Global Optimization of Truss Structure Design

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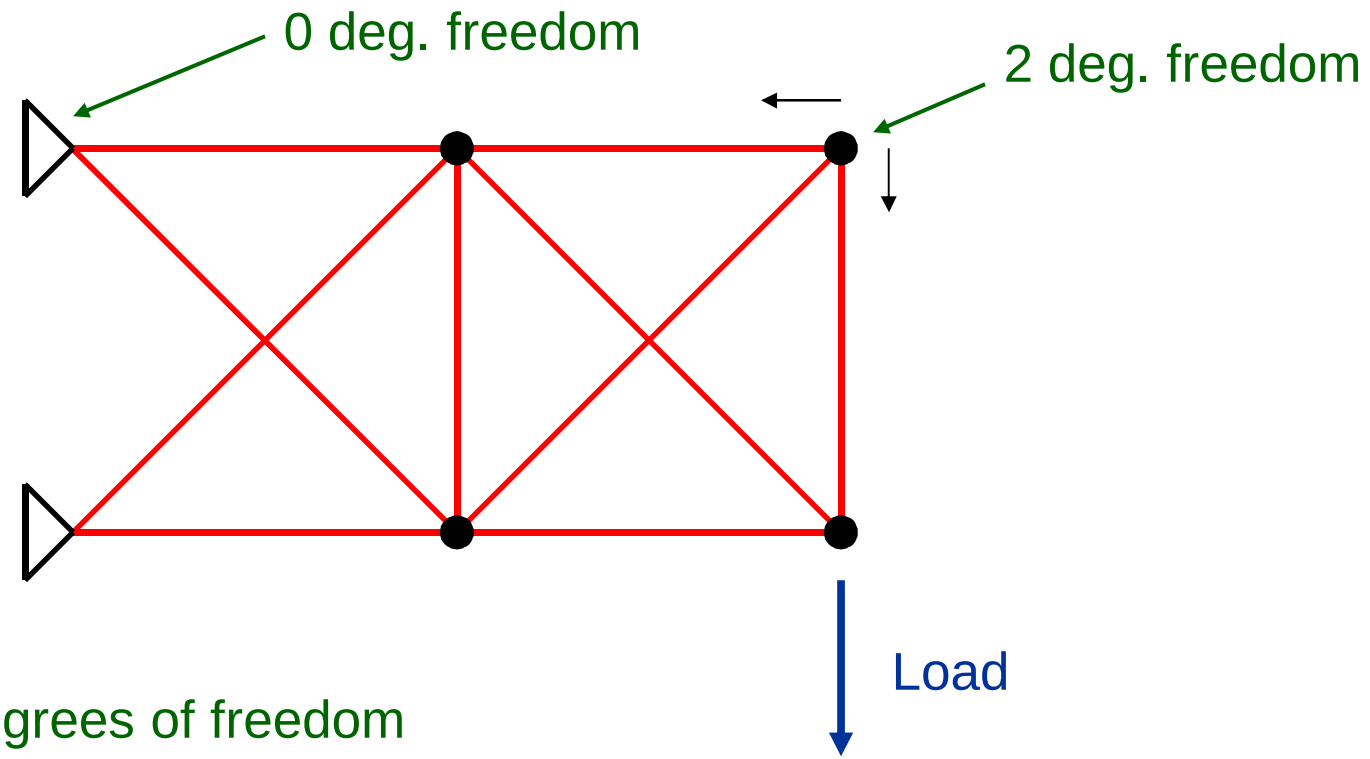
Truss Structure Design

Select size of each bar (possibly zero) to support the load while minimizing weight. Bar sizes are **discrete**.

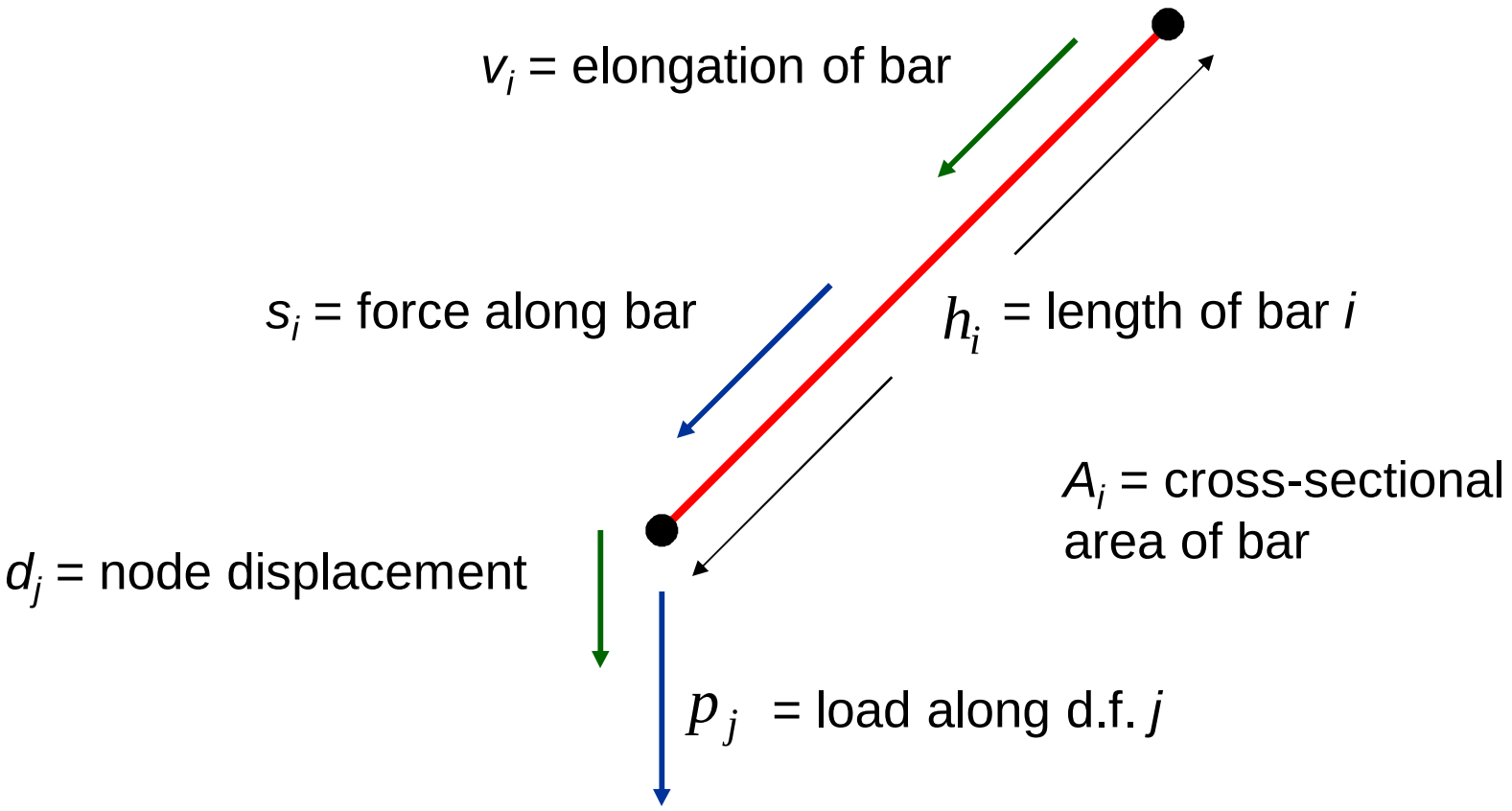
10-bar cantilever truss



Total 8 degrees of freedom

Truss Structure Design

Notation



Truss Structure Design

$$\min \quad \sum_i h_i A_i$$ } Minimize total weight

$$s.t. \quad \sum_i \cos\theta_{ij} s_i = p_j, \text{ all } j$$ } Equilibrium

$$\sum_j \cos\theta_{ij} d_j = v_i, \text{ all } i$$ } Compatibility

nonlinear →

$$\frac{E_i}{h_i} A_i v_i = s_i, \text{ all } i$$ } Hooke's law

$$v_i^L \le v_i \le v_i^U, \text{ all } i$$ } Elongation bounds

$$d_j^L \le d_j \le d_j^U, \text{ all } j$$ } Displacement bounds

$$\bigvee_k (A_i = A_{ik})$$ } Logical disjunction

Area must be one of several discrete values A_{ik}

Constraints can be imposed for multiple loading conditions

Truss Structure Design

Introducing new variables linearizes the problem but makes it much larger.

MILP model

$$\min \quad \sum_i h_i \sum_k A_{ik} y_{ik}$$

$$s.t. \quad \sum_i \cos\theta_{ij} s_i = p_j, \text{ all } j$$

$$\sum_j \cos\theta_{ij} d_j = \sum_k v_{ik}, \text{ all } i$$

$$\frac{E_i}{h_i} \sum_k A_{ik} v_{ik} = s_i, \text{ all } i$$

$$v_i^L \le v_i \le v_i^U, \text{ all } i$$

$$d_j^L \le d_j \le d_j^U, \text{ all } j$$

$$\sum_k y_{ik} = 1, \text{ all } i$$

0-1 variables indicating size of bar *i*

Elongation variable disaggregated by bar size

Hooke's law becomes linear

Solution Approach

- Solve with **SIMPL**, an integrated solver for optimization.
- SIMPL integrates MILP, constraint programming, global optimization in a **unified** approach.

SIMPL Integration Principles

- **Low-level** integration with **high-level** modeling.

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  - Model communicates **problem structure** to the solver.

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- General **search-infer-relax** solution algorithm.
  - Enumerate problem restrictions.
  - Branching or logic-based Benders.
  - Underlying search/inference and search/relaxation dualities.

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- General **search-infer-relax** solution algorithm.
  - Enumerate problem restrictions.
  - Branching or logic-based Benders.
  - Underlying search/inference and search/relaxation dualities
- **Constraint-based** control.
  - Filtering, relaxation, branching.

Classical solution methods

- CP solver
  - **Search:** Branching
  - **Inference:** Flltering
  - **Relaxation:** Domain store
- MILP solver
  - **Search:** Branching
  - **Inference:** Cutting planes, presolve, reduced cost variable fixing
  - **Relaxation:** LP
- Benders
  - **Search:** Enumerate subproblems.
  - **Inference:** Benders cuts
  - **Relaxation:** Master problem

Classical solution methods

- Global optimization
  - **Search:** Enumerate boxes
  - **Inference:** Domain reduction, dual-based variable bounding
  - **Relaxation:** Convexification
- SAT
  - **Search:** Branching
  - **Inference:** Conflict clauses
  - **Relaxation:** Same as restriction
- Local search
  - **Search:** Enumerate neighborhoods.
  - **Inference:** Tabu list, etc.
  - **Relaxation:** Same as restriction

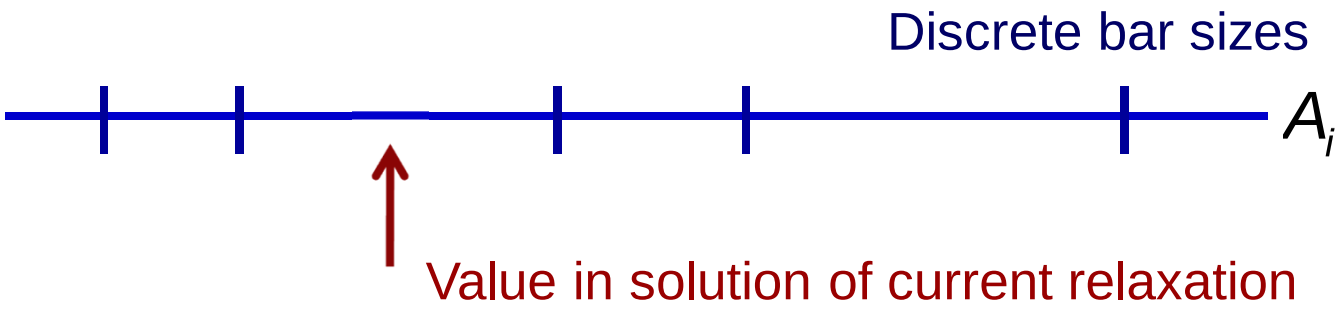
Truss Structure Design

Integrated approach

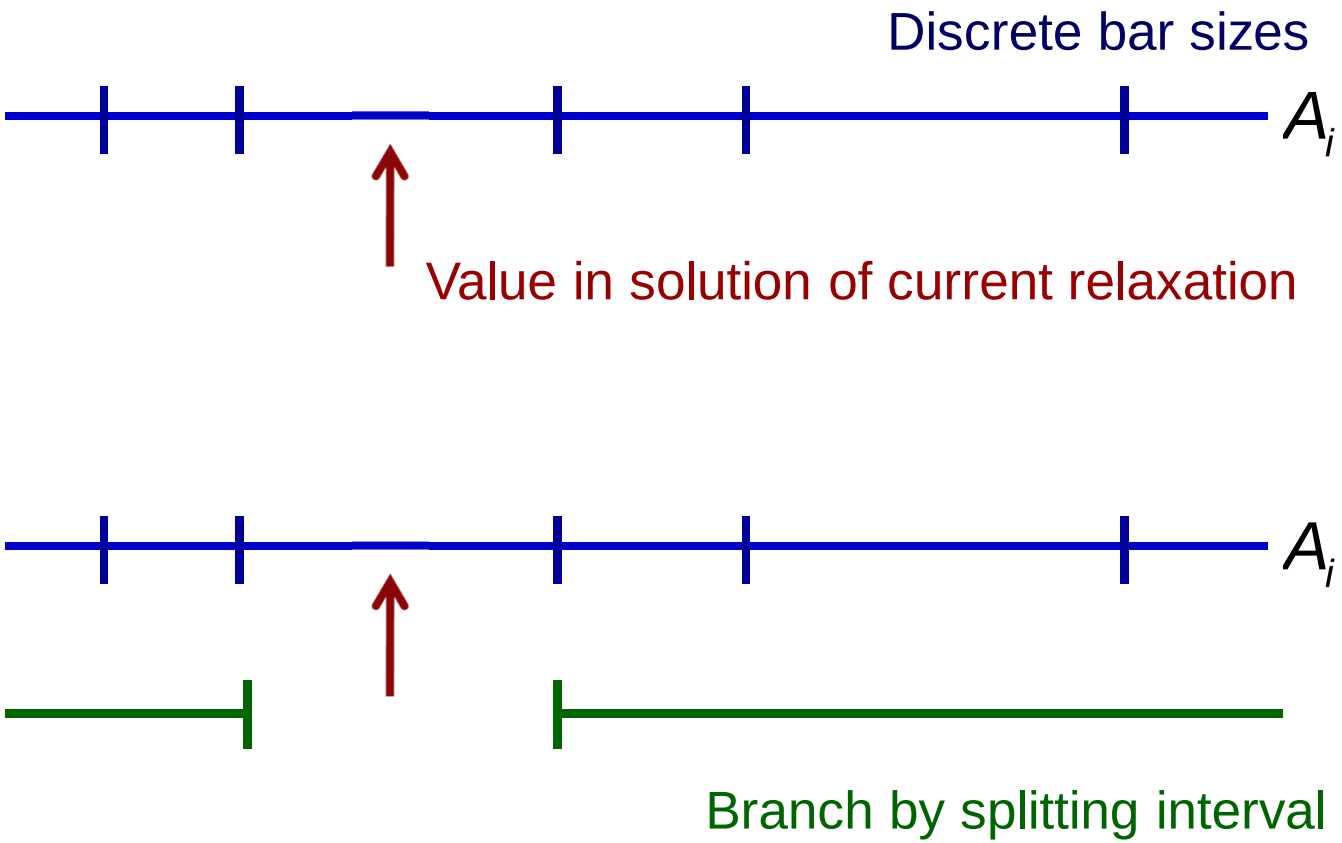
- Use the **original model** (don't introduce new variables)
- Branch by **splitting** the range of areas A_i
    - No 0-1 or integer variables!
- Generate a linear **quasi-relaxation**, which is much smaller than MILP model.
- Use **logic cuts**.

Original hand-coded method: Bollapragada, Ghattas, and JNH 2001.

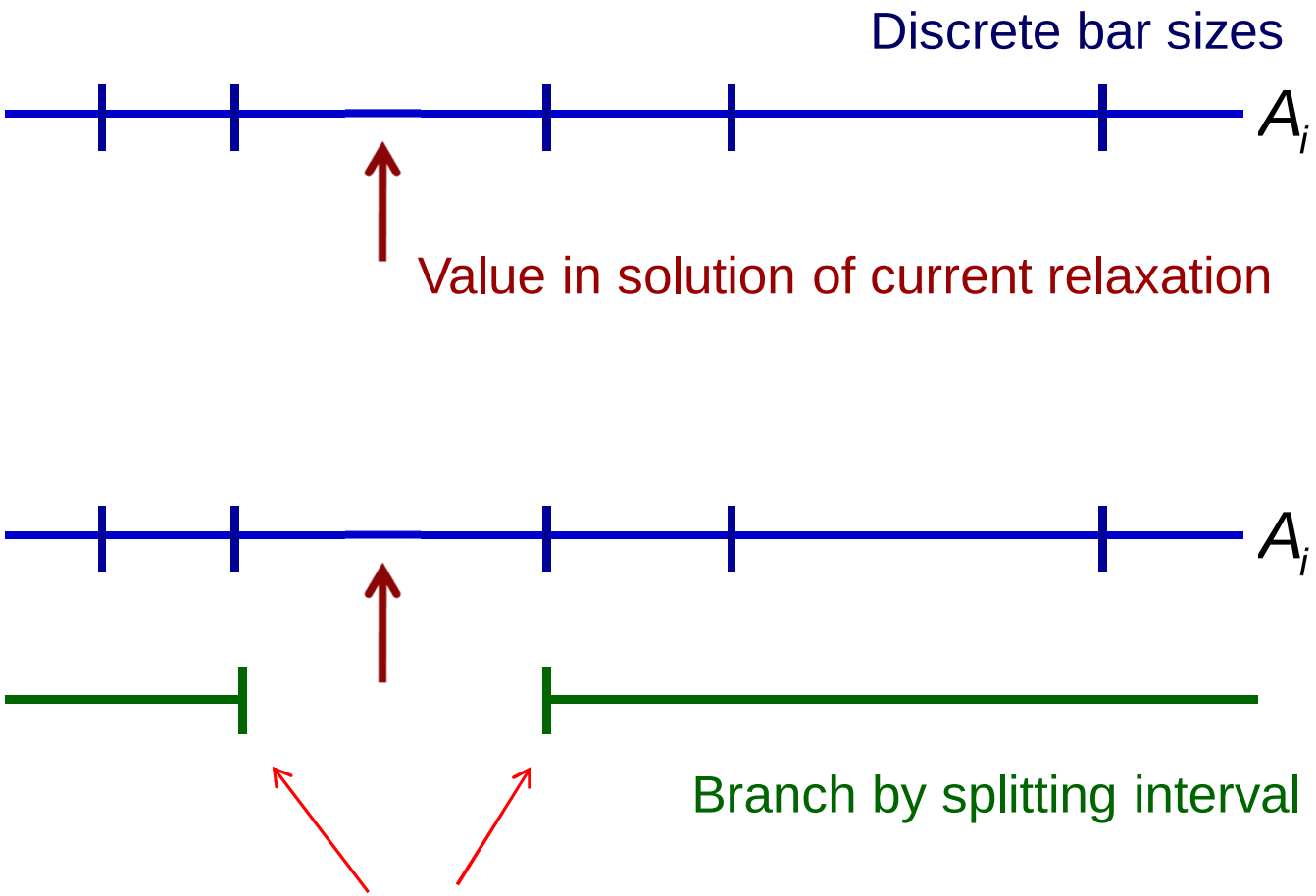
Branching


Discrete bar sizes
A_i
Value in solution of current relaxation

Branching


Discrete bar sizes
A_i
Value in solution of current relaxation
A_i
Branch by splitting interval

Branching



Solution of next relaxation likely to be at an endpoint.
This branching intelligence unavailable in 0-1 model.

Quasi-relaxation

Given problem $\min_{x \in S} \{ f(x) \}$

The problem $\min_{x' \in S'} \{ f'(x') \}$ is a **quasi-relaxation** if

for any $x \in S$, there is an $x' \in S'$ with $f'(x') \leq f(x)$.

A quasi-relaxation need not be a valid relaxation.

But its **optimal value** is a **valid lower bound** on the optimal value of the original problem.

Quasi-relaxation

Consider the problem

$$\min f(x)$$
$$g^j(x, y_j) \leq 0, \text{ all } j$$
$$x \in \mathbb{R}^n, y_j \text{ discrete}$$

Quasi-relaxation

Consider the problem

$$\min f(x)$$

$$g^j(x, y_j) \le 0, \text{ all } j$$

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Each g^j is a vector of functions

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Relaxing the problem by making y_j continuous could result in a **nonconvex** problem.

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But suppose the problem becomes convex when each y_j is fixed to a **constant**.

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Relaxing the problem by making y_j continuous could result in a **nonconvex** problem.

But suppose the problem becomes convex when each y_j is fixed to a **constant**.

Then we may be able to write a **convex quasi-relaxation**.

Quasi-relaxation

Consider the problem

$$\min f(x)$$
$$g^j(x, y_j) \le 0, \text{ all } j$$
$$x \in \mathbb{R}^n, y_j \text{ discrete}$$

Theorem (JNH 2005)

If each $g^j(x,y)$ is semihomogeneous in x and concave in scalar y_j, then the following is a **quasi-relaxation**:

$$\min f(x)$$
$$g^j(x^1, y_j^L) + g^j(x^2, y_j^U) \le 0, \text{ all } j$$
$$\alpha x^L \le x^1 \le \alpha x^U$$
$$(1-\alpha)x^L \le x^2 \le (1-\alpha)x^U$$
$$x = x^1 + x^2$$
$$\alpha \in [0,1]$$

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$g(\alpha x, y) \le \alpha g(x, y)$ for all x, y and $\alpha \in [0,1]$

$g(0, y) = 0$ for all y

Quasi-relaxation

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Bounds on y_i

Quasi-relaxation

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Bounds on x

Quasi-relaxation

Why?

Take any feasible solution $\bar{x}, \bar{y}$

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$$g^j(x^1, y_j^L) + g^j(x^2, y_j^U) \le 0, \text{ all } j$$

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Quasi-relaxation

Why?

Take any feasible solution $\bar{x}, \bar{y}$

Choose α so that

$$\bar{y}_j = \alpha_j y_j^L + (1-\alpha_j) y_j^U$$

Set $x^{j1} = \alpha_j \bar{x}, \quad x^{j2} = (1-\alpha_j)\bar{x}$

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$$\bar{y}_j = \alpha_j y_j^L + (1-\alpha_j) y_j^U$$

Set $x^{j1} = \alpha_j \bar{x}, \quad x^{j2} = (1-\alpha_j)\bar{x}$

Then for each component *i* of g^j we have

$$g_i^j(x^{j1}, y_j^L) + g_i^j(x^{j2}, y_j^U) = g_i^j\left(\alpha_j \bar{x}, y_j^L\right) + g_i^j\left((1-\alpha_j)\bar{x}, y_j^U\right)$$
$$= \alpha_j g_i^j(x, y_j^L) + (1-\alpha_j) g_i^j(x, y_j^U) \le g_i^j\left(x, \alpha_j y_j^L + (1-\alpha_j) y_j^U\right) = g_i^j(x, y_j)$$

homogeneity

concavity

Quasi-relaxation

$$\min f(x)$$
$$g^j(x, y_j) \le 0, \text{ all } j$$
$$x \in \mathbb{R}^n, y_j \text{ discrete}$$

So we have a feasible solution of the quasi-relaxation with value that is less than or equal to (in fact equal to) that of the original problem.

satisfied, by above argument

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satisfied, by construction

Quasi-relaxation

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$\frac{E_i}{h_i} A_i v_i = s_i$ has the form $g(x,y) = 0$ with g semihomogenous in x and concave (linear) in y because we can write it

$$\frac{E_i}{h_i} A_i v_i - s_i = 0$$

with $x = (v_i, s_i)$, $y = A_i$.

Truss Structure Design

So we have a quasi-relaxation of the truss problem:

$$\min \quad \sum_i h_i [A_i^L y_i + A_i^U (1 - y_i)]$$

$$s.t. \quad \sum_i \cos\theta_{ij} s_i = p_j, \text{ all } j$$

$$\sum_j \cos\theta_{ij} d_j = v_{i0} + v_{i1}, \text{ all } i$$

$$\frac{E_i}{h_i}(A_i^L v_{i0} + A_i^U v_{i1}) = s_i, \text{ all } i$$

$$v_i^L y_i \le v_{i0} \le v_i^U y_i, \text{ all } i$$

$$v_i^L (1 - y_i) \le v_{i1} \le v_i^U (1 - y_i), \text{ all } i$$

$$d_j^L \le d_j \le d_j^U, \text{ all } j$$

$$0 \le y_i \le 1, \text{ all } i$$

Hooke's law is linearized

Elongation bounds split into 2 sets of bounds

Truss Structure Design

Logic cuts

v_{i0} and v_{i1} must have same sign in a feasible solution.

If not, we branch by adding logic cuts

$$v_{i0}, v_{i1} \leq 0, \qquad v_{i0}, v_{i1} \geq 0$$

Truss Structure Design

In general, we can have a **metaconstraint** to represent the semihomogeneous constraint $g(x,y) \leq 0$.

This tells the solver to generate a quasi-relaxation.

Truss Structure Design

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This tells the solver to generate a quasi-relaxation.

Since a bilinear constraint $xy = \alpha$ is always semihomogeneous, we will use a **bilinear** metaconstraint with a quasi-relaxation option.

Truss Structure Design

SIMPL model

Recognized as linear systems

```
01. OBJECTIVE
02.   maximize sum i of c[i]*h[i]*A[i]
03. CONSTRAINTS
04.   equilibrium means {
05.     sum i of b[i,j]*s[i,l] = p[j,l] forall j,l
06.     relaxation = { lp } }
07.   compatibility means {
08.     sum j of b[i,j]*d[j,l] = v[i,l] forall i,l
09.     relaxation = { lp } }
10.   hooke means {
11.     E[i]/h[i]*A[i]*v[i,l] = s[i,l] forall i
12.     relaxation = { lp:quasi } }
13. SEARCH
14.   type = { bb:bestdive }
15.   branching = { hooke:first:quasicut, A:splitup }
```

Truss Structure Design

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Recognized as bilinear system

Truss Structure Design

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Generate quasi-relaxation for semihomogenous function

Truss Structure Design

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Branch first on violated logic cuts for quasi-relaxation

Truss Structure Design

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```

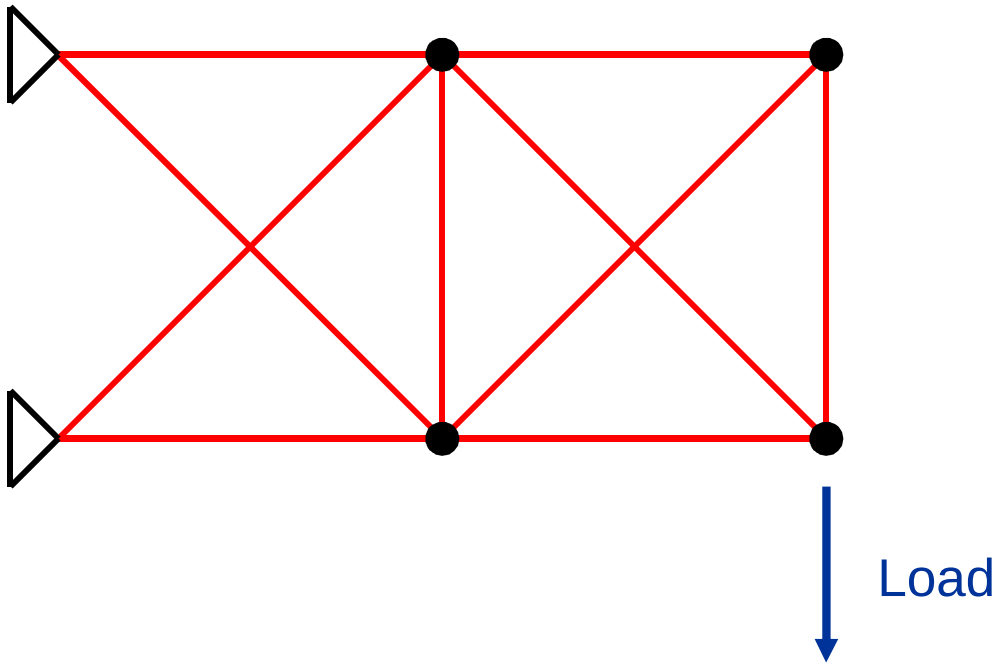
Then branch on A_i in-domain constraint.

Violated when A_i is not one of the discrete bar sizes.

Take upper branch first.

Truss Structure Design

10-bar cantilever truss



Truss Structure Design

Hand-coded integrated method

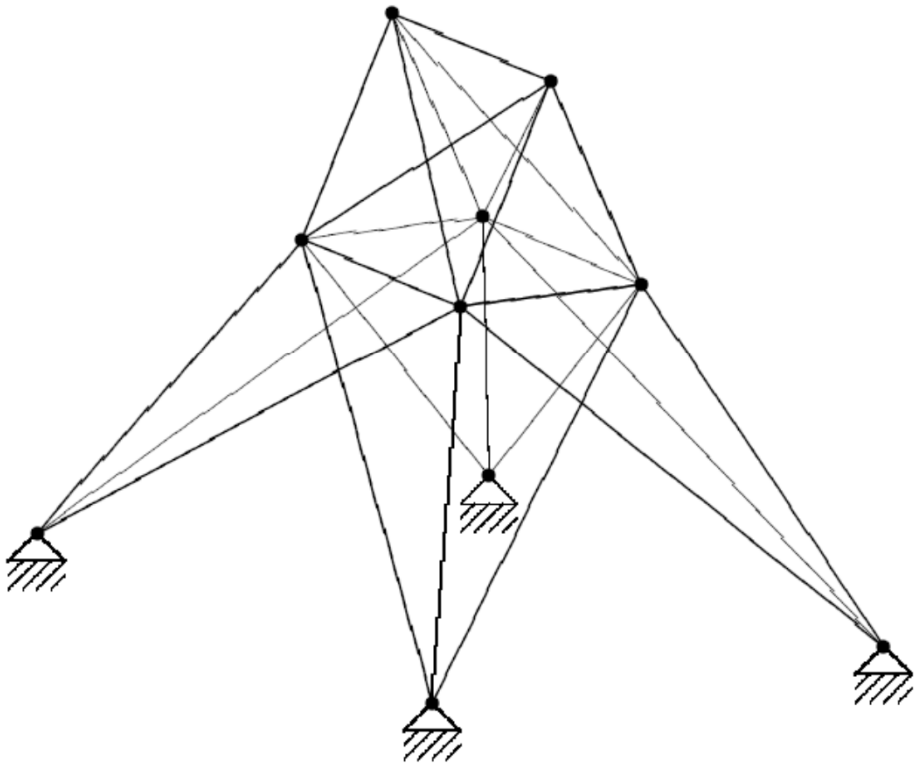
Computational results (seconds)

No. bars	Loads	BARON	CPLEX 11	Hand coded	SIMPL
10	1	5.3	0.40	0.03	0.08
10	1	3.8	0.26	0.02	0.07
10	1	8.1	0.83	0.16	0.49
10	1	8.8	1.2	0.22	0.63
10	2	24	4.9	0.64	1.84
10	2*	327	146	145	65
10	2*	2067	1087	600	651

*plus displacement bounds

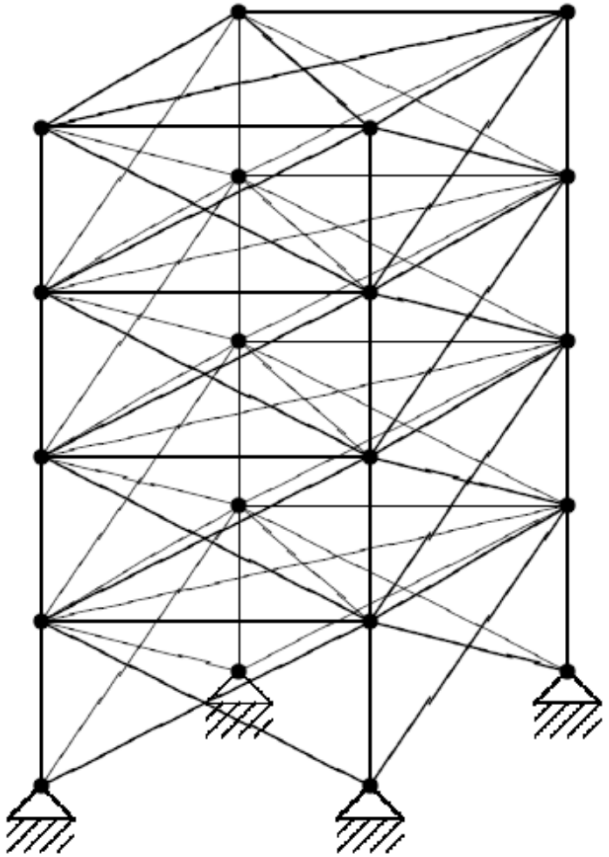
Truss Structure Design

25-bar problem



Truss Structure Design

72-bar problem



Truss Structure Design

Hand-coded integrated method

Computational results (seconds)

No. bars	Loads	BARON	CPLEX 11	Hand coded	SIMPL
25	2	3,302	44	44	20
72	2	3,376	208	33	28
90	2	21,011	570	131	92
108	2	> 24 hr*	3208	1907	1720
200	2	> 24 hr*	> 24 hr*	> 24 hr**	> 24 hr***

* no feasible solution found

** best feasible solution has cost 32,748

*** best feasible solution has cost 32,700

Current Version of SIMPL

- To download:
  - Click the link to SIMPL on John Hooker's website.
  - See readme file for complete instructions.
  - Download executable and associated files
- Operational on GNU/Linux only
- Requires subsidiary solvers
  - CPLEX
  - Eclipse (free download)
- Download problem instances
  - Including all reported in this talk.