Dynamic Programming Bounds from Decision Diagrams

John Hooker

ISAIM 2018

Objective

- Find a general method to **relax dynamic programming models.**
  - **Job sequencing** problems in particular.
  - With **state dependent** processing times.

Objective

- Find a general method to **relax dynamic programming models.**
  - **Job sequencing** problems in particular.
  - With **state dependent** processing times.
- **Why?** To obtain **bounds** on the optimal value.
  - Useful for **heuristics** and **exact methods**

Objective

- Find a general method to **relax dynamic programming models.**
  - **Job sequencing** problems in particular.
  - With **state dependent** processing times.
- **Why?** To obtain **bounds** on the optimal value.
  - Useful for **heuristics** and **exact methods**
- **How?** Using relaxed **decision diagrams**
  - Constructed with **node merger**

Why Dynamic Programming?

- Highly **flexible** modeling.
  - Costs and constraints need not be convex, linear, or even in closed form.
  - Exploits recursive structure.

Why Dynamic Programming?

- Highly **flexible** modeling.
  - Costs and constraints need not be convex, linear, or even in closed form.
  - Exploits recursive structure.
- **Good relaxations** often **unavailable**.
  - Due to the very generality of the model.

Why Dynamic Programming?

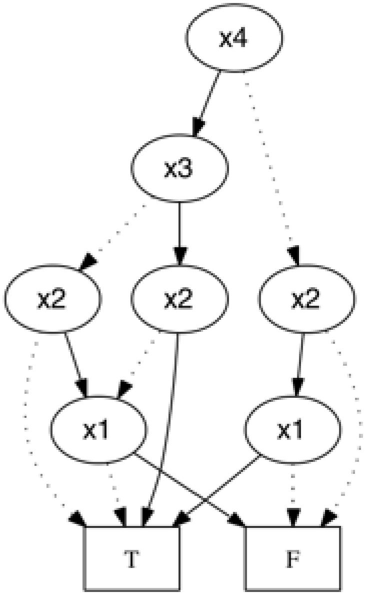
- Highly **flexible** modeling.
  - Costs and constraints need not be convex, linear, or even in closed form.
  - Exploits recursive structure.
- **Good relaxations** often **unavailable**.
  - Due to the very generality of the model.
- Focus on discrete, **deterministic** DP.
  - Extension to **stochastic** DP possible.

Why Decision Diagrams?

- A potentially useful **discrete relaxation**.
  - Obtained by node **splitting** or node **merger**.
  - We focus on node **merger**.
- Can provide relaxations where **none previously existed**.
  - As in job sequencing problems with **state-dependent processing times**..

Decision Diagrams

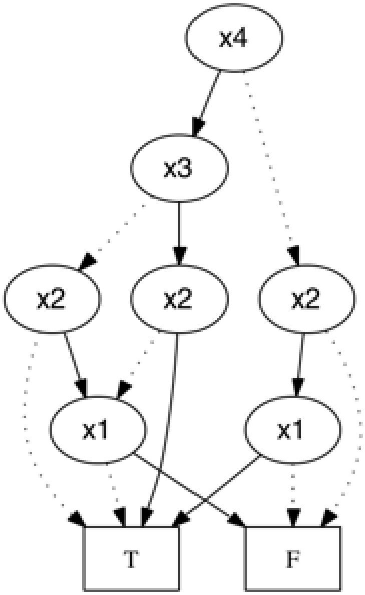
- Graphical encoding of a boolean function
  - Historically used for circuit design & verification
  - **Binary diagrams** easily extended to **multivalued diagrams**.
  - Unique **reduced** diagram for a given variable ordering.



Lee (1959), Bryant (1986)

Decision Diagrams

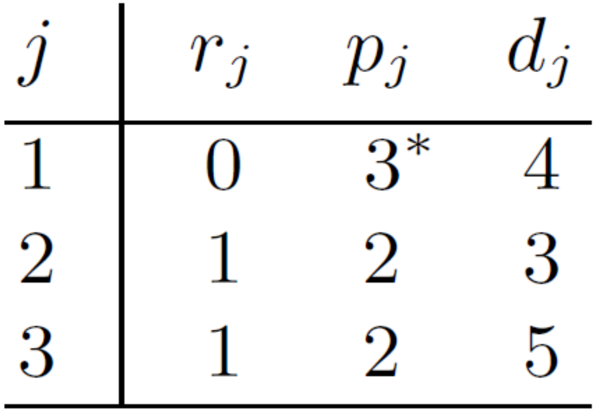
- Adapt to optimization and constraint programming
  - **Paths** from top to bottom (T) represent **feasible solutions**
  - Path **lengths** represent **costs**.
  - **Shortest** path is **optimal** solution.



Hadžić and JH (2006, 2007)

Job Sequencing

- Problem: sequence jobs with given processing times
  - Minimize **tardiness** subject to **time windows**
  - Processing time may depend on previous jobs
    - For example, some necessary components may have been made for previous jobs

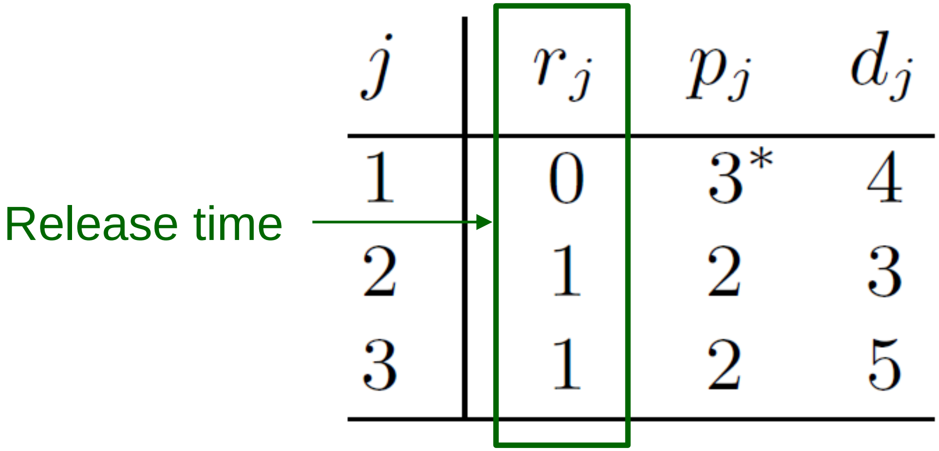


j	r_j	p_j	d_j
1	0	3^*	4
2	1	2	3
3	1	2	5

*2 when job 2 has previously been processed.

Job Sequencing

- Problem: sequence jobs with given processing times
  - Minimize **tardiness** subject to **time windows**
  - Processing time may depend on previous jobs
    - For example, some necessary components may have been made for previous jobs



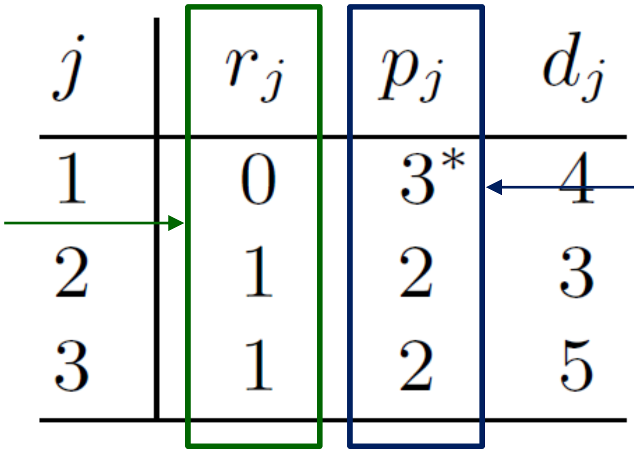
Release time

j	r_j	p_j	d_j
1	0	3^*	4
2	1	2	3
3	1	2	5

*2 when job 2 has previously been processed.

Job Sequencing

- Problem: sequence jobs with given processing times
  - Minimize **tardiness** subject to **time windows**
  - Processing time may depend on previous jobs
    - For example, some necessary components may have been made for previous jobs



j	r_j	p_j	d_j
1	0	3^*	4
2	1	2	3
3	1	2	5

Release time

Processing time

*2 when job 2 has previously been processed.

Job Sequencing

- Problem: sequence jobs with given processing times
  - Minimize **tardiness** subject to **time windows**
  - Processing time may depend on previous jobs
    - For example, some necessary components may have been made for previous jobs

j	r_j	p_j	d_j
1	0	3^*	4
2	1	2	3
3	1	2	5

Release time

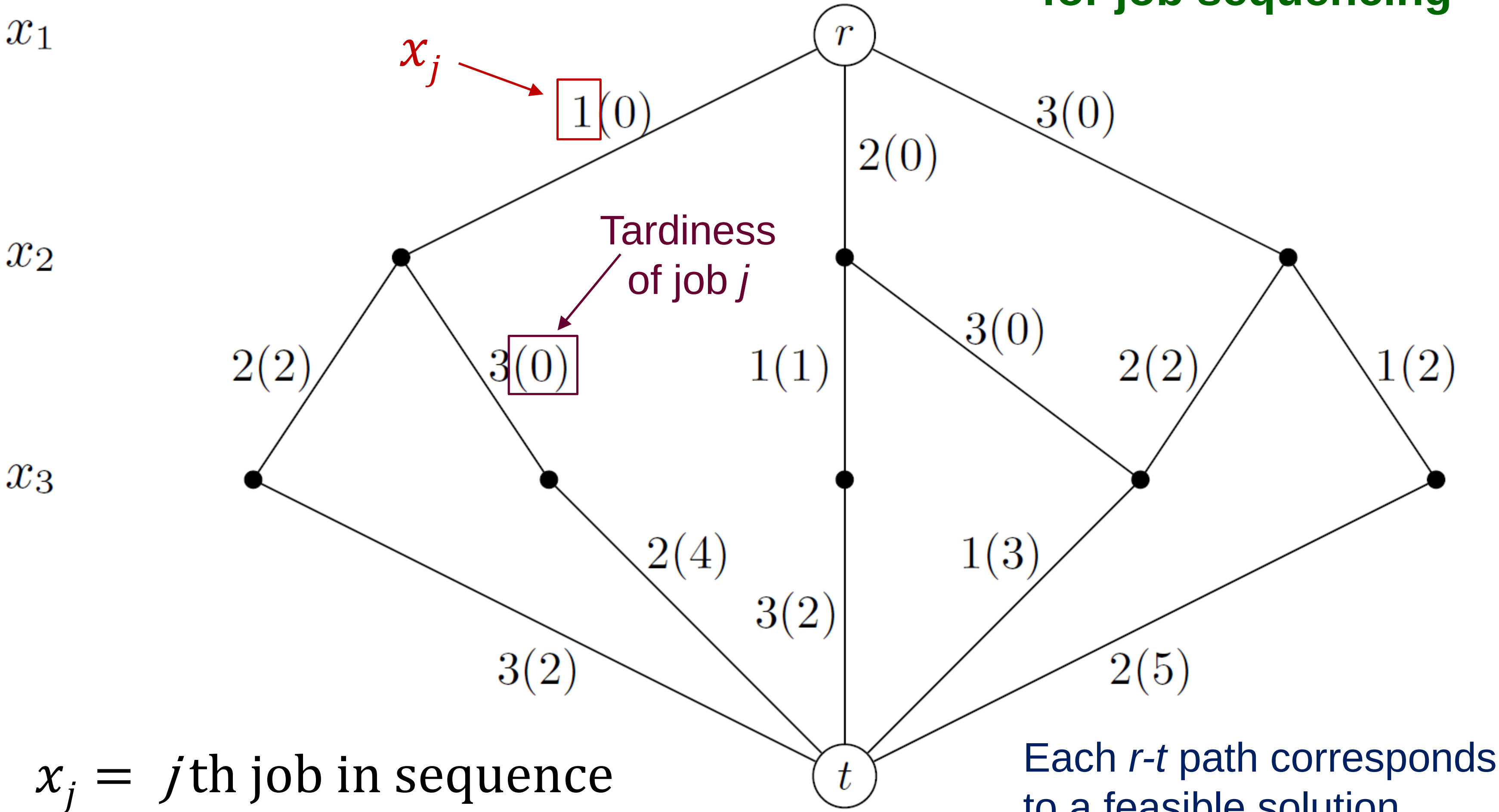
Processing time

Due date

*2 when job 2 has previously been processed.

Job Sequencing

Decision diagram
for job sequencing

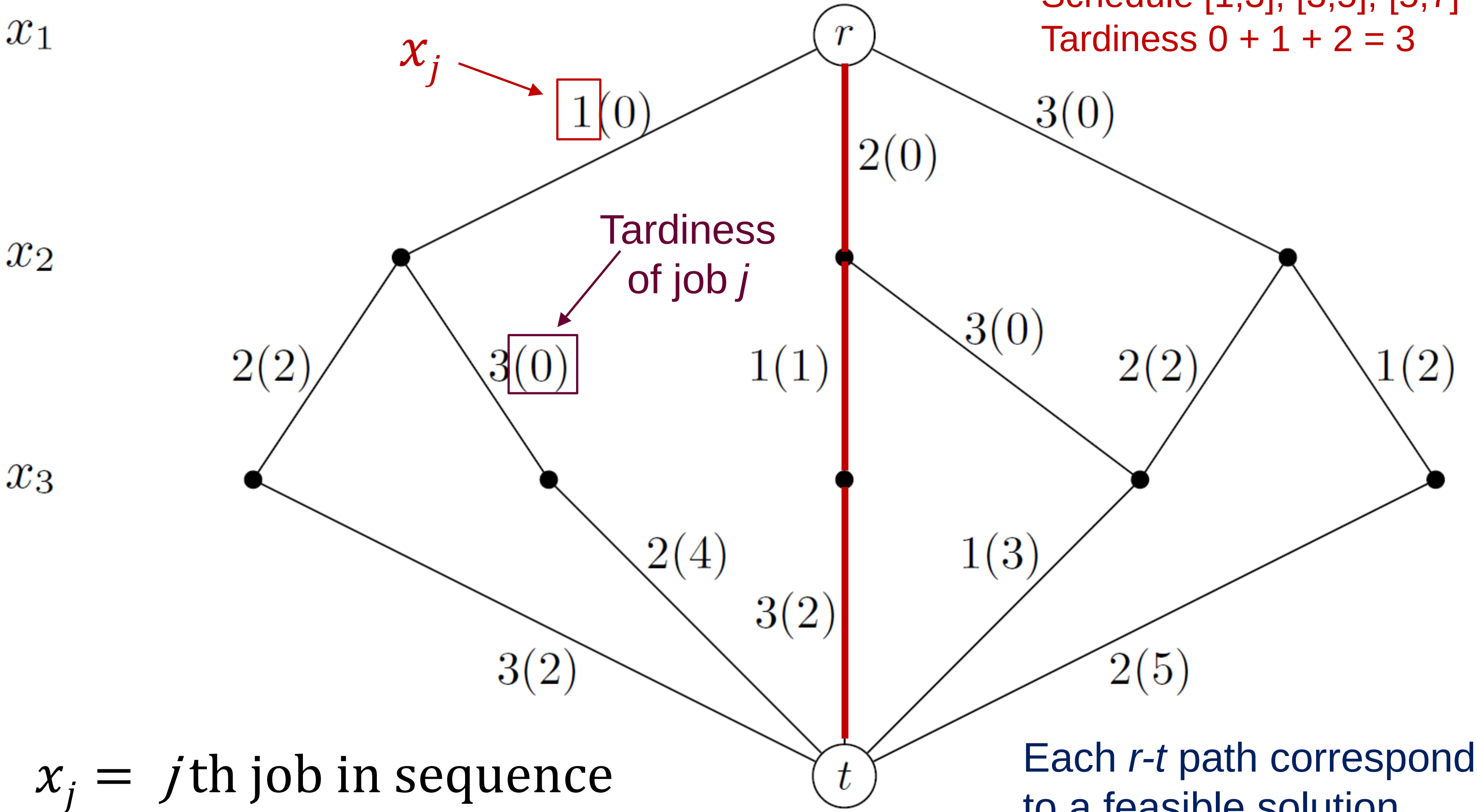


$x_j =$ jth job in sequence

Each *r*-*t* path corresponds to a feasible solution

Job Sequencing

An optimal solution:
Sequence 2-1-3
Schedule [1,3], [3,5], [5,7]
Tardiness 0 + 1 + 2 = 3



x_j

Tardiness of job *j*

$x_j =$ *j*th job in sequence

Each *r*-*t* path corresponds to a feasible solution

Building a Decision Diagram

- Our approach:
  - Associate dynamic programming **states** with nodes..
  - …as in a state transition graph.

Dynamic Programming Model

- Our approach:
  - Associate dynamic programming **states** with nodes..
  - …as in a state transition graph.

General recursive model

$$h_j(\boxed{S}) = \min_{x_j \in X_j(S)} \left\{ c_j(S, x_j) + h_{j+1}\Big((\phi_j(S, x_j))\Big) \right\}$$

$$\uparrow$$

State in stage *j*

Dynamic Programming Model

- Our approach:
  - Associate dynamic programming **states** with nodes..
  - …as in a state transition graph.

General recursive model

$$h_j(\boxed{S}) = \min_{x_j \in \boxed{X_j(S)}} \left\{ c_j(S, x_j) + h_{j+1}\Big((\phi_j(S, x_j))\Big) \right\}$$

State in stage *j*

Set of possible controls

Dynamic Programming Model

- Our approach:
  - Associate dynamic programming **states** with nodes..
  - …as in a state transition graph.

General recursive model

$$h_j(\boxed{S}) = \min_{x_j \in \boxed{X_j(S)}} \left\{ \boxed{c_j(S, x_j)} + h_{j+1}\Big((\phi_j(S, x_j)\Big) \right\}$$

State in stage *j*

Set of possible controls

Immediate cost

Dynamic Programming Model

- Our approach:
  - Associate dynamic programming **states** with nodes..
  - …as in a state transition graph.

General recursive model

State transition function

$$h_j(\boxed{S}) = \min_{x_j \in \boxed{X_j(S)}} \left\{ \boxed{c_j(S, x_j)} + h_{j+1}\left(\left(\boxed{\phi_j}(S, x_j)\right)\right) \right\}$$

State in stage *j*

Immediate cost

Set of possible controls

Dynamic Programming Model

- Our approach:
  - Associate dynamic programming **states** with nodes..
  - …as in a state transition graph.

General recursive model

State transition function

$$h_j(\boxed{S}) = \min_{x_j \in \boxed{X_j(S)}} \left\{ \boxed{c_j(S, x_j)} + \boxed{h_{j+1}((\boxed{\phi_j}(S, x_j))} \right\}$$

State in stage *j*

Set of possible controls

Immediate cost

Cost to go

DP Model for Job Sequencing

Set of jobs scheduled so far

Initial state = $(\emptyset, 0)$

State: $S = (\boxed{V,}\boxed{f})$

Finish time of last job scheduled

$$h_j(\boxed{S}) = \min_{x_j \in \boxed{X_j(S)}} \left\{ \boxed{c_j(S, x_j)} + \boxed{h_{j+1}((\phi_j(S, x_j)) \right\}}$$

State in stage *j*

Set of possible controls

Immediate cost

Cost to go

DP Model for Job Sequencing

Set of jobs scheduled so far

State: $S = (\boxed{V,}\boxed{f})$

Initial state = $(\emptyset, 0)$

Finish time of last job scheduled

Controls: $x_j(V, f) = \{1, \ldots, n\} \setminus V$

$$h_j(\boxed{S}) = \min_{x_j \in \boxed{X_j(S)}} \left\{ \boxed{c_j(S, x_j)} + \boxed{h_{j+1}((\phi_j(S, x_j))} \right\}$$

State in stage *j*

Set of possible controls

Immediate cost

Cost to go

DP Model for Job Sequencing

Set of jobs scheduled so far

State: $S = (\boxed{V,}\boxed{f})$

Initial state = $(\emptyset, 0)$

Finish time of last job scheduled

Controls: $x_j(V, f) = \{1, \ldots, n\} \setminus V$

Immediate cost: $c_j((V, f), x_j) = \left(\max\{r_{x_j}, f\} + \boxed{p_{x_j}(V)} - d_{x_j}\right)^+$

State-dependent processing time

$$h_j(\boxed{S}) = \min_{x_j \in \boxed{X_j(S)}} \left\{ \boxed{c_j(S, x_j)} + \boxed{h_{j+1}((\phi_j(S, x_j))} \right\}$$

State in stage *j*

Set of possible controls

Immediate cost

Cost to go

DP Model for Job Sequencing

Set of jobs scheduled so far

Initial state = $(\emptyset, 0)$

State: $S = (\boxed{V,}\boxed{f})$

Finish time of last job scheduled

Controls: $x_j(V, f) = \{1, \ldots, n\} \setminus V$

Immediate cost: $c_j\big((V, f), x_j\big) = \big(\max\{r_{x_j}, f\} + p_{x_j}(V) - d_{x_j}\big)^+$

Transition: $\phi_j\big((V, f), x_j\big) = \big(V \cup \{x_j\}, \max\{r_{x_j}, f\} + p_{x_j}(V)\big)$

$$h_j(\boxed{S}) = \min_{x_j \in \boxed{X_j(S)}} \Big\{\boxed{c_j(S, x_j)} + \boxed{h_{j+1}\big((\phi_j(S, x_j)\big)\Big\}}$$

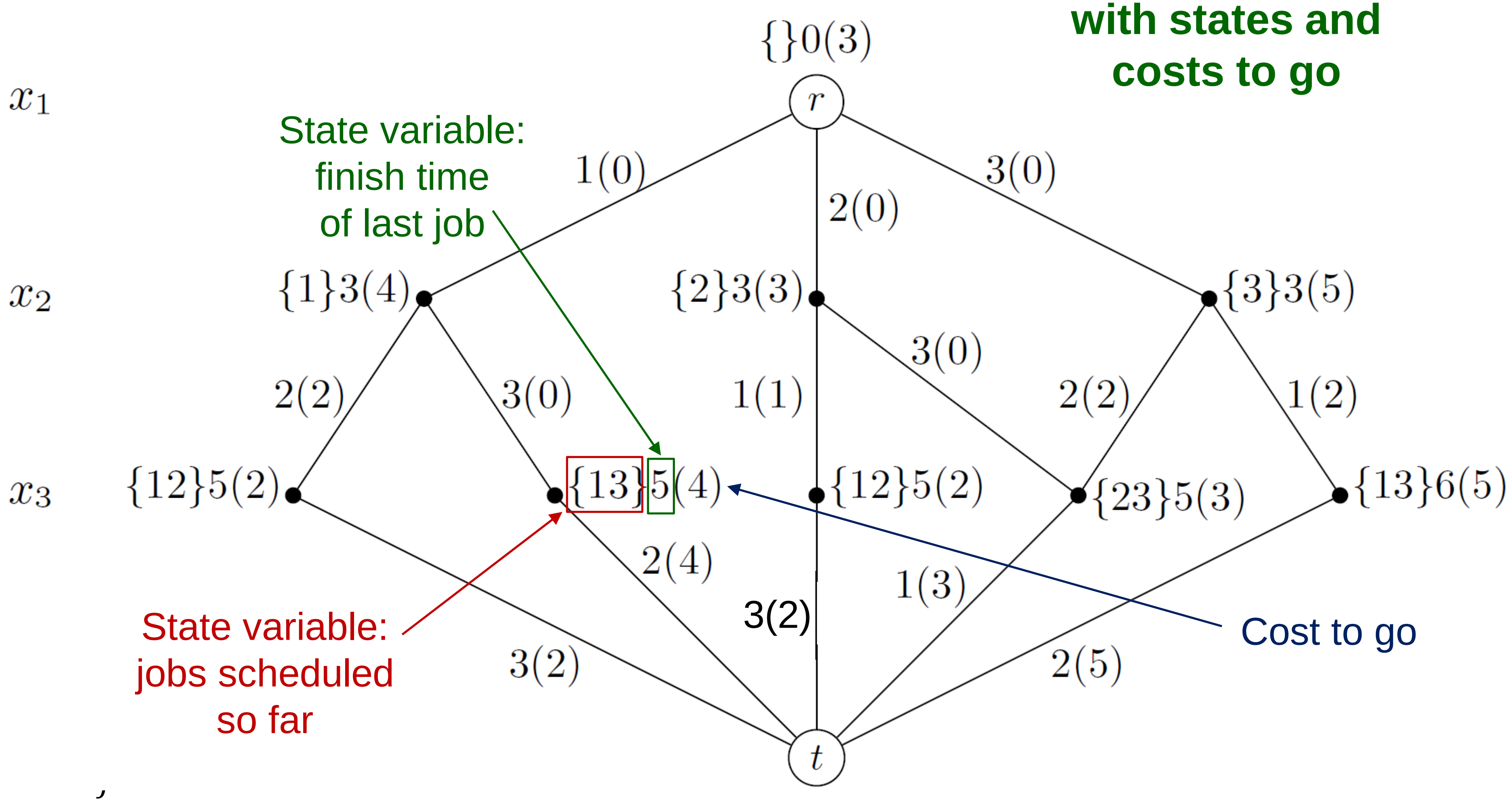
State in stage *j*

Set of possible controls

Immediate cost

Cost to go

Job Sequencing Diagram



Decision diagram with states and costs to go

State variable: finish time of last job

State variable: jobs scheduled so far

Cost to go

Relaxed Decision Diagram

- Definition
  - Every *r*-*t* path of the original diagram appears in the relaxed diagram with equal or smaller cost.
  - So a relaxed diagram **may represent some infeasible solutions**.
- Motivation
  - **Shortest path** in the relaxed diagram provides a **lower bound** on the optimal value.

Andersen, Hadžić, JH, Tiedemanmn (2007)

Building a Relaxed Diagram

- Node splitting
  - Start with a diagram that represents **all** solutions (feasible and infeasible) and **refine** it.

Andersen, Hadžić, JH, Tiedemanmn (2007)

Ciré and van Hoeve (2013)

Building a Relaxed Diagram

- Node splitting
  - Start with a diagram that represents **all** solutions (feasible and infeasible) and **refine** it.
- Node merger – examined here
  - **Merge** some nodes in the **exact** diagram.
  - …to make the diagram **smaller** while excluding no feasible solutions and introducing some infeasible low-cost solutions.

Andersen, Hadžić, JH, Tiedemanmn (2007)

Ciré and van Hoeve (2013)

Bergman, Ciré, van Hoeve, JH (2013)

Node Merger

- Don’t begin with exact diagram
  - It is too large
- Merge nodes as the diagram is constructed
  - In particular, nodes that are not likely to lie on an optimal path.
  - Combine states of the merged nodes in a way that yields a valid relaxation.
  - This often requires **additional state variables**.
  - As in the job sequencing case.

Bergman, Ciré, van Hoeve, JH (2013, 2016)

Relaxed DP Model

Set of jobs scheduled in **all** feasible solutions so far

$$S = (\boxed{V,}\boxed{U,}\boxed{f)}$$

Initial state = $(\emptyset, \emptyset, 0)$

New state variable: set of jobs scheduled in **some** feasible solution so far

Earliest possible finish time of immediately previous job

$$h_j(S) = \min_{x_j \in X_j(S)} \left\{ c_j(S, x_j) + h_{j+1}\big((\phi_j(S, x_j))\big) \right\}$$

Relaxed DP Model

Set of jobs scheduled in **all** feasible solutions so far

$$S = (\boxed{V,}\boxed{U,}\boxed{f)}$$

Initial state = $(\emptyset, \emptyset, 0)$

New state variable: set of jobs scheduled in **some** feasible solution so far

Earliest possible finish time of immediately previous job

Transition:

$$\phi_j\big((V,U,f),x_j\big) = \big(V \cup \{x_j\}, U \cup \{x_j\}, \max\{r_{x_j}, f\} + \boxed{p_{x_j}(U)}\big)$$

Processing time depends on *U*, not *V*
(state variable *V* can be dropped if desired)

$$h_j(S) = \min_{x_j \in X_j(S)} \Big\{ c_j(S, x_j) + h_{j+1}\big((\phi_j(S, x_j))\big) \Big\}$$

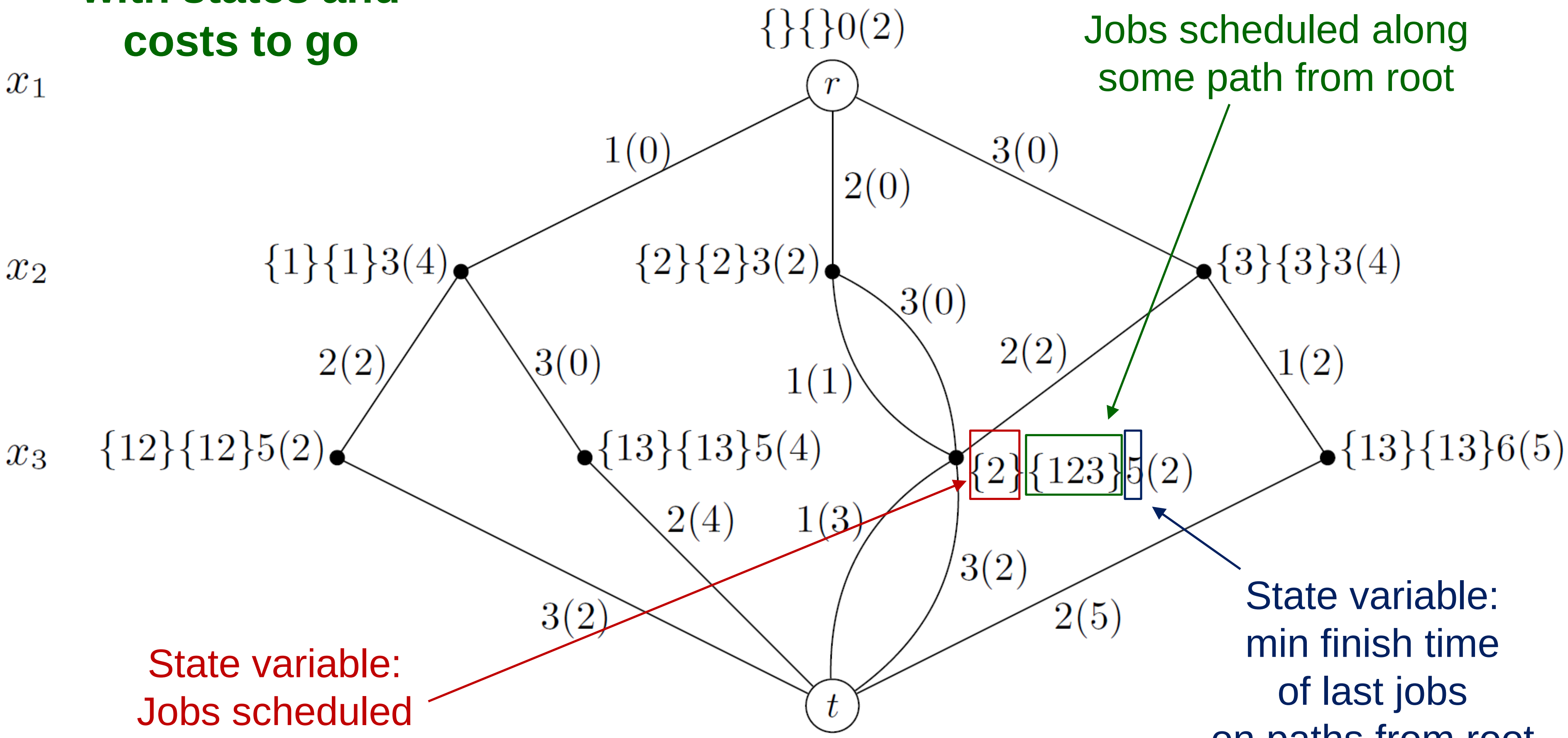
Node Merger in Relaxation

- Merge nodes as the diagram is constructed
  - States *S*, *T* merge to form state $S \oplus T$
- Merger operation must yield valid relaxation
  - In state-dependent job sequencing,

$$(V, U, f) \oplus (V', U', f') = \big(V \cap V', U \cup U', \min\{f, f'\}\big)$$

Job Sequencing Relaxed Diagram

Relaxed decision diagram with states and costs to go



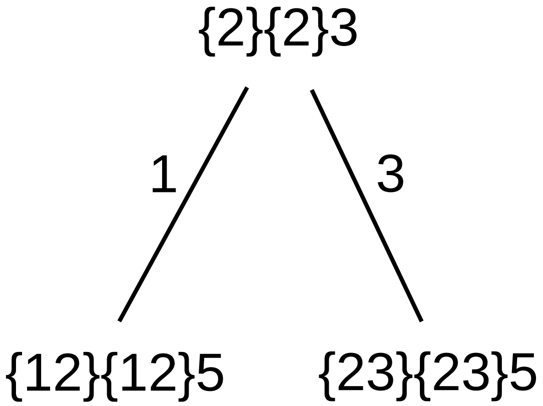
State variable: Jobs scheduled along some path from root

State variable: min finish time of last jobs on paths from root

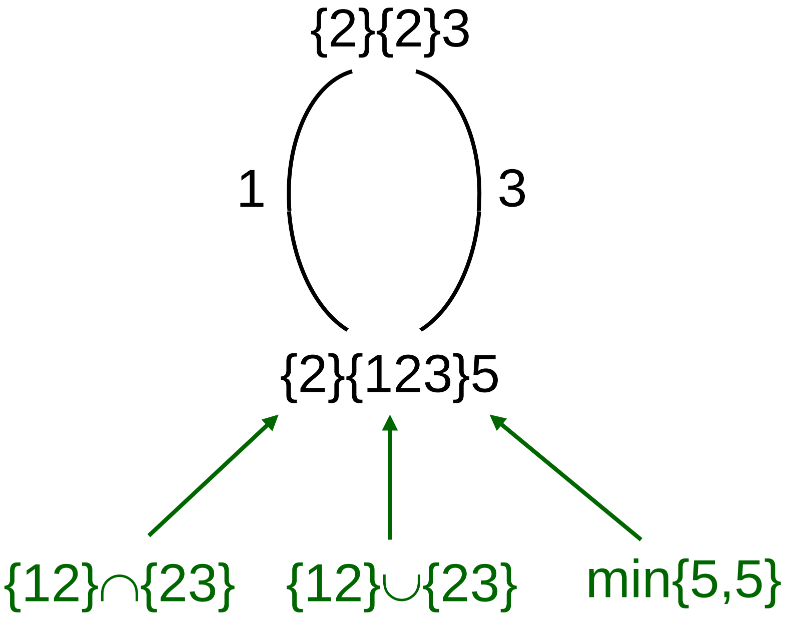
State variable: Jobs scheduled along all paths from root

Job Sequencing Node Merger

Without merger



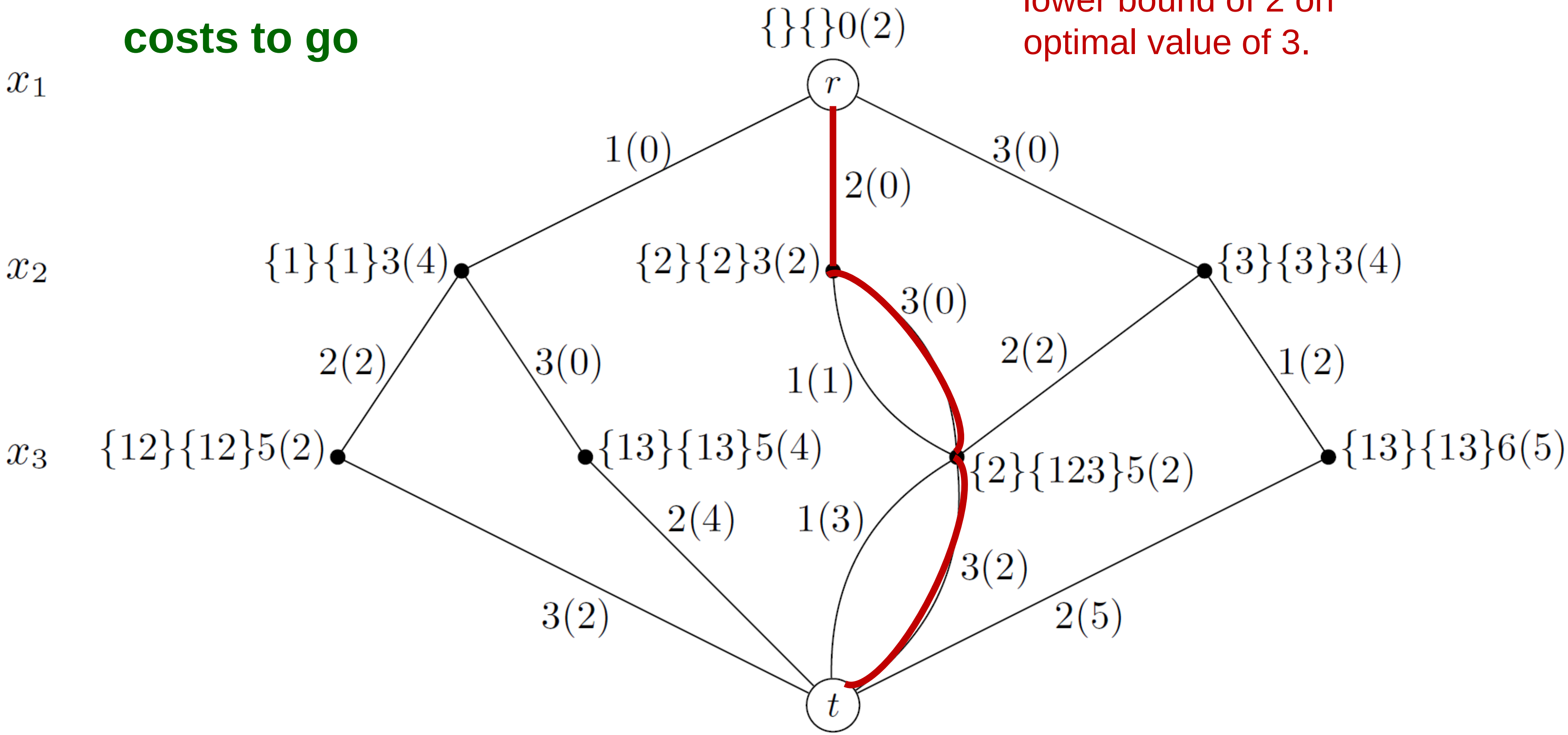
With merger



Job Sequencing Relaxed Diagram

Relaxed decision diagram with states and costs to go

Shortest path yields a lower bound of 2 on optimal value of 3.



Conditions for a Valid Relaxation

- There are two **jointly sufficient conditions** for obtaining a relaxed diagram from node merger.
  - A condition on the **transition function**
  - And a condition on the **merger operation** $S \oplus T$

Conditions for a Valid Relaxation

- There are two **jointly sufficient conditions** for obtaining a relaxed diagram from node merger.
  - A condition on the **transition function**
  - And a condition on the **merger operation** $S \oplus T$
- First we need a definition: state *S*′ **relaxes** state *S* in the same stage if
  - Every control feasible in *S* is feasible in *S*′

$$X_j(S) \subseteq X_j(S')$$

  - The immediate cost of a control feasible in *S* is no greater in *S*′.

$$c_j(S, x_j) \geq c_j(S', x_j), \quad \text{all } x_j \in X_j(S)$$

Conditions for a Valid Relaxation

Theorem. The merger of states *S* and *T* in layer *j* of diagram *D* yields a relaxation of *D* if:

- S′ relaxes S implies that $\phi_j(S', x_j)$ relaxes $\phi_j(S, x_j)$ for any control x_j feasible in *S*.

- $S \oplus T$ relaxes both *S* and *T*.

Proof by induction.

This generalizes to stochastic decision diagrams, where the conditions are much more complicated.

Conditions for a Valid Relaxation

Check whether node merger for **job sequencing** satisfies the conditions.

- S′ relaxes S implies that $\phi_j(S', x_j)$ relaxes $\phi_j(S, x_j)$ for any control x_j feasible in *S*.

- $S \oplus T$ relaxes both *S* and *T*.

Conditions for a Valid Relaxation

Check whether node merger for **job sequencing** satisfies the conditions.

- S′ relaxes S implies that $\phi_j(S', x_j)$ relaxes $\phi_j(S, x_j)$ for any control x_j feasible in S.

(V', U', f') relaxes (V, U, f) when $$\begin{array}{l} V' \subseteq V \\ U' \supseteq U \\ f' \leq f \end{array}$$ Now…

$\phi_j\big((V', U', f'), x_j\big)$ relaxes $\phi_j\big((V, U, f), x_j\big)$ because

$$\begin{array}{l} V' \cup \{x_j\} \subseteq V \cup \{x_j\} \\ U' \cup \{x_j\} \supseteq U \cup \{x_j\} \\ \min\{r_{x_j}, f'\} + p_{x_j}(U') \leq \min\{r_{x_j}, f\} + p_{x_j}(U) \end{array}$$

Conditions for a Valid Relaxation

Check whether node merger for **job sequencing** satisfies the conditions.

- $S \oplus T$ relaxes both *S* and *T*.

$$(V, U, f) \oplus (V', U', f) \text{ relaxes } (V, U, f)$$

because
$$\begin{aligned} &V \cap V' \subseteq V \\ &U \cup U' \supseteq U \\ &\min\{f, f'\} \leq f \end{aligned}$$

…and similarly for (V', U', f)

Merger Heuristics

- Goal:
  - Merge nodes that are **not likely to lie on short paths**.
  - This reduces the likelihood of creating a superoptimal path, which would weaken the bound.
  - Keep merging nodes until desired width is obtained.

Merger Heuristics

- Goal:
  - Merge nodes that are **not likely to lie on short paths**.
  - This reduces the likelihood of creating a superoptimal path, which would weaken the bound.
  - Keep merging nodes until desired width is obtained.
- Underlying idea
  - Preserve accuracy in the region of the diagram that is likely to contain the best solutions.
  - Analogous to using smaller finite elements in models of the atmosphere in regions with more activity.

Merger Heuristics

- Finish time heuristic
  - Merge nodes whose **last finish time states are large**.
  - Paths through these nodes are likely to be longer.
  - Information readily available in the state description.

Merger Heuristics

- Finish time heuristic
  - Merge nodes whose **last finish time states are large**.
  - Paths through these nodes are likely to be longer.
  - Information readily available in the state description.
- Shortest path heuristic
  - Merge nodes whose **shortest-path distances from root are large.**
  - This requires recursively computing shortest paths to root as we go along

Merger Heuristics

- Finish time heuristic
  - Merge nodes whose **last finish time states are large**.
  - Paths through these nodes are likely to be longer.
  - Information readily available in the state description.
- Shortest path heuristic
  - Merge nodes whose **shortest-path distances from root are large.**
  - This requires recursively computing shortest paths to root as we go along

– Random heuristic

  - Randomly choose nodes for merger..

Experimental Design

- Question 1
  - What is the relationship between the width of the relaxed diagram and the quality of the bound?
- Question 2
  - Which node merger heuristic is best?

Experimental Design

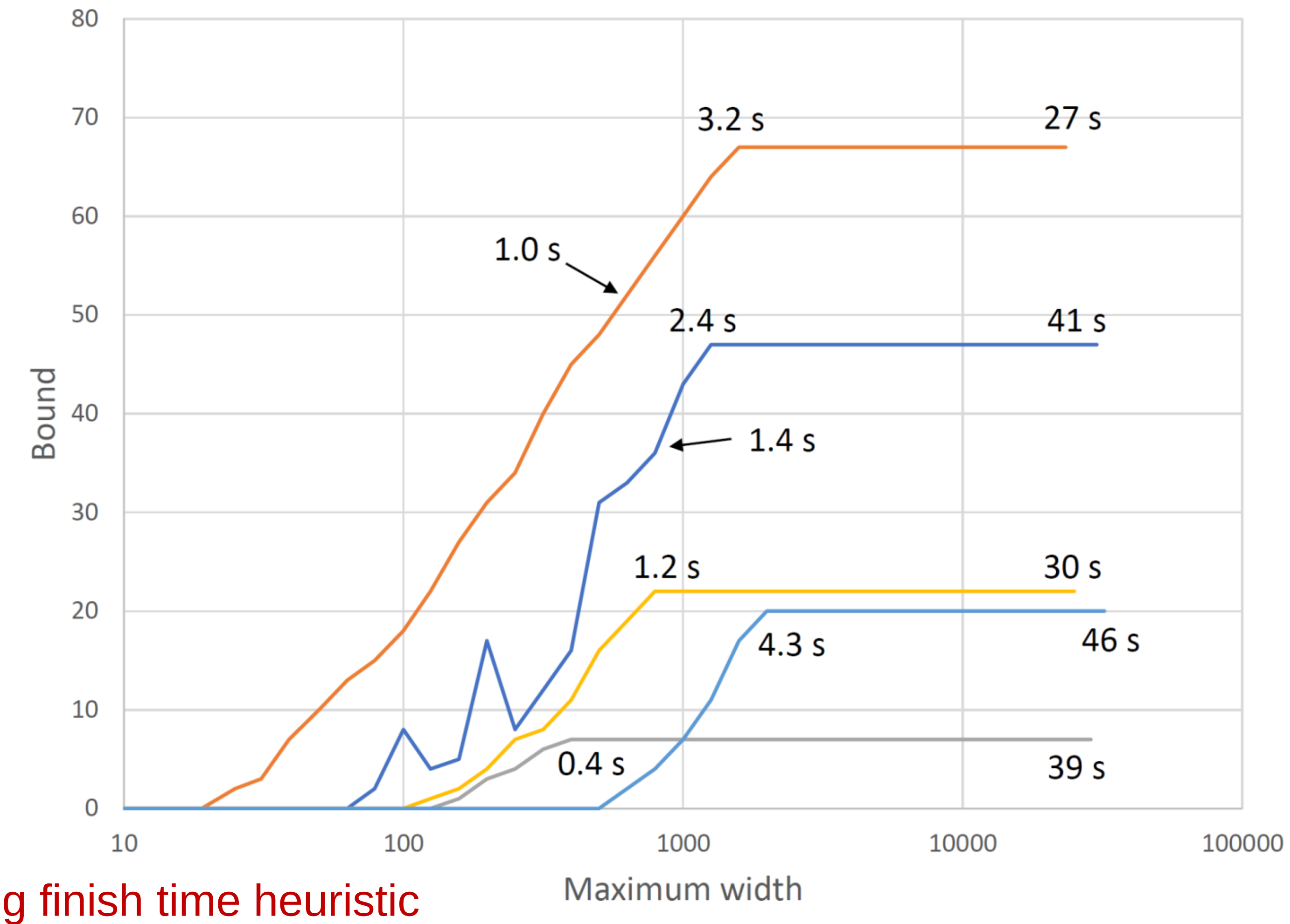
- Problem instances
  - State-dependent processing times.
  - In particular, processing times depend on state variable *U*
- Benchmark instances?
  - Apparently, none exist for **state-dependent** processing times.
- Random instances?
  - Must generate random instances.

Experimental Design

- Optimal solutions
  - Must have optimal value to assess quality of bound.
  - Because instances frequently have min tardiness = zero.
- How to obtain optimal solutions?
  - Dynamic programming is the only practical method for **state-dependent** processing times.
  - Due to state-space explosion, instances with more than 15 jobs are very hard to solve.
  - Solution with DP is equivalent to generating an **exact** diagram.
  - Solve instances with 12 and 14 jobs.

Computational Results

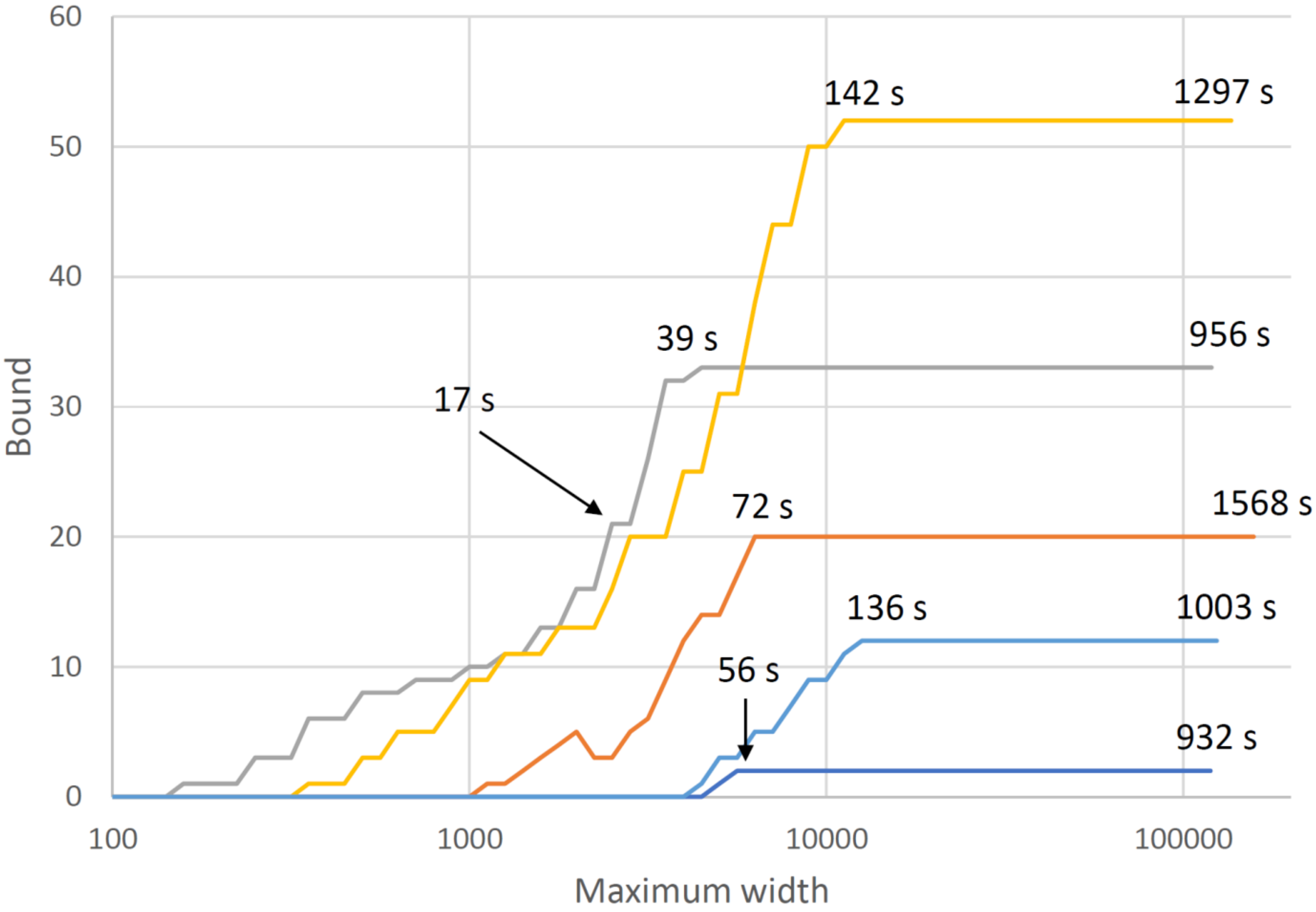
12 jobs



Using finish time heuristic

Computational Results

14 jobs



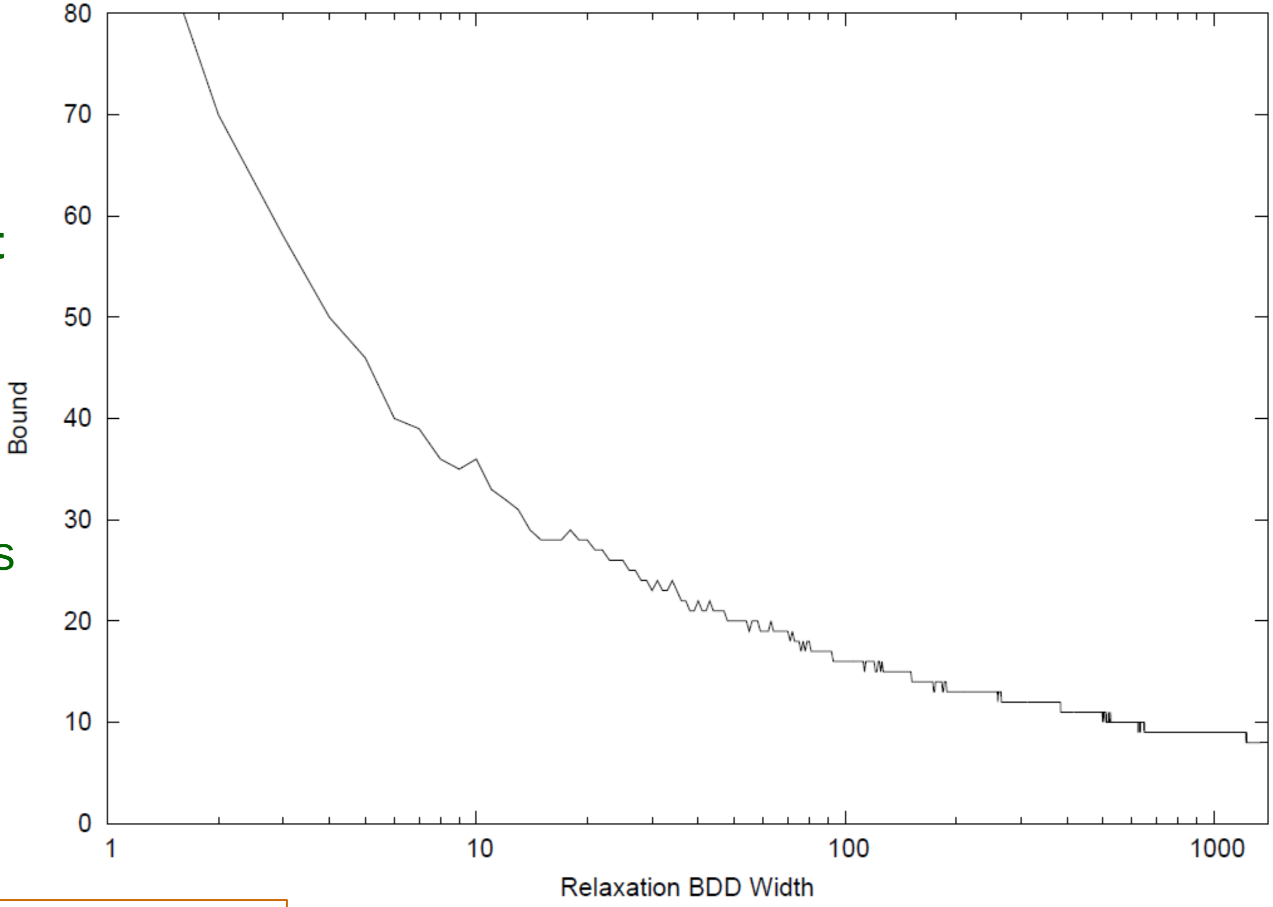
Using finish time heuristic

Computational Results

- Consistent results
  - …across instances. May extend to larger instances.
- Early convergence to optimality
  - For 12 jobs, optimal value achieved when relaxed diagram is 1/32 to 1/15 width of exact diagram.
  - For 14 jobs, optimal value achieved when relaxed diagram is 1/26 to 1/10 width of exact diagram.
- …rather than asymptotic improvement
  - As in studies on other types of problems.

Computational Results

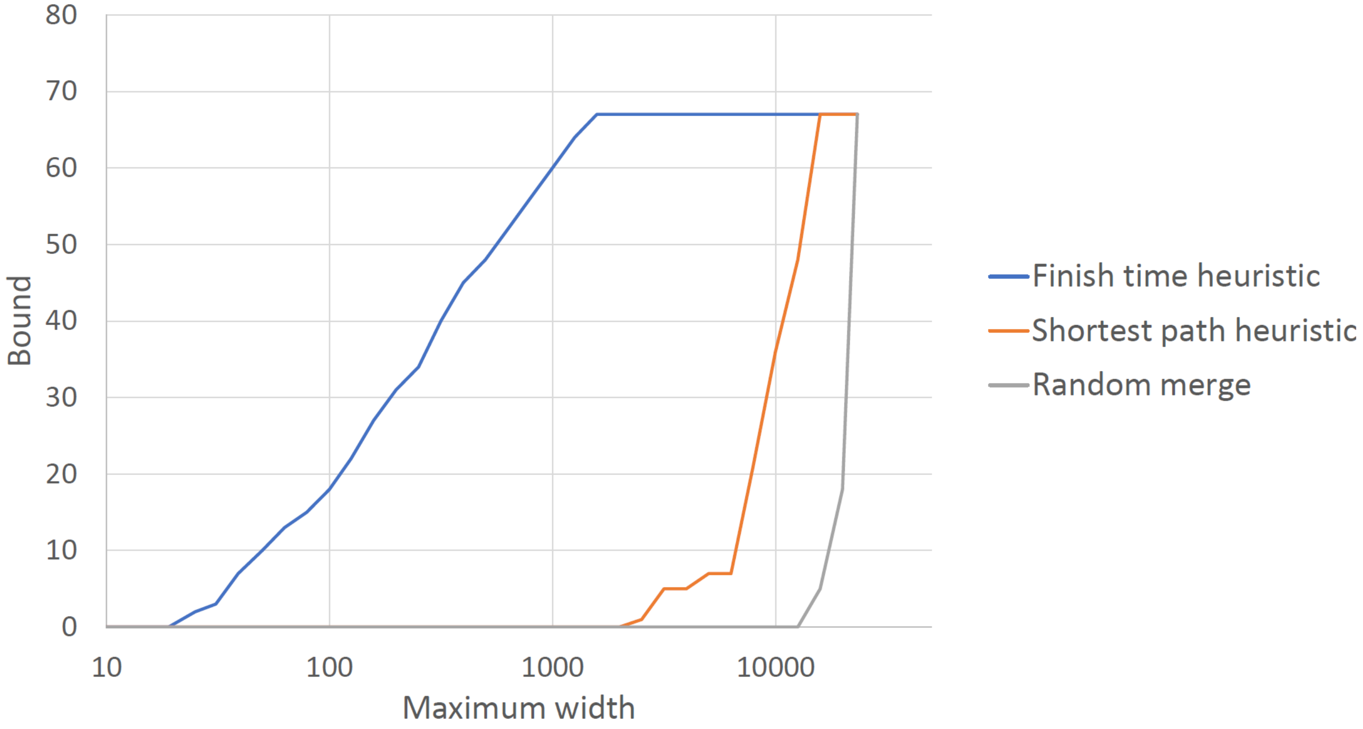
- Asymptotic convergence to optimality
  - Here, **max independent set** problem.
  - We want an **upper** bound
  - Typical of **other** types of problems



Bergman, Ciré, van Hoeve, JH (2013)

Computational Results

Comparison of node merger heuristics on a 12-job instance



Computational Results

- **Finish time heuristic** is vastly superior to others.
  - **Shortest path heuristic** fails because shortest distance to root is usually zero in the upper part of the diagram.
  - This means the heuristic provides no guidance until it is too late.
  - **Random heuristic** provides useless bound.
  - This confirms the importance of a good heuristic.
  - …and potential for further improvement.

Future Work

- Problem: diagrams of a **fixed size** lose their ability to generate bounds as instances scale up.
  - Bound does not rise above zero until relaxed diagram width is 1/1000 to 1/25 that of exact diagram

Future Work

- Problem: diagrams of a **fixed size** lose their ability to generate bounds as instances scale up.
  - Bound does not rise above zero until relaxed diagram width is 1/1000 to 1/25 that of exact diagram
- This suggests a combination with other bounding techniques
  - …that can yield a nonzero bound in smaller relaxed diagrams.
  - Such as **Lagrangean relaxation** obtained by modifying costs in the diagram..

Bergman, Ciré, van Hoeve (2015)

Future Work

- Bounds for **stochastic dynamic programming**
  - From **stochastic diagrams**.
  - Node merger can again provide a valid relaxation.
  - A theoretical result is available.
  - Awaiting good merger heuristics and computational tests.