

Stochastic Planning and Scheduling with Logic-Based Benders Decomposition

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November 2021

Abstract

We apply logic-based Benders decomposition (LBB) to two-stage stochastic planning and scheduling problems in which the second stage is a scheduling task. We solve the master problem with mixed integer/linear programming and the subproblem with constraint programming. As Benders cuts, we use simple nogood cuts as well as analytical logic-based cuts we develop for this application. We find that LBB is computationally superior to the integer L-shaped method, with a branch-and-check variant of LBB faster by several orders of magnitude, allowing significantly larger instances to be solved. This is due primarily to computational overhead incurred by the integer L-shaped method while generating classical Benders cuts from a continuous relaxation of an integer programming subproblem. To our knowledge, this is the first application of LBB to two-stage stochastic optimization with a scheduling second-stage problem, and the first comparison of LBB with the integer L-shaped method. The results suggest that LBB could be a promising approach to other stochastic and robust optimization problems with integer or combinatorial recourse.

1 Introduction

Benders decomposition has seen many successful applications to two-stage stochastic optimization, where it typically takes the form of the *L-shaped method* [6, 41]. It offers the advantage that the second-stage problem decouples into a separate problem for each possible scenario, allowing much faster computation of the recourse decision.

A limitation of classical Benders decomposition, however, is that the subproblem must be a linear programming problem, or a continuous nonlinear programming problem in the case of “generalized” Benders decomposition [19]. This is necessary because the Benders cuts are derived from dual multipliers (or Lagrange multipliers) in the subproblem. Yet in many problems, the recourse decision is a combinatorial optimization problem that does not yield dual multipliers. This issue has been addressed by the *integer L-shaped method* [29], which formulates the subproblem as a mixed integer/linear programming (MILP) problem and obtains dual multipliers from its linear programming (LP) relaxation. To ensure finite convergence,

classical Benders cuts from the LP relaxation are augmented with “integer cuts” that simply exclude the master problem solutions enumerated so far.

Unfortunately, a combinatorial subproblem may be difficult to model as an MILP, in the sense that many variables are required, or the LP relaxation is weak. This is particularly the case when the recourse decision poses a scheduling problem. We therefore investigate the option of applying *logic-based Benders decomposition* (LBB) to problems with a second-stage scheduling decision [23, 27], because it does not require dual multipliers to obtain Benders cuts. Rather, the cuts are obtained from an “inference dual” that is based on a structural analysis of the subproblem. This allows the subproblem to be solved by a method that is best suited to compute optimal schedules, without having to reformulate it as an MILP.

We investigate the LBB option by observing its behavior on a generic planning and scheduling problem in which scheduling takes place after the random events have been observed. The planning element is an assignment of jobs to facilities that occurs in the first stage. Jobs assigned to each facility are then scheduled in the second stage subject to time windows. We assume that the job processing time is a random variable, but the LBB approach is easily modified to accommodate other random elements, such as the release time. The subproblem decouples into a separate scheduling problem for each facility and each scenario. For greater generality, we suppose the recourse decision is a *cumulative* scheduling problem in which multiple jobs can run simultaneously on a single facility, subject to a limit on total resource consumption at any one time.

We solve the first-stage problem by MILP, which is well suited for assignment problems. More relevant to the present study is our choice to solve the scheduling subproblem by constraint programming (CP), which has proved to be effective on a variety of scheduling problems, perhaps the state of the art in many cases. We therefore formulate the subproblem in a CP modeling language rather than as an MILP. In view of the past success of LBB on a number of deterministic planning and scheduling problems, we test the hypothesis that it can obtain similar success on stochastic problems with many scenarios. We perform computational experiments while minimizing makespan, total tardiness, and total assignment cost. We also derive new logic-based Benders cuts for the minimum makespan problem that have not been used in previous work.

In addition to standard LBB, we experiment with *branch and check*, a variation of LBB that solves the master problem only once and generates Benders cuts on the fly during the MILP branching process [23, 40]. We find that both versions of LBB are superior to the integer L-shaped method. In particular, branch and check is faster by several orders of magnitude, allowing significantly larger instances to be solved. We also conduct a variety of tests to identify factors that explain the superior performance of LBB, the relative effectiveness of various Benders cuts, and the impact of modifying the integer L-shaped method in various ways. To our knowledge, this is the first computational comparison between LBB and the integer L-shaped method on any kind of stochastic optimization problem. It also appears to be the first application of LBB to two-stage stochastic optimization with a scheduling second-stage problem.

The remainder of this paper is organized as follows. We introduce the stochastic planning and scheduling problem in Section 3. This is followed by Section 4 where we propose the logic-

based Benders decomposition based solution methods for solving three variants of the stochastic planning and scheduling problem. We present the computational results in Section 7 and give our concluding remarks in Section 8. Additional computational experiments and details on the data set can be found in the electronic companion to the paper.

2 Previous Work

A wide range of problems can be formulated as two-stage stochastic programs. For theory and various applications, we refer the reader to [8], [39], [35], and the references therein. Allowing discrete decisions in the second-stage problem significantly expands the applicability of the two-stage stochastic framework, as for example to last-mile relief network design [34] and vehicle routing with stochastic travel times [30].

Benders decomposition [6] has long been applied to large-scale optimization problems [20, 14, 7, 13]. [36] provide an excellent survey of enhancements to the classical method. In particular, it has been applied to two-stage stochastic programs with linear recourse by means of the L-shaped method [41]. Its applicability was extended to integer recourse by the integer L-shaped method of [29], which was recently revisited and improved by [3] and [31]. Other Benders-type algorithms that have been proposed for integer recourse include disjunctive decomposition [38] and decomposition with parametric Gomory cuts [18]. The essence of these two methods is to convexify the integer second-stage problem using disjunctive cuts and Gomory cuts, respectively. Still other decomposition-based methods in the literature include progressive hedging for multi-stage stochastic convex programs [37] and a dual decomposition method for multi-stage stochastic programs with mixed-integer variables [11]. We refer the reader to [28] for a review of two-stage stochastic mixed-integer programming.

Logic-based Benders decomposition was introduced by [23] and further developed in [27]. Branch and check, a variant of LBBD, was also introduced by [23] and first tested computationally by [40], who coined the term “branch and check.” A general exposition of both standard LBBD and branch and check, with an extensive survey of applications, can be found in [26]. A number of these applications have basically the same mathematical structure as the planning and scheduling problem studied here, albeit generally without a stochastic element.

In more recent work, [4] focus on a one-stage stochastic model for single-machine scheduling in which they minimize the value-at-risk of several random performance measures. [10] consider a two-stage chance-constrained mean-risk stochastic programming model for single-machine scheduling problem, but the scheduling decisions do not occur in the second stage. Rather, the second-stage problem is a simple optimal timing problem that can be solved very rapidly. The deterministic version of the planning and scheduling problem we consider here is solved by LBBD in [24] and [12]. We rely on some techniques from these studies.

We are aware of three prior applications of LBBD to stochastic optimization. [32] use LBBD to assign computational tasks to chips and to schedule the tasks assigned to each chip. In this application, the scheduling problem parameters are not random, and the expected recourse has an analytic solution. The authors reformulate the problem as a single-stage stochastic program using the analytical solution. [17] solve a stochastic location-routing problem with LBBD, but

there is no actual recourse decision in the second stage, which only penalizes vehicles if the route determined by first-stage decisions exceeds their threshold capacity. [21] use LBB to schedule patients in operating rooms, where the random element is the surgery duration. Here the scheduling takes place in the master problem, where patients are assigned to operating rooms and surgery dates. The subproblem checks whether there is time during the day to perform all the surgeries assigned to a given operating room, and if not, finds a cost-minimizing selection of surgeries to cancel on that day. Unstrengthened nogood cuts are used as LBB cuts, along with classical Benders cuts derived from a network flow model of the subproblem that is obtained from a binary decision diagram.

The present study therefore appears to be the first application of LBB to two-stage stochastic optimization with scheduling in the second stage. It is also the first to compare any application of stochastic LBB with the integer L-shaped method.

3 The Problem

We study a two-stage stochastic programming problem that, in general, has the following form:

$$\min_{\mathbf{x} \in X} \{f(\mathbf{x}) + \mathbb{E}_\omega[Q(\mathbf{x}, \omega)]\} \quad (1)$$

where $Q(\mathbf{x}, \omega)$ is the optimal value of the second-stage problem:

$$Q(\mathbf{x}, \omega) = \min_{\mathbf{y} \in Y(\omega)} \{g(\mathbf{y})\} \quad (2)$$

Variable \mathbf{x} represents the first-stage decisions, while \mathbf{y} represents second-stage decisions that are made after the random variable ω is realized. We suppose that ω ranges over a finite set Ω of possible scenarios, where each scenario ω has probability π_ω . The first-stage problem (1) may therefore be written as

$$\min_{\mathbf{x} \in X} \left\{ f(\mathbf{x}) + \sum_{\omega \in \Omega} \pi_\omega Q(\mathbf{x}, \omega) \right\}$$

We consider a generic planning and scheduling problem in which the first stage assigns jobs to facilities, and the second stage schedules the jobs assigned to each facility. The objective is to minimize expected makespan or expected total tardiness. We assume that only the processing times are random in the second stage, but a slight modification of the model allows for random release times and/or deadlines as well.

We therefore suppose that each job j has a processing time p_{ij}^ω on facility i in scenario ω and must be processed during the interval $[r_j, d_j]$. For greater generality, we allow for cumulative scheduling (Aggoun and Beldiceanu 1993, Baptiste et al. 2001), where each job j consumes resources c_{ij} on facility i , and the total resource consumption must not exceed K_i .

To formulate the problem we let variable x_j be the facility to which job $j \in J$ is assigned. The first-stage problem is

$$\min_{\mathbf{x}} \left\{ g(\mathbf{x}) + \sum_{\omega \in \Omega} \pi_\omega Q(\mathbf{x}, \omega) \mid x_j \in I, \text{ all } j \in J \right\} \quad (3)$$

where I indexes the facilities. In the second-stage problem, we let s_j be the time at which job j starts processing. We also let $J_i(\mathbf{x})$ be the set of jobs assigned to facility i , so that $J_i(\mathbf{x}) = \{j \in J \mid x_j = i\}$. Thus

$$Q(\mathbf{x}, \omega) = \min_{\mathbf{s}} \left\{ h(\mathbf{s}, \mathbf{x}, \omega) \mid s_j \in [r_j, d_j - p_{x_j j}^\omega], \text{ all } j \in J; \sum_{\substack{j \in J_i(\mathbf{x}) \\ 0 \leq t \leq s_j + p_{ij}^\omega}} c_{ij} \leq K_i, \text{ all } i \in I, \text{ all } t \right\}$$

where $h(\mathbf{s}, \mathbf{x}, \omega)$ denotes the second-stage objective function given the first-stage decision \mathbf{x} and scenario ω .

The two-stage problem (1) is risk-neutral in the sense that it is concerned with minimizing expectation. However, the LBBB approach presented here can be adapted to a more general class of problems that incorporate a dispersion statistic \mathbb{D}_ω that measures risk, such as variance, as in the classical [33] model. Then the problem (1) becomes

$$\min_{\mathbf{x} \in X} \left\{ f(\mathbf{x}) + (1 - \lambda) \mathbb{E}_\omega [Q(\mathbf{x}, \omega)] + \lambda \mathbb{D}_\omega [Q(\mathbf{x}, \omega)] \right\} \quad (4)$$

and the first-stage planning and scheduling problem (3) becomes

$$\min_{\mathbf{x}} \left\{ g(\mathbf{x}) + (1 - \lambda) \sum_{\omega \in \Omega} \pi_\omega Q(\mathbf{x}, \omega) + \lambda \mathbb{D}_\omega [Q(\mathbf{x}, \omega)] \mid x_j \in I, \text{ all } j \in J \right\} \quad (5)$$

Formulations (4) and (5) also accommodate robust optimization, as for example when $\lambda = 1$ and

$$\mathbb{D}_\omega(Q(\mathbf{x}, \omega)) = \max_{\omega \in \Omega} \{Q(\mathbf{x}, \omega)\}$$

and Ω is an uncertainty set. See [2] for a discussion of various tractable and intractable risk measures.

4 Logic-based Benders Decomposition

Logic-based Benders decomposition (LBBB) is designed for problems of the form

$$\min_{\mathbf{x}, \mathbf{y}} \left\{ f(\mathbf{x}, \mathbf{y}) \mid C(\mathbf{x}, \mathbf{y}), \mathbf{x} \in D_x, \mathbf{y} \in D_y \right\} \quad (6)$$

where $C(\mathbf{x}, \mathbf{y})$ denotes a set of constraints that contain variables \mathbf{x} and \mathbf{y} , and D_y and D_x represent variable domains. The rationale behind dividing the variables into two groups is that once some of the decisions are fixed by setting $\mathbf{x} = \bar{\mathbf{x}}$, the remaining *subproblem* becomes much easier to solve, perhaps by decoupling into smaller problems. In our study, the smaller problems will correspond to scenarios and facilities. The subproblem has the form

$$\text{SP}(\bar{\mathbf{x}}) = \min_{\mathbf{y}} \left\{ f(\bar{\mathbf{x}}, \mathbf{y}) \mid C(\bar{\mathbf{x}}, \mathbf{y}), \mathbf{y} \in D_y \right\} \quad (7)$$

The key to LBBB is analyzing the subproblem solution so as to find a function $B_{\bar{\mathbf{x}}}(\mathbf{x})$ that provides a lower bound on $f(\mathbf{x}, \mathbf{y})$ for any given $\mathbf{x} \in D_x$. The bound must be sharp for $\mathbf{x} = \bar{\mathbf{x}}$; that is, $B_{\bar{\mathbf{x}}}(\bar{\mathbf{x}}) = \text{SP}(\bar{\mathbf{x}})$. The bounding function is derived from the *inference dual* of the

subproblem in a manner discussed below. In classical Benders decomposition, the subproblem is an LP problem, and the inference dual is the LP dual.

Each iteration of the LBB algorithm begins by solving a *master problem*:

$$\text{MP}(\bar{\mathbf{X}}) = \min_{\mathbf{x}, \beta} \{ \beta \mid \beta \geq B_{\bar{\mathbf{x}}}(\mathbf{x}), \text{ all } \bar{\mathbf{x}} \in \bar{\mathbf{X}}; \mathbf{x} \in D_x \} \quad (8)$$

where the inequalities $\beta \geq B_{\bar{\mathbf{x}}}(\mathbf{x})$ are *Benders cuts* obtained from previous solutions $\bar{\mathbf{x}}$ of the subproblem. There may be several cuts for a given $\bar{\mathbf{x}}$, but for simplicity we assume in this section there is only one. Initially, the set $\bar{\mathbf{X}}$ can be empty, or it can contain a few solutions obtained heuristically to implement a “warm start.” The optimal value $\text{MP}(\bar{\mathbf{X}})$ of the master problem is a lower bound on the optimal value of the original problem (6). If $\bar{\mathbf{x}}$ is an optimal solution of the master problem, the corresponding subproblem is then solved to obtain $\text{SP}(\bar{\mathbf{x}})$, which is an upper bound on the optimal value of (6). A new Benders cut $\beta \geq B_{\bar{\mathbf{x}}}(\mathbf{x})$ is generated for the master problem and $\bar{\mathbf{x}}$ added to $\bar{\mathbf{X}}$ in (8). The process repeats until the lower and upper bounds provided by the master problem and subproblem converge; that is, until $\text{MP}(\bar{\mathbf{X}}) = \min_{\bar{\mathbf{x}} \in \bar{\mathbf{X}}} \{ \text{SP}(\bar{\mathbf{x}}) \}$. The following is proved in [23]:

Theorem 1. *If D_x is finite, the LBB algorithm converges to an optimal solution of (6) after a finite number of iterations.*

The inference dual of the subproblem seeks the tightest bound on the objective function that can be inferred from the constraints. Thus the inference dual is

$$\text{DSP}(\bar{\mathbf{x}}) = \max_{P \in \mathcal{P}} \left\{ \gamma \mid (C(\bar{\mathbf{x}}, \mathbf{y}), \mathbf{y} \in D_y) \stackrel{P}{\Rightarrow} (f(\bar{\mathbf{x}}, \mathbf{y}) \geq \gamma) \right\} \quad (9)$$

where $A \stackrel{P}{\Rightarrow} B$ indicates that proof P deduces B from A . The inference dual is always defined with respect to set \mathcal{P} of valid proofs. In classical linear programming duality, valid proofs consist of nonnegative linear combinations of the inequality constraints in the problem. We assume a strong dual, meaning that $\text{SP}(\bar{\mathbf{x}}) = \text{DSP}(\bar{\mathbf{x}})$. The dual is strong when the inference method is complete. For example, the classical Farkas Lemma implies that nonnegative linear combination is a complete inference method for linear inequalities. Indeed, any exact optimization method is associated with a complete inference method that it uses to prove optimality, perhaps one that involves branching, cutting planes, constraint propagation, and so forth.

In the context of LBB, the proof P that solves the dual (9) is the proof of optimality the solver obtains for the subproblem (7). The bounding function $B_{\bar{\mathbf{x}}}(\mathbf{x})$ is derived by observing what bound on the optimal value *this same proof P* can logically deduce for a given \mathbf{x} , whence the description “logic-based.” In practice, the solver may not reveal how it proved optimality, or the proof may be too complicated to build a useful cut. One option in such cases is to tease out the structure of the proof by re-solving the subproblem for several values of \mathbf{x} and observing the optimal value that results. This information can be used to design *strengthened no-good cuts* that provide useful bounds for many values of \mathbf{x} other than $\bar{\mathbf{x}}$. Another approach is to use *analytical Benders cuts*, which deduce bounds on the optimal value when $\bar{\mathbf{x}}$ is changed in certain ways, based on structural characteristics of the subproblem and its current solution. We will employ both of these options.

Branch and check is a variation of LBBDD that solves the master problem only once and generates Benders cuts on the fly. It is most naturally applied when the master problem is solved by branching. Whenever the branching process discovers a solution $\bar{\mathbf{x}}$ that is feasible in the current master problem, the corresponding subproblem is solved to obtain one or more Benders cuts, which are added to the master problem. Branching then continues and terminates in the normal fashion, all the while satisfying Benders cuts as they accumulate. Branch and check can be superior to standard LBBDD when the master problem is much harder to solve than the subproblems.

A common enhancement of LBBDD and other Benders methods is a *warm start*, which includes initial Benders cuts in the master problem. Recent studies that benefit from this technique include [3], [15], and [22]. Benders cuts can also be aggregated before being added to the master problem, a technique first explored in [9]. A particularly useful enhancement for LBBDD is to include a relaxation of the subproblem in the master problem, where the relaxation is written in terms of the master problem variables [24, 16]. We employ this technique in the present study.

5 Benders Formulation of Planning and Scheduling

We apply LBBDD to the generic planning and scheduling problem by placing the assignment decision in the master problem and the scheduling decision in the subproblem. The master problem is therefore

$$\min_{\mathbf{x}} \left\{ g(\mathbf{x}) + \sum_{\omega \in \Omega} \pi_{\omega} \beta_{\omega} \mid \text{Benders cuts; subproblem relaxation; } x_j \in I, \text{ all } j \in J \right\}$$

where β_{ω} is an auxiliary variable that captures second-stage objective function value under scenario ω . The Benders cuts provide lower bounds on each β_{ω} . The cuts and subproblem relaxation are somewhat different for each variant of the problem we consider below. The scheduling subproblem decouples into a separate problem for each facility and scenario. If $\bar{\mathbf{x}}$ is an optimal solution of the master problem, the scheduling problem for facility i and scenario ω is

$$\text{SP}_{i\omega}(\bar{\mathbf{x}}) = \min_{\mathbf{s}} \left\{ h_i(\mathbf{s}, \bar{\mathbf{x}}, \omega) \mid s_j \in [r_j, d_j - p_{ij}^{\omega}], \text{ all } j \in J_i(\bar{\mathbf{x}}); \sum_{\substack{j \in J_i(\bar{\mathbf{x}}) \\ 0 \leq t \leq s_j + p_{ij}^{\omega}}} c_{ij} \leq K_i, \text{ all } t \right\}$$

We solve the master problem and subproblem by formulating the former as an MILP problem and the latter as a CP problem. In the master problem, we let variable $x_{ij} = 1$ when job j is assigned to facility i . The master problem becomes

$$\begin{aligned} & \text{minimize} && \hat{g}(\mathbf{x}) + \sum_{\omega \in \Omega} \pi_{\omega} \beta_{\omega} \\ & \text{subject to} && \sum_{i \in I} x_{ij} = 1, \quad j \in J \\ & && \text{Benders cuts} \\ & && \text{subproblem relaxation} \\ & && x_{ij} \in \{0, 1\}, \quad i \in I, \quad j \in J \end{aligned} \tag{10}$$

where \mathbf{x} now denotes the matrix of variables x_{ij} . If $\bar{\mathbf{x}}$ is an optimal solution of the master problem, the subproblem for each facility i and scenario ω becomes

$$\begin{aligned} & \text{minimize} && \hat{h}_i(\mathbf{s}, \bar{\mathbf{x}}, \omega) \\ & \text{subject to} && \text{cumulative} \left((s_j \mid j \in J_i(\bar{\mathbf{x}})), (p_{ij}^\omega \mid j \in J_i(\bar{\mathbf{x}})), (c_{ij} \mid j \in J_i(\bar{\mathbf{x}})), K_i \right) \\ & && s_j \in [r_j, d_i - p_{ij}^\omega], j \in J_i(\bar{\mathbf{x}}) \end{aligned} \quad (11)$$

The optimal value of (11) is again $\text{SP}_{i\omega}(\bar{\mathbf{x}})$. The cumulative global constraint in (11) is a standard feature of CP models and requires that the total resource consumption at any time on facility i be at most K_i .

To solve a problem (5) that incorporates risk, one need only replace the objective function of (10) with

$$\hat{g}(\mathbf{x}) + (1 - \lambda) \sum_{\omega \in \Omega} \pi_\omega \beta_\omega + \lambda \mathbb{D}_\omega[\beta_\omega]$$

and otherwise proceed as in the risk-neutral case.

5.1 Minimum Makespan Problem

We begin by considering a minimum makespan problem in which the jobs have release times and no deadlines. The first-stage objective function is $g(\mathbf{x}) = 0$, and so we have $\hat{g}(\mathbf{x}) = 0$ in the MILP model (10). The second-stage objective function is the finish time of the last job to finish:

$$h(\mathbf{s}, \mathbf{x}, \omega) = \max_{j \in J(\mathbf{x})} \left\{ s_j + p_{x_j j}^\omega \right\}$$

This objective function is incorporated into the CP problem (11) by setting $\hat{h}_i(\mathbf{s}, \bar{\mathbf{x}}, \omega) = M$ and adding to (11) the constraints $M \geq s_j + p_{ij}^\omega$ for all $j \in J_i(\bar{\mathbf{x}})$. Since there are no deadlines, we assume $d_j = \infty$ for all $j \in J$.

Both strengthened nogood cuts and analytic Benders cuts can be developed for this problem. A simple nogood cut for scenario ω can take the form of a set of inequalities

$$\beta_\omega \geq \beta_{i\omega}, i \in I \quad (12)$$

where each $\beta_{i\omega}$ is bounded by

$$\beta_{i\omega} \geq \text{SP}_{i\omega}(\bar{\mathbf{x}}) \left(\sum_{j \in J_i(\bar{\mathbf{x}})} x_{ij} - |J_i(\bar{\mathbf{x}})| + 1 \right) \quad (13)$$

and where $\bar{\mathbf{x}}$ is the solution of the current master problem and $|J_i(\bar{\mathbf{x}})|$ denotes the cardinality of set $J_i(\bar{\mathbf{x}})$. The cut says that if all the jobs in $J_i(\bar{\mathbf{x}})$ are assigned to facility i , possibly among other jobs, then the makespan of facility i in scenario ω is at least the current makespan $\text{SP}_{i\omega}(\bar{\mathbf{x}})$. The cut is weak, however, because if even one job in $J_i(\bar{\mathbf{x}})$ is not assigned to i , the bound in (13) becomes useless. The cut can be strengthened by heuristically assigning proper subsets of the jobs in $J_i(\bar{\mathbf{x}})$ to facility i , and re-computing the minimum makespan for each subset, to discover a smaller set of jobs that yields the same makespan. This partially reveals which job assignments serve as premises of the optimality proof. Then $J_i(\bar{\mathbf{x}})$ in (13) is replaced with this

smaller set to strengthen the cut. This simple scheme, and variations of it, can be effective when the makespan problem solves quickly [24].

A stronger cut can be obtained without re-solving the makespan problem by using an analytical Benders cut. We introduce a cut based on the following lemma:

Lemma 1. *Consider a minimum makespan problem P in which each job $j \in J$ has release time r_j and processing time p_j , with no deadlines. Let M^* denote the minimum makespan for P , and \hat{M} the minimum makespan for the problem \hat{P} that is identical to P except that the jobs in a nonempty set $\hat{J} \subset J$ are removed. Then*

$$M^* - \hat{M} \leq \Delta + r^+ - r^- \quad (14)$$

where $\Delta = \sum_{j \in \hat{J}} p_j$, and $r^+ = \max_{j \in J} r_j$ and $r^- = \min_{j \in J} r_j$ are the latest and earliest release times of the jobs in set J .

Proof. Proof. Consider any solution of \hat{P} with makespan \hat{M} . We will construct a feasible solution for P by extending this solution. If $\hat{M} > r^+$, we schedule all the jobs in \hat{J} sequentially starting from time \hat{M} , resulting in makespan $\hat{M} + \Delta$. This is a feasible solution for P , and we have $M^* \leq \hat{M} + \Delta$. The lemma follows because $r^+ - r^-$ is nonnegative. If $\hat{M} < r^+$, we schedule all the jobs in \hat{J} sequentially starting from time r^+ to obtain a solution with makespan of $r^+ + \Delta$. Again this is a feasible solution for P , and we have $M^* \leq r^+ + \Delta$. This implies

$$M^* - \hat{M} \leq r^+ - \hat{M} + \Delta \quad (15)$$

Because \hat{M} is at least r^- , (15) implies (14), and the lemma follows. \square

We can now derive a valid analytical cut:

Theorem 2. *A valid Benders cut for scenario ω can be obtained by adding inequalities (12) and the following to the master problem:*

$$\beta_{i\omega} \geq \left\{ \begin{array}{ll} \text{SP}_{i\omega}(\bar{\mathbf{x}}) - \left(\sum_{j \in J_i(\bar{\mathbf{x}})} (1 - x_{ij}) p_{ij}^\omega + r^+ - r^- \right), & \text{if } x_{ij} = 0 \text{ for some } j \in J_i(\bar{\mathbf{x}}) \\ \text{SP}_{i\omega}(\bar{\mathbf{x}}), & \text{otherwise} \end{array} \right\}, \quad i \in I \quad (16)$$

where $r^+ = \max_{j \in J_i(\bar{\mathbf{x}})} r_j$ and $r^- = \min_{j \in J_i(\bar{\mathbf{x}})} r_j$.

Proof. Proof. The cut clearly provides a sharp bound $\max_{i \in I} \{\text{SP}_{i\omega}(\bar{\mathbf{x}})\}$ when $\mathbf{x} = \bar{\mathbf{x}}$, because the second line of (16) applies in this case. The validity of the cut follows immediately from Lemma 1. \square

We linearize the cut (16) as follows:

$$\begin{aligned} \beta_{i\omega} &\geq \text{SP}_{i\omega}(\bar{\mathbf{x}}) - \sum_{j \in J_i(\bar{\mathbf{x}})} (1 - x_{ij}) (p_{ij}^\omega + r^+ - r^-) & (a) \\ \beta_{i\omega} &\geq \text{SP}_{i\omega}(\bar{\mathbf{x}}) - \sum_{j \in J_i(\bar{\mathbf{x}})} (1 - x_{ij}) p_{ij}^\omega - (r^+ - r^-) & (b) \end{aligned} \quad (17)$$

The Benders cut (16) is inserted into the master problem by including inequalities (17) for each $i \in I$ and $\omega \in \Omega$, along with the inequalities (12).

Corollary 1. *The inequalities (17) yield a valid Benders cut equivalent to (16).*

Proof. Proof. Let $k = \sum_{j \in J_i(\bar{x})} (1 - x_{ij})$. If $k = 0$, (17a) is identical to the second line of (16), while (17b) is implied by (17a) and therefore valid. If $k = 1$, both (17a) and (17b) are identical to the first line of (16). If $k \geq 2$, (17b) is identical to the first line of (16), while (17a) is implied by (17b) and therefore valid. \square

A Benders cut can also be derived for the case in which all release times are equal and the jobs have deadlines. The following is an immediate consequence of a theorem proved in [24]:

Theorem 3. *Suppose all release times are $r_j = 0$, and each job j has a deadline d_j . A valid Benders cut for scenario ω can be obtained by adding inequalities (12) and the following to the master problem:*

$$\beta_{i\omega} \geq \left\{ \begin{array}{ll} \text{SP}_{i\omega}(\bar{x}) - \left(\sum_{j \in J_i(\bar{x})} (1 - x_{ij}) p_{ij}^\omega + d^+ - d^- \right), & \text{if } x_{ij} = 0 \text{ for some } j \in J_i(\bar{x}) \\ \text{SP}_{i\omega}(\bar{x}), & \text{otherwise} \end{array} \right\}, \quad i \in I \quad (18)$$

where $d^+ = \max_{j \in J_i(\bar{x})} \{d_j\}$ and $d^- = \min_{j \in J_i(\bar{x})} \{d_j\}$.

The linearization provided in [24] for this cut introduces a continuous variable for each job j . This adds a considerable computational burden for the stochastic problem, since it requires a new continuous variable for each scenario ω and each facility i . However, we can avoid additional variables by formulating a linearization parallel to (17) that uses the following inequalities:

$$\begin{aligned} \beta_{i\omega} &\geq \text{SP}_{i\omega}(\bar{x}) - \sum_{j \in J_i(\bar{x})} (1 - x_{ij}) (p_{ij}^\omega + d^+ - d^-) & (a) \\ \beta_{i\omega} &\geq \text{SP}_{i\omega}(\bar{x}) - \sum_{j \in J_i(\bar{x})} (1 - x_{ij}) p_{ij}^\omega - (d^+ - d^-) & (b) \end{aligned} \quad (19)$$

The Benders cut is inserted into the master problem by including inequalities (19) for each $i \in I$ and $\omega \in \Omega$, along with the inequalities (12). The proof of validity is similar to the proof of Corollary 1.

Corollary 2. *The inequalities (19) yield a valid Benders cut equivalent to (18).*

Finally, we add a subproblem relaxation to the master problem. We use a relaxation from [24], modified to be scenario-specific:

$$\beta_{i\omega} \geq \frac{1}{K_i} \sum_{j \in J} c_{ij} p_{ij}^\omega x_{ij}, \quad i \in I, \omega \in \Omega \quad (20)$$

This relaxation is valid for arbitrary release times and deadlines.

5.2 Minimum cost problem

In the minimum cost problem, there is only a fixed cost ϕ_{ij} associated with assigning job j to facility i . So we have

$$\hat{g}(\mathbf{x}) = \sum_{i \in I} \sum_{j \in J} \phi_{ij} x_{ij}$$

in the MILP master problem (10), and we set $\beta_\omega = 0$ for $\omega \in \Omega$. The subproblem decouples into a feasibility problem for each i and ω , because $\hat{h}_i(\mathbf{s}, \bar{\mathbf{x}}, \omega) = 0$.

A Benders cut is generated for each i and ω when the corresponding scheduling problem (11) is infeasible. A simple nogood cut is

$$\sum_{j \in J_i(\bar{\mathbf{x}})} (1 - x_{ij}) \geq 1 \quad (21)$$

We strengthen the cut heuristically by re-solving the scheduling problem $|J_i(\bar{\mathbf{x}})|$ times, each time removing a different job j from $J_i(\bar{\mathbf{x}})$. We add the nogood cut (21), with j removed from $J_i(\bar{\mathbf{x}})$, whenever the scheduling subproblem is infeasible.

To create a subproblem relaxation for the master problem, one can exploit the fact that we now have two-sided time windows $[r_j, d_j]$. Let $J(t_1, t_2)$ be the set of jobs j for which $[r_j, d_j] \subseteq [t_1, t_2]$. Adapting an approach from [24], one can add the following inequalities to the master problem for each $i \in I$:

$$\frac{1}{K_i} \sum_{j \in J(t_1, t_2)} p_{ij}^{\min} c_{ij} x_{ij} \leq t_2 - t_1, \quad t_1 \in \{\bar{r}_1, \dots, \bar{r}_{n'}\}, \quad t_2 \in \{\bar{d}_1, \dots, \bar{d}_{n''}\} \quad (22)$$

where $\bar{r}_1, \dots, \bar{r}_{n'}$ are the distinct release times among r_1, \dots, r_n , and $\bar{d}_1, \dots, \bar{d}_{n''}$ the distinct deadlines among d_1, \dots, d_n . Some of these inequalities may be redundant, and a method for detecting them is presented in [24]. Because the relaxation must be valid across all scenarios, the processing time is set to $p_{ij}^{\min} = \min_{\omega \in \Omega} \{p_{ij}^\omega\}$.

5.3 Minimum tardiness problem

In this section, we consider a minimum tardiness problem in which jobs are all released at time zero but have different due dates \bar{d}_j . There are no hard deadlines, and so we let $d_j = \infty$ for all $j \in J$. As in the minimum makespan problem, there is no first-stage cost, so that $\hat{g}(\mathbf{x}) = 0$ in the MILP model (10). The second-stage objective function is expected total tardiness, and we have

$$\hat{h}_i(\mathbf{s}, \mathbf{x}, \omega) = \sum_{j \in J_i(\mathbf{x})} (s_j + p_{ij}^\omega - \bar{d}_j)^+$$

in the CP scheduling problem (11). Here $\alpha^+ = \max\{0, \alpha\}$.

The following analytic Benders cut can be adapted from [25]:

$$\beta_\omega \geq \sum_{i \in I} \left(\text{SP}_{i\omega}(\bar{\mathbf{x}}) - \sum_{j \in J_i(\bar{\mathbf{x}})} \left(\sum_{j' \in J_i(\bar{\mathbf{x}})} p_{ij'}^\omega - \bar{d}_j \right)^+ (1 - x_{ij}) \right) \quad (23)$$

The cut is added to (10) for each $\omega \in \Omega$. Strengthened nogood cuts similar to those developed for the makespan problem can also be used.

Two subproblem relaxations can be adapted from [24]. The simpler one is analogous to (22) and adds the following inequalities to (10) for each i and ω

$$\beta_{i\omega} \geq \frac{1}{K_i} \sum_{j' \in J(0, \bar{d}_j)} p_{ij'}^\omega c_{ij'} x_{ij'} - \bar{d}_j, \quad j \in J \quad (24)$$

along with the bounds $\beta_{i\omega} \geq 0$. A second relaxation more deeply exploits the structure of the subproblem. For each facility i and scenario ω , let τ_i^ω be a permutation of $\{1, \dots, n\}$ such that $p_{i\tau_i^\omega(1)}^\omega c_{i\tau_i^\omega(1)} \leq \dots \leq p_{i\tau_i^\omega(n)}^\omega c_{i\tau_i^\omega(n)}$. We also assume that jobs are indexed so that $\bar{d}_1 \leq \dots \leq \bar{d}_n$. Then we add the following inequalities to the master problem (10) for each i and ω :

$$\beta_{i\omega} \geq \frac{1}{K_i} \sum_{j' \in J} p_{i\tau_i^\omega(j')}^\omega c_{i\tau_i^\omega(j')} x_{i\tau_i^\omega(j')} - \bar{d}_j - (1 - x_{ij}) U_{ij\omega}, \quad j \in J$$

where

$$U_{ij\omega} = \frac{1}{K_i} \sum_{j' \in J} p_{i\tau_i^\omega(j')}^\omega c_{i\tau_i^\omega(j')} - \bar{d}_j$$

6 The Integer L-Shaped Method

The integer L-Shaped method is a Benders-based algorithm proposed by [29] to solve two-stage stochastic integer programs. It terminates in finitely many iterations when the problem has complete recourse and binary first-stage variables. It is similar to branch and check in that Benders cuts are generated while solving the first-stage problem by branching. It differs in that it uses subgradient cuts derived from a linear programming relaxation of the subproblem rather than combinatorial cuts derived from the original subproblem. It also uses a simple integer nogood cut to ensure convergence, but the cut is quite weak and does not exploit the structure of the subproblem as does branch and check. We describe the integer L-shaped method as it applies to minimizing makespan in the planning and scheduling problem.

We first state an MILP model of the deterministic equivalent problem, as it will play a benchmarking role in computational testing. We index discrete times by $t \in T$ and introduce a

0–1 variable z_{ijt}^ω that is 1 if job j starts at time t on facility i in scenario ω . The model is

$$\begin{aligned}
& \text{minimize} && \sum_{\omega \in \Omega} \pi_\omega \beta_\omega && (a) \\
& \text{subject to} && \sum_{i \in I} x_{ij} = 1, \quad j \in J && (b) \\
& && \beta_\omega \geq \beta_{i\omega}, \quad i \in I, \omega \in \Omega && (c) \\
& && x_{ij} \in \{0, 1\}, \quad i \in I, j \in J && (d) \\
& && \beta_{i\omega} \geq \sum_{t \in T} (t + p_{ij}^\omega) z_{ijt}^\omega, \quad i \in I, j \in J, \omega \in \Omega && (e) \\
& && z_{ijt}^\omega \leq x_{ij}, \quad i \in I, j \in J, t \in T, \omega \in \Omega && (f) \\
& && \sum_{i \in I} \sum_{t \in T} z_{ijt}^\omega = 1, \quad j \in J, \omega \in \Omega && (g) \\
& && \sum_{j \in J} \sum_{t' \in T_{tij}^\omega} c_{ij} z_{ijt'}^\omega \leq K_i, \quad i \in I, t \in T, \omega \in \Omega && (h) \\
& && z_{ijt}^\omega = 0, \quad i \in I, \omega \in \Omega, j \in J, \text{ all } t \in T \text{ with } t < r_j && (i) \\
& && z_{ijt}^\omega \in \{0, 1\}, \quad i \in I, j \in J, t \in T, \omega \in \Omega && (j)
\end{aligned} \tag{25}$$

where $T_{tij}^\omega = \{t' \mid 0 \leq t' \leq t - p_{ij}^\omega\}$. In the integer L-shaped method, the first stage minimizes (25a) subject to (25b)–(25d) and Benders cuts that provide bounds on β_ω . The Benders cuts consist of classical Benders cuts derived from the linear relaxation of the second-stage scheduling problem for each i and ω , as well as integer cuts. If \bar{x} is an optimal solution of the first-stage problem, the second-stage problem for facility i and scenario ω is

$$\begin{aligned}
& \text{minimize} && M \\
& \text{subject to} && M \geq \sum_{t \in T} (t + p_{ij}^\omega) z_{ijt}^\omega, \quad j \in J_i(\bar{x}) \\
& && \sum_{t \in T} z_{ijt}^\omega = 1, \quad j \in J_i(\bar{x}) \\
& && \sum_{j \in J} \sum_{t' \in T_{tij}^\omega} c_{ij} z_{ijt'}^\omega \leq K_i, \quad t \in T \\
& && z_{ijt}^\omega \in \{0, 1\}, \quad j \in J_i(\bar{x}), t \in T \\
& && z_{ijt}^\omega = 0, \quad j \in J_i(\bar{x}), \text{ all } t \in T \text{ with } t < r_j \\
& && z_{ijt}^\omega \in \{0, 1\}, \quad j \in J_i(\bar{x}), t \in T
\end{aligned} \tag{26}$$

The following integer L-shaped cut is used for each ω to ensure convergence:

$$\beta_\omega \geq (\text{SP}_\omega(\bar{x}) - \text{LB}_\omega) \left(\sum_{j \in S(\bar{x})} x_{ij} - \sum_{j \notin S(\bar{x})} x_{ij} - |S(\bar{x})| + 1 \right) + \text{LB}_\omega \tag{27}$$

where $S(\bar{x}) := \{i : x_i = 1\}$ and LB_ω is a global lower bound on makespan under scenario ω . We obtain LB_ω by solving the LP relaxation of (25) for fixed scenario ω . The same lower bound is used to strengthen the initial master problem in LBB and branch-and-check methods by adding bounds of the form

$$\beta_\omega \geq \text{LB}_\omega, \quad \omega \in \Omega. \tag{28}$$

7 Computational Study

In this section, we describe computational experiments we conducted for all three objective functions described in Section 5. In addition, we use the minimum makespan problem to test the effect of several modifications to the LBBD and integer L-shaped methods.

One effect that has been observed in previous work (Hooker 2007, Ciré et al. 2016) is that the relative advantage of LBBD for planning and scheduling tends to increase with the number of facilities, for a given fixed number of jobs. In particular, the advantage of LBBD is much less pronounced when there are only two facilities. This is because a larger number of facilities results in more decoupling of the subproblem and a smaller number of jobs assigned to each facility (the complexity of the scheduling problem is highly sensitive to the number of assigned jobs). Stochastic planning and scheduling is similar in that the subproblem is smaller when there are more facilities, but there is a difference as well: the scheduling problem size remains constant as the number of *scenarios* increases. We might therefore expect that computational tests will show the relative advantage of stochastic LBBD to be less with two facilities than with a greater number, while it is an open question how its advantage will vary with the number of scenarios. To test the former hypothesis, we design experiments with two and four facilities. To investigate the latter question, we run tests with a wide range of scenario counts (1 to 500).

All experiments are conducted on a personal computer with a 2.80 GHz Intel[®] Core[™] i7-7600 processor and 24 GB memory running on a Microsoft Windows 10 Pro. All MILP and CP formulations are solved in C++ using the CPLEX and CP Optimizer engines of IBM[®] ILOG[®] CPLEX[®] 12.7 Optimization Studio, respectively. We use only use a single thread in all computational experiments. We modify CP Optimizer parameters to execute an extended filtering and DFS search. The rest of the parameters are set to their default values for both CPLEX and CP Optimizer engines. Lastly, we use the Lazy Constraint Callback function of CPLEX to implement branch and check.

7.1 Minimum Makespan Problem

For the minimum makespan problem, we generate problem instances by combining ideas from [24] and [4]. We first generate the deterministic problem as in [24]. Let $|I| = m$ and $|J| = n$. The capacity limits of the facilities is set to $K_i = 10$ for all $i \in I$, and integer capacity requirements of jobs are drawn from a uniform distribution on $[1, 10]$. Integer release times are drawn from a uniform distribution on $[0, 2.5n(m+1)/m]$. For each facility $i \in I$, integer mean processing times \bar{p}_{ij} are drawn from a uniform distribution on $[2, 25 - 10(i-1)/(m-1)]$. This causes facilities with a higher index to process jobs more rapidly.

We then follow [4] by perturbing the mean processing times to obtain a set of scenarios. In particular, we first divide the jobs into two groups, one group containing jobs i for which $0 < \bar{p}_{ij} \leq 16$, and the other group containing the remainder of the jobs. We then generate a perturbation parameter ϵ^ω for each scenario $\omega \in \Omega$ from a mixture of uniform distributions. Specifically, for jobs in the first group, ϵ^ω is distributed uniformly on the interval $[-0.1, 0.5]$ with

Table 1: Computation times in seconds (averaged over 3 instances) of the integer L-shaped and branch-and-check methods for 2 and 4 facilities.

Tasks	Scenarios	2 facilities			4 facilities		
		Integer L-shaped method	B&Ch nogood cuts	B&Ch analytic cuts	Integer L-shaped method	B&Ch nogood cuts	B&Ch analytic cuts
10	1	127.3	1.5	0.9	2405.0 ^{††}	0.5	0.4
	5	839.2	4.6	2.1	*	2.5	2.4
	10	2316.9 [†]	6.1	3.1	*	3.9	3.7
	50	*	28.4	17.3	*	24.4	17.4
	100	*	56.1	36.7	*	41.9	28.6
	500	*	375.6	279.1	*	268.3	164.8
14	1	1831.6 [†]	4.3	3.8	3016.6 ^{††}	0.9	1.7
	5	*	16.3	16.9	*	5.6	4.8
	10	*	37.7	26.1	*	12.4	7.3
	50	*	186.0	134.4	*	88.0	34.2
	100	*	411.0	357.9	*	189.5	83.2
	500	*	2431.9 [†]	2424.3 [†]	*	854.7	443.9
18	1	2416.2 ^{††}	208.3	155.5	2404.7 ^{††}	1.5	1.0
	5	*	1102.4	988.4	*	57.0	20.7
	10	*	2184.0	1888.3 [†]	*	116.4	39.5
	50	*	*	*	*	458.2	187.8
	100	*	*	*	*	943.7	332.7
	500	*	*	*	*	2804.5 ^{††}	2825.9 ^{††}

[†]Average excludes one instance that exceeded an hour in computation time.

^{††}Average excludes two instances that exceeded an hour.

*All three instances exceeded an hour.

probability 0.9 and on the interval $[2.0, 3.0]$ with probability 0.1. For jobs in the second group, ϵ^ω is distributed uniformly on the interval $[-0.1, 0.5]$ with probability 0.99 and on the interval $[1.0, 1.5]$ with probability 0.01. Finally, we generate the processing times under scenario $\omega \in \Omega$ by letting $p_{ij}^\omega = \lceil \bar{p}_{ij}(1 + \epsilon^\omega) \rceil$. All problem instances used throughout this paper are publicly available (see electronic companion C).

Table 1 summarizes the relative performance of LBBD and the integer L-shaped method on instances of various sizes. The table focuses on the branch-and-check variant of LBBD because we found it to be superior to standard LBBD. Statistics for standard LBBD and other variants are reported in subsequent tables. The specific methods compared are as follows:

- *Integer L-shaped method.* We decouple the second-stage problem by facility and scenario, and we solve the resulting problems and their LP relaxations using the MILP engine of CPLEX whenever a candidate incumbent solution is identified. We then add the integer cut (27), as well as the classical Benders cut from the LP relaxation for each scenario. The

initial bounds (28) are included in the master problem, even though they are not standard, because previous experience indicates that they significantly enhance performance. The subproblem relaxation (20) is likewise included in the master problem for fair comparison with LBB and branch and check, where it is standard.

- *Branch and check with nogood cuts.* We use (12) and unstrengthened nogood cuts (13). We solve the decoupled subproblems by `CP Optimizer`. The initial bounds (28) are included in the master problem.
- *Branch and check with analytical cuts.* We use (12) and analytical cuts (17) rather than nogood cuts. The decoupled subproblems are solved by `CP Optimizer`. The initial bounds (28) are again included in the master problem.

The results indicate that branch and check is clearly superior to the L-shaped method. It is already orders of magnitude faster in those few smaller instances where the L-shaped method could solve the problem within an hour. Perhaps not surprisingly, the analytic Benders cuts are almost always more effective than the nogood cuts. These data also confirm the hypothesis that the advantage of branch and check is greater when there are 4 facilities rather than 2, indeed dramatically greater as instance size increases.

Table 2 probes algorithmic performance more deeply by comparing computation times and optimality gaps for seven algorithms. Three of the methods are described above, and the remaining four are as follows:

- *Deterministic equivalent MILP.* We solve the deterministic equivalent model (25) using the MILP engine in `CPLEX`, which we also use to solve the first stage of the other six models.
- *Integer L-shaped method with CP.* We modify the standard method by solving the second-stage subproblems with CP rather than MILP. Integer cuts are as before, and classical Benders cuts are derived from the LP relaxation of the MILP model as before. The initial bounds (28) and subproblem relaxation (20) are again included in the master problem.
- *Standard LBB with nogood cuts.* We use (12) and unstrengthened nogood cuts (13). We solve the decoupled subproblems by `CP Optimizer`. The initial bounds (28) are included in the master problem for comparability with the integer L-shaped method.
- *Standard LBB with analytical cuts.* We use (12) and analytical cuts (17) rather than nogood cuts. The decoupled subproblems are solved by `CP Optimizer`. The initial bounds (28) are again included in the master problem.

In addition to average computation time (in seconds), Table 2 reports the optimality gap obtained for each solution method, defined as $(UB - LB)/UB$. For the deterministic equivalent and branch-and-check methods, UB and LB are, respectively, the upper and lower bounds obtained from `CPLEX` upon solution of the master problem. For standard LBB, UB and LB are, respectively, the smallest subproblem optimal value and the largest master problem optimal value obtained during the Benders algorithm.

Table 2: Average computation time in seconds over 3 instances (upper half of table) and average relative optimality gap (lower half) for various solution methods, based on 10 jobs and 2 facilities.

Scenarios	Determ.	Integer	Integer	LBBD	LBBD	B&Ch	B&Ch
	equiv. MILP	L-shaped method	L-shaped with CP	Nogood cuts	Analytic cuts	Nogood cuts	Analytic cuts
1	2.4	127.3	27.9	2.0	0.6	1.5	0.9
5	475.8 ^{††}	839.2	149.3	12.1	3.0	4.6	2.1
10	*	2316.9 [†]	437.8	27.4	7.3	6.1	3.1
50	*	*	2517.8 ^{††}	243.1	42.8	28.4	17.3
100	*	*	*	952.8	118.8	56.1	36.7
500	*	*	*	*	900.9	375.6	279.1
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	7.8	0.0	0.0	0.0	0.0	0.0	0.0
10	12.4	3.8	0.0	0.0	0.0	0.0	0.0
50	17.4	21.7	13.9	0.0	0.0	0.0	0.0
100	25.4	25.4	21.7	0.0	0.0	0.0	0.0
500	44.5	25.8	25.4	13.5	0.0	0.0	0.0

[†]Average excludes one instance that exceeded an hour in computation time.

^{††}Average excludes two instances that exceeded an hour.

*All three instances exceeded an hour.

As one might expect, the integer L-shaped implementations are faster than solving the deterministic equivalent MILP, because they exploit the scenario-based block structure of two-stage stochastic programs. We also see that the integer L-shaped method can be significantly accelerated by solving the exact subproblem with CP rather than MILP (to obtain upper bounds and generate the integer cut), since CP is more effective for this type of scheduling problem.

It is clear from Table 2 that all four implementations of LBBD substantially outperform the integer L-shaped method, even when the latter uses CP. Furthermore, the two branch-and-check implementations scale much better than standard LBBD, due mainly to time spent in solving the master problem in standard LBBD. This confirms the rule of thumb that branch and check is superior when solving the master problem takes significantly longer than solving the subproblems. The results also indicate that analytical Benders cuts are more effective than unstrengthened nogood cuts in both standard LBBD and branch and check.

Table 3 provides a more detailed comparison of the integer L-shaped method with the branch-and-check implementations. The L-shaped method with CP is shown, as we have seen that it is faster than solving the subproblem with MILP. Interestingly, solving a CP formulation of the subproblem is much faster than solving the LP relaxation of an MILP formulation. This illustrates the computational cost of using the larger MILP formulation. We also see that the stronger analytical cuts reduce the number of times the subproblem must be solved, and therefore the number of cuts generated and the resulting size of the master problem. Furthermore, the number of subproblem calls is roughly constant as the number of scenarios increases. Finally, the subproblem solutions consume about half of the total computation time in the branch-and-cut

Table 3: Analysis of the integer L-shaped method with CP subproblems and two branch-and-check algorithms. Each number is an average over 3 problem instances.

Scenarios	Integer L-shaped with CP					B&C with nogood cuts				B&C with analytical cuts			
	Time (sec)		Statistics			Time (sec)		Statistics		Time (sec)		Statistics	
	Total	CPsub	LPsub	Cuts	Calls	Total	CPsub	Cuts	Calls	Total	CPsub	Cuts	Calls
1	27.9	0.9	2.1	450	452	1.5	0.7	282	150	0.9	0.4	180	95
5	149.3	5.6	16.4	2692	541	4.6	2.1	1289	144	2.1	0.8	658	79
10	437.8	15.8	73.4	5114	515	6.1	3.1	2390	134	3.1	1.3	1243	75
50	2517.8 [†]	97.3 [†]	500.2 [†]	20002 [†]	401 [†]	28.4	14.8	12616	148	17.3	7.6	7684	94
100	*	*	*	*	*	56.1	29.7	25880	152	36.7	16.0	15800	99
500	*	*	*	*	*	375.6	169.0	127404	150	279.1	76.9	76029	96

[†]Average excludes two instances that exceeded an hour.

*Computation terminated for all 3 instances after one hour.

algorithms. Previous experience suggests that for best results, the computation time should, in fact, be about equally split between the master problem and subproblem (Ciré et al. 2016).

Given the computational burden of solving the LP relaxation of the MILP subproblem, we experimented with running the integer L-shaped method with only integer cuts. This obviates the necessity of solving the LP relaxation of an MILP model. The results appear in Table 4. The three implementations shown in the table are exactly the same except for the cuts used and therefore permit a direct comparison of the effectiveness of the cuts. The integer L-shaped method actually runs faster using only integer cuts, without any classical Benders cuts obtained from the LP relaxation. We also see that the analytical cuts are much more effective than integer cuts, which are quite weak.

Finally, these data allow us to address the question, posed earlier, as to whether the advantage of branch and check relative to the L-shaped method increases with the number of scenarios. The advantage appears to be roughly constant for 2 facilities and perhaps increasing for 4 facilities, although the latter is uncertain because the L-shaped method (even with no LP relaxation) quickly times out.

We also experimented with a different distribution of processing times. We simulated a situation in which processing proceeds normally except when there is a delay due to mechanical breakdown or other causes. Accordingly, we defined processing time to be a random variable that is equal to the mean quantity specified above with 80% probability, but 1.5 times as large with 15% probability, and 4 times as large as 5% probability. The results appear in Table 5. Comparison with Table 4 reveals that the relative advantage of branch and check is even greater with this processing time distribution than with the original one.

7.2 Minimum Cost Problem

In this section, we present the results of the computational experiments on the minimum cost problem. We use the same instances we used for the minimum makespan problem, with the

Table 4: Performance of the integer L-shaped method with integer cuts only (no cuts from the LP relaxation).

Tasks	Scenarios	2 facilities			4 facilities		
		Integer L-shaped method	L-shaped integer cuts only	B&Ch analytic cuts	Integer L-shaped method	L-shaped integer cuts only	B&Ch analytic cuts
10	1	127.3	2.8	0.9	2405.0 ^{††}	78.1	0.4
	5	839.2	9.0	2.1	*	906.5	2.4
	10	2316.9 [†]	16.0	3.1	*	2213.0 [†]	3.7
	50	*	87.2	17.3	*	*	17.4
	100	*	209.5	36.7	*	*	28.6
	500	*	1166.6	279.1	*	*	164.8
14	1	1831.6 [†]	48.9	3.8	3016.6 ^{††}	2403.9 ^{††}	1.7
	5	*	229.5	16.9	*	2402.8 ^{††}	4.8
	10	*	284.7	26.1	*	*	7.3
	50	*	1850.6	134.4	*	*	34.2
	100	*	2810.4 ^{††}	357.9	*	*	83.2
	500	*	*	2424.3 [†]	*	*	443.9
18	1	2416.2 ^{††}	1358.6 [†]	155.5	2404.7 ^{††}	1346.8 [†]	1.0
	5	*	3048.4 ^{††}	988.4	*	2405.5 ^{††}	20.7
	10	*	3477.2 ^{††}	1888.3 [†]	*	*	39.5
	50	*	*	*	*	*	187.8
	100	*	*	*	*	*	332.7
	500	*	*	*	*	*	2825.9 ^{††}

[†]Average excludes one instance that exceeded an hour in computation time.

^{††}Average excludes two instances that exceeded an hour.

*All three instances exceeded an hour.

addition of costs and deadlines. The fixed cost associated with assigning jobs to facility i is drawn from a uniform distribution on the interval $[400/\alpha, 800/\alpha]$, where $\alpha = 25 - (i - 1)(10/(m - 1))$, so that the faster facilities tend to be more expensive. The deadline d_j of job j is obtained as follows. Let $L = 20 \times n/m$. We set $d_j = r_j + \beta$ where r_j is the release time of job j and β is drawn from a uniform distribution on $[0.75 \times \alpha L, 1.25 \times \alpha L]$ with $\alpha = 2/3$.

We compare LBB performance solely with the deterministic equivalent MILP formulation, since the type of integer cut used in the L-shaped method is an optimality cut and is not defined for infeasible subproblems. The MILP model is the same as (25) except that the objective function is replaced by $\sum_{i \in I} \sum_{j \in J} \phi_{ij} x_{ij}$, β variables are eliminated, and constraint (i) is modified to reflect two-sided time windows.

As is evident in Table 6, the deterministic equivalent MILP performs better on this problem than on the minimum makespan problem. However, the branch-and-check method scales better than the MILP formulation and is superior for solving the larger instances.

Table 5: Average computation time for the makespan problem with alternate processing times.

Jobs	Scenarios	2 facilities		4 facilities	
		L-shaped integer cuts only	B&C analytic cuts	L-shaped integer cuts only	B&C analytic cuts
10	1	18.5	0.3	3246.3	0.9
	5	141.3	2.6	*	1.9
	10	292.3	2.7	*	2.8
	50	2425.0	11.6	*	15.1
	100	*	22.3	*	21.5
	500	*	129.3	*	149.1
14	1	1971.2	5.2	1368.6	0.8
	5	*	18.1	*	3.7
	10	*	43.3	*	5.6
	50	*	241.8	*	26.2
	100	*	704.4	*	60.1
	500	*	3503.4	*	407.4
18	1	*	213.3	1770.1	2.7
	5	*	2190.4	*	10.1
	10	*	2932.3	*	31.5
	50	*	*	*	316.6
	100	*	*	*	634.3
	500	*	*	*	3162.8

[†]Average excludes one instance that exceeded an hour in computation time.

^{††}Average excludes two instances that exceeded an hour.

*All three instances exceeded an hour.

7.3 Minimum Tardiness Problem

In this section, we present the results of the computational experiments on the minimum tardiness problem. We use the same instances as for the minimum makespan problem, with due dates added. The due dates are obtained in the same fashion as the deadlines for the minimum cost problem, except that we set α equal to $1/3$ rather than $2/3$.

The MILP model we used for the integer L-shaped method is the same as (25) except that constraint (i) is modified to reflect two-sided time windows, constraints (c) and (e) are replaced by the following:

$$\beta_{\omega} \geq \sum_{i \in I} \sum_{j \in J} \beta_{ij\omega}, \quad \omega \in \Omega \quad (c)$$

$$\beta_{ij\omega} \geq \sum_{t \in T} (t + p_{ij}^{\omega}) z_{ijt}^{\omega} - d_j, \quad \beta_{ij\omega} \geq 0, \quad i \in I, j \in J, \omega \in \Omega \quad (e)$$

The implementation of the integer L-shaped method is otherwise identical to the one applied to the minimum makespan problem. We use the logic-based cuts and the simpler subproblem

Table 6: Average computation time in seconds over 3 instances for the minimum cost problem.

Jobs	Scenarios	2 facilities		4 facilities	
		Determ. equiv. MILP	B&Ch analytic cuts	Determ. equiv. MILP	B&Ch analytic cuts
10	1	0.3	0.1	0.2	0.2
	5	4.9	0.8	1.2	1.2
	10	3.7	1.7	2.7	1.7
	50	6.5	4.9	15.1	9.0
	100	19.2	6.1	18.5	15.5
	500	353.1	31.9	201.5	82.5
14	1	4.1	0.3	0.6	0.3
	5	19.0	1.5	2.8	1.5
	10	46.6	3.3	4.9	2.6
	50	85.5	9.1	24.7	10.5
	100	445.4	19.9	75.4	20.6
	500	*	156.9	844.0	130.0
18	1	23.2	32.8	2.8	1.8
	5	56.5	97.7	17.6	4.8
	10	416.8	128.6	38.4	11.3
	50	2527.9 ^{††}	587.1	107.2	64.7
	100	3348.2 ^{††}	1425.1 [†]	335.1	133.9
	500	*	2526.0 ^{††}	*	716.9

[†] Average excludes one instance that exceeded an hour in computation time.

^{††} Average excludes two instances that exceeded an hour.

* All three instances exceeded an hour.

Table 7: Average computation time in seconds over 3 instances for the minimum tardiness problem.

Jobs	Scenarios	2 facilities			4 facilities		
		Determ. equiv. MILP	L-shaped integer cuts only	B&Ch analytic cuts	Determ. equiv. MILP	L-shaped integer cuts only	B&Ch analytic cuts
10	1	3.1	4.0	2.7	2.1	*	7.5
	5	36.7	8.5	4.8	7.9	*	31.6
	10	868.4	19.7	10.8	29.1	*	60.2
	50	2614.5 ^{††}	100.3	55.1	248.4	*	280.0
	100	2787.5 ^{††}	215.8	110.1	1312.3	*	1108.8
	500	*	1262.3	641.9	*	*	*
14	1	4.7	7.9	2.6	2.9	*	9.9
	5	1307.9	71.5	14.6	21.4	*	25.5
	10	2431.8 ^{††}	94.1	16.3	45.4	*	36.9
	50	*	387.6	92.0	1574.1 [†]	*	374.3
	100	*	969.1	152.4	2945.2 ^{††}	*	782.7
	500	*	2765.1 ^{††}	762.7	*	*	2605.2 [†]
18	1	5.8	18.1	5.3	3.6	*	7.8
	5	250.0	12.8	6.3	162.6	*	24.3
	10	599.5	94.3	25.4	1092.6	*	37.3
	50	*	2415.1 ^{††}	1075.9	*	*	158.2
	100	*	2408.1 ^{††}	1573.3 [†]	*	*	382.8
	500	*	2429.6 ^{††}	2532.0 ^{††}	*	*	2390.4 [†]

[†]Average excludes one instance that exceeded an hour in computation time.

^{††}Average excludes two instances that exceeded an hour.

*All three instances exceeded an hour.

relaxation (24) presented in Section 5.3.

The results in Table 7 shows that the branch-and-check method clearly outperforms both of the benchmark methods. Interestingly, the integer L-shaped method is unable to solve any of the instances even without the overhead created by an LP relaxation.

8 Conclusion

In this study, we applied logic-based Benders decomposition (LBBD) to two-stage stochastic optimization with a scheduling task in the second stage. While Benders decomposition is often applied to such problems, notably in the integer L-shaped method, the necessity of generating classical Benders cuts requires that the subproblem be formulated as a mixed integer/linear programming problem and cuts generated from its continuous relaxation. We observed that this process incurs substantial computational overhead that LBBD avoids by generating logic-based cuts directly from a constraint programming model of the scheduling subproblem. Although the integer cuts used with the L-shaped method can be regarded as a special case of logic-based Benders cuts, they are extremely weak, even weaker than simple nogood cuts often used in an LBBD context. Furthermore, the type of subproblem analysis that has been used for past applications of LBBD permits much stronger logic-based cuts to be derived, again without the overhead of obtaining a continuous relaxation.

Computational experiments found that, due to these factors, LBBD solves a generic stochastic planning and scheduling problem much more rapidly than the integer L-shaped method. The speedup is several orders of magnitude for the minimum makespan problem when a branch-and-check variant of LBBD is used. Branch and check is also superior when minimizing assignment cost or total tardiness, although its advantage for the minimum cost problem is less pronounced. These outcomes suggest that LBBD could be a promising approach to other two-stage stochastic and robust optimization problems with integer or combinatorial recourse, particularly when the subproblem is relatively difficult to model as an integer programming problem.

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A CP Parameters

In this section, we test the impact of using different CP parameters on the overall solution time of the LBB algorithm for solving the minimum makespan problem. We use the same instances used in Tables 1 – 4. The results are presented in Table 8.

Table 8: Average computation time in seconds over 3 instances for different CP parameter values, based on 10 tasks and 2 facilities.

		INT-L				LBB				
		Default	Extended	Default	Extended	Default	Extended	Default	Extended	
		Default	Default	DFS	DFS	Default	Default	DFS	DFS	
Scenarios	Inference level									
	Search type									
		1	2.25	2.26	1.77	2.84	1.44	1.58	1.22	0.91
		5	11.85	13.40	8.43	9.00	3.73	4.33	3.17	2.12
		10	19.75	23.92	15.92	16.00	4.65	5.44	5.54	3.09
		50	116.16	116.97	92.09	87.19	23.92	26.46	23.16	17.33
	100	247.71	254.05	192.95	209.51	50.78	55.73	45.29	36.72	
	500	1430.95	1322.95	1151.25	1166.55	363.55	450.47	391.01	279.07	

We see from Table 8 that using different CP parameters does not change the overall picture of the performance of the two solution methods. Extended inference level and DFS search look

Table 9: Average computation time in seconds over 3 instances for two different lower bounds, based on 10 tasks and 2 facilities.

Scenarios	integer bounding			relaxed bounding		
	Preprocessing time	INT-L time	total	Preprocessing time	INT-L time	total
1	2.2	0.9	3.0	0.2	1.9	2.1
5	9.0	8.2	17.2	0.6	8.1	8.7
10	14.0	12.0	26.0	1.1	15.0	16.1
50	64.1	80.2	144.3	5.3	88.4	93.6
100	139.7	186.2	325.9	12.0	212.3	224.2
500	707.4	1227.3	1934.6	53.3	1473.7	1527.0

to be good choices for the CP solver, therefore, we use these setting in all the experiments we perform in this paper.

B Lower bound for the Integer L-shaped Method

In this section, we perform experiments to see the impact of using better lower bounds on the performance of the integer L-shaped method. We tested two different sets of lower bounds as shown in Table 9. The results in the “integer bounding” column correspond to the global lower bound obtained by solving (25) for fixed scenario ω without relaxing the integrality constraints. The results in the “relaxed bounding” column correspond to the results where we use the bounds obtained by solving the LP relaxation of (25). We again use the same makespan instances used in Tables 1 – 4.

We see from Table 9 that better lower bounds from integer programming yield modest improvements in the average solution times (indicated in columns labeled “INT-L time”). Yet this improvement is substantially offset by the much longer time required to compute the integer programming bound. We therefore opted to use relaxed bounding in all computational experiments.

C Accessing Problem Instances

The readers can access all the problem instances used in our computational experiments via <https://github.com/ozgunelci/Stochastic-Scheduling-With-LBBD>.