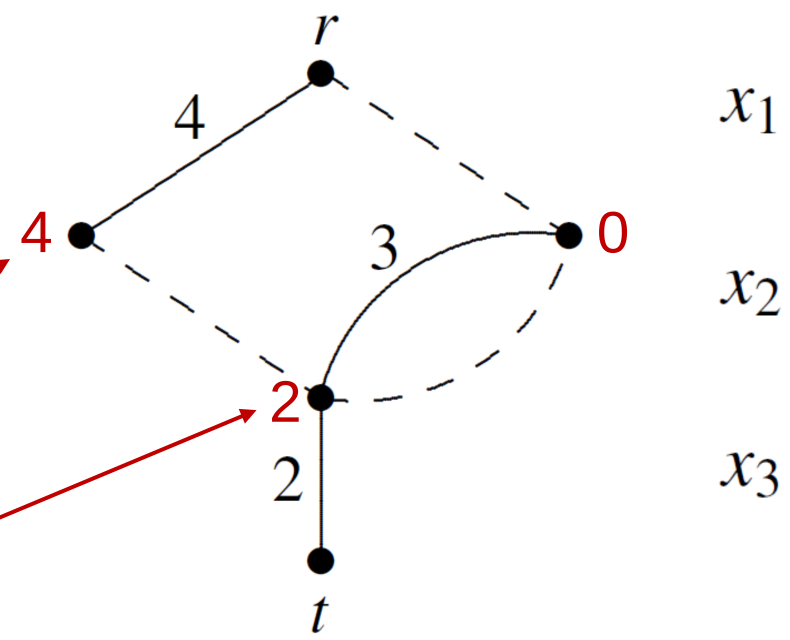
Optimal value = 2

Contains spurious solution *x* = (1,1,1)
Its value = 9 > 2 + Δ

Postoptimality Queries

Example: What values can x_2 take when $\delta = 2$?

First use **original** DD
for $\Delta = 6$.



Shortest distance to root

Shortest distance to terminus

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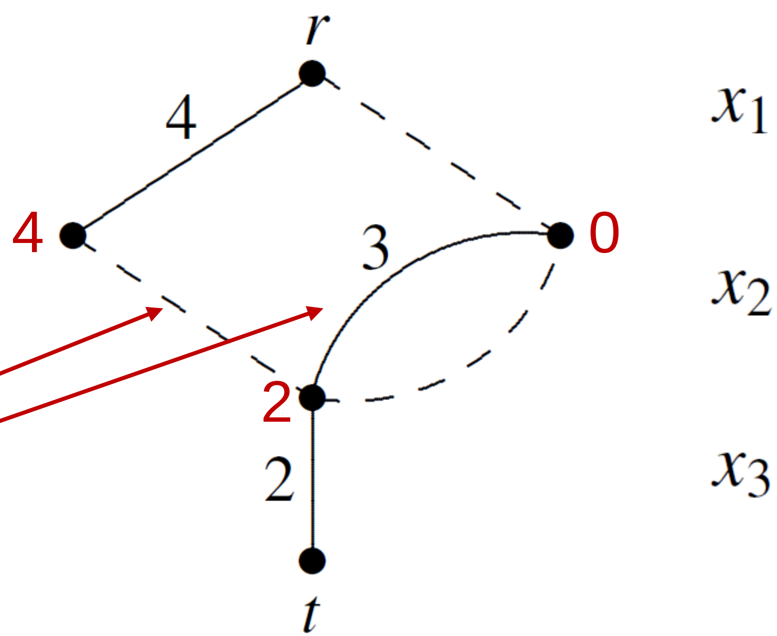

These arcs are removed because $4 + 0 + 2 > 2 + \delta$

$0 + 3 + 2 > 2 + \delta$

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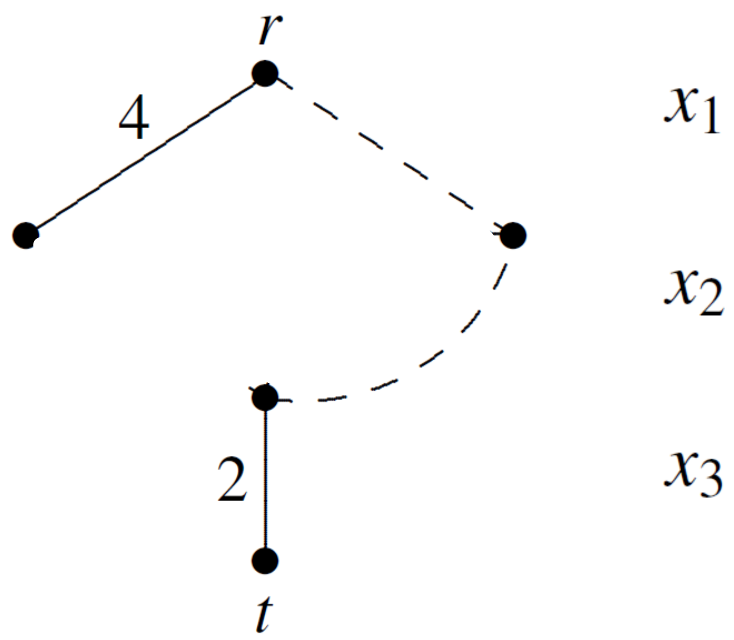

x_2 can take only value 0 when $\delta = 2$.

Postoptimality Queries

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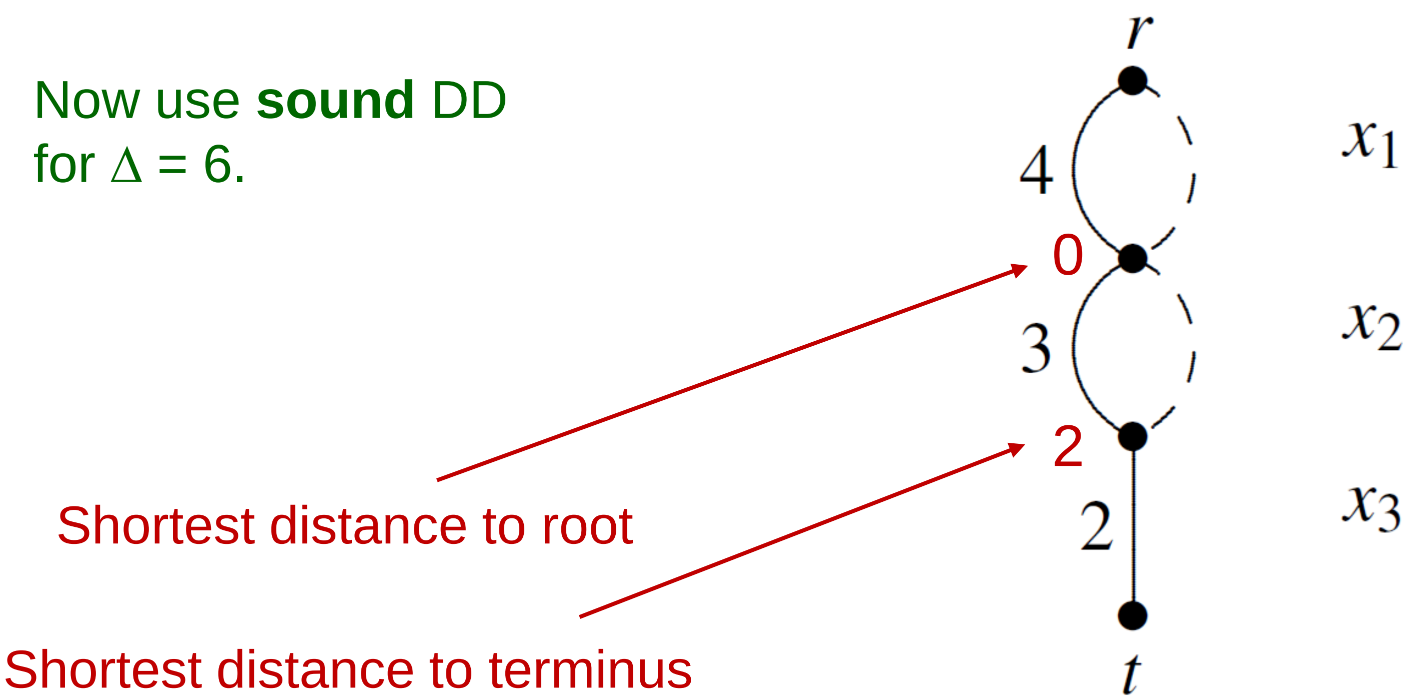

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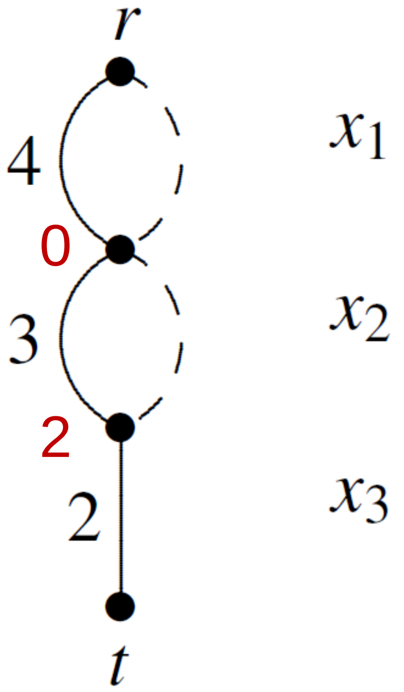


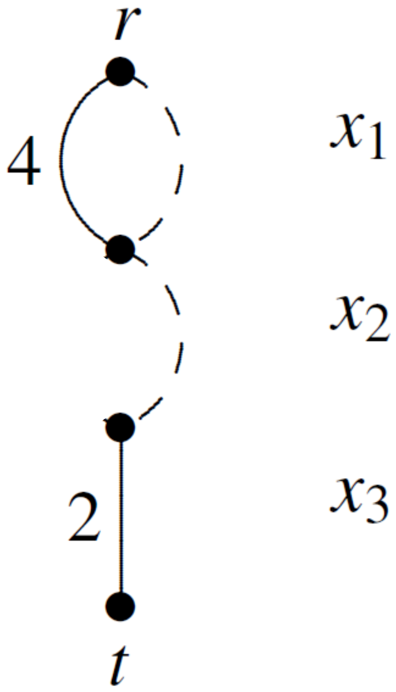
This arc is removed
because $0 + 3 + 2 > 2 + \delta$

Postoptimality Queries

Example: What values can x_2 take when $\delta = 2$?

Now use **sound** DD for $\Delta = 6$.

Spurious solution (1,1,1) is screened out in the process.



x_2 can take only value 0 when $\delta = 2$.
Spurious solution has no effect on the analysis.

Minimal Sound DDs

- A sound DD is **minimal** if no arcs/nodes can be removed without destroying soundness.
  - Easy to check whether an arc/node can be removed.

Theorem. A minimal sound DD for $\Delta = 0$ never contains spurious solutions.

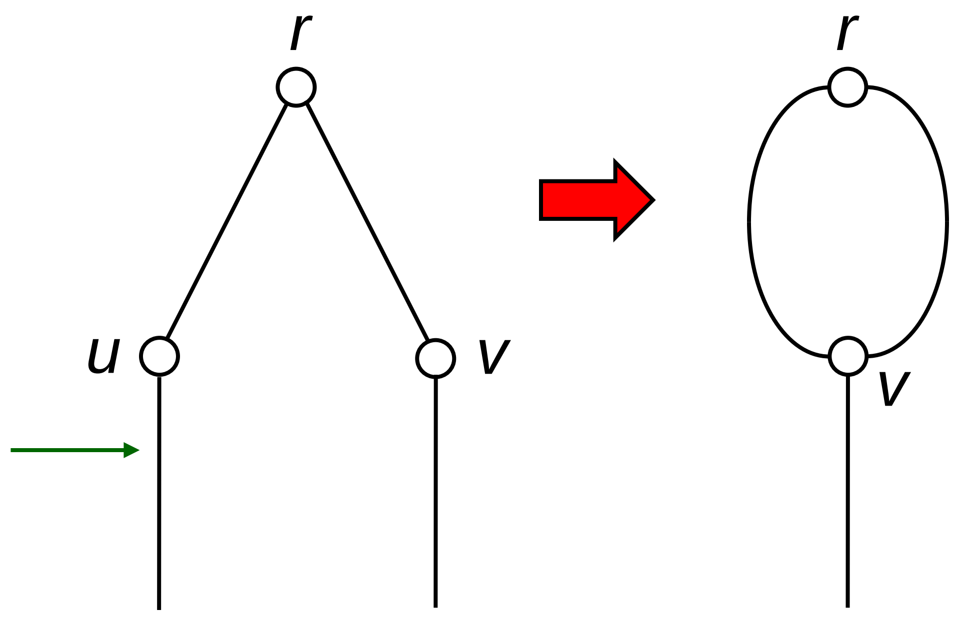
So there is no point in using sound DDs for **optimal solutions only**. (Not so for MIP.)

Serra and JH (2018)

Sound Reduction

- We can **sound reduce** node ***u*** into node ***v*** when this removes no Δ-optimal solutions and introduces only spurious solutions.
  - Can be checked recursively while building diagram.

Arcs out of *u* are dropped



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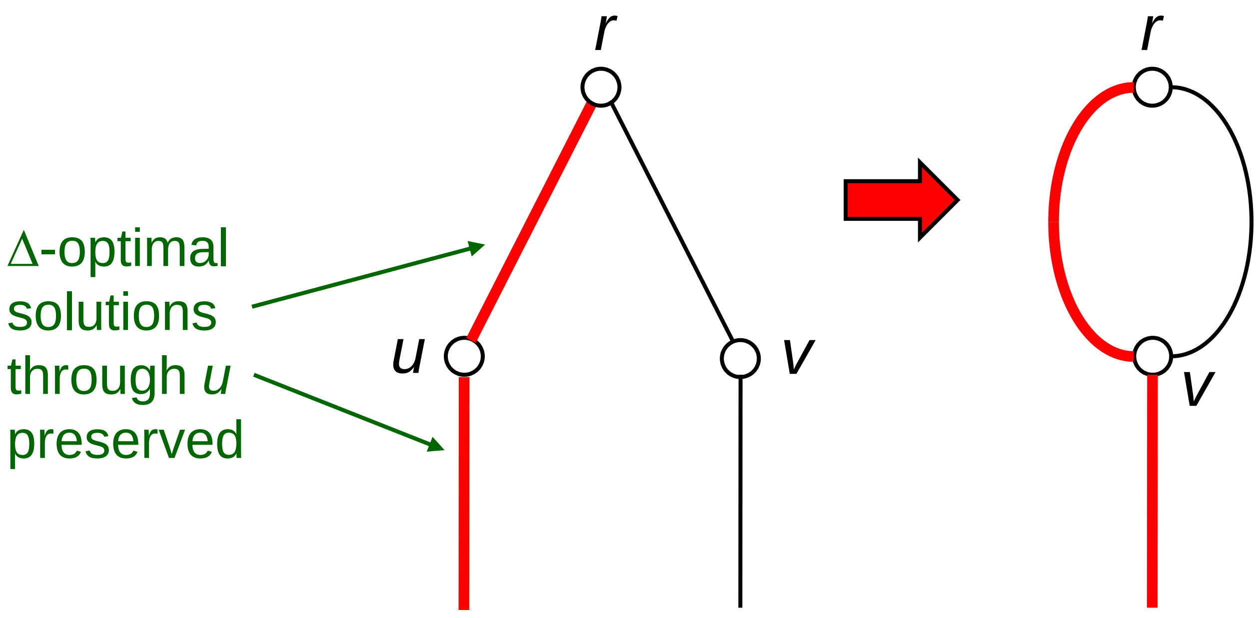

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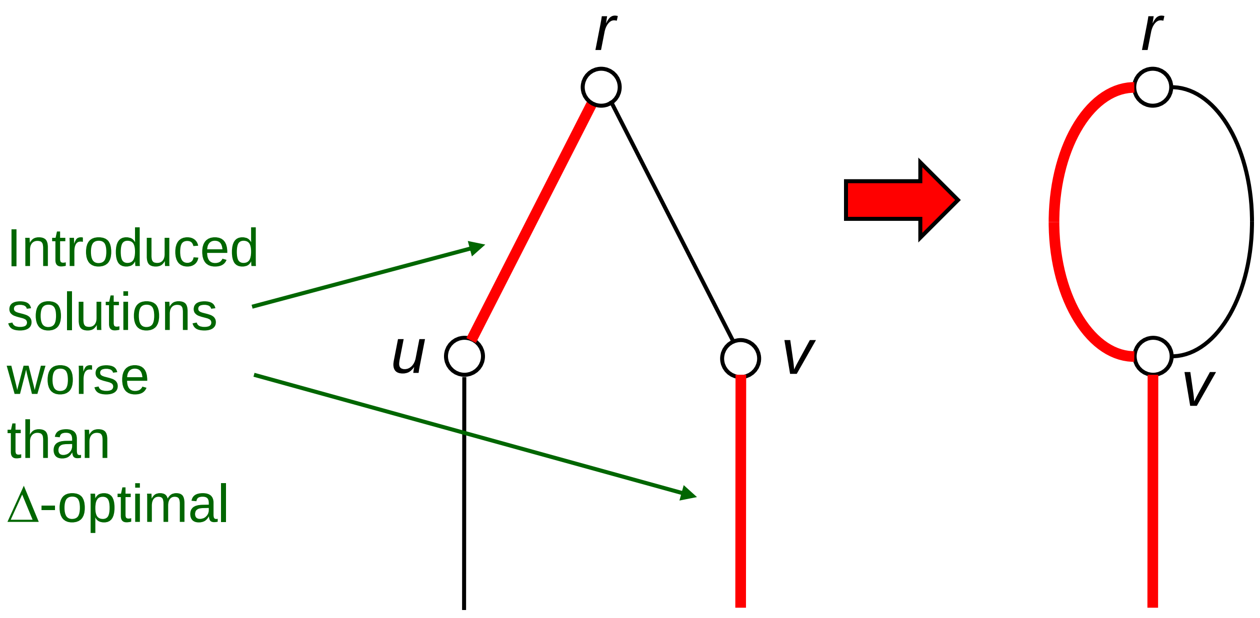
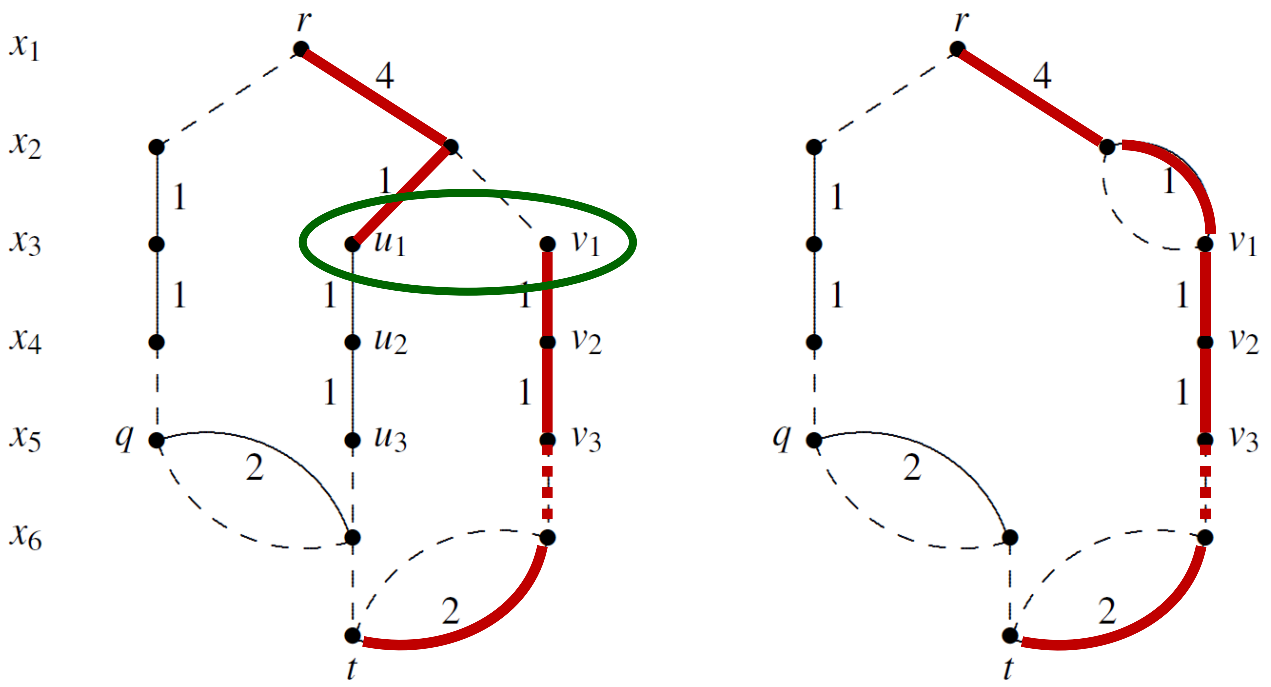


Sound Reduction

Optimal value = 2, Δ = 6.
Sound-reduce u_1 into v_1



Introduced solution is spurious, value = 9

Sound Reduction

Theorem. Repeated application of the sound reduction operation (in any order) yields a **sound** DD of **minimum size**.

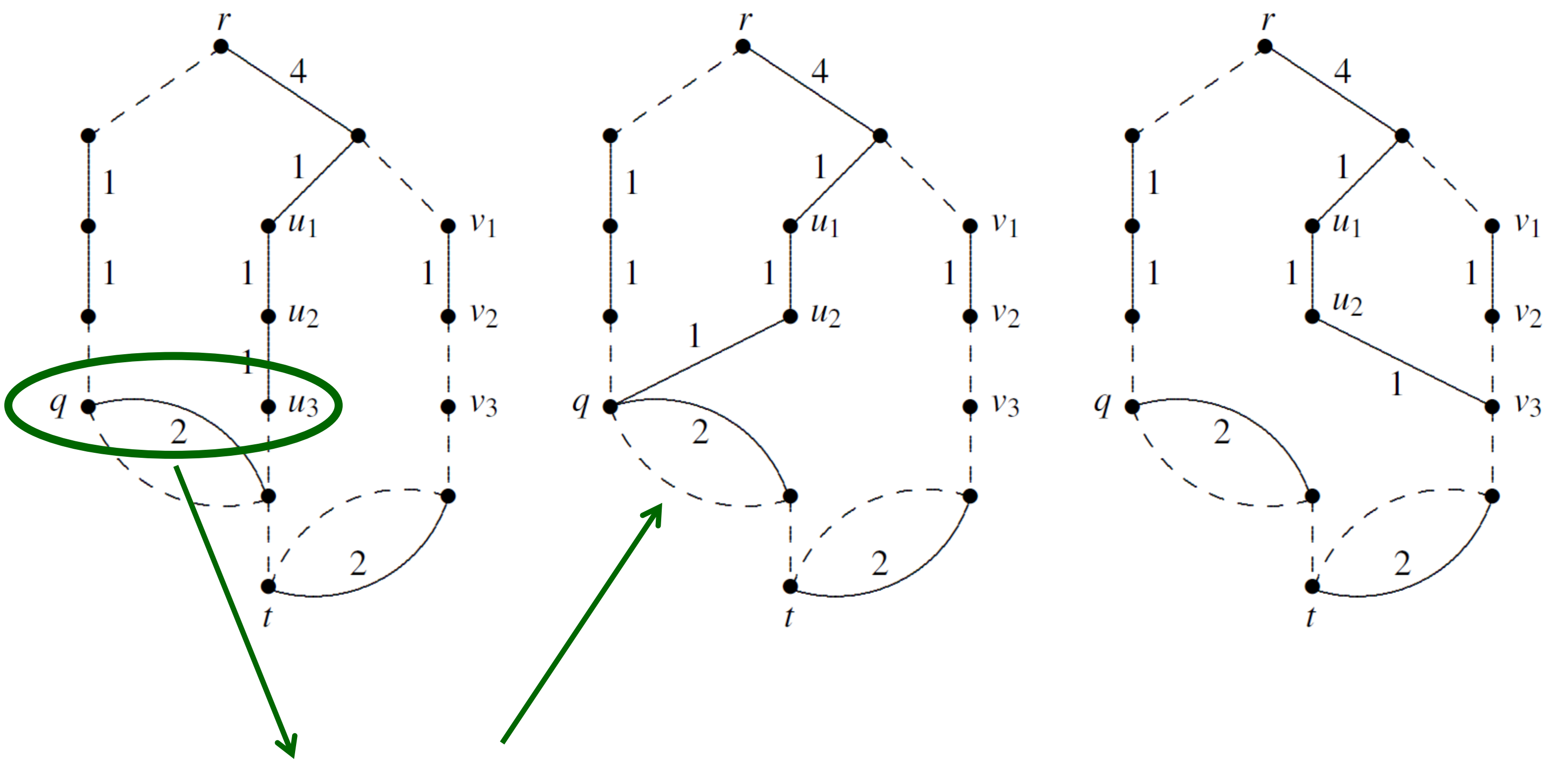
Different reduction orders can yield different diagrams, but **they all have the same size!**

(Does not hold for MILP.)

Serra and JH (2018)

Sound Reduction

Mergers yield 2 different sound-reduced diagrams,
but of the same size



This merger yields one sound-reduced diagram

Sound Reduction

Mergers yield 2 different sound-reduced diagrams,
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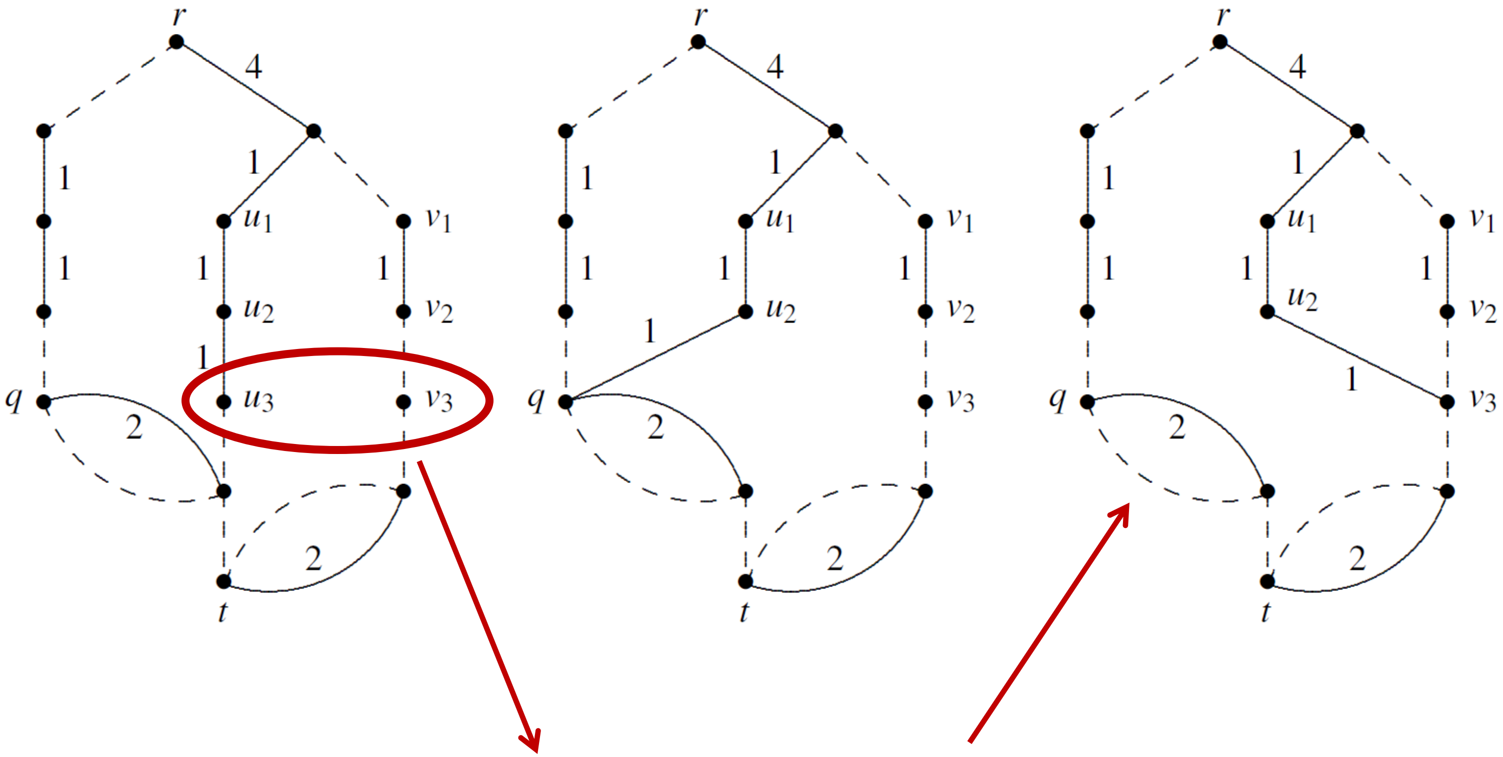


This merger yields another, of the same minimum size

Building a Sound DD

- Two options:
- Stand-alone approach
  - Build the sound DD and identify Δ-optimal solutions simultaneously.
  - Use branching search with backtracking.
    - Use only the optimal value, obtained from a solver.
    - Identify nodes and sound-reduce nodes when possible.
- Solver-assisted approach
  - Obtain Δ-optimal solutions from a solver.
  - Use similar backtracking algorithm, without search for solutions.

Building a Sound DD

Technical conditions for sound-reducing *u* into *v*:

Shortest *r*–*u* distance

$$\boxed{w(r,u)} + \mathrm{LCDS}(u,v) > \boxed{z^*} + \Delta$$

Optimal value

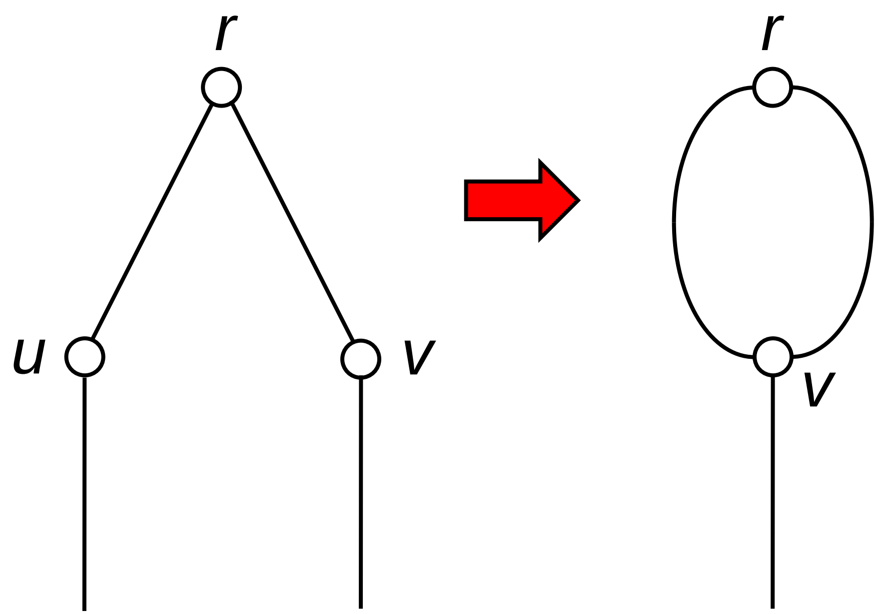
$$w(r,u) + \boxed{\mathrm{LCDS}}(v,u) > z^* + \Delta$$

Least-cost differing suffix
= min cost of suffix of *v* that is not a suffix of *u*

Suffix = path to terminus

Computed recursively while backtracking.

LCDS is known when we have backtracked to both *u and* v.



Compression

- Sound reduction can significantly compress a diagram that represents near-optimal solutions.
  - We investigate compression **for a large Δ**, larger than needed in practice.
    - For some instances, Δ is large enough to include all feasible solutions.
  - Same diagram used for **multiple queries**, using different tolerances $\delta < \Delta$.

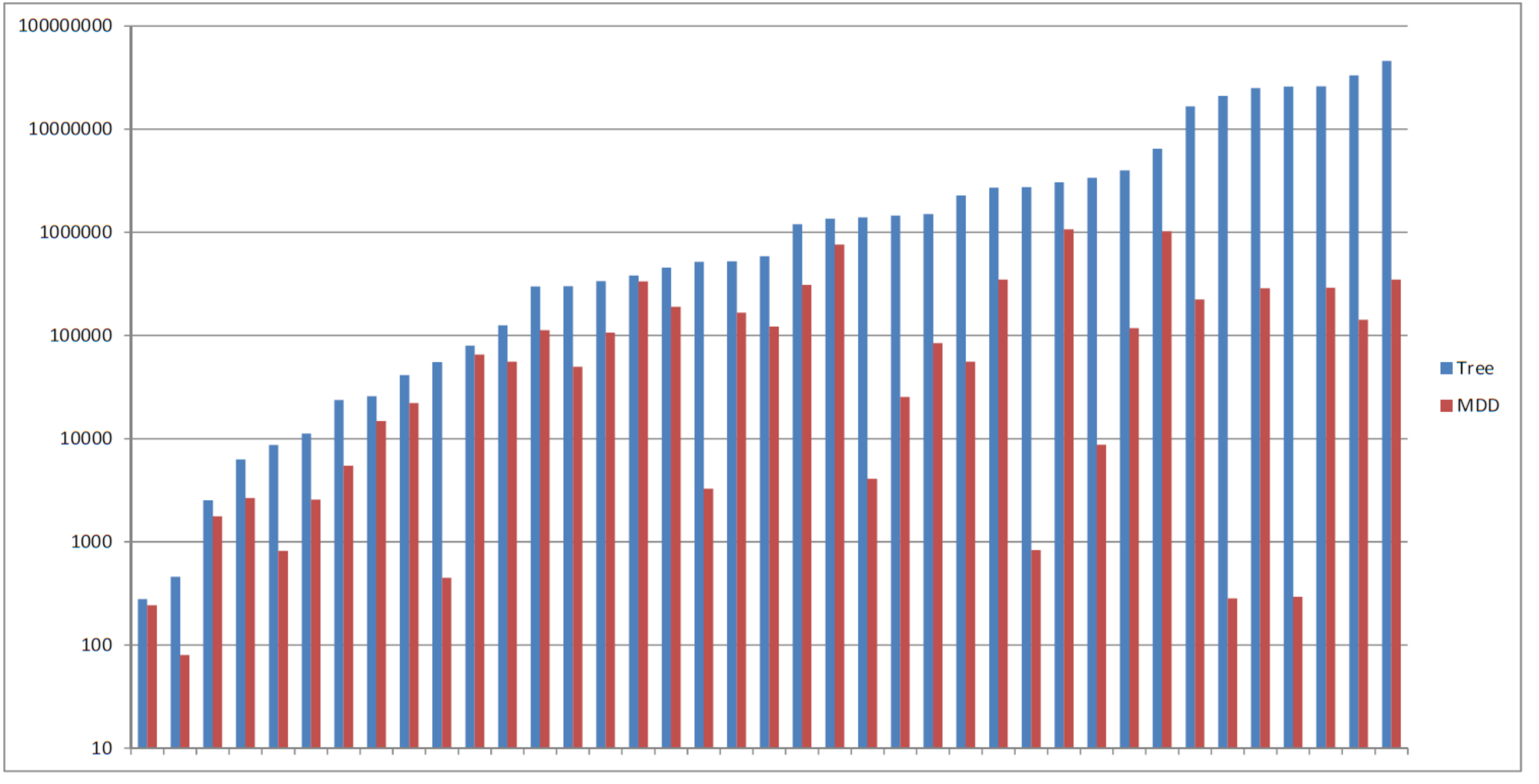
Compression

- We measure:
  - Size of **tree** representation of Δ-optimal solutions.
    - Smaller than a list.
  - Size of **reduced DD** and **sound-reduced DD**.
  - **Computation times**, including search time for Δ-optimal solutions.
    - Stand-alone method
    - CPLEX-assisted method

DD Compression

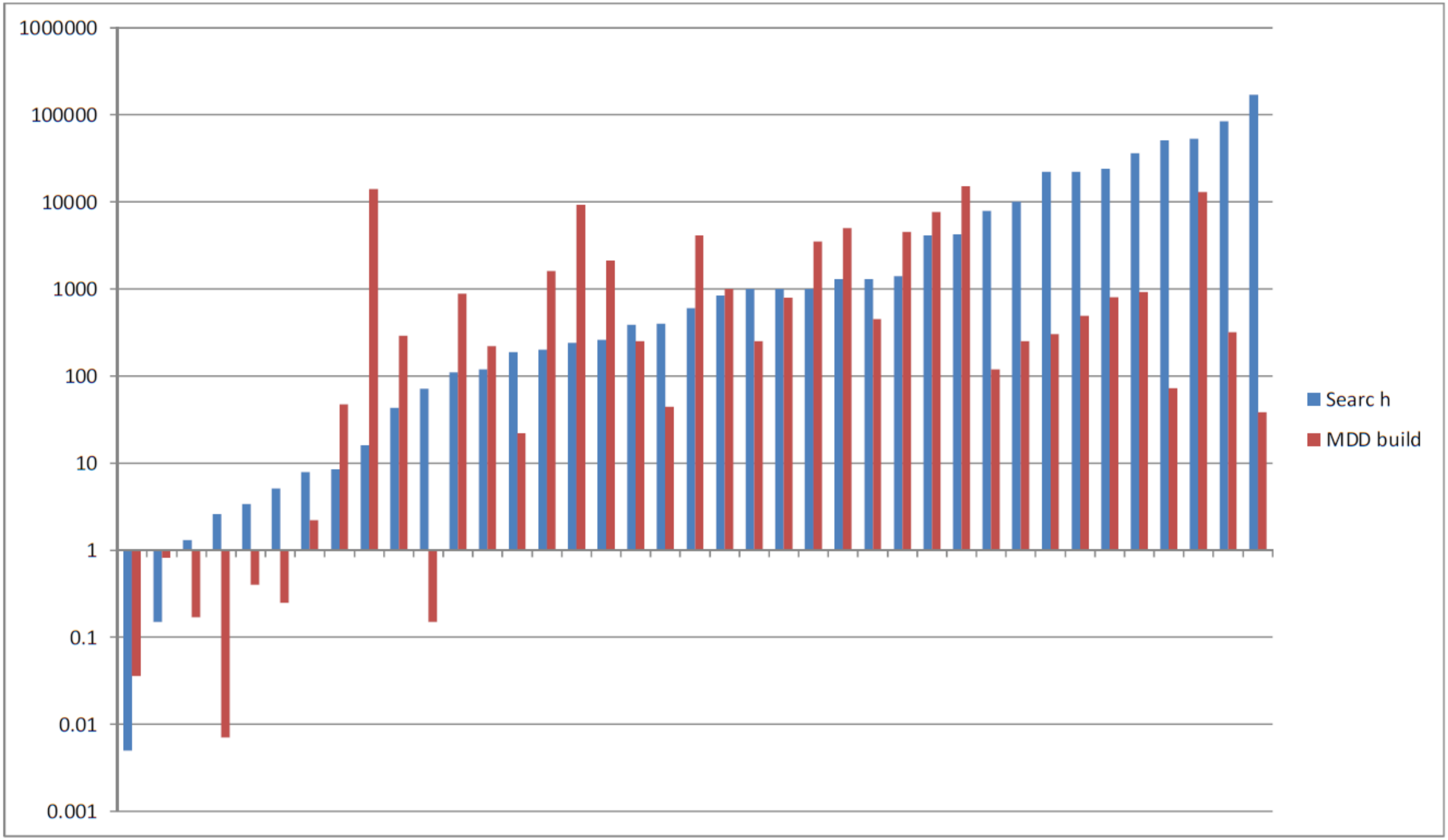
Tree size & sound-reduced DD size for large Δ

39 IP instances from MIPLIB 3.0 and MIPLIB 2010

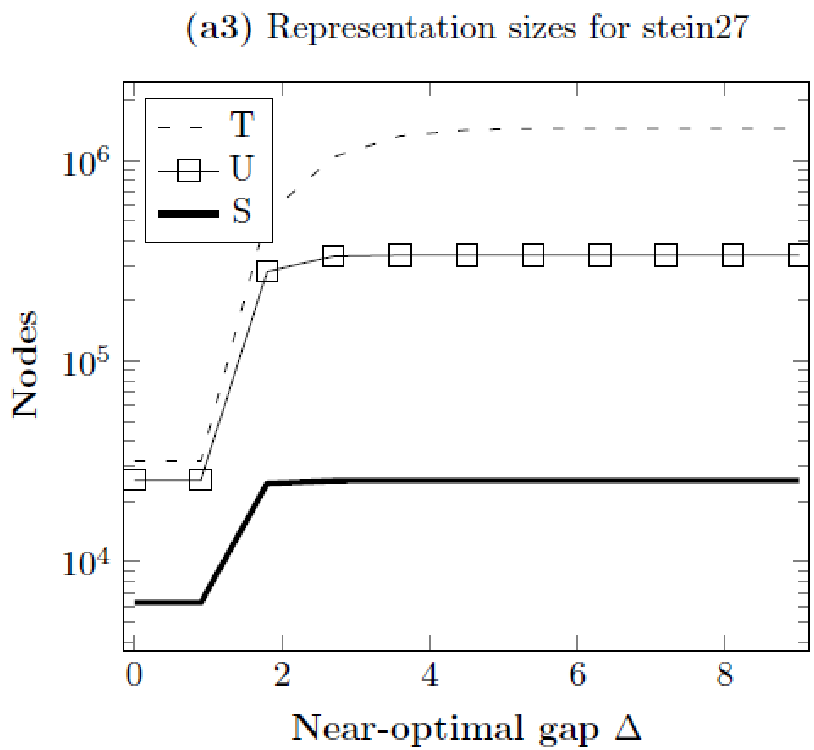


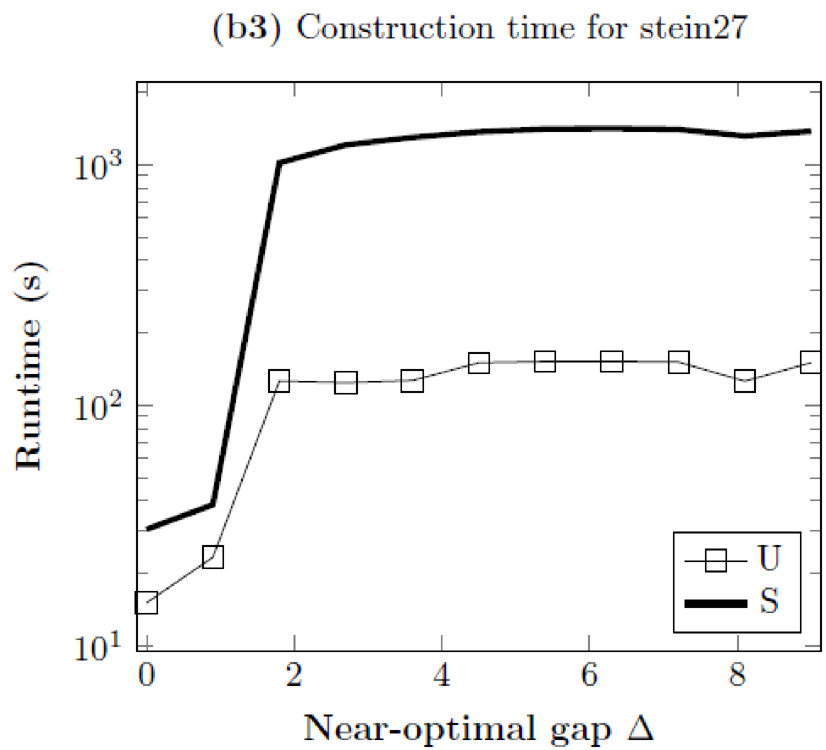
Search & Compression Time

CPLEX search time & DD build time (sec) for large Δ



DD Compression & Time vs Δ

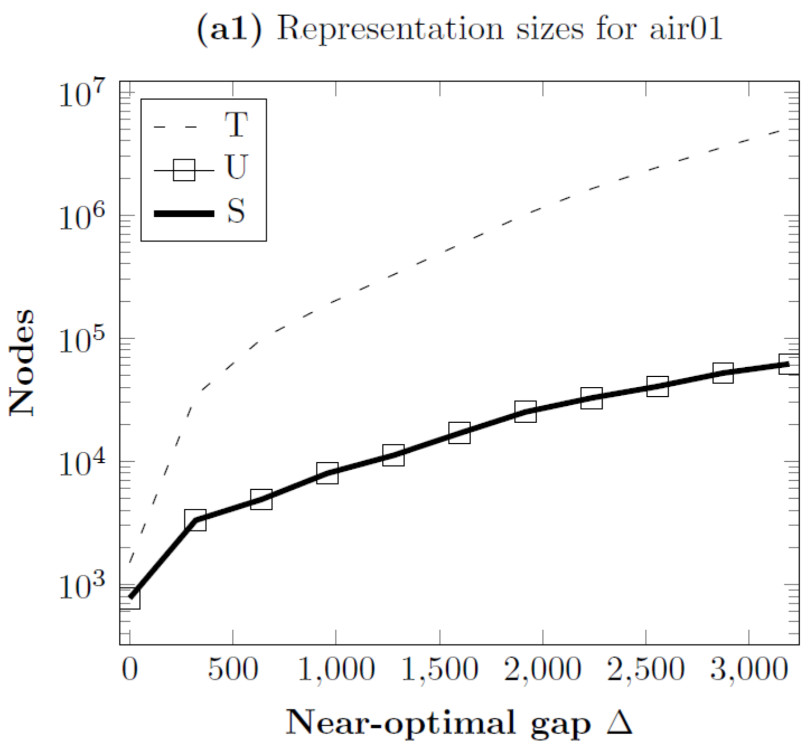


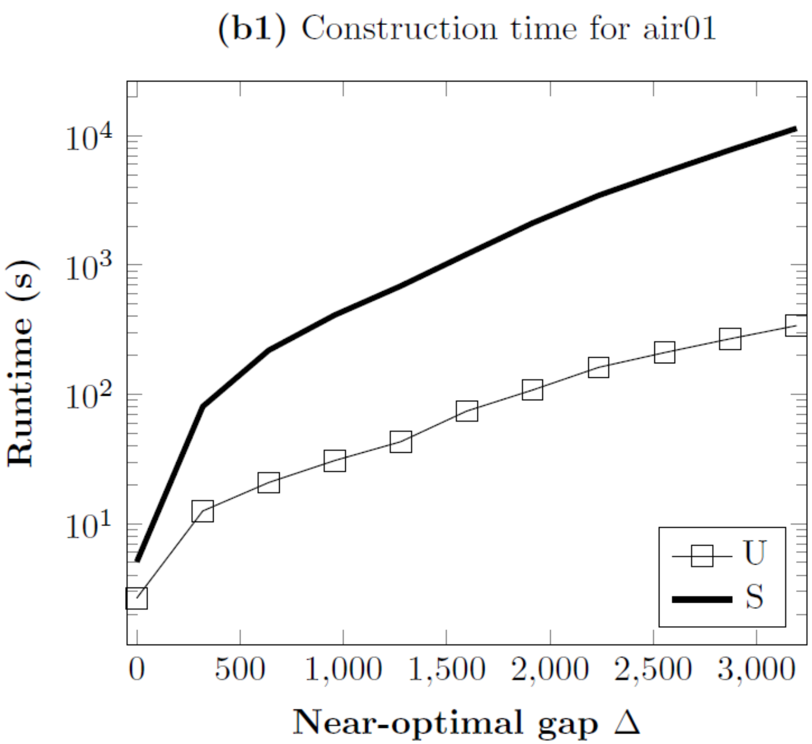


Stand-alone method

T = tree representation
U = reduced DD
S = sound-reduced DD

DD Compression & Time vs Δ





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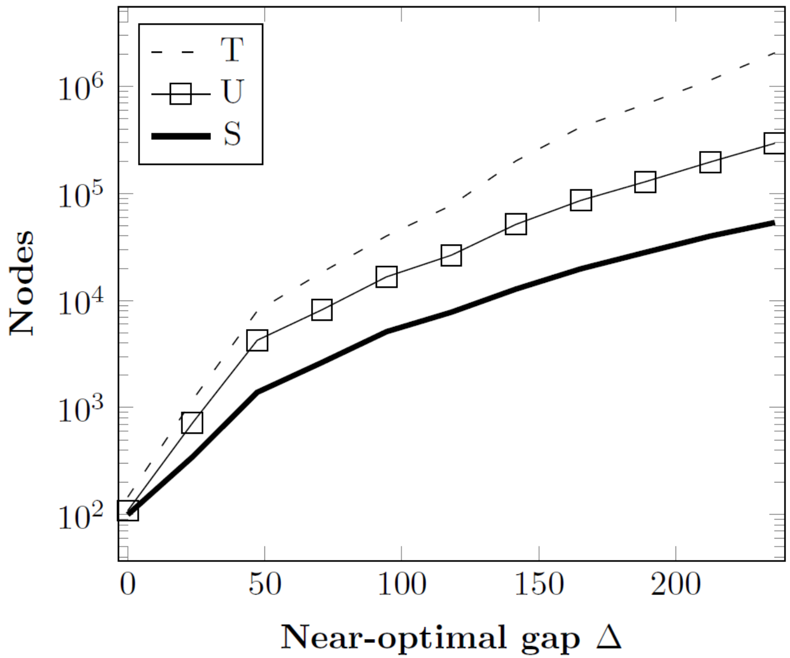
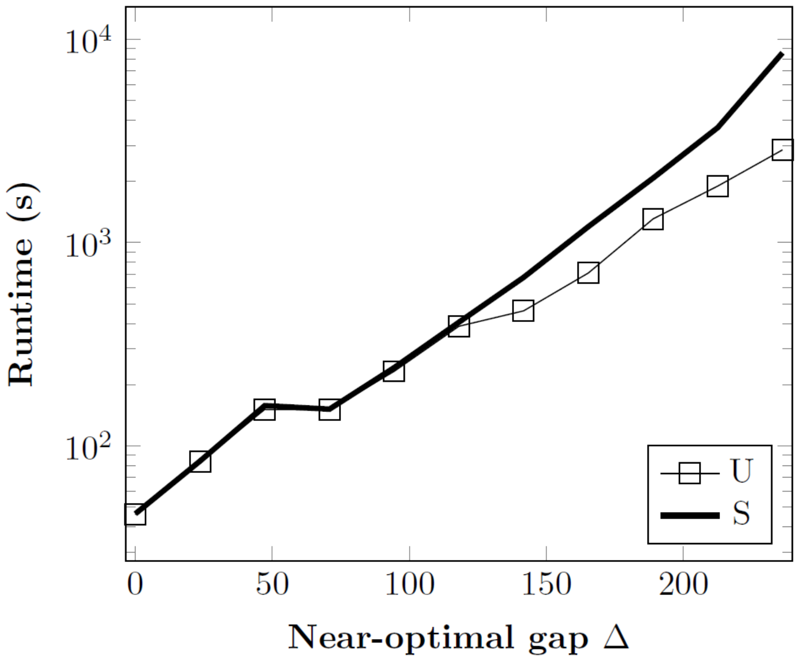
DD Compression & Time vs Δ

(a2) Representation sizes for lseu



(b2) Construction time for lseu



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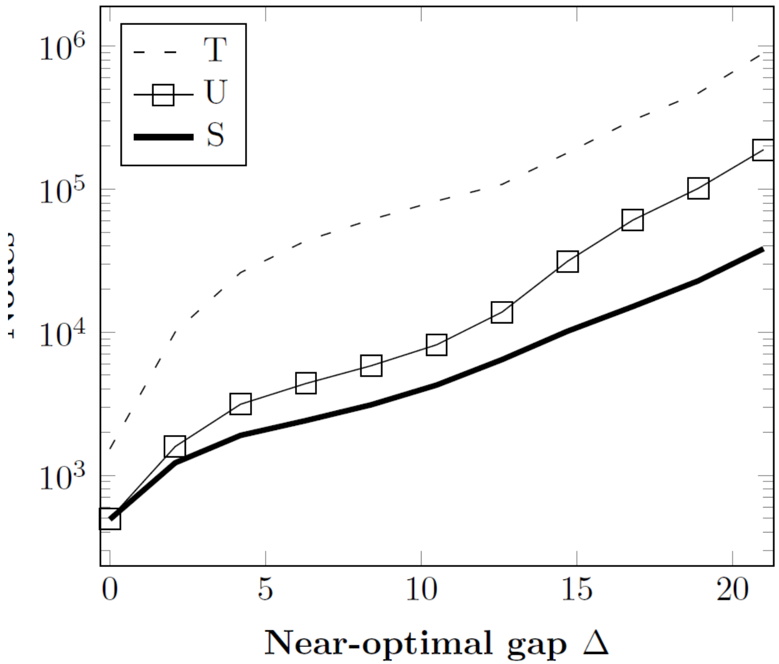
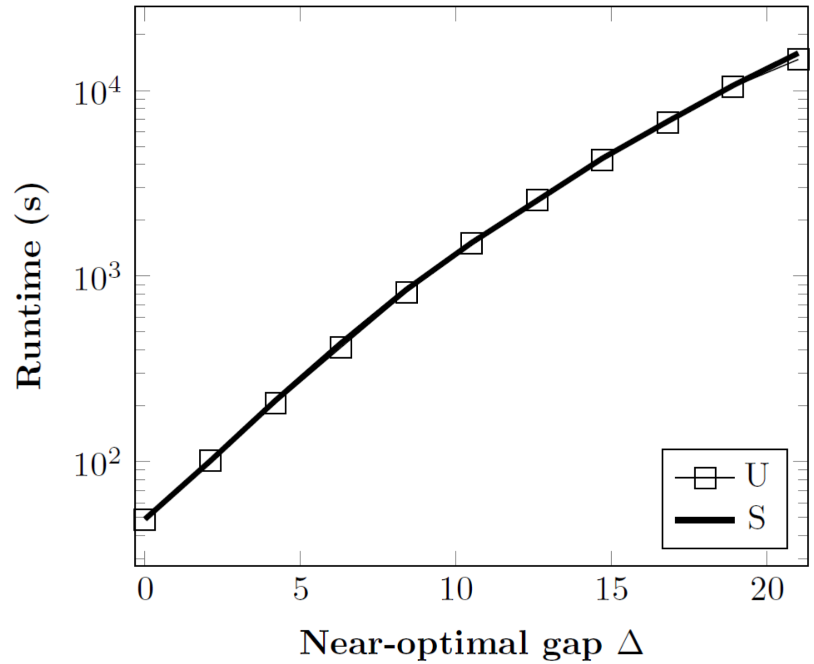
DD Compression & Time vs Δ

(a3) Representation sizes for mod008



(b3) Construction time for mod008



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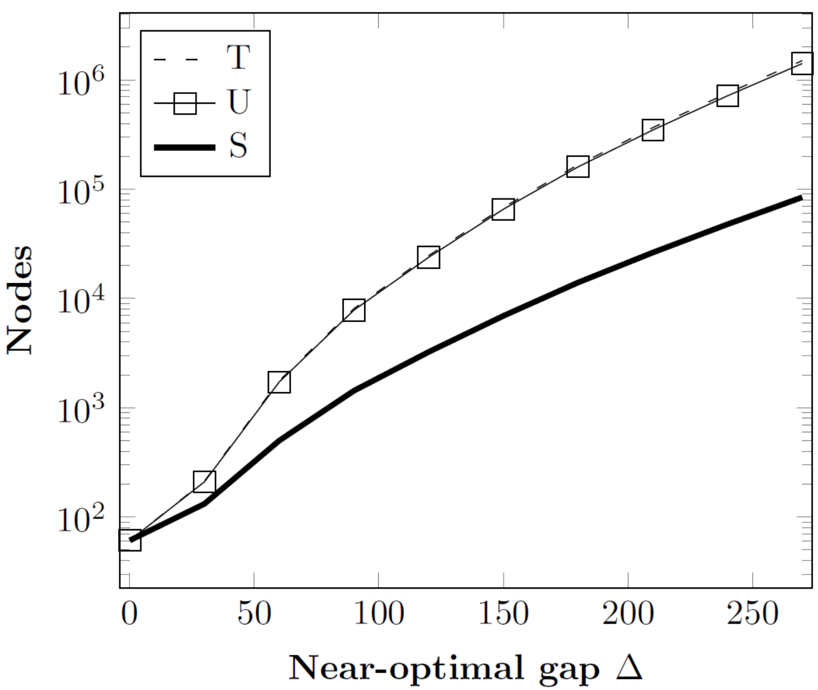
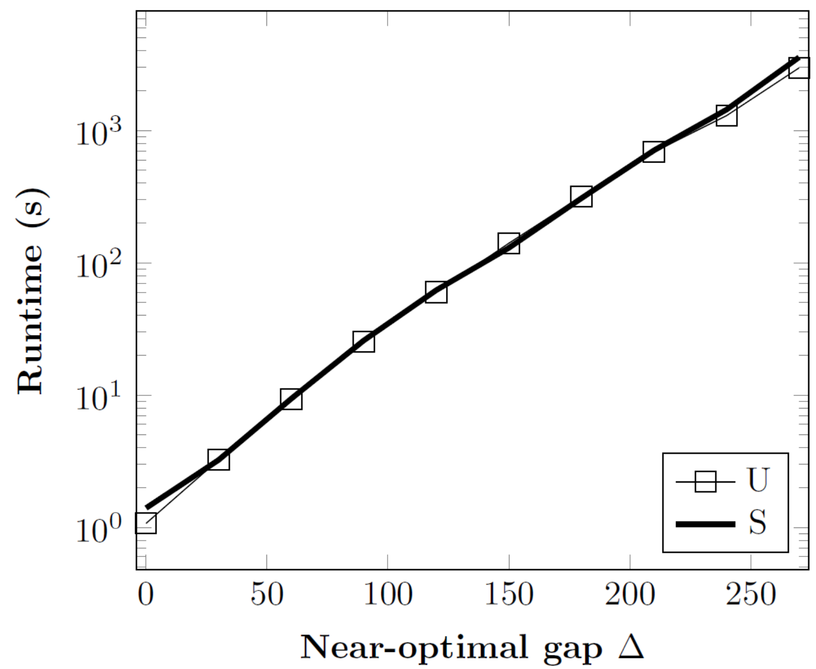
DD Compression & Time vs Δ

(a) Representation sizes for sentoy



(c) Construction time for sentoy



Stand-alone method

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Extension to MILP

- DD representation of MILP
  - DD represents **only integer solutions**.
  - Distinguish:
    - path **length** = cost of integer variables on path
    - path **cost** = value of **LP relaxation** after fixing integer variables on path

Extension to MILP

- DD representation of MILP
  - DD represents **only integer solutions.**
  - Distinguish:
    - path **length** = cost of integer variables on path
    - path **cost** = value of **LP relaxation** after fixing integer variables on path
- Two basic strategies
  - Merge nodes with **equivalent** states
    - More effective for MILP than IP
  - Shrink DD by introducing **spurious solutions.**
    - By **dualizing** constraints to obtain node equivalence.
    - By **sound reduction**, as in IP but modified.

Soundness

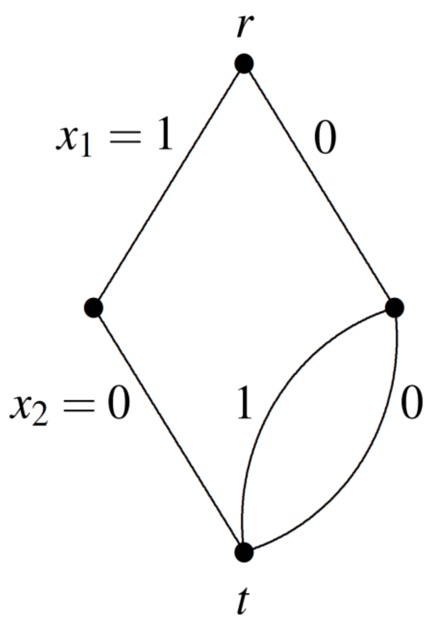
- Soundness defined as before
  - Admit spurious solutions with cost greater than $z^* + \Delta$
  - Spurious solutions can be feasible or infeasible.

Soundness

- Soundness defined as before
  - Admit spurious solutions with cost greater than $z^* + \Delta$
  - Spurious solutions can be feasible or infeasible.
- Sound reduction now useful for **optimal** as well as near-optimal solutions.
  - A minimal sound DD can contain spurious solutions even when $\Delta = 0$.

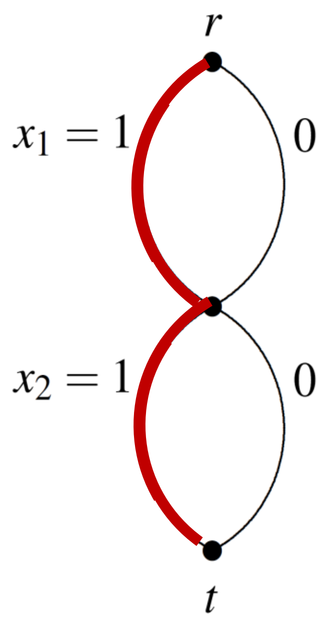
$$\min\left\{z = x_1 + x_2 + y_1 \;\middle|\; y_1 \geq 1 - x_1 - x_2,\ x_1, x_2 \in \{0,1\},\ y_1 \geq 0\right\}$$

Exact DD
for $\Delta = 0$



3 optimal solutions
with $z^* = 1$

Minimal sound DD
for $\Delta = 0$



Minimal because
every arc is part of a
Δ-optimal solution

Spurious solution
is suboptimal.

DD is smaller.

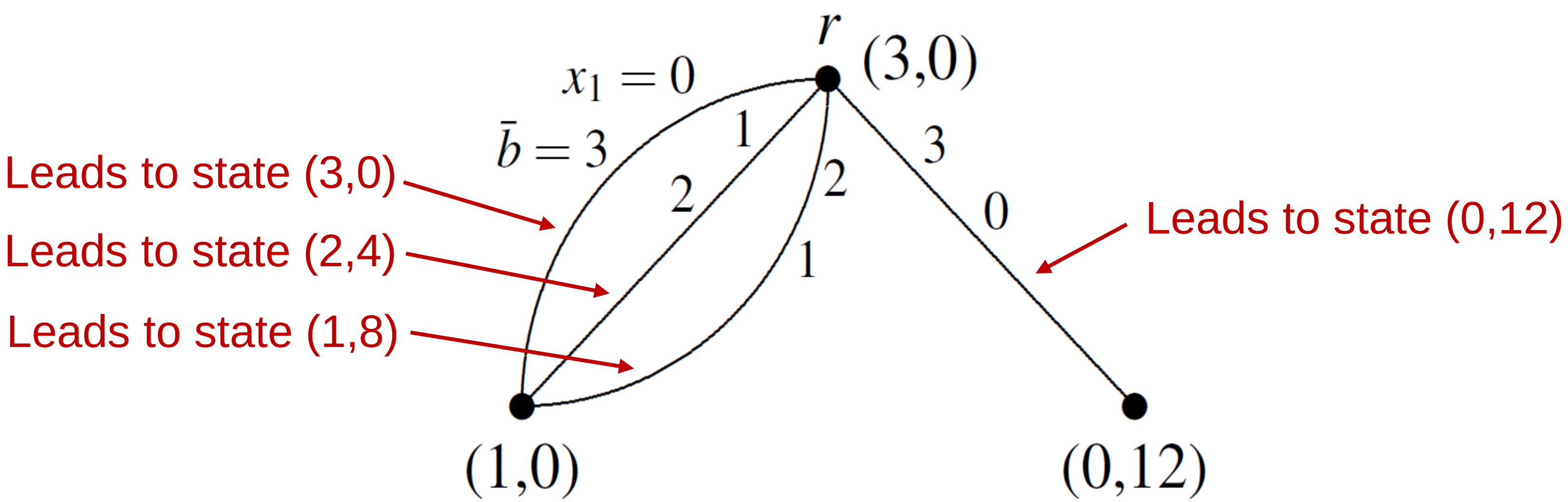
Equivalent States

- Nodes with **equivalent RHS** states $\bar{b}$ can be identified.
  - Node state is $(\bar{b}, v)$

Modified RHS after variables along path down to current node are fixed (in general, a tuple of RHSs)

Length (not cost) of shortest path from root to current node

$$\min\left\{ z = 4x_1 + 5x_2 - y_1 \;\middle|\; x_1 + 3x_2 - y_1 \geq 3,\ x_1, x_2 \in \{0,1,2,3\},\ y_1 \geq 0 \right\}$$



All RHS states $\bar{b} \in [\varepsilon, 3]$ are equivalent because they allow the same values of x_1, x_2. So arcs leading to (3,0), (2,4), (1,8) can lead to the same node with state (min{3,2,1}, min{0,4,8}) = (1,0).

We say $[\varepsilon, 3]$ is an **equivalency range for** $\bar{b}$.

Equivalent States

- MILP states are more often equivalent than IP states.
  - Presence of continuous variables often leads to equivalency range $[-\infty, \infty]$.
  - So many constraints have same equivalency range.

$$ax + d_1 y_1 - d_2 y_2 \geq \beta, \quad d_1, d_2 > 0$$

$$a'x - d_1' y_3 + d_2' y_4 \geq \beta', \quad d_1', d_2' > 0$$

Both constraints have equivalency range $[-\infty, \infty]$

Deleting Arcs

- An arc can be deleted when it cannot be part of a Δ-optimal solution.
  - Based on LP bound $L_j(\bar{b})$ of cost between node at the end of the arc and the terminus.

If the MILP is

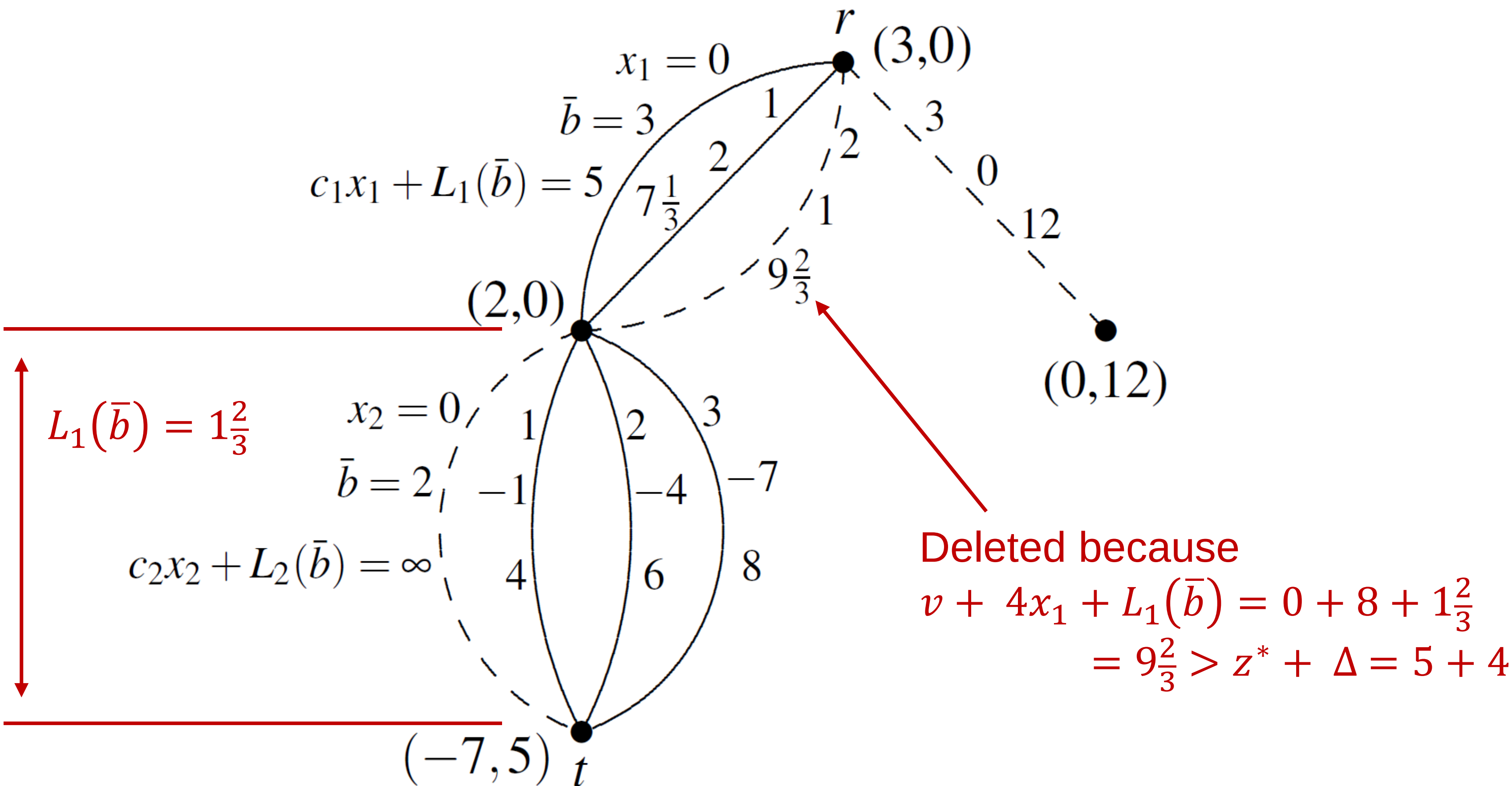
$$\min\left\{cx + dy \;\middle|\; Ax + By \geq b,\ x_j \in \{L_j, \ldots, U_j\},\ \text{all } j\right\}$$

Then

$$L_j(\bar{b}) = \min\left\{ \sum_{i=j+1}^{n} c_i x_i + dy \;\middle|\; \sum_{i=j+1}^{n} A_i x_i + By \geq \bar{b},\right.$$
$$\left. L_i \leq x_i \leq U_j,\ i = j+1, \ldots, n \right\}$$

$$\min\left\{z = 4x_1 + 5x_2 - y_1 \;\middle|\; x_1 + 3x_2 - y_1 \geq 3,\ x_1, x_2 \in \{0,1,2,3\},\ y_1 \geq 0\right\}$$

Dashed arcs can be deleted when Δ = 4.



Top-Down Compilation

Build a DD by top-down branching, identifying equivalent states, and deleting arcs when possible.

Theorem. This procedure results in a sound DD for a given Δ.

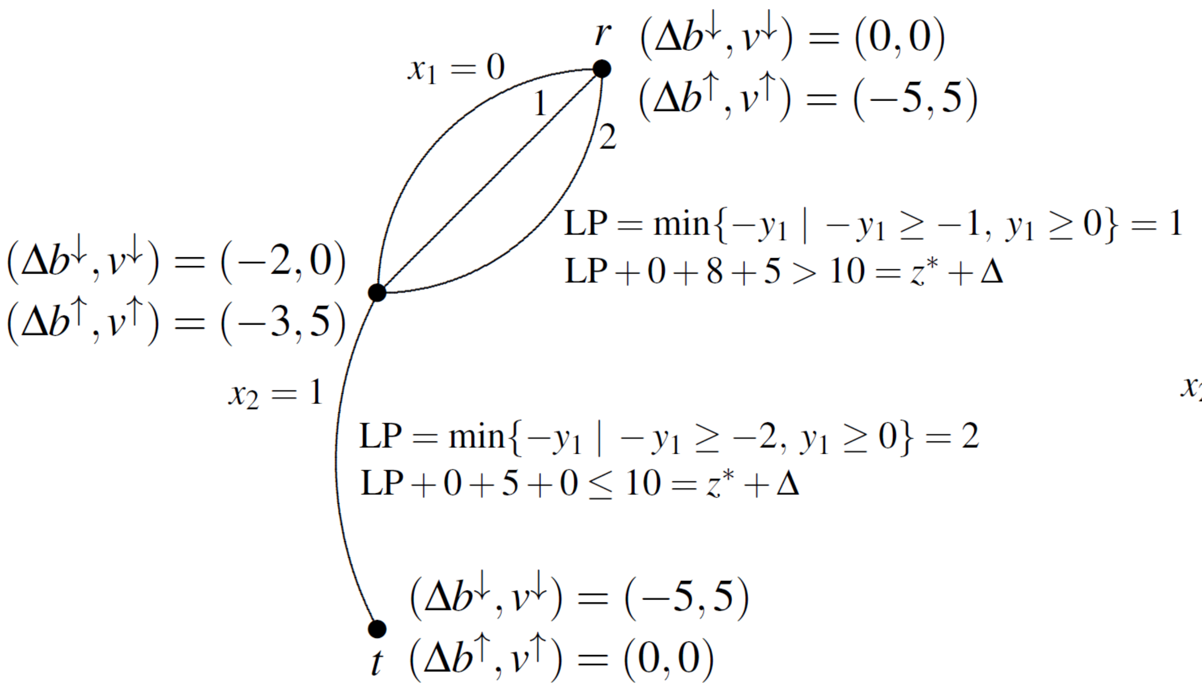
When identifying nodes, use the state with the smallest LP bound on cost, rather than taking mins.

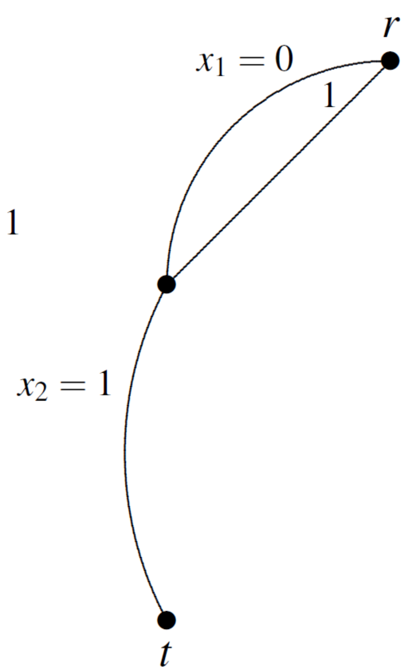
Theorem. This can result in a smaller sound DD.

Theorem. A bottom-up pass can yield a still smaller DD.

$$\min\left\{z = 4x_1 + 5x_2 - y_1 \;\middle|\; x_1 + 3x_2 - y_1 \geq 3,\ x_1, x_2 \in \{0,1,2,3\},\ y_1 \geq 0\right\}$$

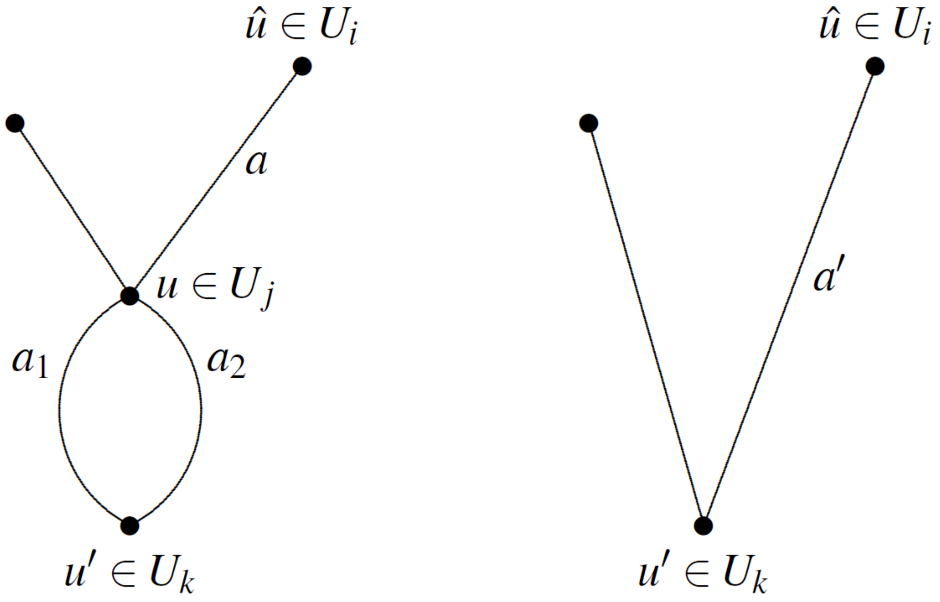
Bottom-up pass deletes one more arc.





Theorem. Arc contraction can delete more arcs while preserving soundness.

Contraction of arcs a_1, a_2



Separable Constraints

- Problem: Because $\bar{b}$ is a tuple, it is hard to prove equivalence.
  - Let a subset *S* of constraints be **separable** when the problem of finding equivalency ranges for the entire constraint set can be decomposed into finding ranges for *S* and its complement separately.
- Dividing constraints into separable subsets can help prove equivalence.
  - Particularly because constraints with continuous variables often have RHS equivalency range $[-\infty, \infty]$.
    - Their RHS states are always equivalent.

Separable Constraints

Theorem. If *S* has no continuous variables in common with other constraints, then *S* is separable.

Corollary. A pure integer constraint is separable and can therefore be analyzed separately.

Corollary. If all constraints in *S* have equivalency range $[-\infty, \infty]$, we can ignore *S* when computing equivalency ranges for the entire constraint set.

Dualizing Constraints

- Constraints that block equivalence can be **dualized**.
  - Given constraint $A_i x + B_i y \geq b_i$ add artificial variable to obtain $A_i x + B_i y + s_i \geq b_i$
  - Add $s_i \geq 0$ to the constraint set and $+M s_i$ to the objective function.
  - Constraint $A_i x + B_i y \geq b_i$ can now be ignored when checking for equivalence.

Theorem. For sufficiently large but bounded *M,* dualizing constraints preserves soundness.

Yet it results in more spurious solutions.

Sound Reduction

- Sound reduction can be defined in parallel with IP.
  - However, the test for sound reduction is harder to pass.
  - It relies on LP bounds rather than path lengths.
  - Finding weaker conditions for sound reduction is a current research issue.

Research Issues

- Applications to:
  - Multiobjective optimization
    - Original application!
  - General mixed discrete/continuous programming
    - Not just MILP

Research Issues

- Applications to:
  - Multiobjective optimization
    - Original application!
  - General mixed discrete/continuous programming
    - Not just MILP
- How to combine DD-based solution with DD-based postoptimality?
  - Partially analogous to "**1-tree**" method for generating near-optimal MILP solutions.
    - Search for all near-optimal solutions using the **same tree**.
  - Use a "**1-DD**" method.
    - Search for all near-optimal solutions in the **same DD**.
    - Result is a **sound DD** representing the solutions, rather than just a **list** as in MILP.