Decision Diagrams and Dynamic Programming

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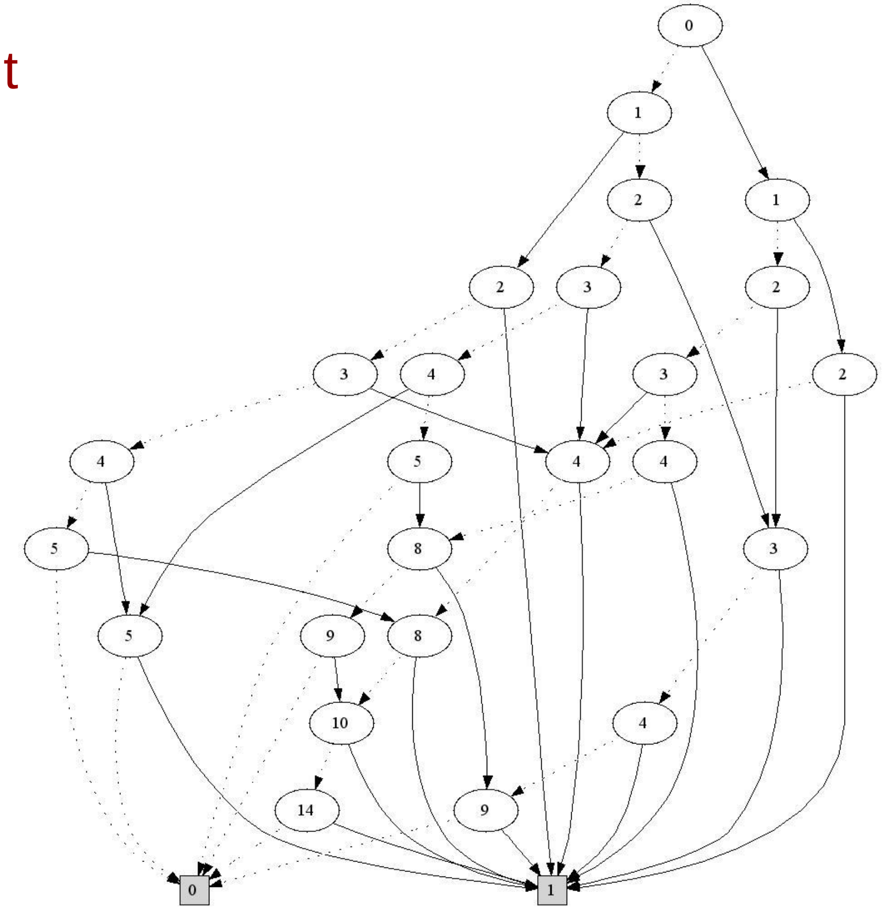
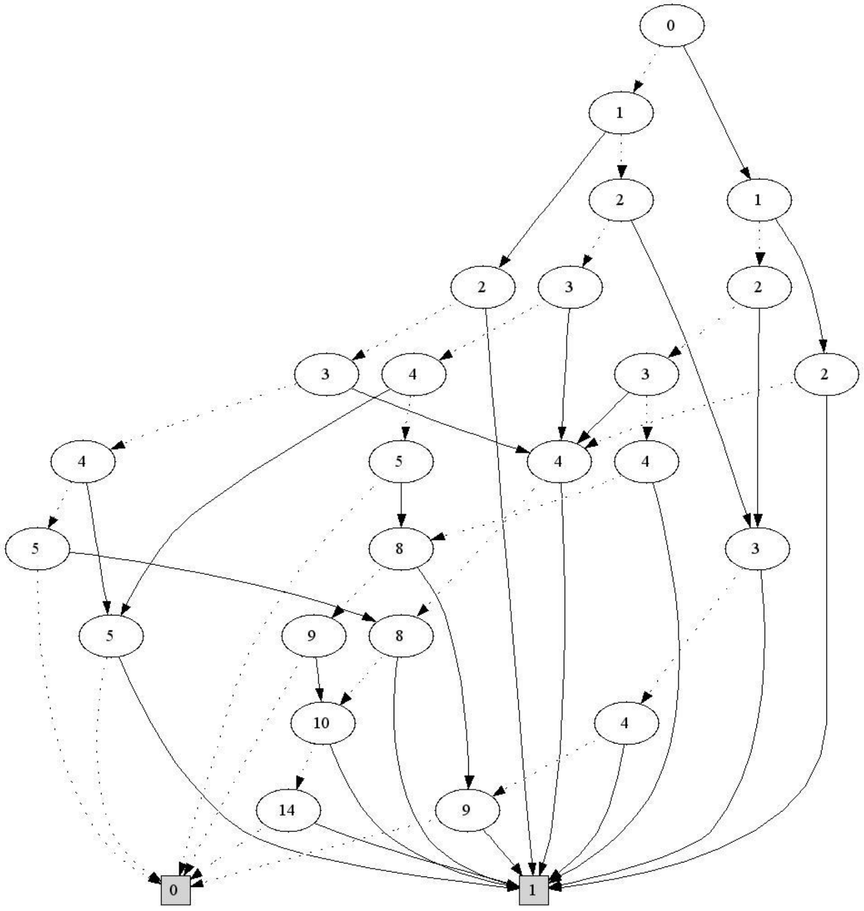
Decision Diagrams & Dynamic Programming

- Binary/multivalued **decision diagrams** are related to **dynamic programming**.
  - But there are important differences.



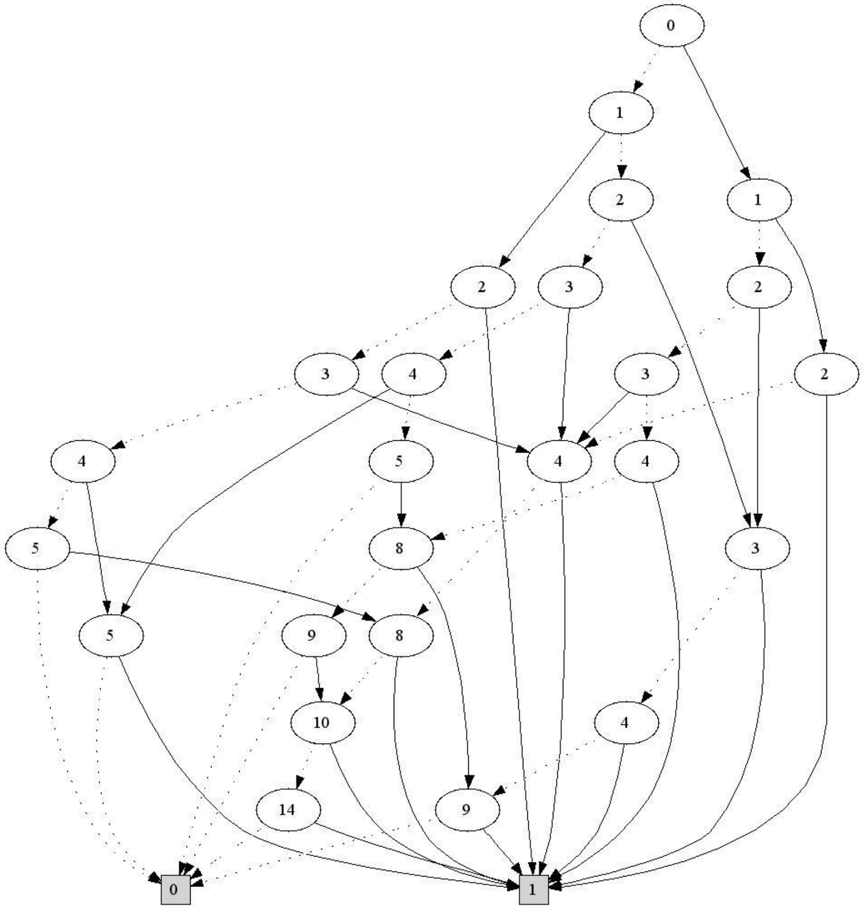
Decision Diagrams & Dynamic Programming

- Binary/multivalued **decision diagrams** are related to **dynamic programming**.
  - But there are important differences.
  - Dynamic programming has **state variables** and **state-dependent costs**.



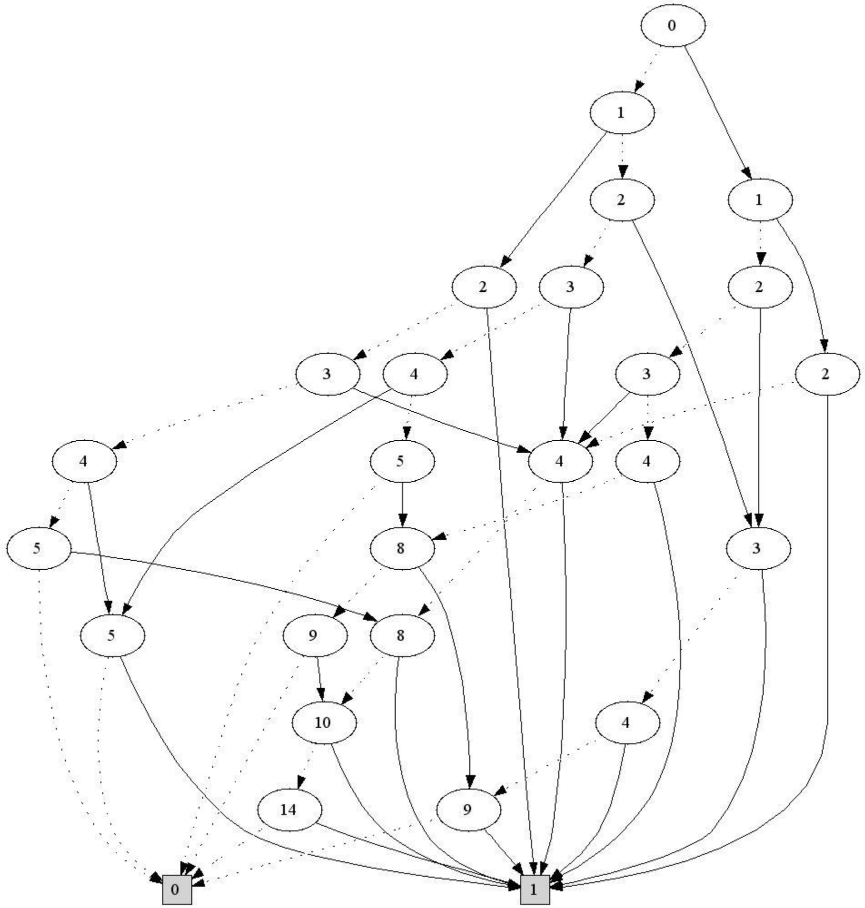
Decision Diagrams & Dynamic Programming

- We **extend** the theory of decision diagrams to accommodate state-dependent-costs.
  - We prove **uniqueness** theorem for weighted DDs using canonical costs.



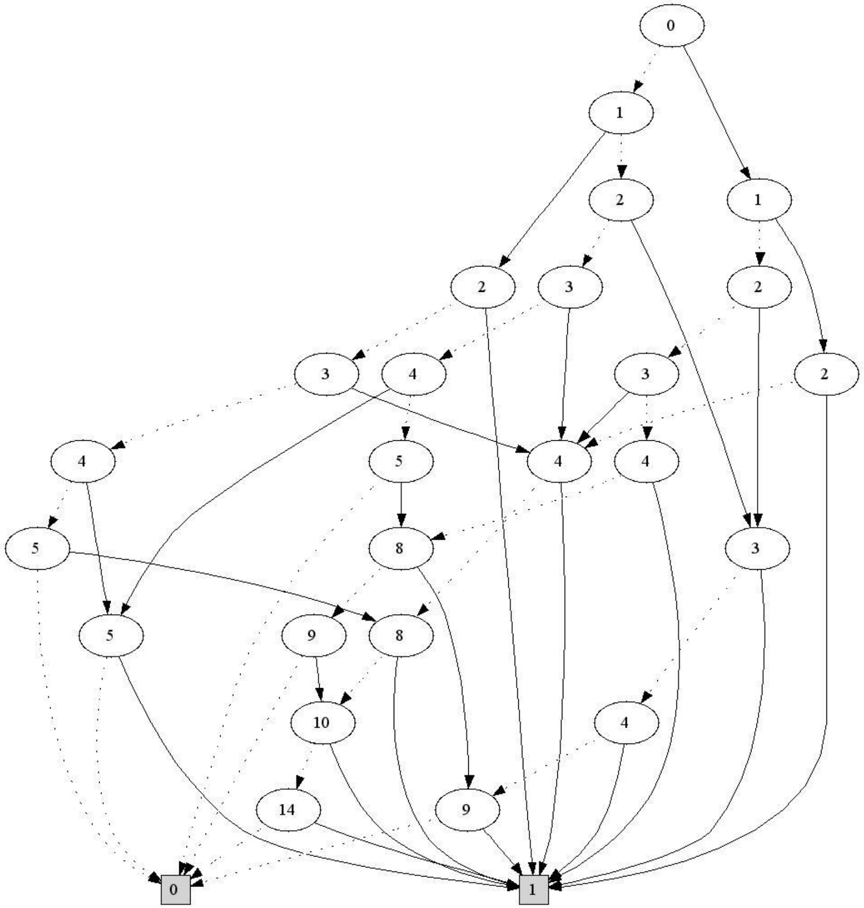
Decision Diagrams & Dynamic Programming

- We **extend** the theory of decision diagrams to accommodate state-dependent-costs.
  - We prove **uniqueness** theorem for weighted DDs using canonical costs.
- We can now view DP state transition graph as a decision diagram.
  - And perhaps **reduce** the decision diagram to simplify the DP model.



Outline

- Dynamic programming example
- Weighted decision diagrams and canonical costs
- Application to the example
- Ongoing research



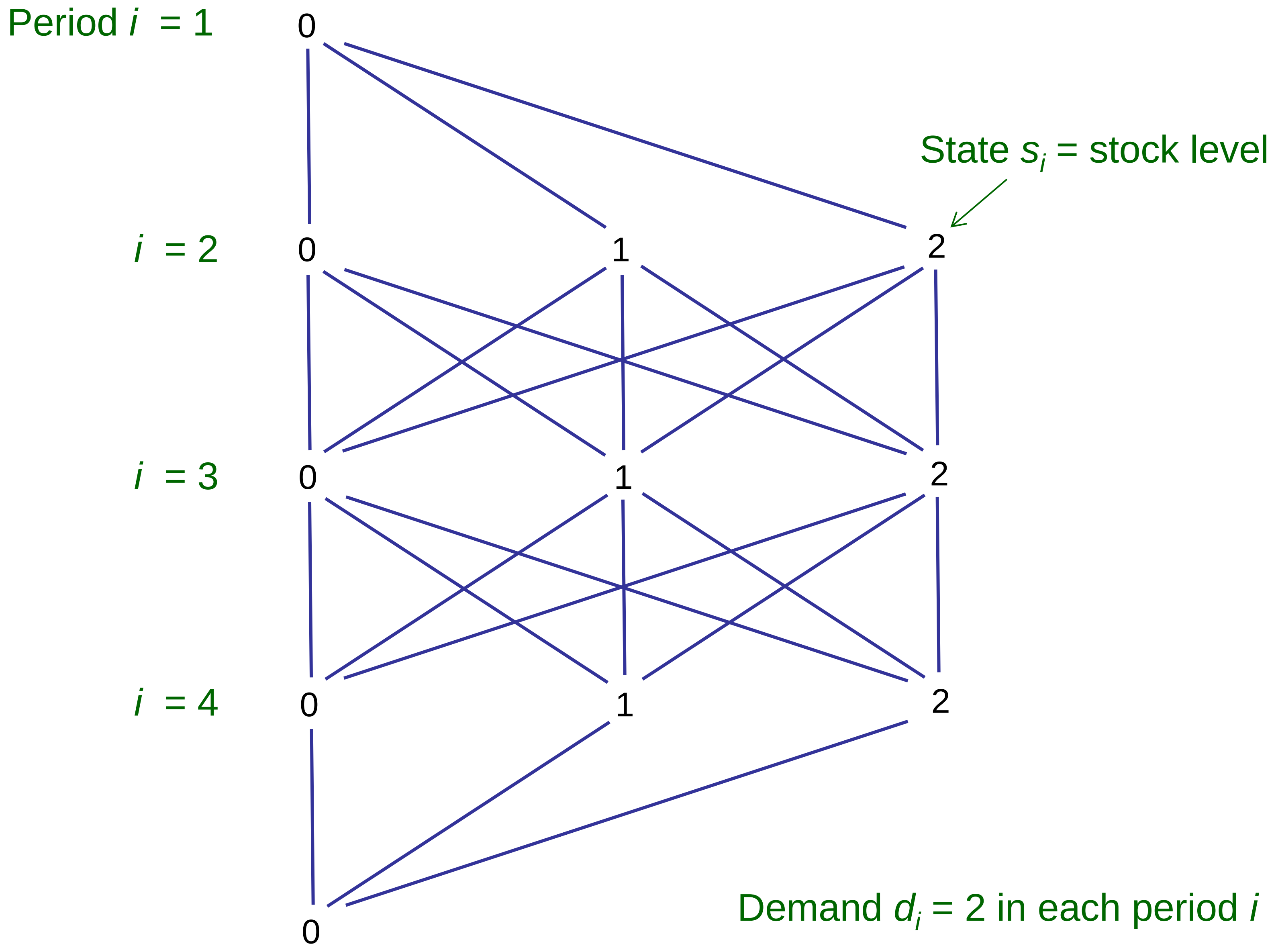
Dynamic Programming

- **Dynamic programming** (including the name) was introduced by Richard Bellman in 1950s.
  - Different concept than decision diagram, caching, etc.
  - But DP state transition graph can be viewed as a weighted decision diagram.
- Illustration: a very basic inventory management problem.
  - In the literature at least 50 years.

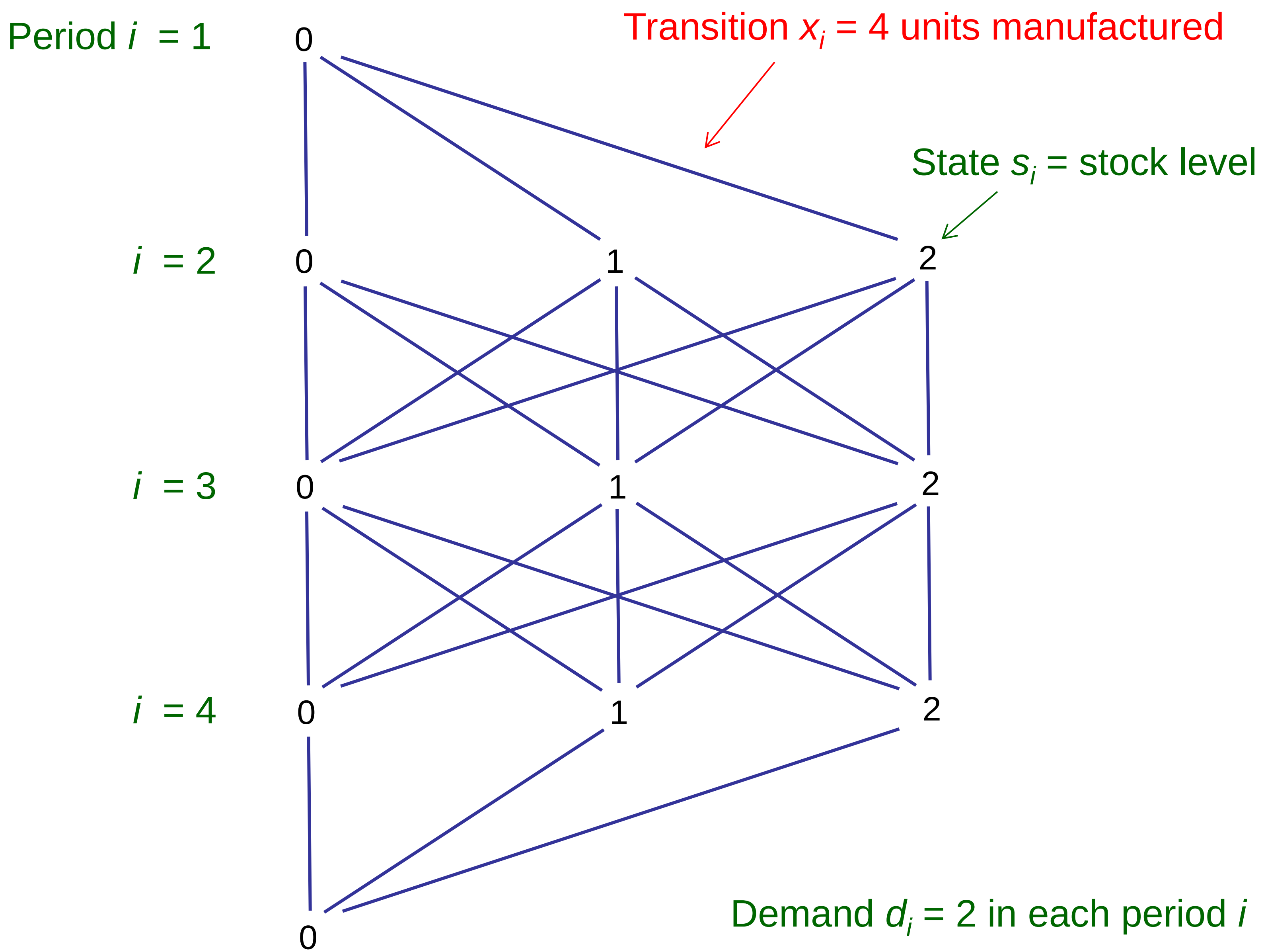
Inventory Management Example

- In each period i, we have:
  - Demand d_i
  - Unit production cost c_i
  - Warehouse space m
  - Unit holding cost h_i
- In each period, we decide:
  - Production level x_i
  - Stock level s_i
- Objective:
  - Meet demand each period while minimizing production and holding costs.

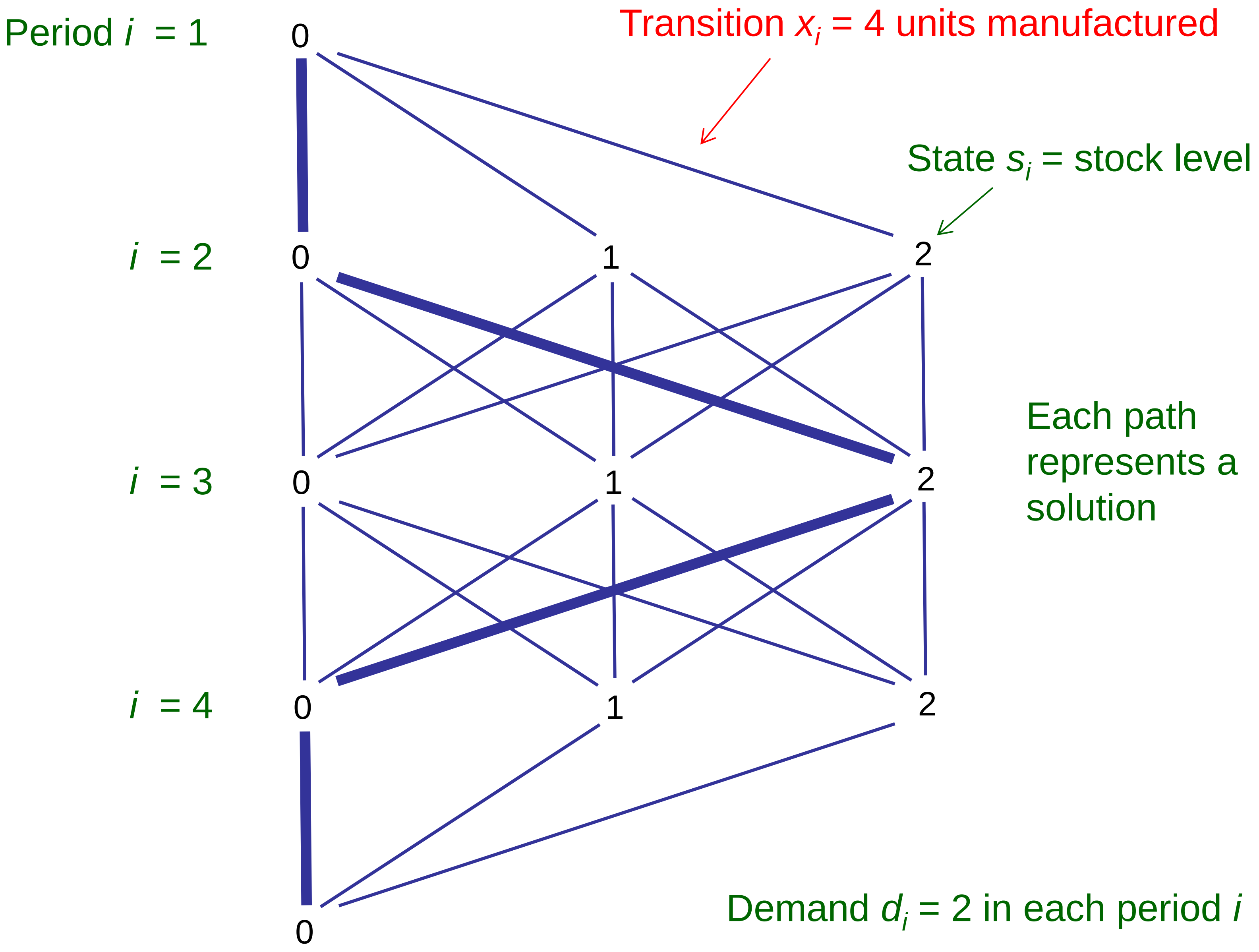
State transition graph



State transition graph



State transition graph



Transition costs

$c_1 = 2$

$h_2 = 2$

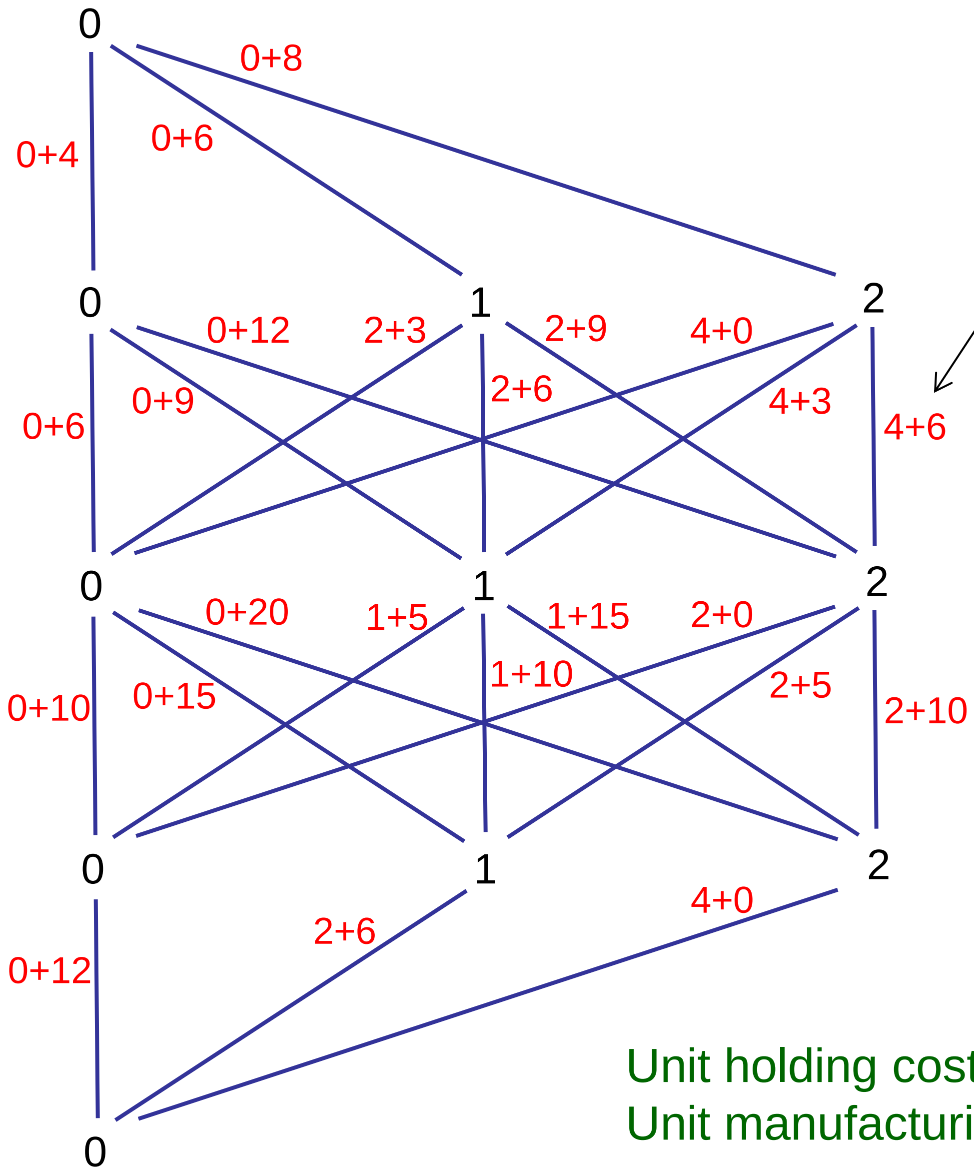
$c_2 = 3$

$h_3 = 1$

$c_3 = 5$

$h_4 = 2$

$c_4 = 6$



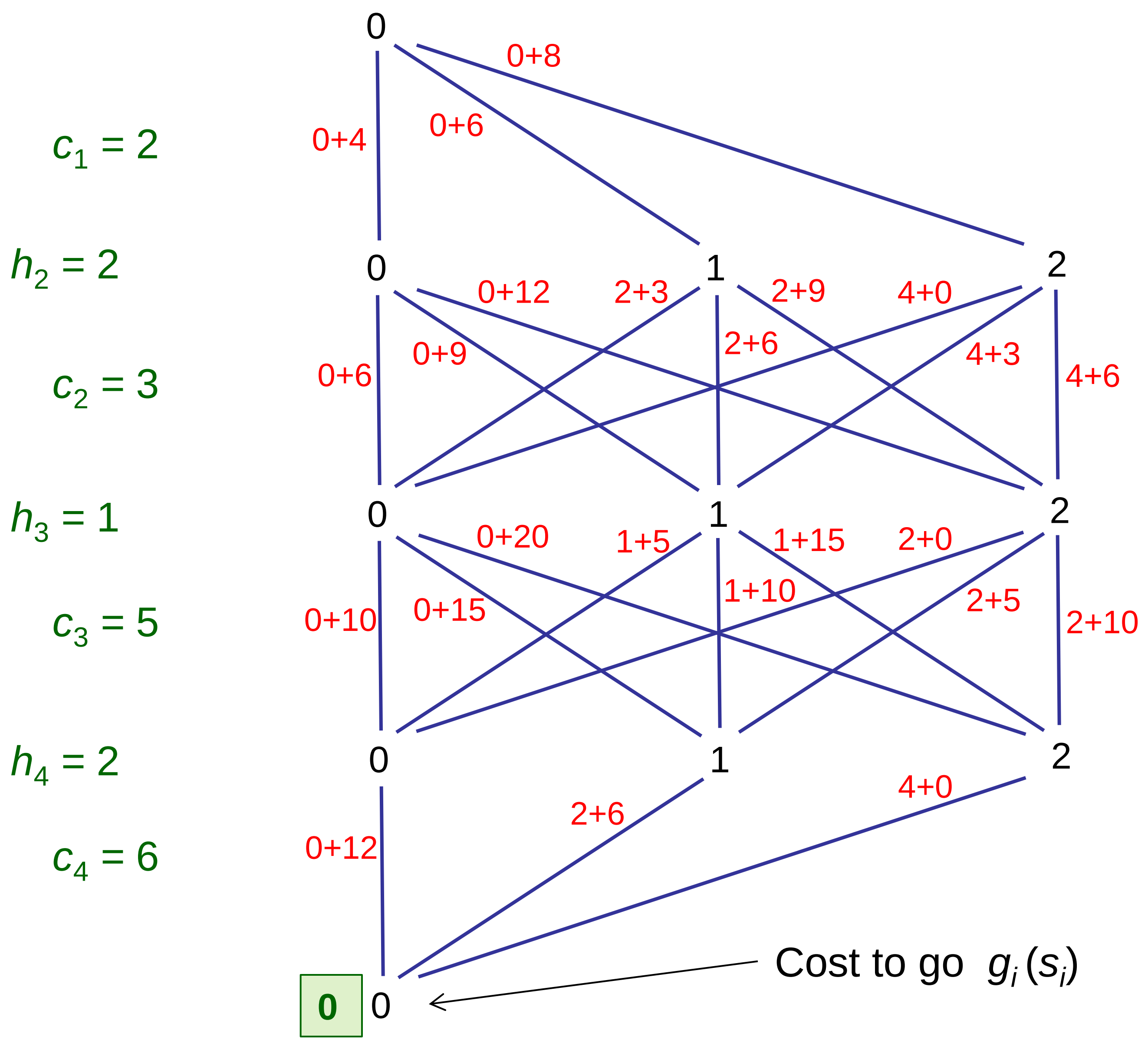
Transition cost
(immediate cost)

$h_i s_i + c_i x_i$

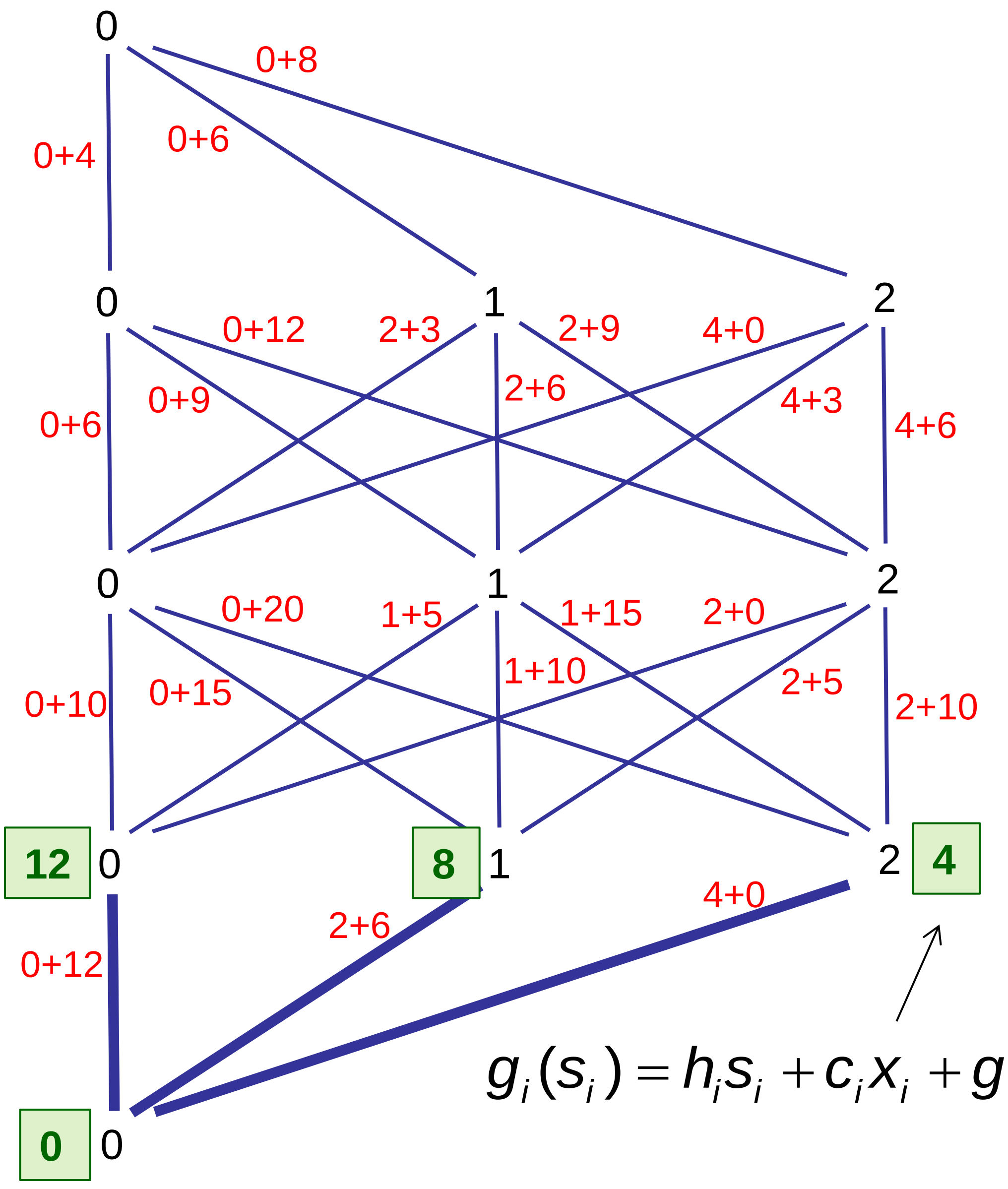
Unit holding cost = h_i

Unit manufacturing cost = c_i

Backward recursion

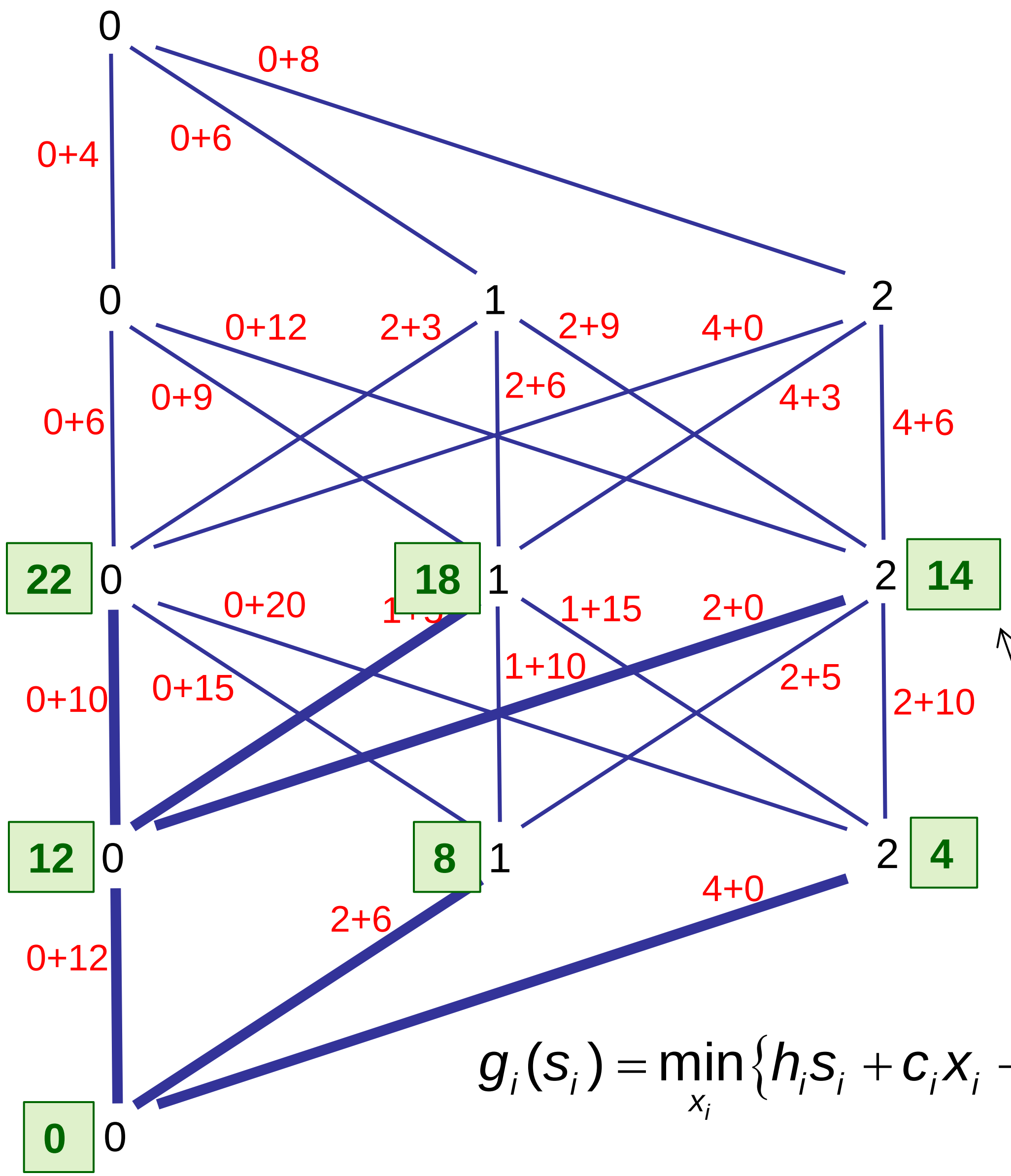


Backward recursion



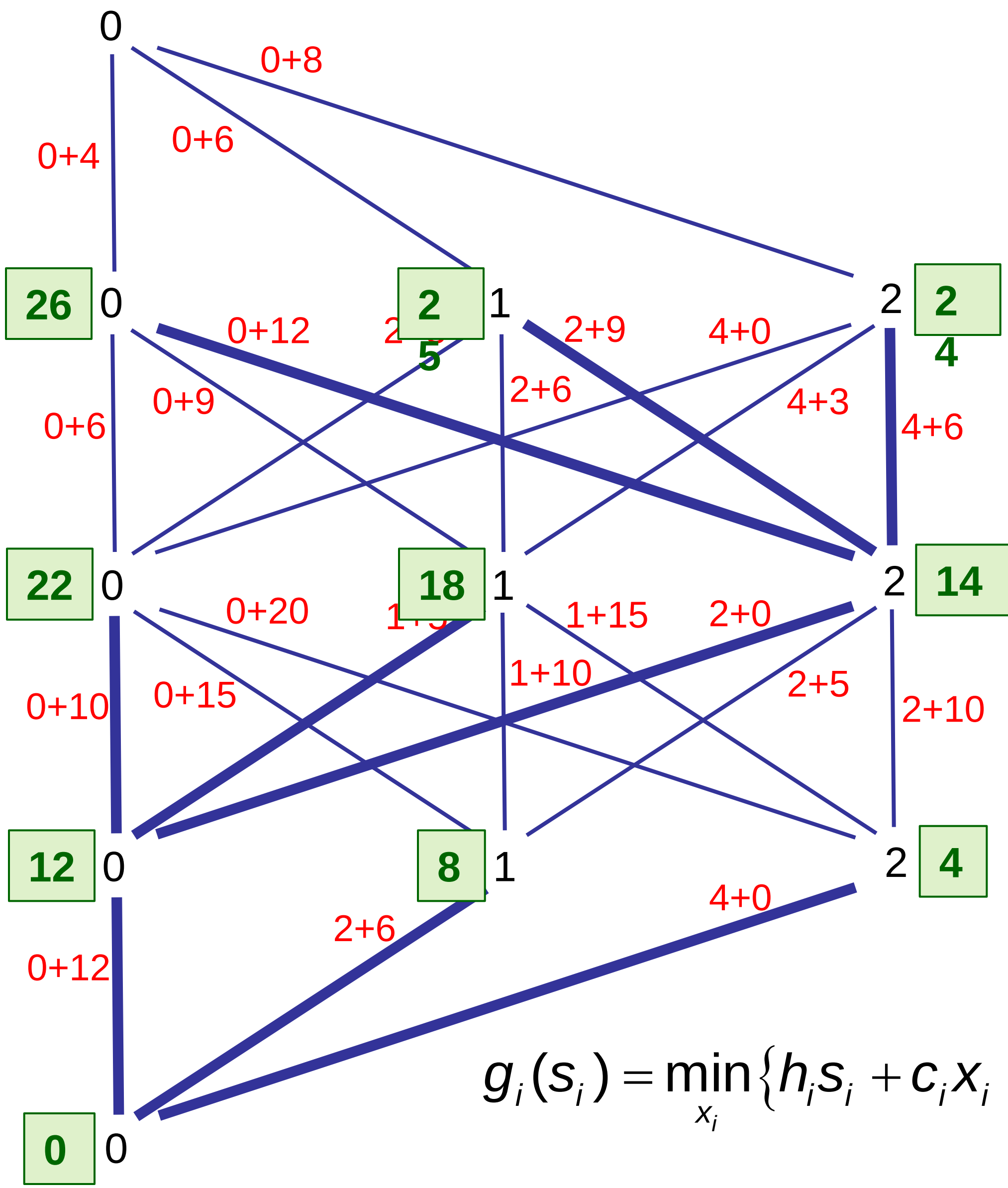
$$g_i(s_i) = h_i s_i + c_i x_i + g_{i+1}(s_i + x_i - d_i)$$

Backward recursion



$$g_i(s_i) = \min_{x_i} \{ h_i s_i + c_i x_i + g_{i+1}(s_i + x_i - d_i) \}$$

Backward recursion



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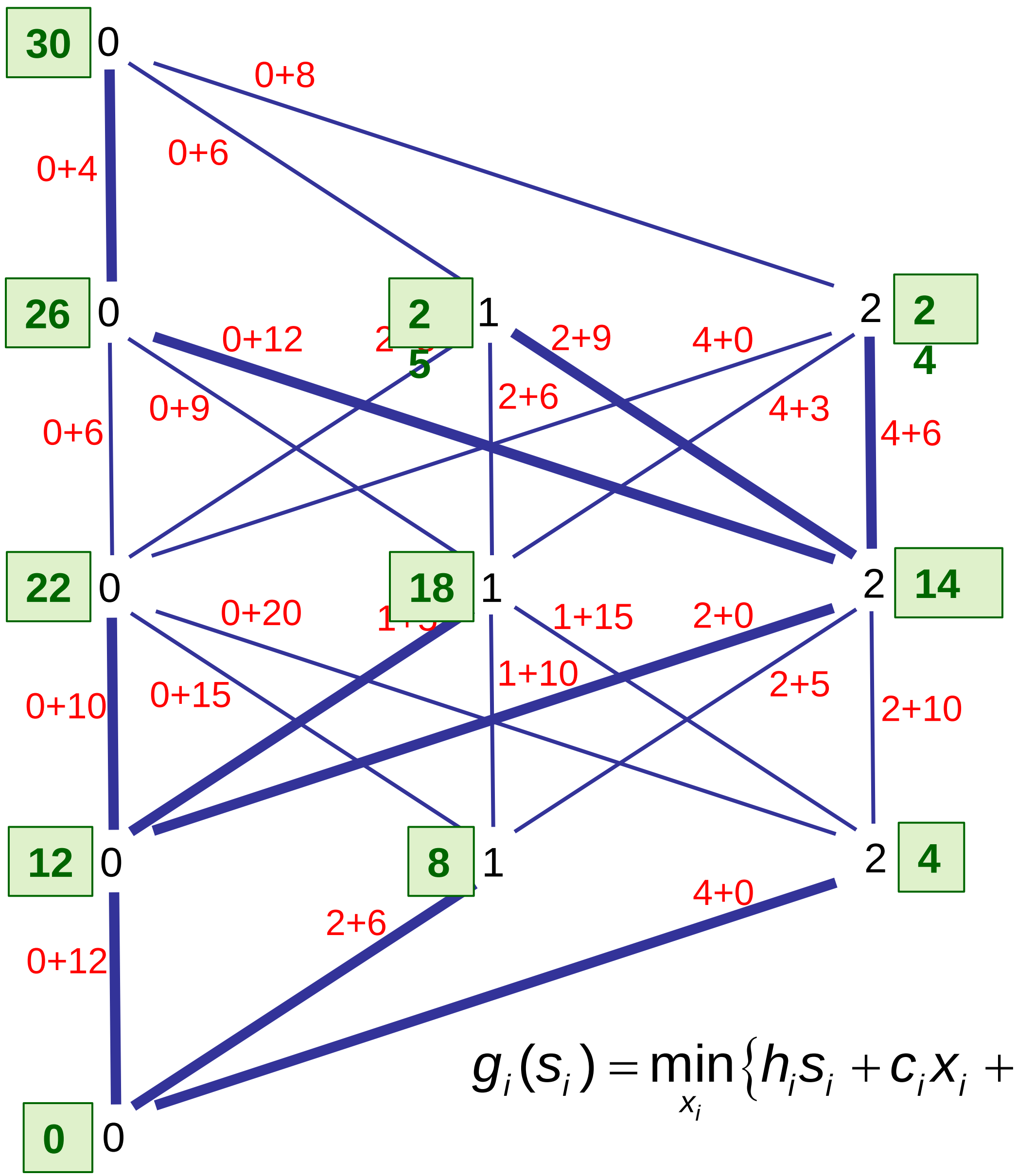
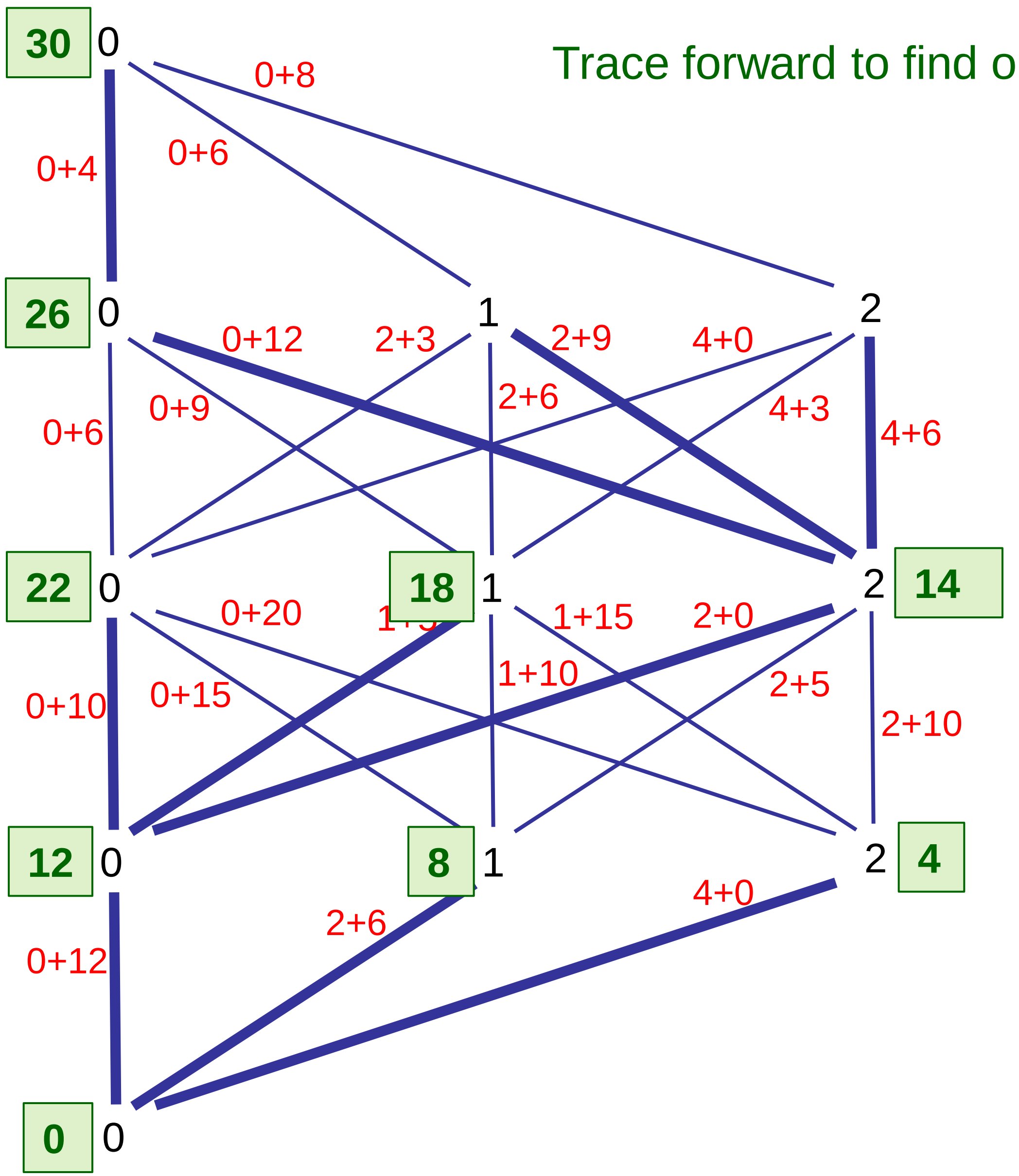
Backward recursion



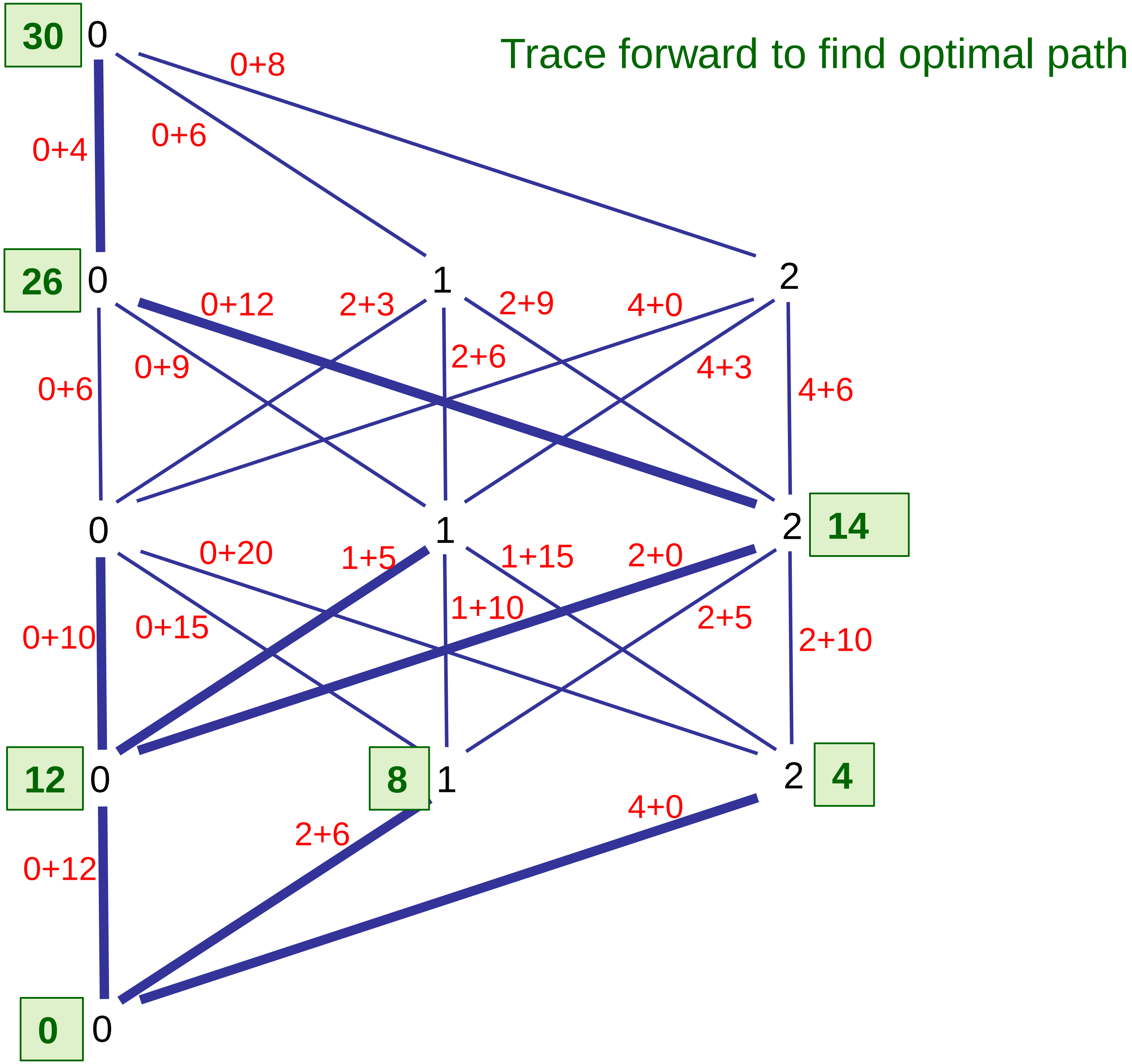
$$g_i(s_i) = \min_{x_i} \{ h_i s_i + c_i x_i + g_{i+1}(s_i + x_i - d_i) \}$$

Optimal solution

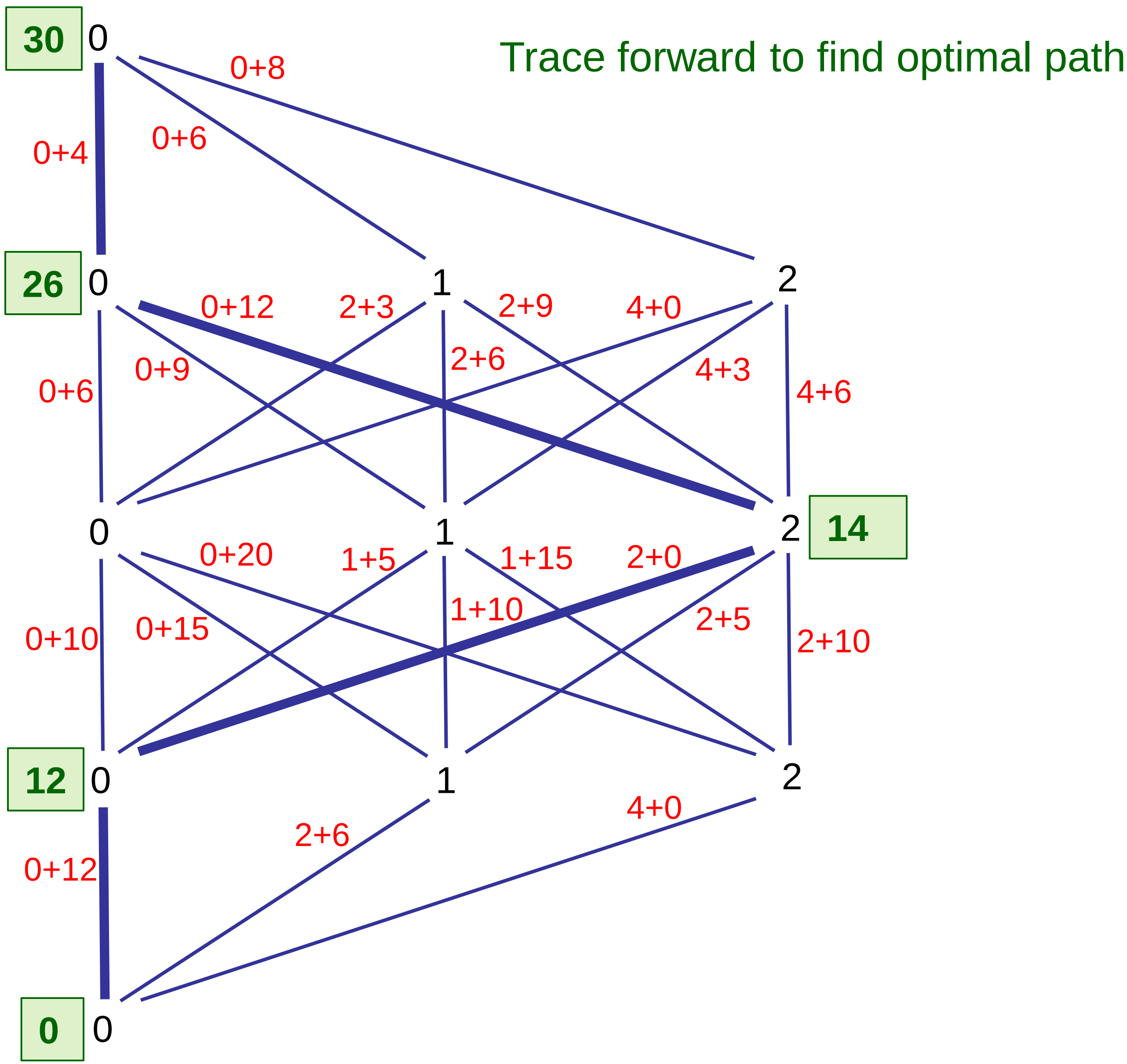
Trace forward to find optimal path



Optimal solution



Optimal solution



Dynamic Programming Recursion

- In general, the state transition is

$$s_{i+1} = \varphi_i(s_i, x_i), \quad i = 1, \ldots, n$$

- Cost is a function of state and control pairs

$$f(x) = \sum_{i=1}^{n} c_i(s_i, x_i)$$

- The recursion is

$$g_i(s_i) = \min_{x_i} \left\{ c_i(s_i, x_i) + g_{i+1}\left(\varphi_i(s_i, x_i)\right) \right\}$$

  – with boundary condition $g_{n+1}(s_{n+1}) = 0, \quad \text{all } s_{n+1}$

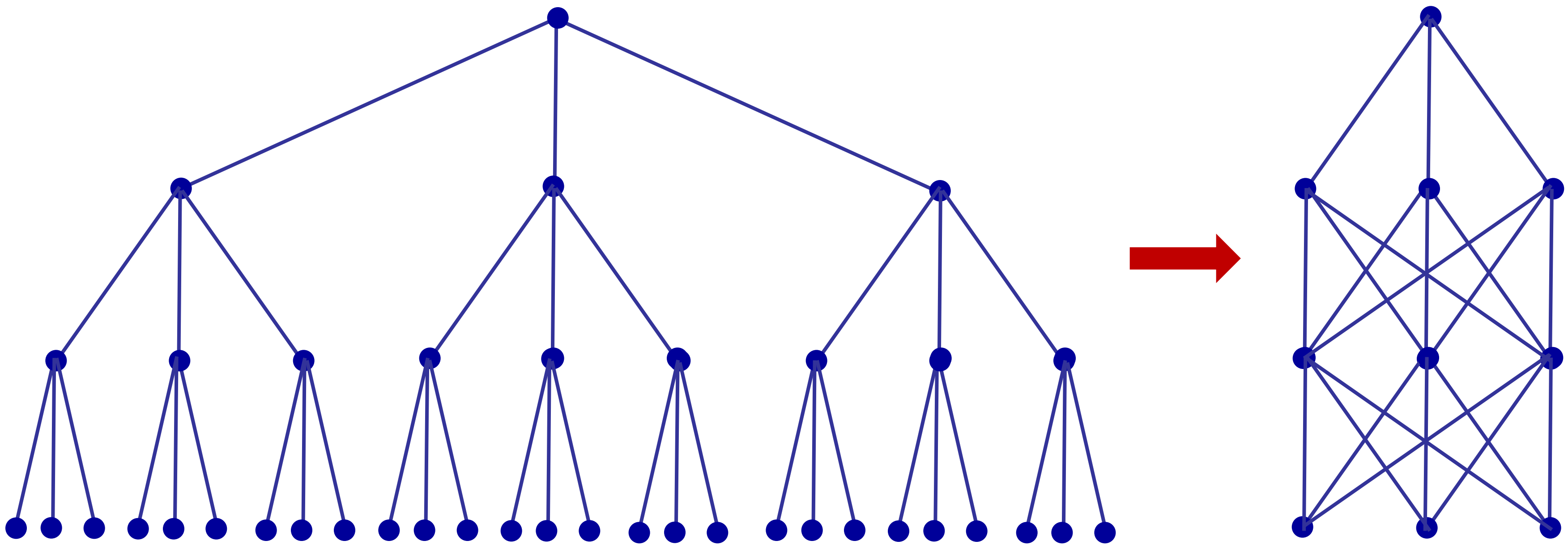
  – and optimal value $g_1(s_1)$ for starting state s_1

Dynamic Programming Characteristics

- There are **state variables** in addition to decision variables.
- **Costs** are function of **state variables** as well as decision variables.
- State transitions are **Markovian**.
  - Current state determines possible transitions and costs.
- Problem is solved **recursively**.
  - Often by moving backward through stages.
- The **art** of dynamic programming:
  - Find a small state description that is Markovian.

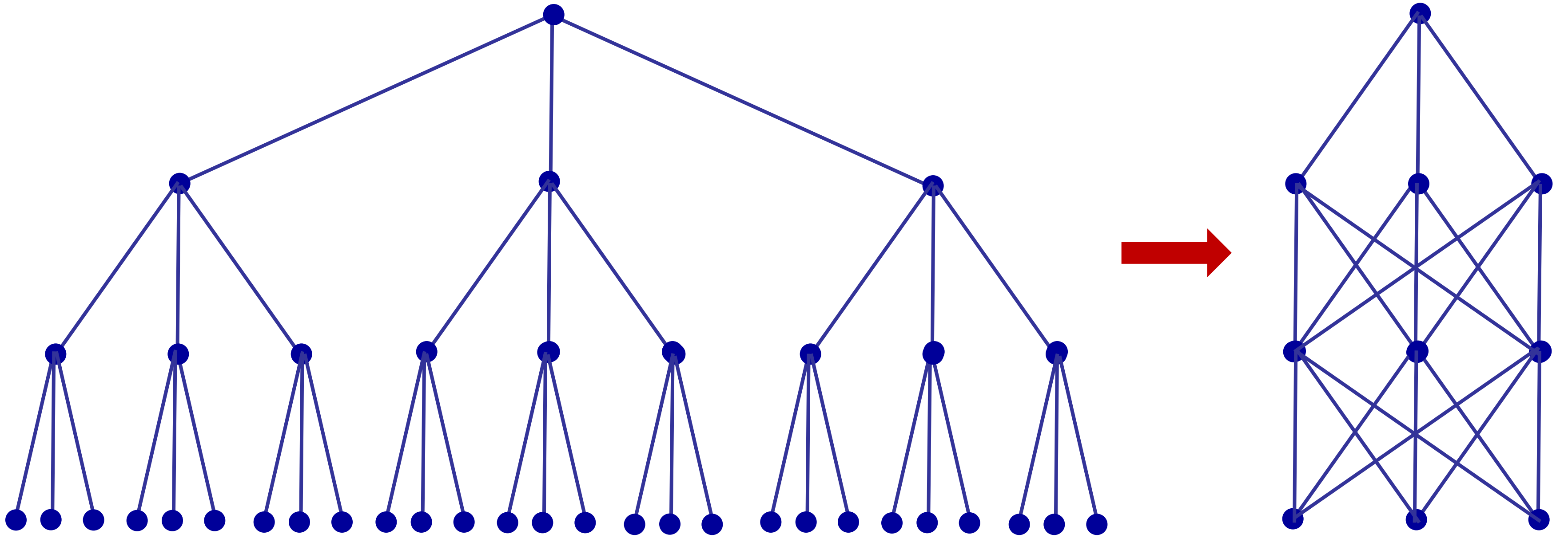
DP vs Caching

- Dynamic programming ≠ caching
  - Yes, DP identifies equivalent subproblems.



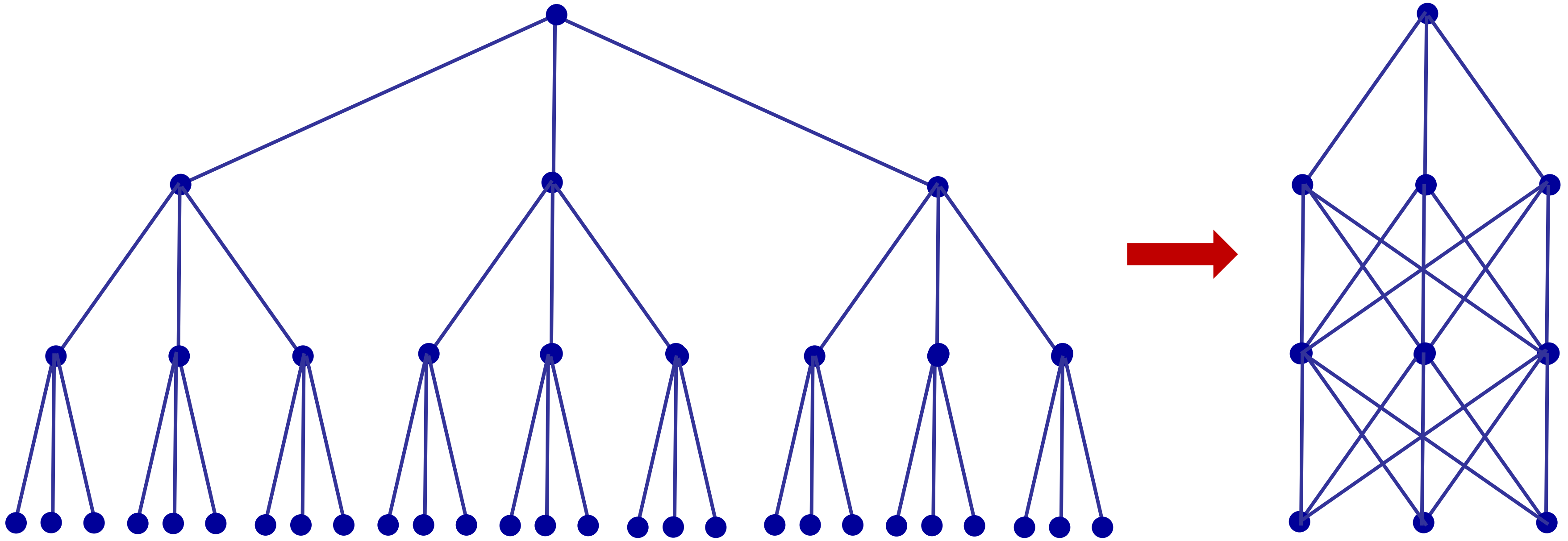
DP vs Caching

- Dynamic programming ≠ caching
  - Yes, DP identifies equivalent subproblems.
  - But **not** by identifying distinct states.
  - All states are treated separately (except in approximate DP).
- The intelligence is in the state description.

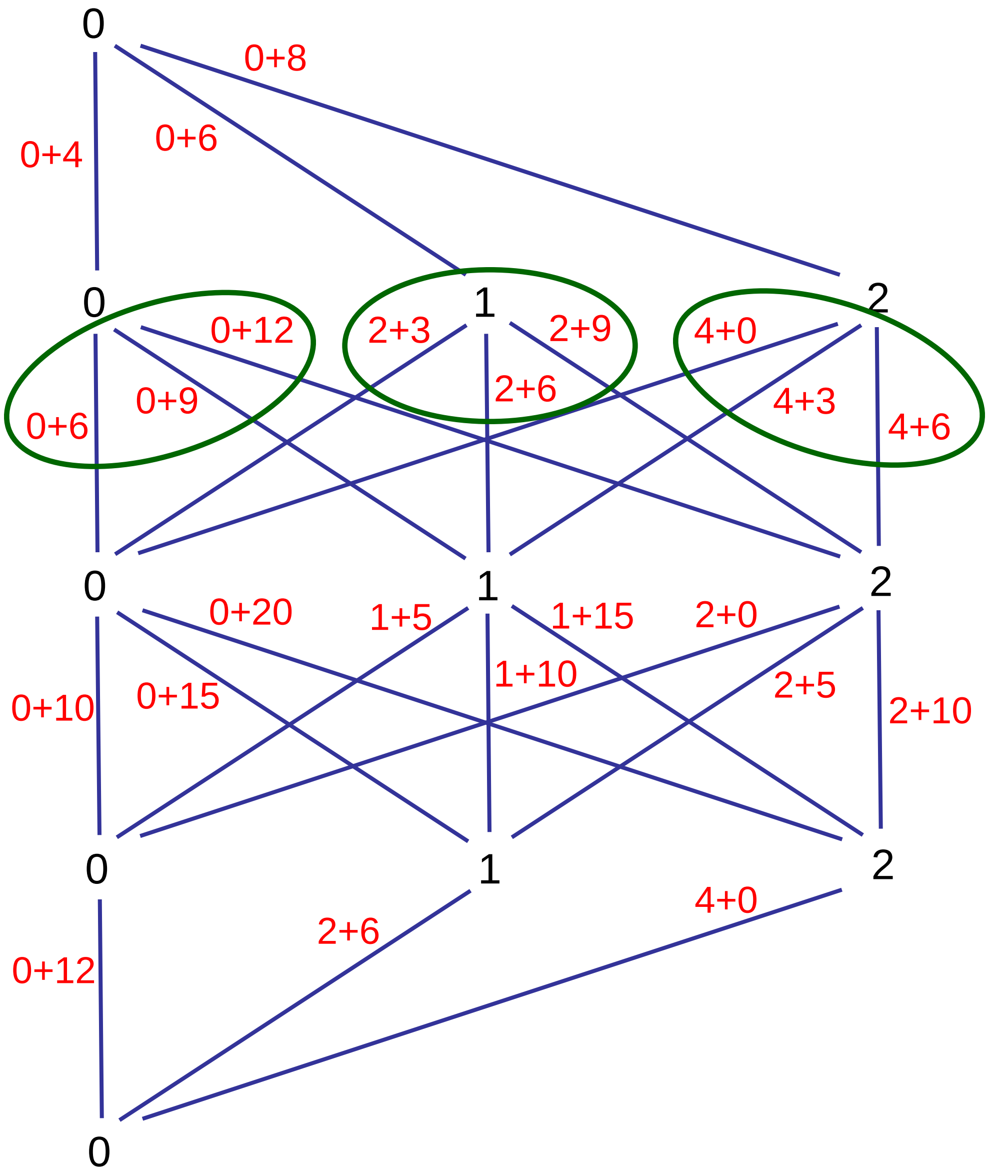


DP vs Caching

- However, caching can be applied **on top of** DP.
  - We will use the concept of **reduced decision diagram** (reduced MDD) to identify equivalent states.
- Problem: how to deal with **state-dependent costs**.



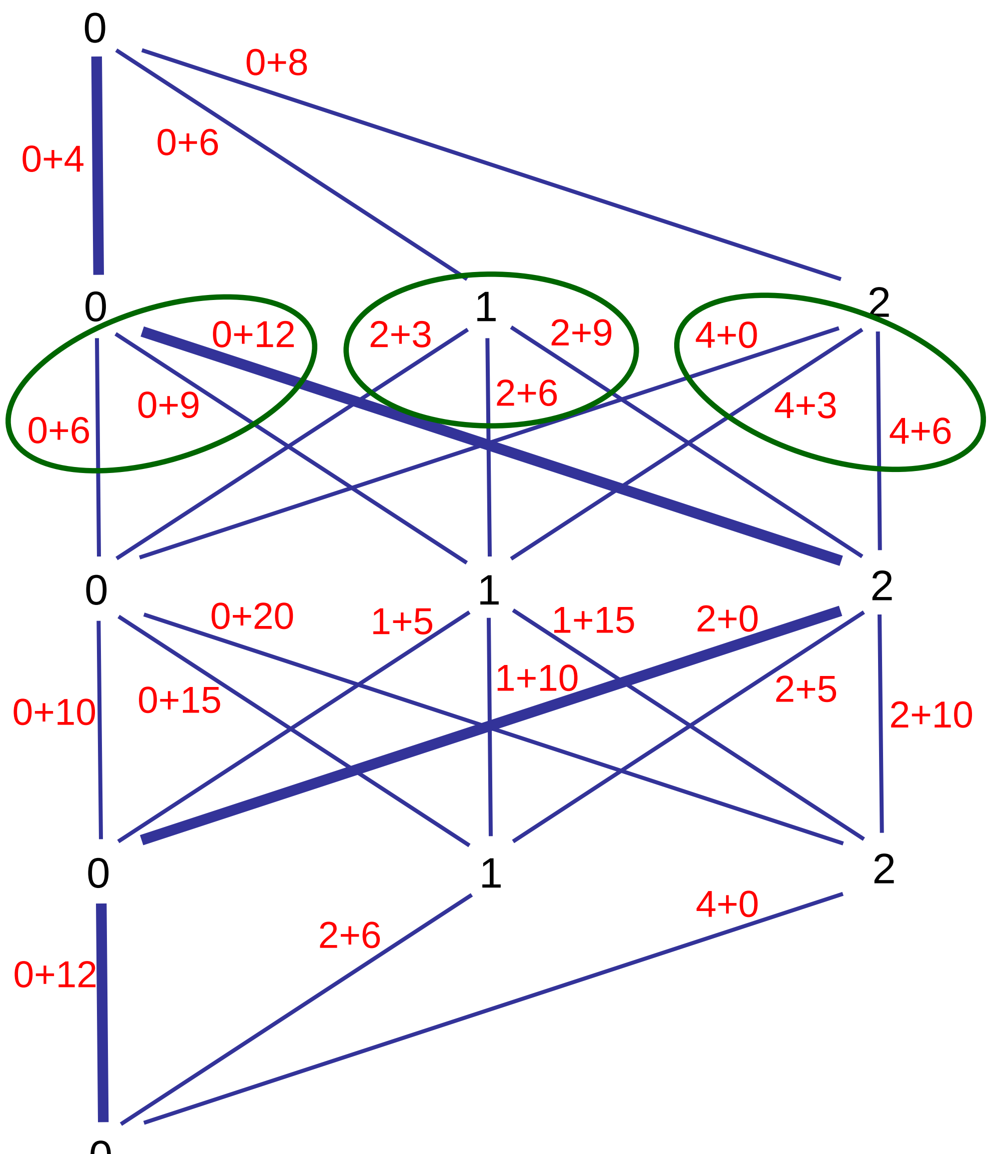
Reducing the Transition Graph



Arcs leaving each node are very similar.

- Transition to the same states.
- Have the same costs, up to an offset.

Reducing the Transition Graph

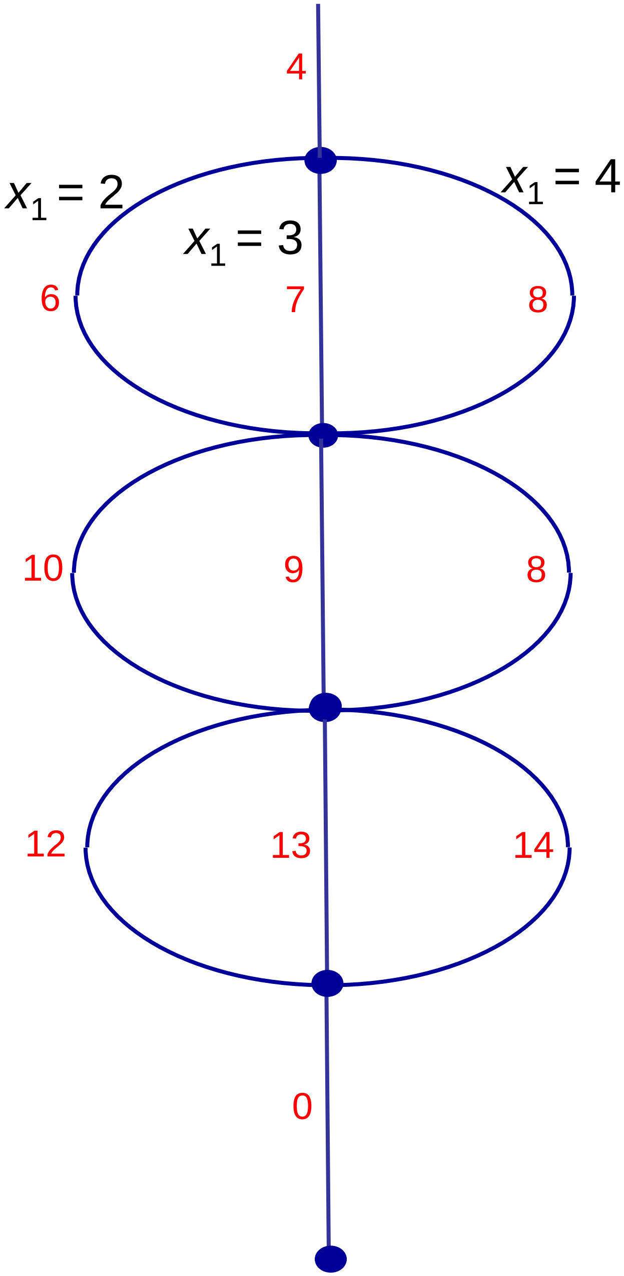


Arcs leaving each node are very similar.

- Transition to the same states.
- Have the same costs, up to an offset.

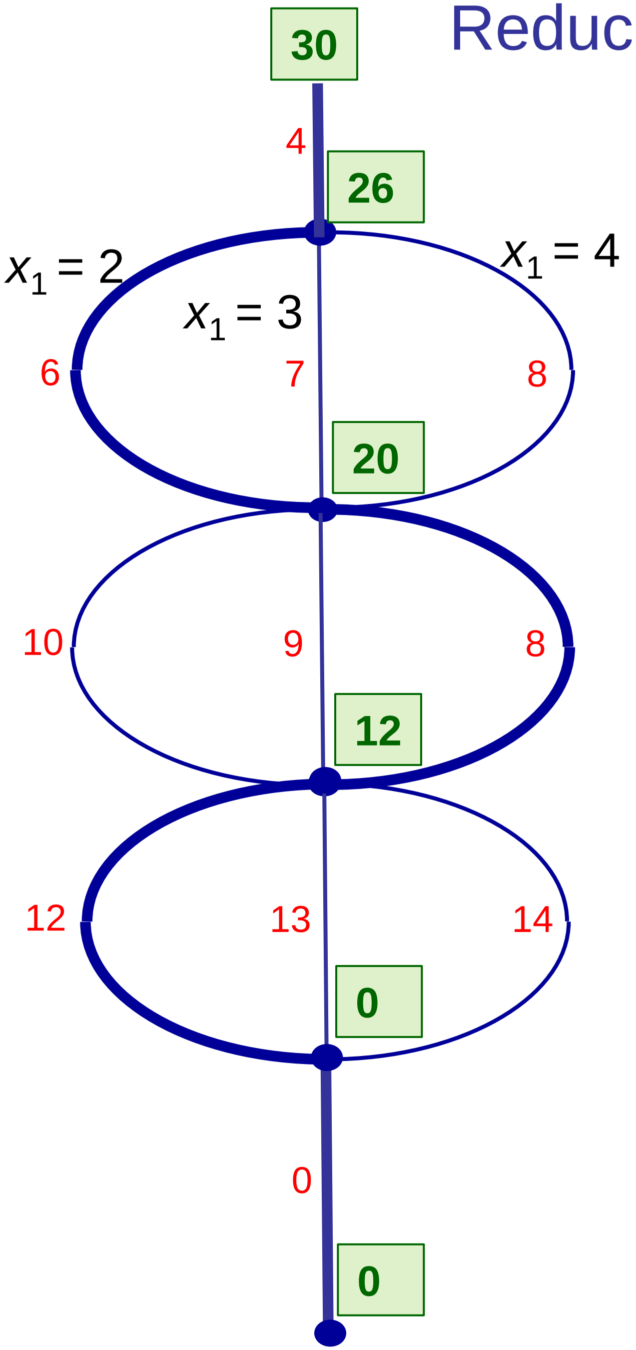
Incidentally, there is also a **bang-bang** solution.

Reducing the Transition Graph



By rearranging the costs, we can collapse the states in each period.

Reducing the Transition Graph



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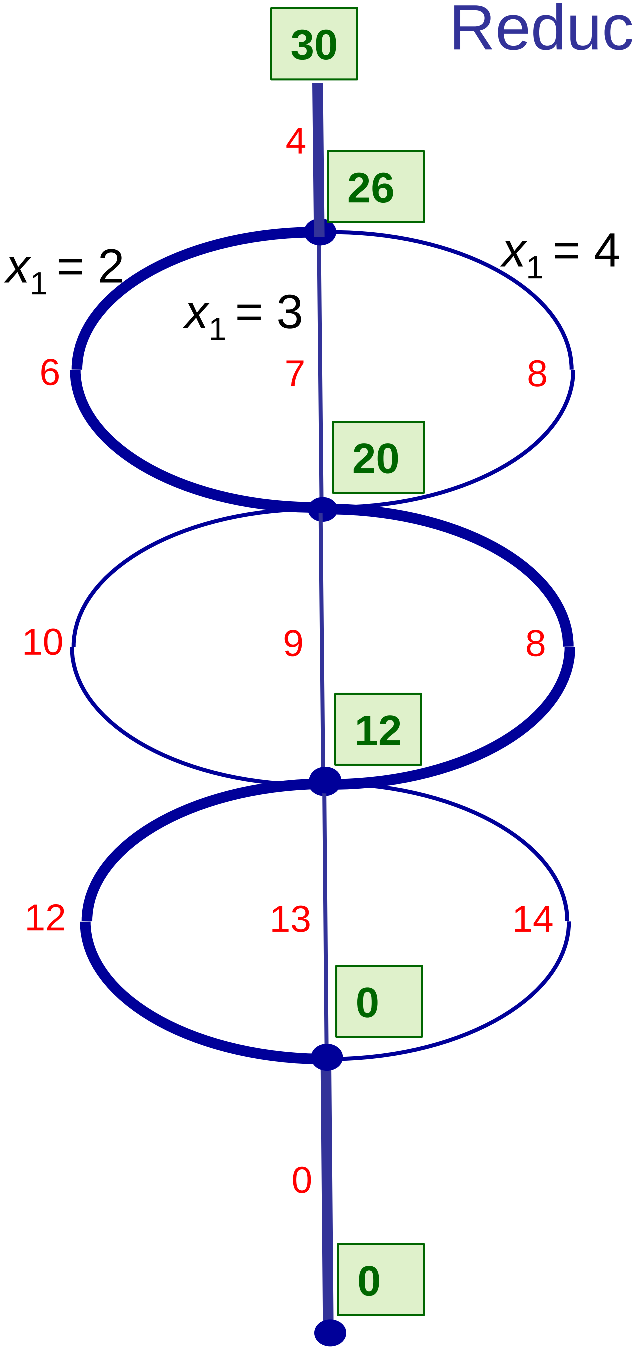
Now it is easier to compute the optimal solution

Reducing the Transition Graph



By rearranging the costs, we can collapse the states in each period.

Now it is easier to compute the optimal solution

This looks like reduction of a **decision diagram** (MDD).

We will develop this idea in general.

Decision Diagrams

Set covering example

Select a minimum-weight family of sets that contain all 4 elements A, B, C, D

	Set i 1	2	3	4
A	●	●		
B	●		●	●
C		●	●	
D		●		●
Weight	3	5	4	6

x_i = 1 when we select set i

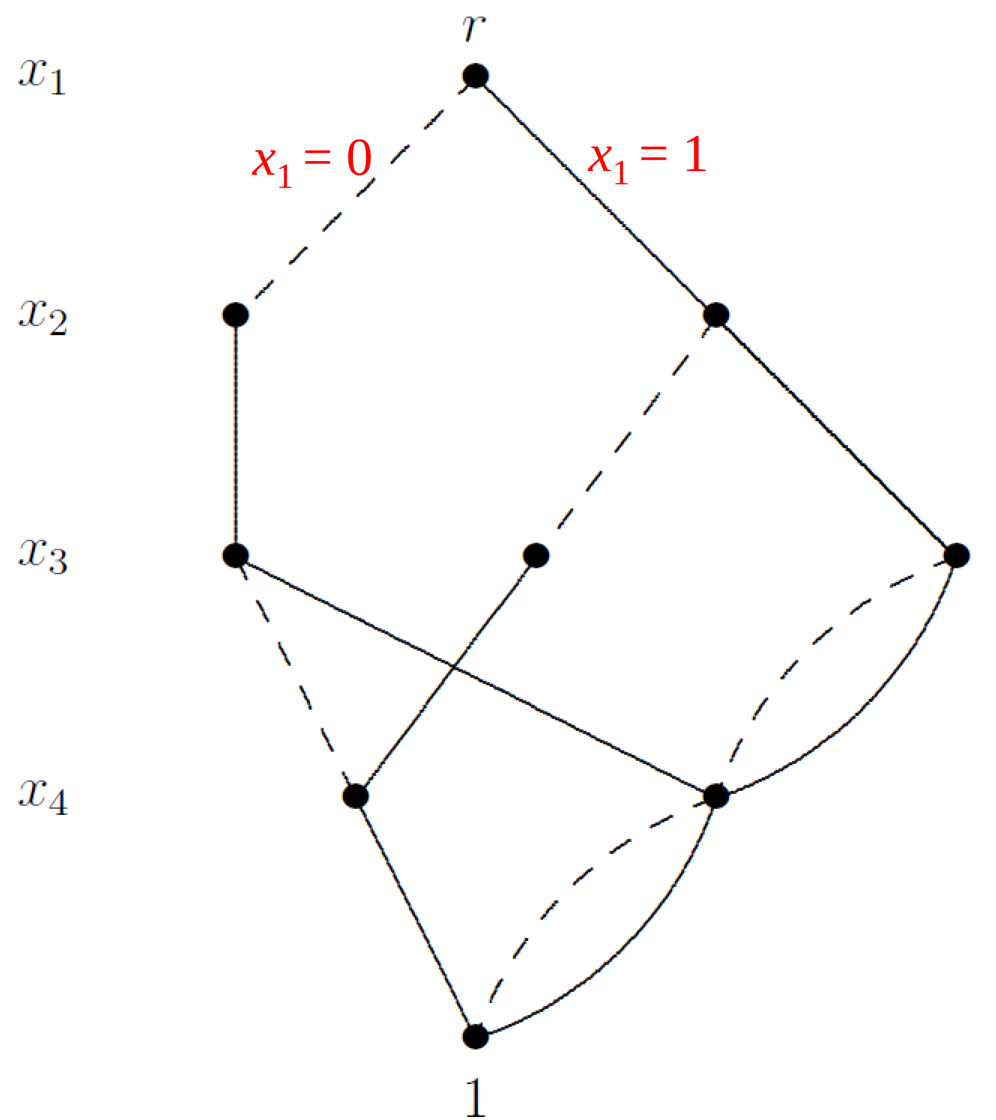
Decision Diagrams

Decision diagram

Each path corresponds to a feasible solution.

	Set i			
	1	2	3	4
A	●	●		
B	●		●	●
C		●	●	
D		●		●
Weight	3	5	4	6

$x_i = 1$ when we select set i



Weighted Decision Diagrams

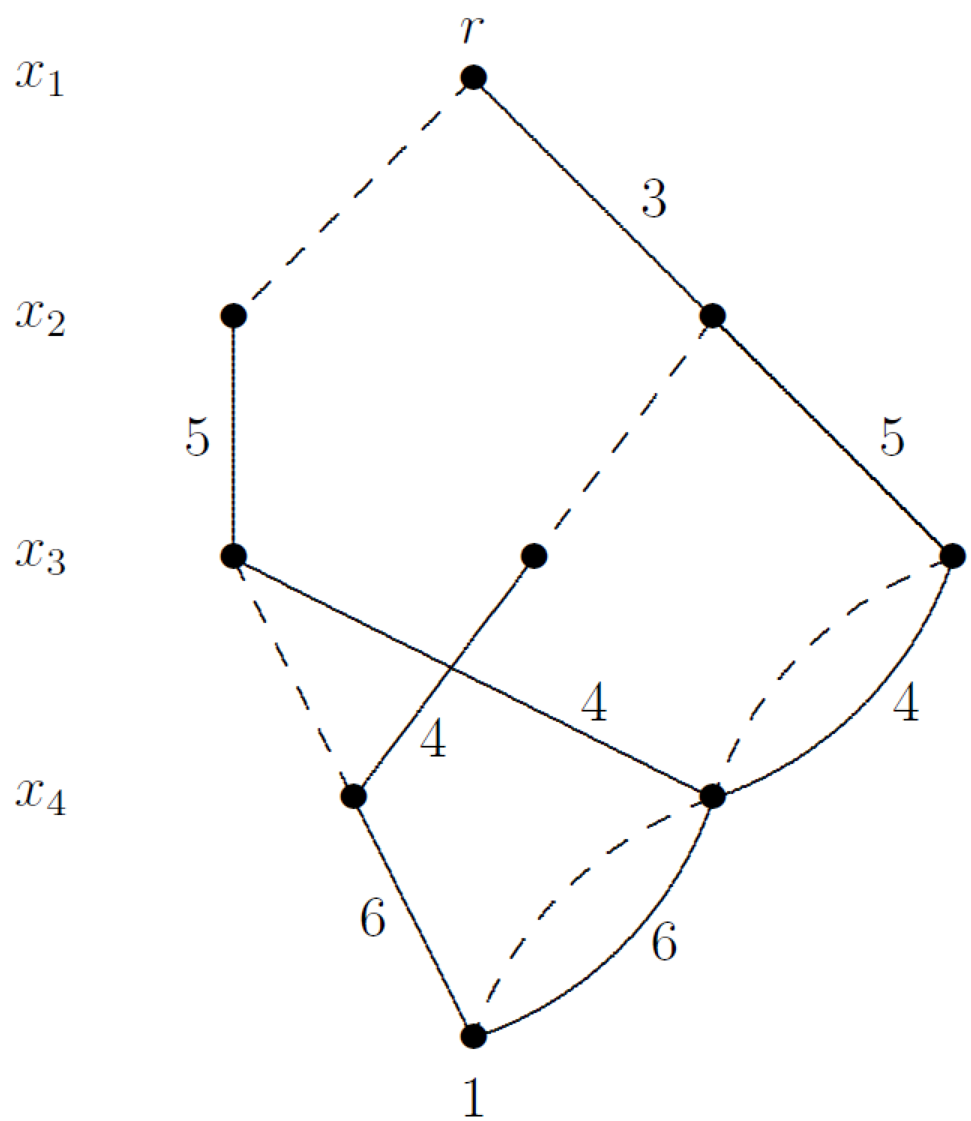
Separable cost function

Just label arcs with weights.

Shortest path corresponds to an optimal solution.

	Set i 1	2	3	4
A	●	●		
B	●		●	●
C		●	●	
D		●		●
Weight	3	5	4	6

x_i = 1 when we select set i



Weighted Decision Diagrams

- State-dependent costs in dynamic programming imply a **nonseparable** cost function:

$$f(x) = \sum_{i=1}^{n} c_i(s_i, x_i)$$

where $s_{i+1} = \varphi_i(s_i, x_i), \quad i = 1, \ldots, n$

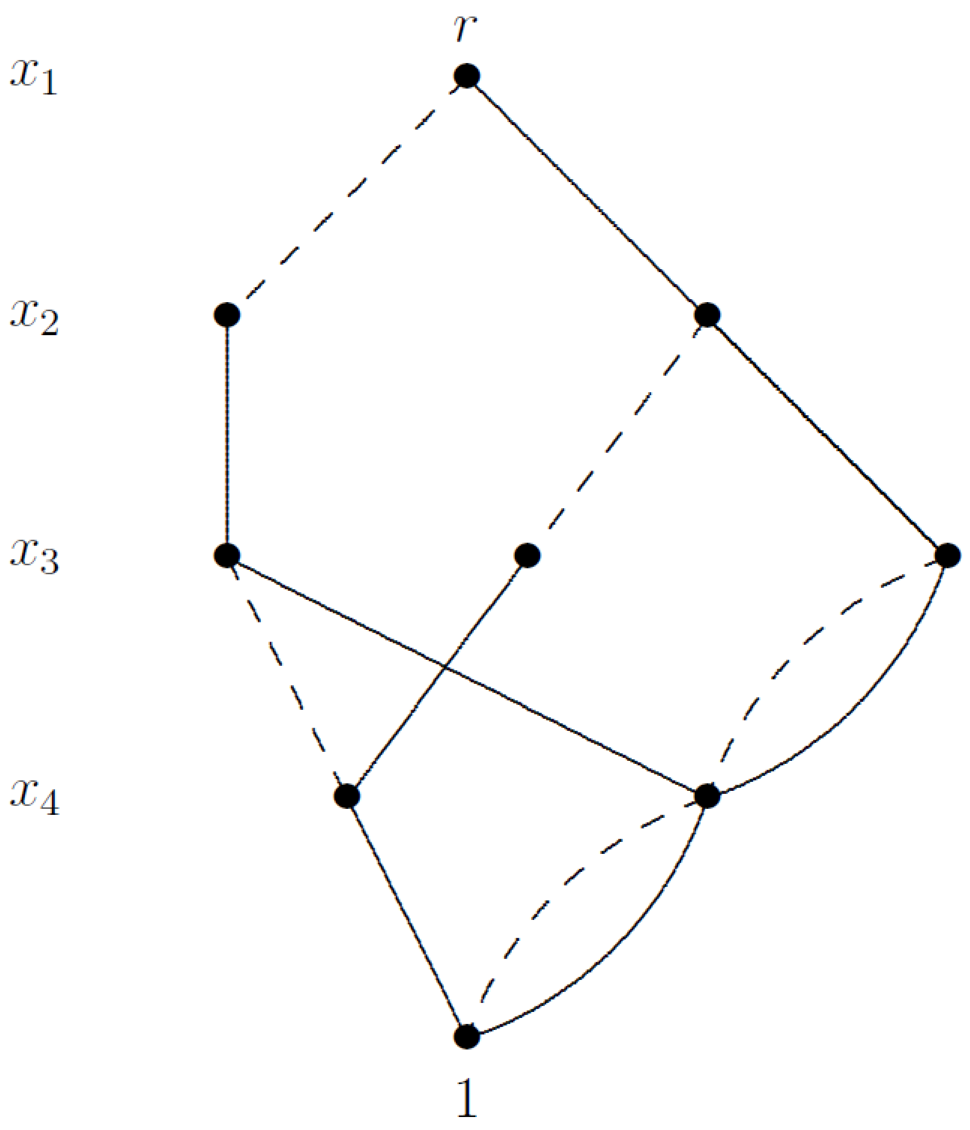
- – We need a theory of decision diagrams that deals with nonseparable costs.

Weighted Decision Diagrams

Nonseparable cost function

Now what?

x	$f(x)$
(0,1,0,1)	6
(0,1,1,0)	7
(0,1,1,1)	8
(1,0,1,1)	5
(1,1,0,0)	6
(1,1,0,1)	8
(1,1,1,0)	7
(1,1,1,1)	9



Weighted Decision Diagrams

Nonseparable cost function

Put costs on leaves
of branching tree.

x	$f(x)$
(0,1,0,1)	6
(0,1,1,0)	7
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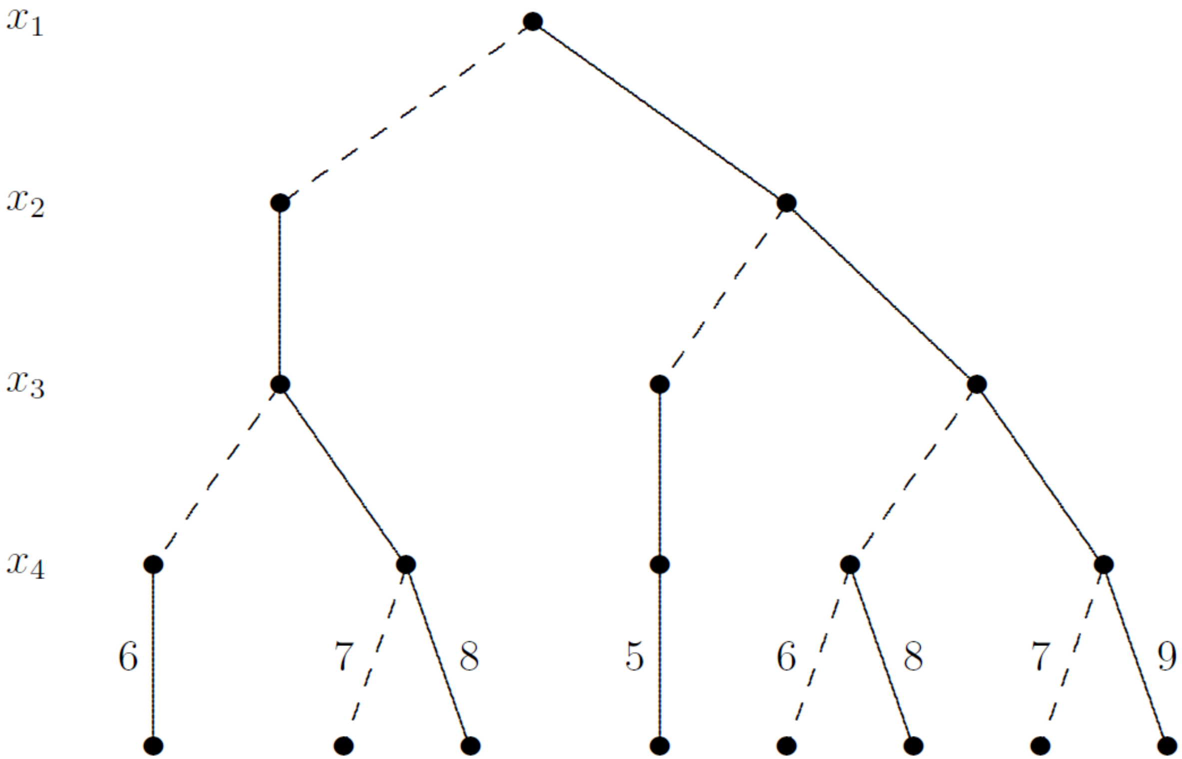
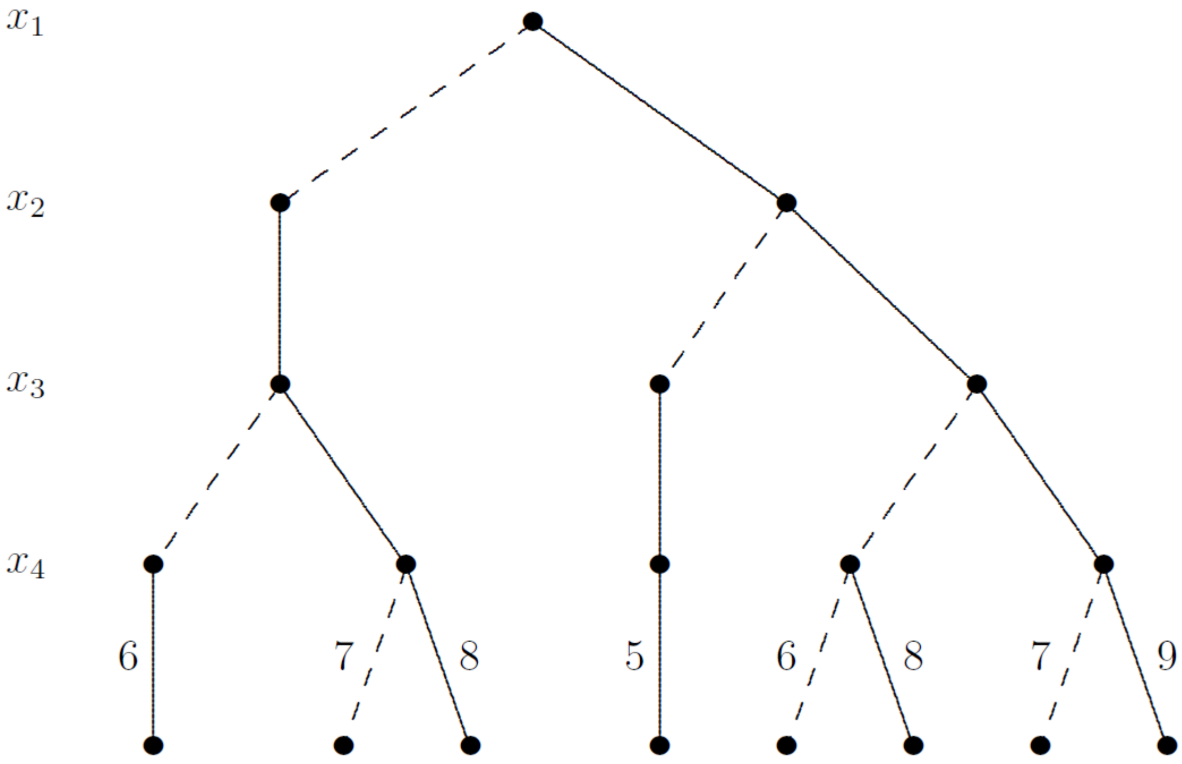


Weighted Decision Diagrams

Nonseparable cost function

Put costs on leaves of branching tree.

But now we can't reduce the tree to an efficient decision diagram.



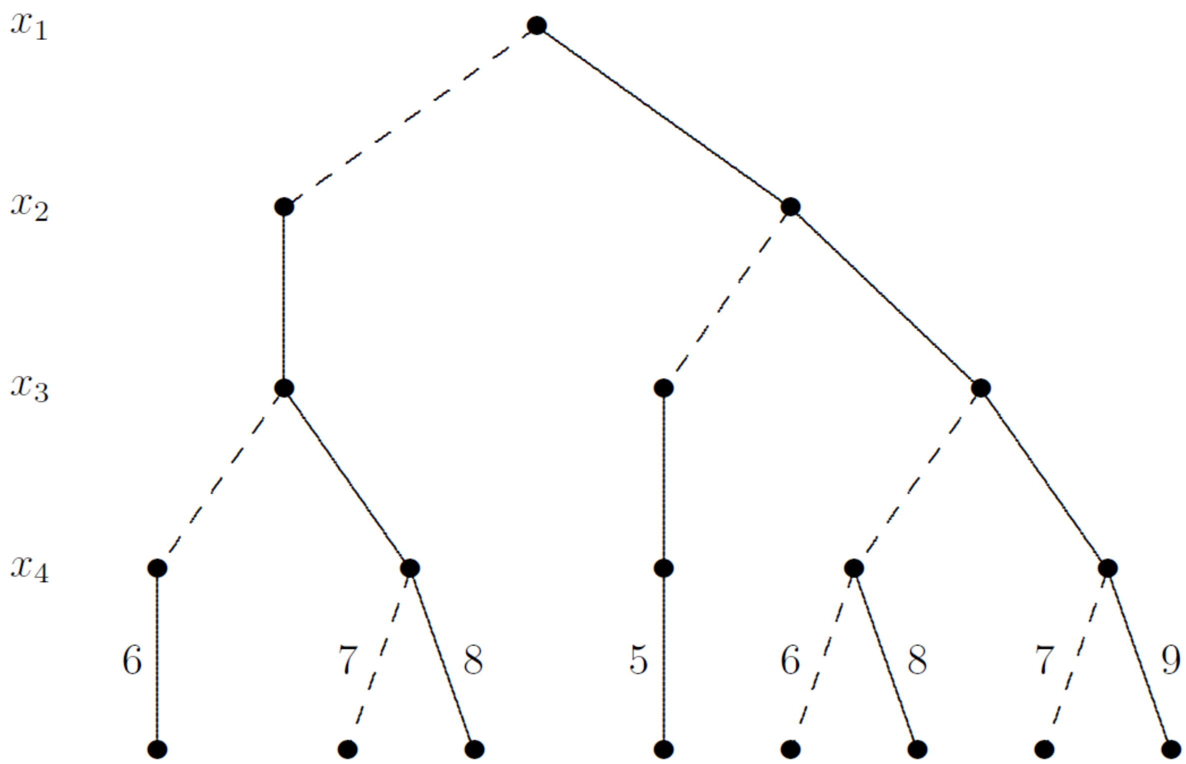
Weighted Decision Diagrams

Nonseparable cost function

Put costs on leaves of branching tree.

But now we can't reduce the tree to an efficient decision diagram.

We will rearrange costs to obtain **canonical costs**.



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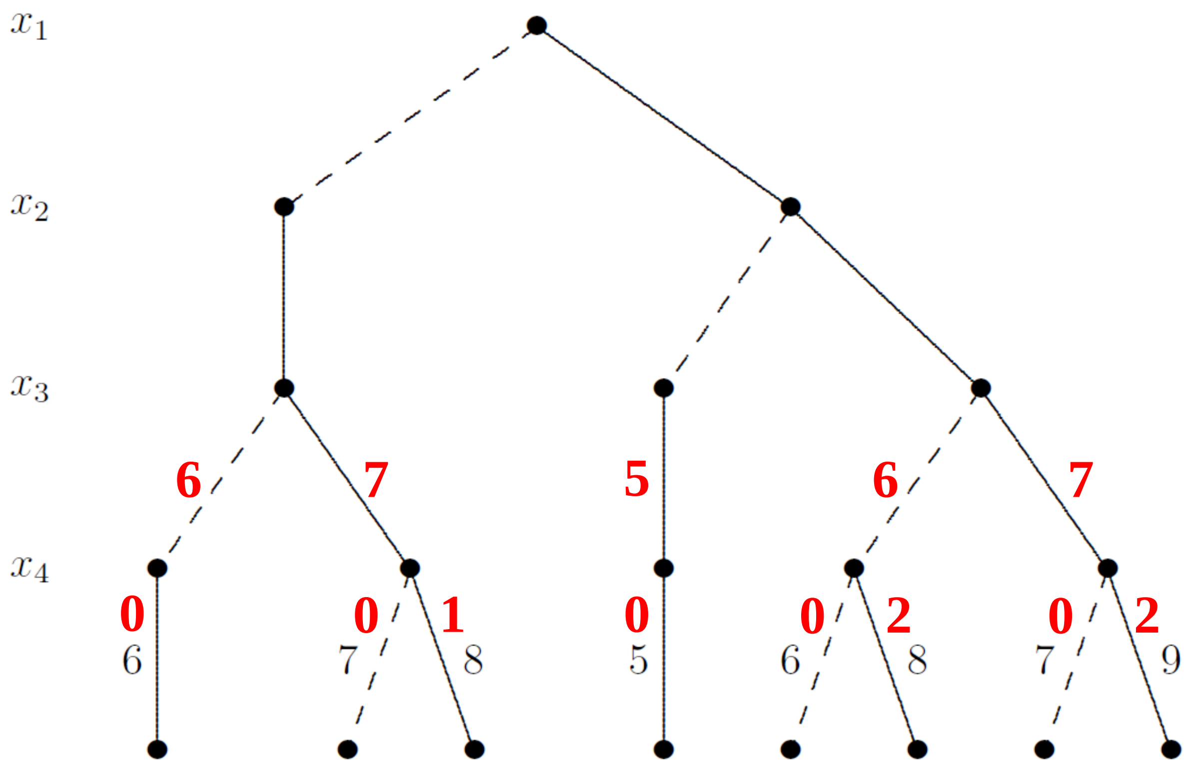

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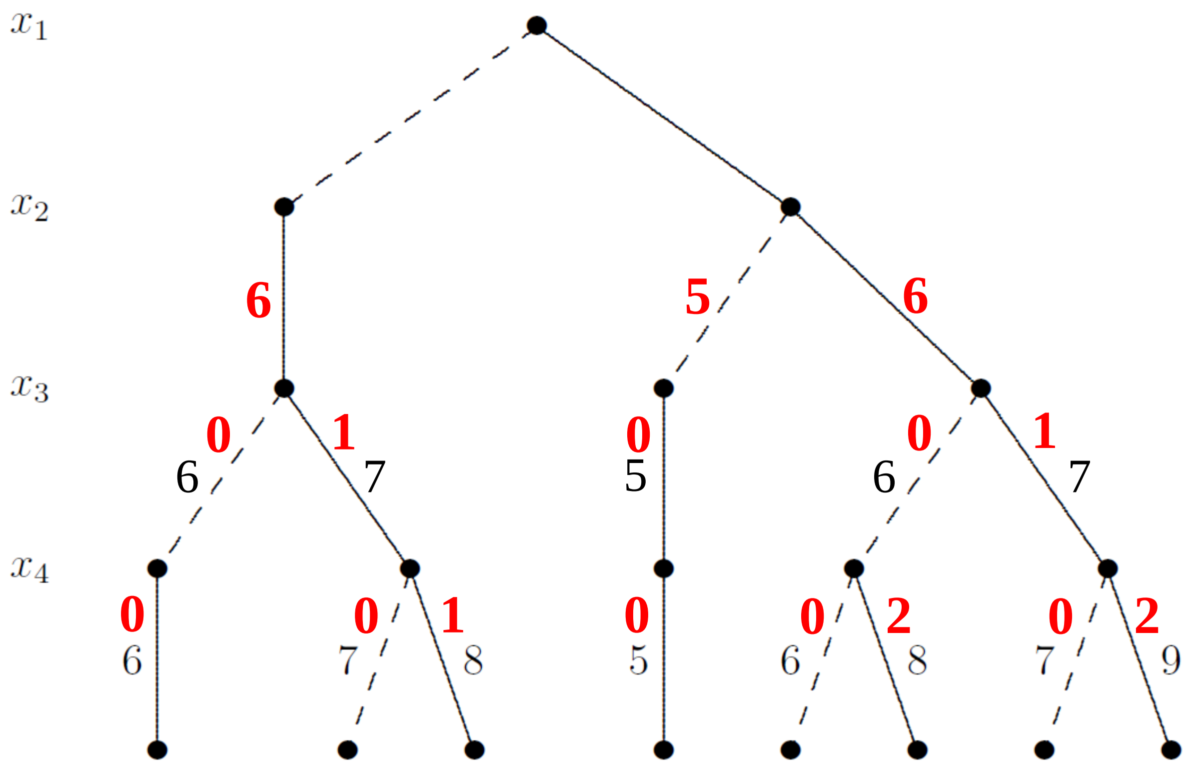

Weighted Decision Diagrams

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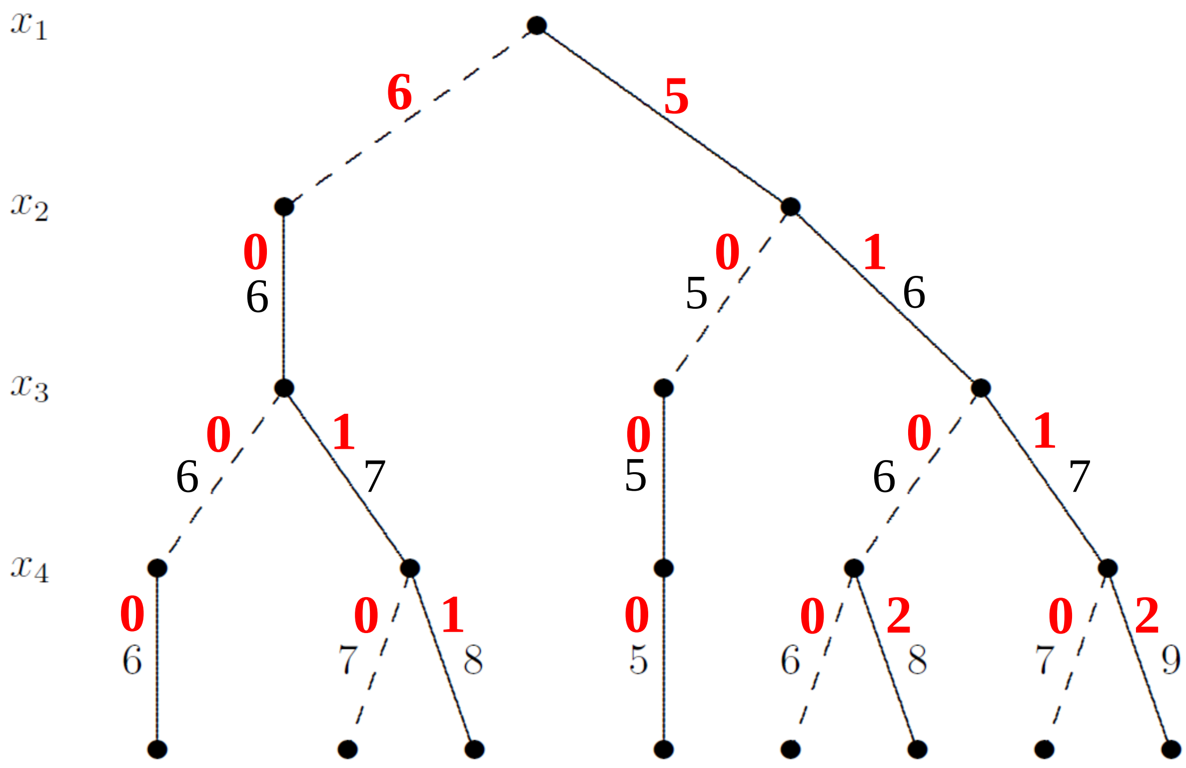
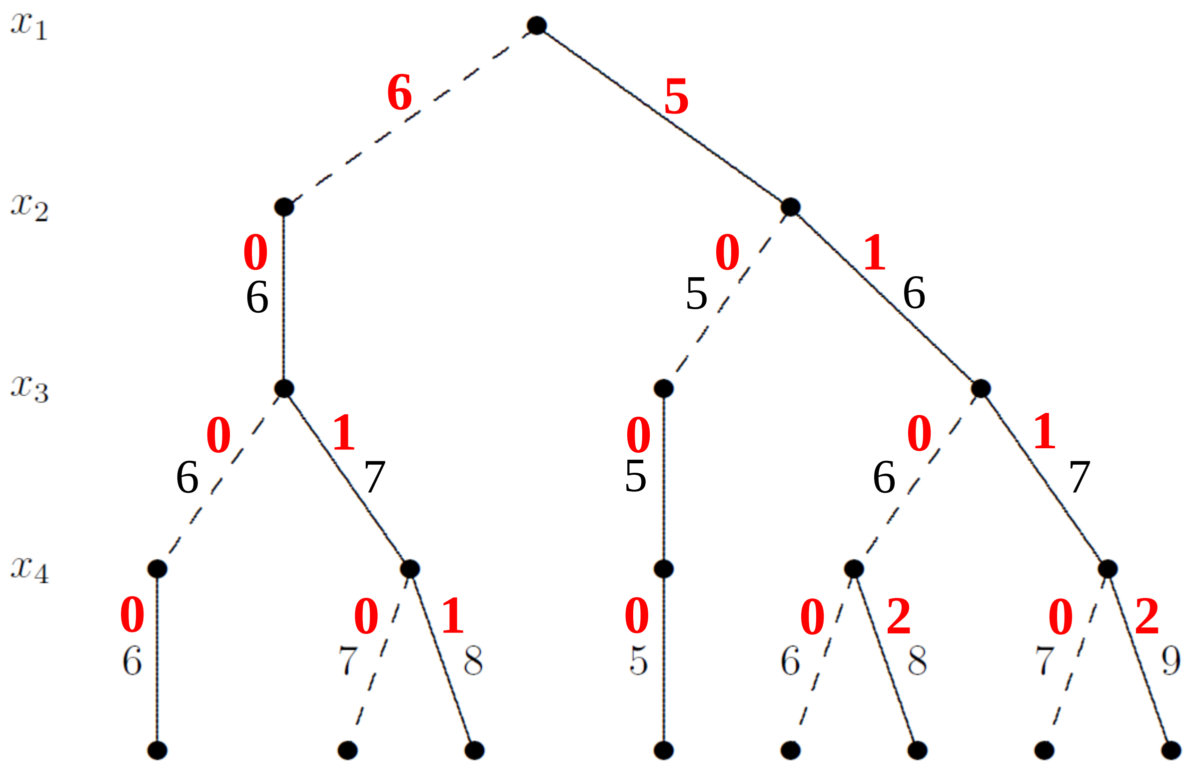


Weighted Decision Diagrams

Nonseparable cost function

Now the tree can be reduced.



Weighted Decision Diagrams

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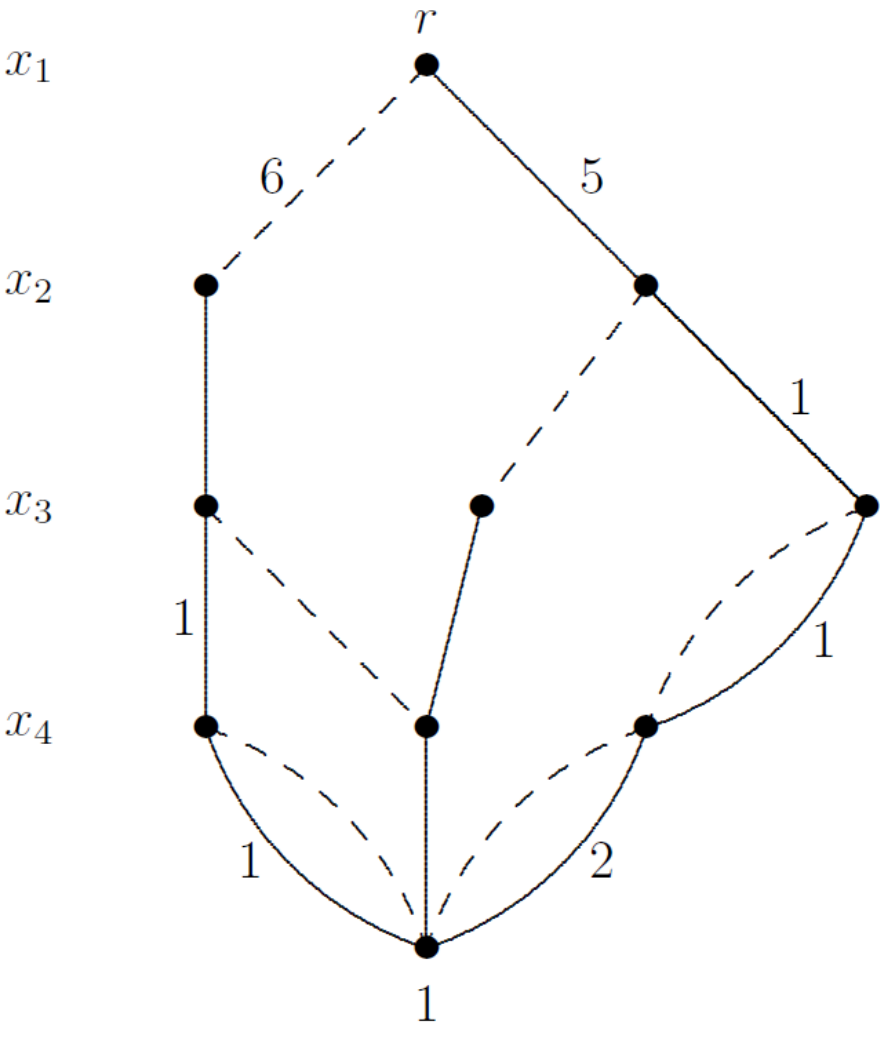
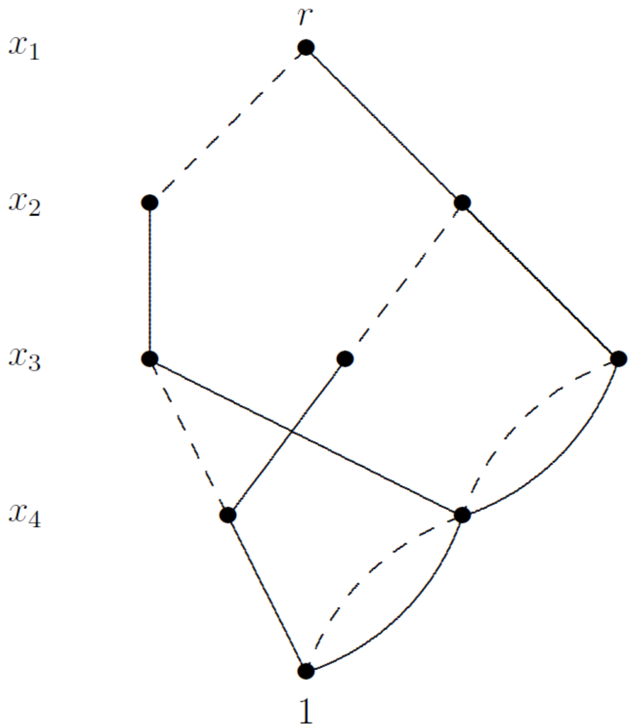
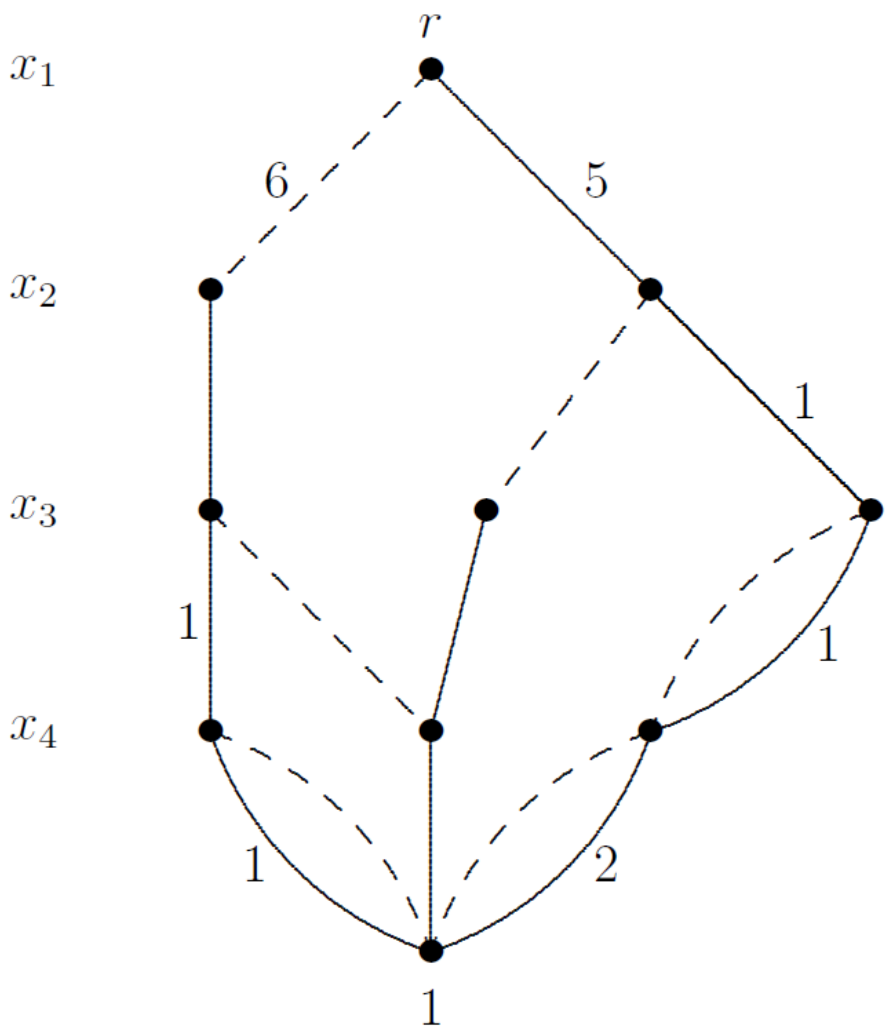


Weighted Decision Diagrams

Nonseparable cost function

Note that DD is larger than reduced unweighted DD, but still compact.



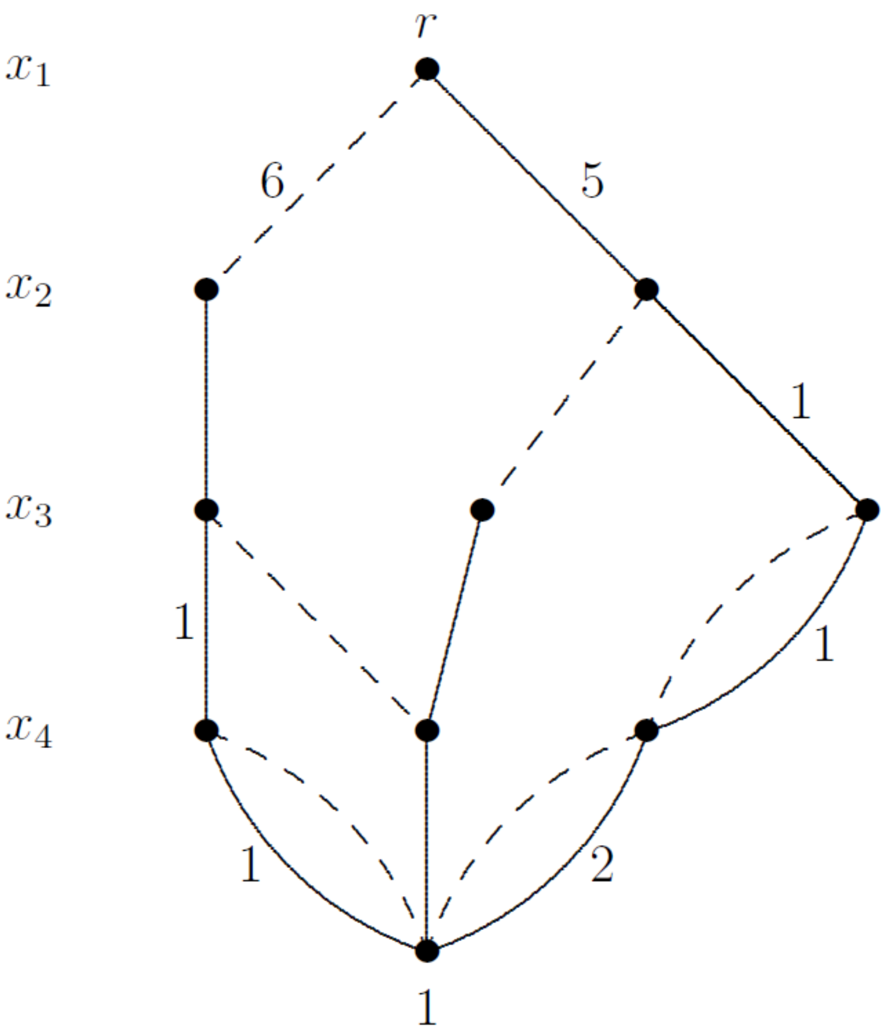


Weighted Decision Diagrams

Nonseparable cost function

We can represent any discrete optimization problem with such a decision diagram…

even if the costs are nonseparable.



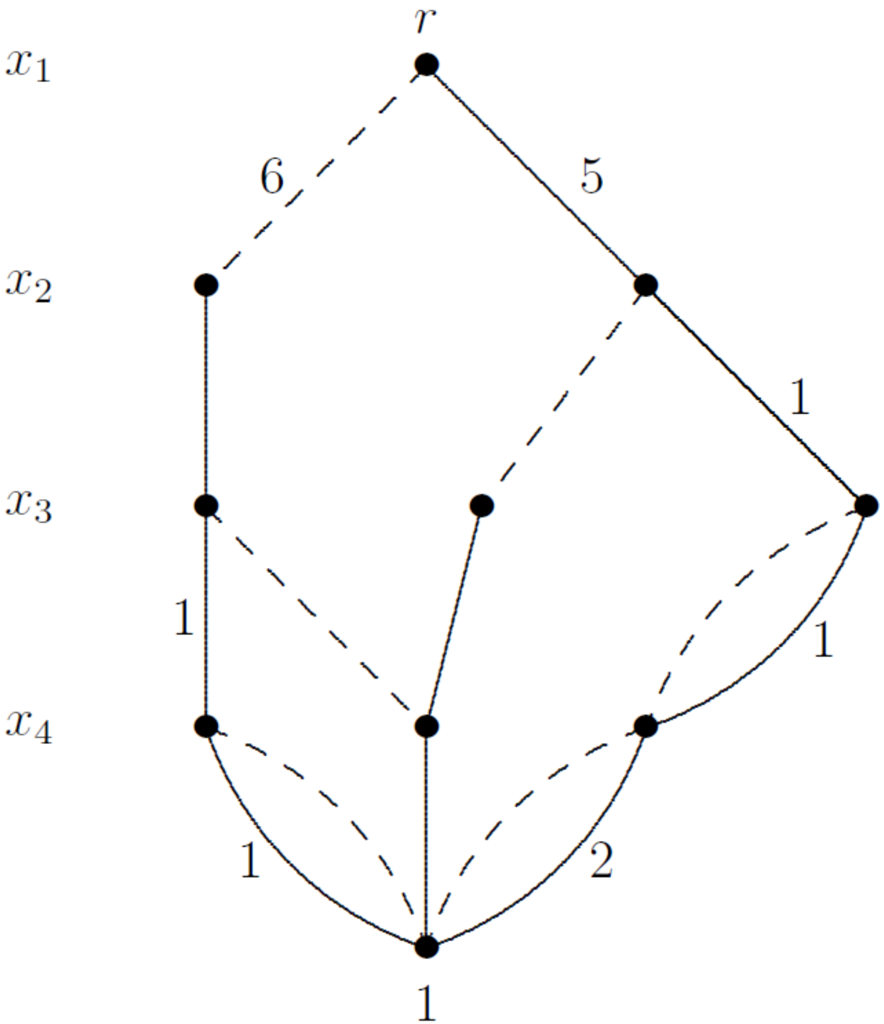
Weighted Decision Diagrams

Nonseparable cost function

We know that without weights, there is a **unique** reduced decision diagram for a given variable ordering.

Is this true for decision diagrams with **canonical** weights?

Yes.



Weighted Decision Diagrams

Definition. Costs on a decision diagram are **canonical** if for every node in layer *i*, the costs c_{ij} leaving that node satisfy

$$\min_j \{c_{ij}\} = \alpha_i$$

for fixed α_i (e.g., 0).

Weighted Decision Diagrams

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Theorem. Any given discrete optimization problem is **uniquely** represented by a weighted decision diagram with canonical costs, for a given variable ordering.

Weighted Decision Diagrams

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Theorem. Any given discrete optimization problem is **uniquely** represented by a weighted decision diagram with canonical costs, for a given variable ordering.

- Similar result proved for Affine Algebraic Decision Diagrams (AADDs) by Sanner and McAllester (IJCAI 2005).
  - Definition of **canonical** is somewhat different.

Weighted Decision Diagrams

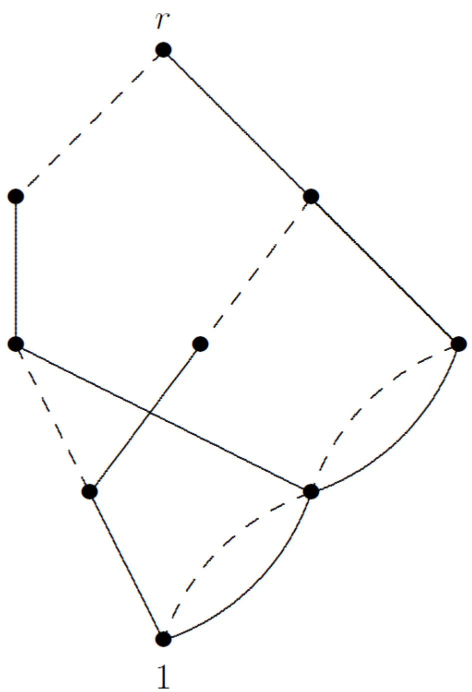
- Converting to canonical costs does not destroy the benefits of separability.

Definition. A decision diagram is **separable** when arc costs represent terms of a separable cost function.

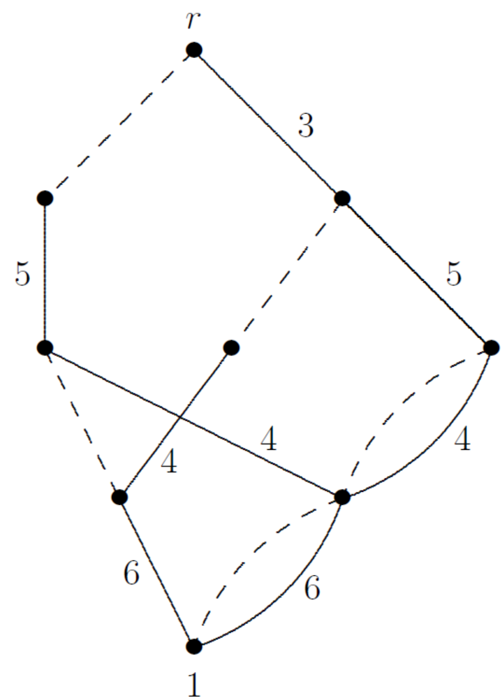
Theorem. A separable decision diagram that is reduced when costs are ignored is also reduced when costs are converted to canonical costs.

Weighted Decision Diagrams

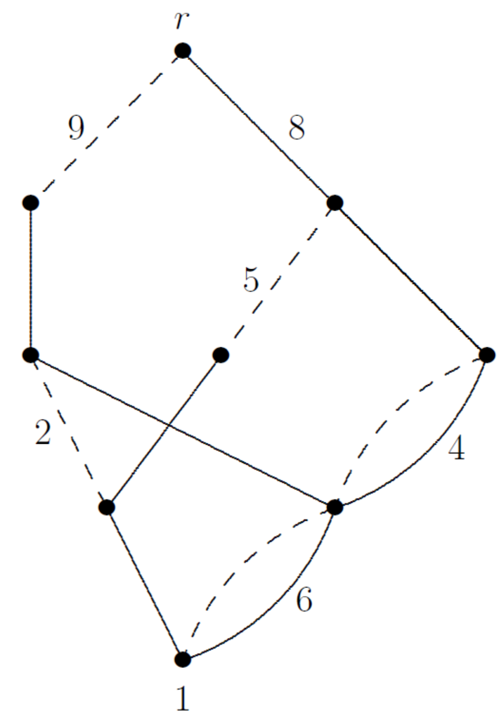
Example



Reduced unweighted DD

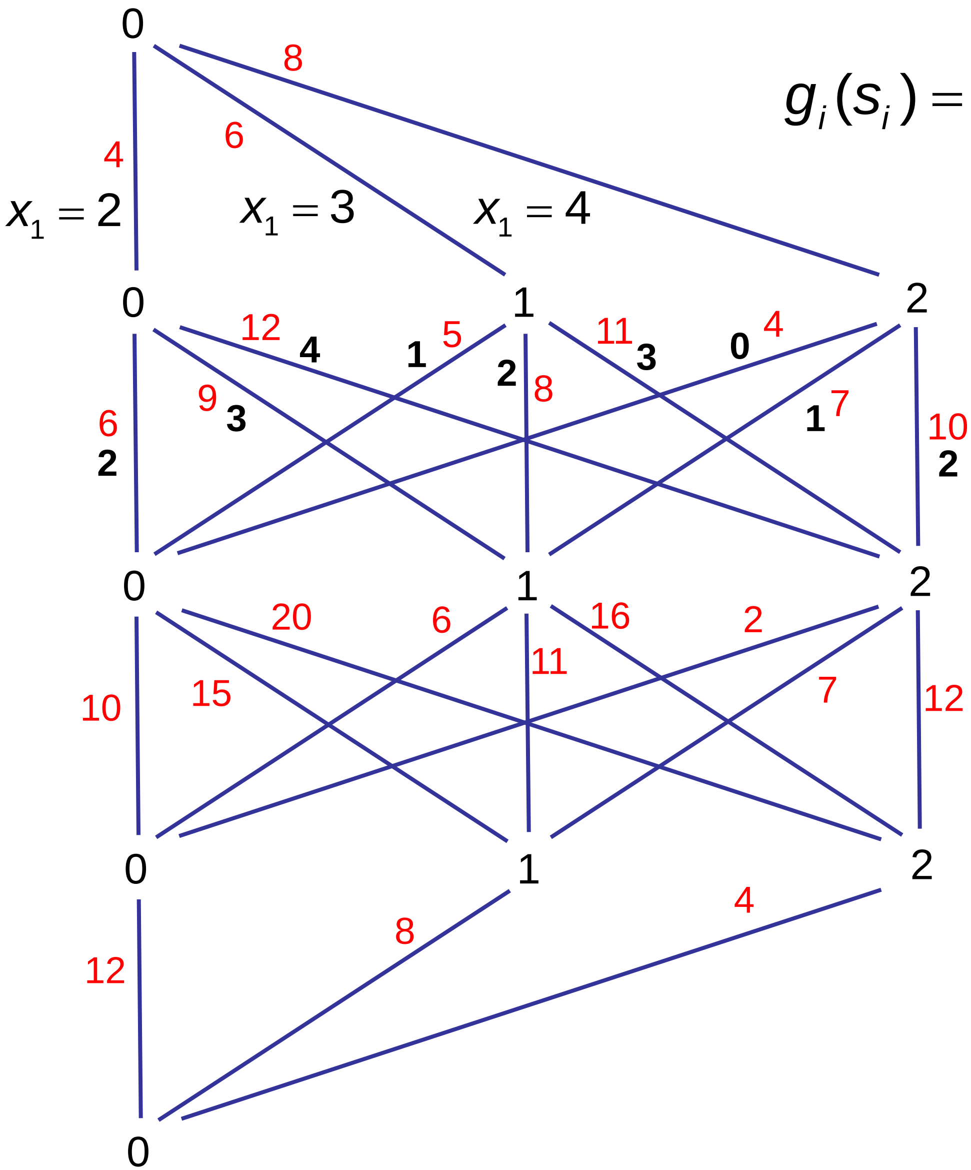


Add separable costs



Reduced weighted DD with canonical costs has same shape

Application to Inventory Problem



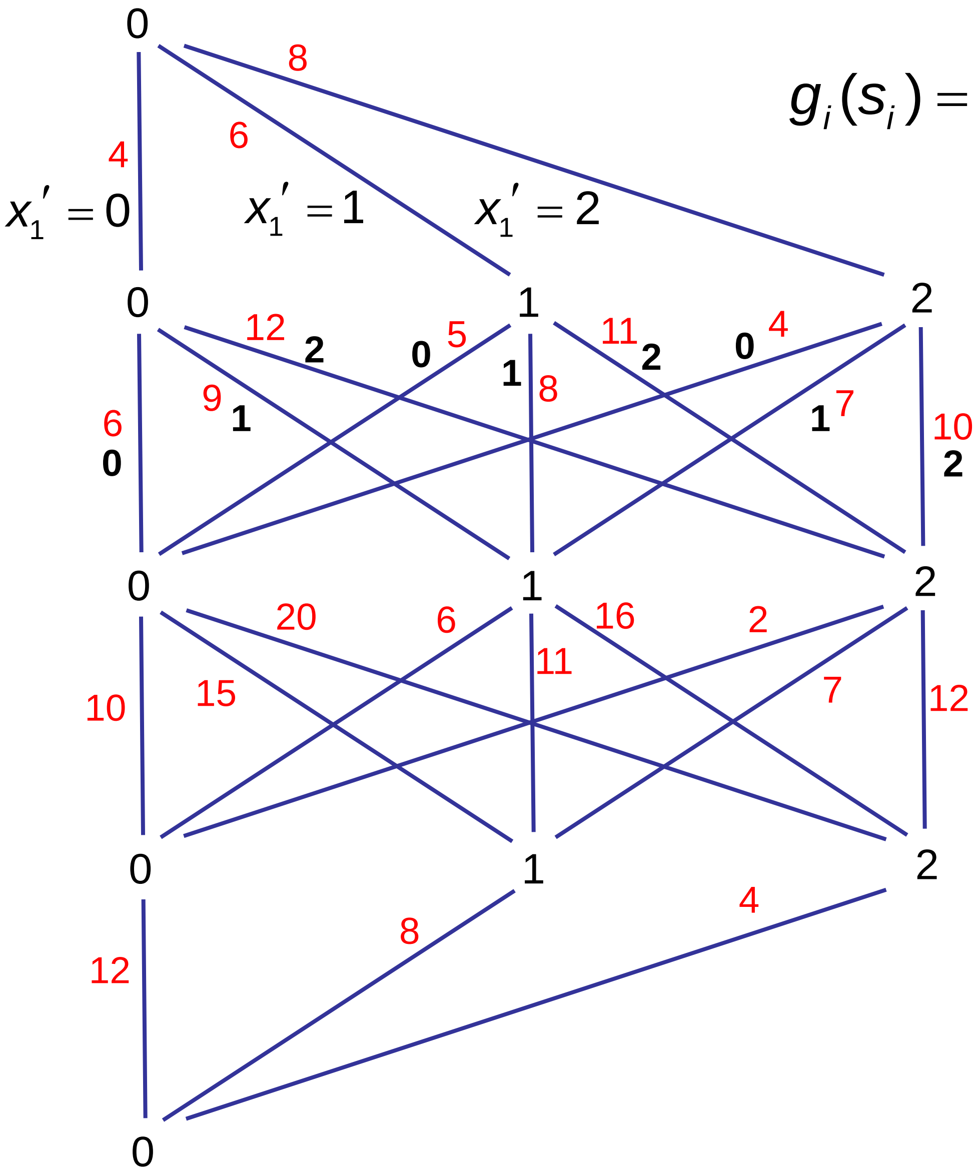
$$g_i(s_i) = \min_{x_i} \{ h_i s_i + c_i x_i + g_{i+1}(s_i + x_i - d_i) \}$$

To equalize controls, let

$$x_i' = s_i + x_i - d_i$$

Be the stock level in next period.

Application to Inventory Problem



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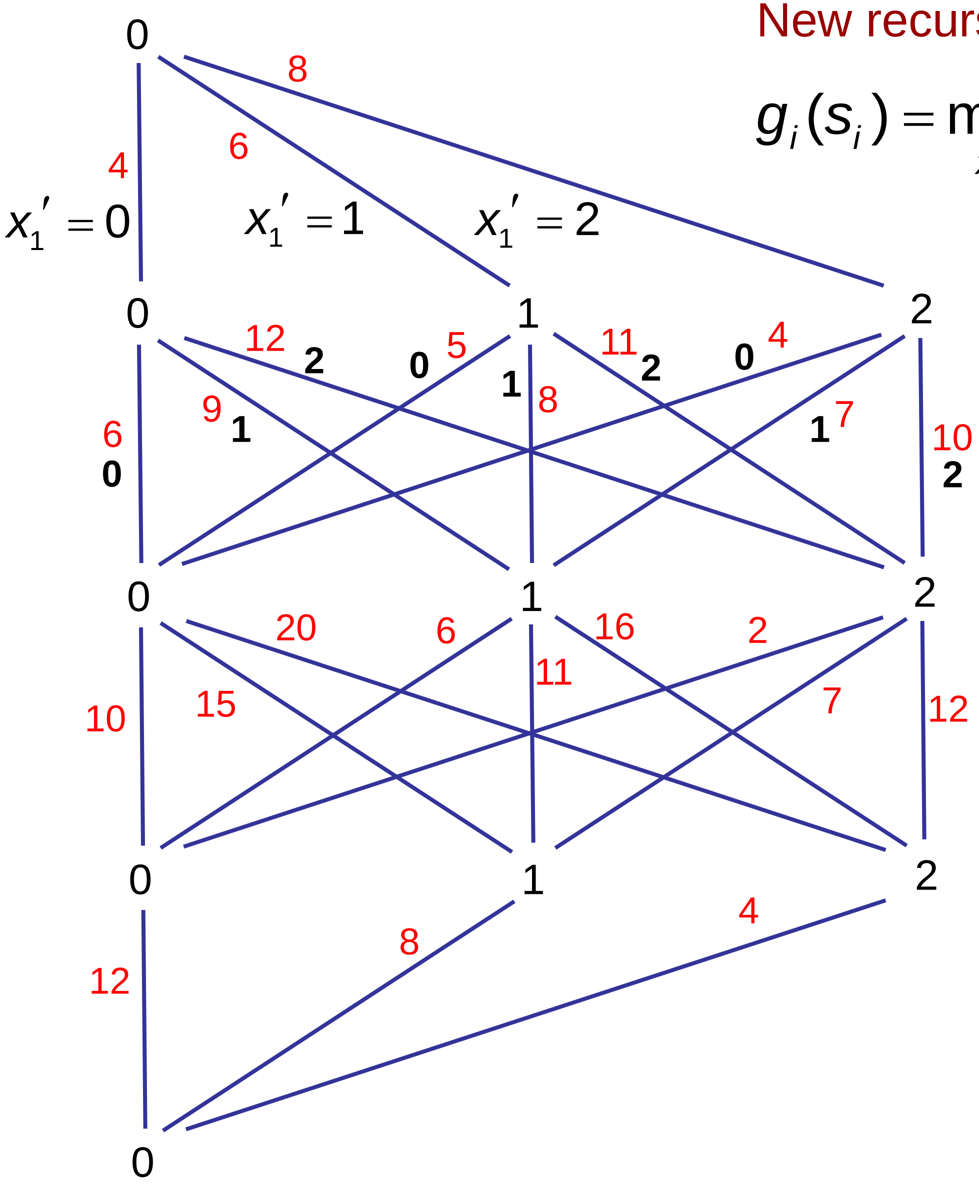


New recursion:

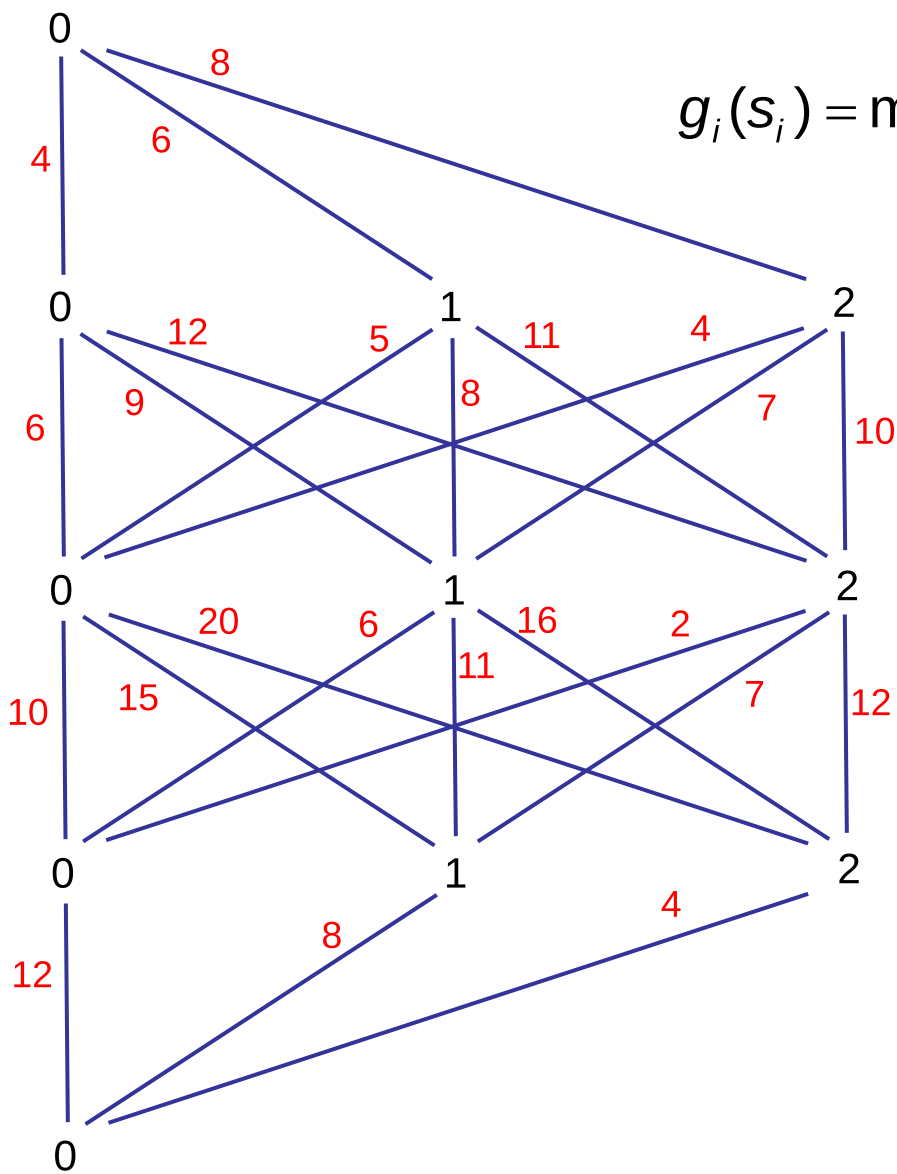
$$g_i(s_i) = \min_{x_i'} \left\{ h_i s_i + c_i(x_i' - s_i + d_i) + g_{i+1}(x_i') \right\}$$

To equalize controls, let

$$x_i' = s_i + x_i - d_i$$

Be the stock level in next period.

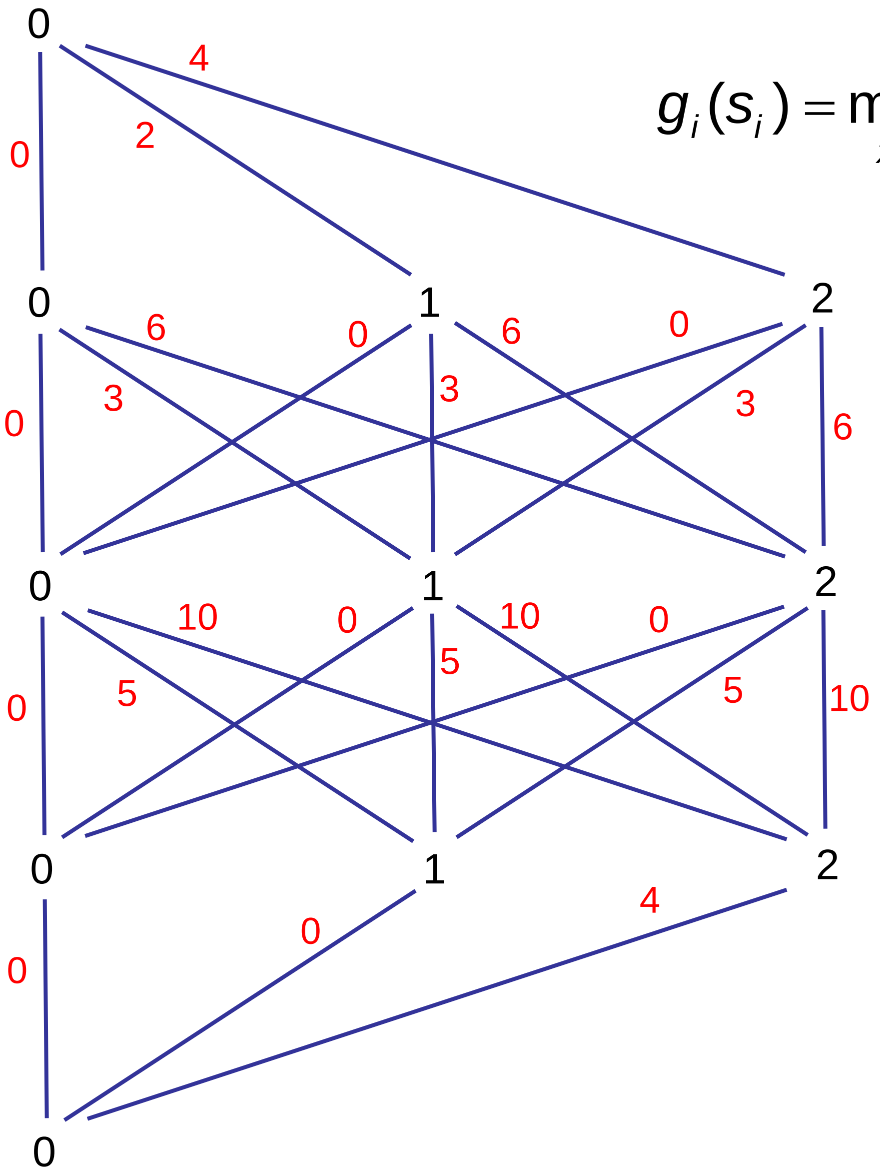
Application to Inventory Problem



$$g_i(s_i) = \min_{x_i'} \left\{ h_i s_i + c_i(x_i' - s_i + d_i) + g_{i+1}(x_i') \right\}$$

To obtain canonical costs, subtract $c_i(m - s_i) + h_i s_i$ from cost on each arc (s_i, s_{i+1}).

Application to Inventory Problem



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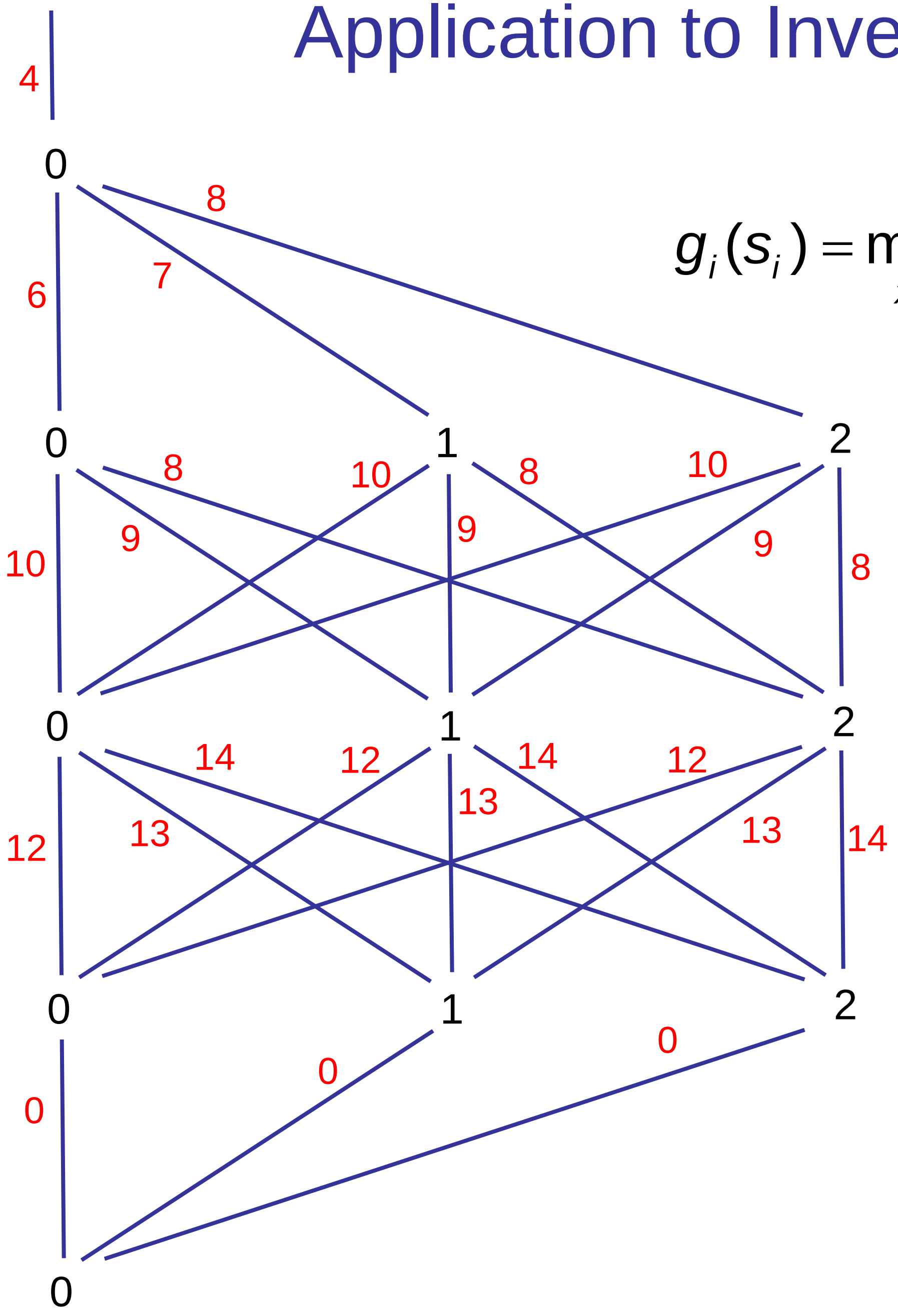
Add these offsets to incoming arcs.

Application to Inventory Problem



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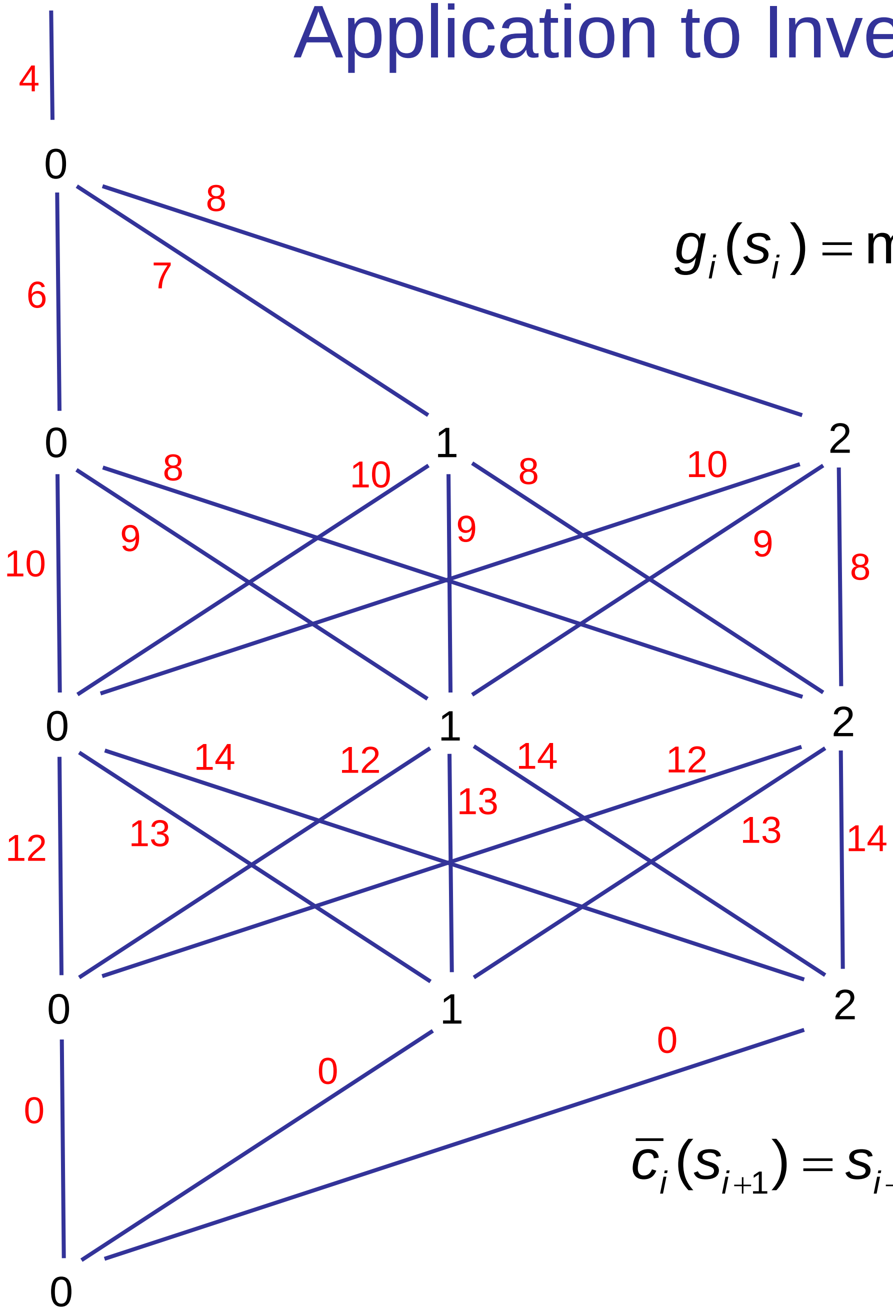

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To obtain canonical costs, subtract $c_i(m - s_i) + h_i s_i$ from cost on each arc (s_i,s_{i+1}).

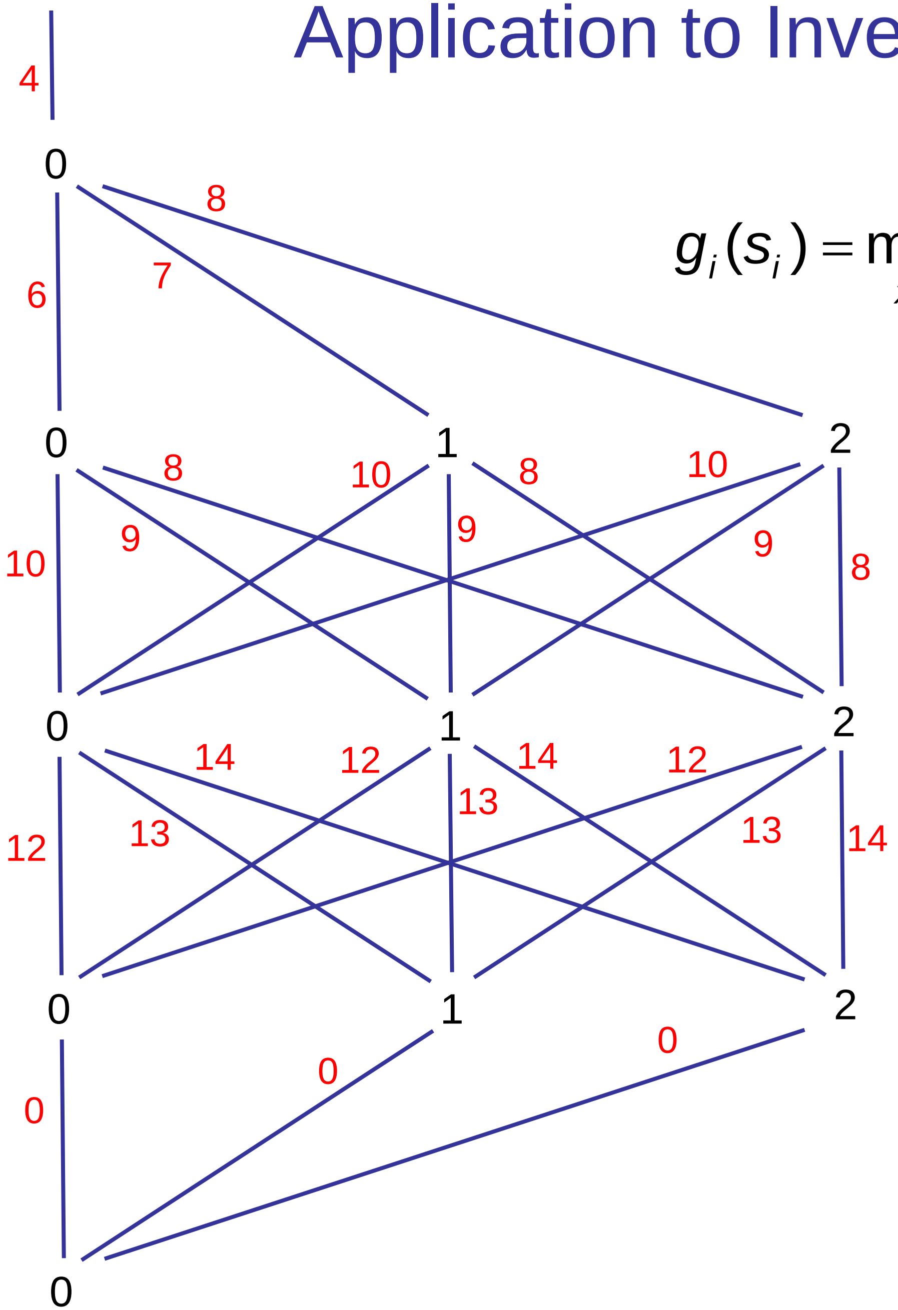
Add these offsets to incoming arcs.

Now outgoing arcs look alike.

And all arcs into state s_i have the same cost

$$\bar{c}_i(s_{i+1}) = s_{i+1}h_{i+1} + c_i(d_i - s_{i+1} - m) + c_{i+1}(m - s_{i+1})$$

Application to Inventory Problem

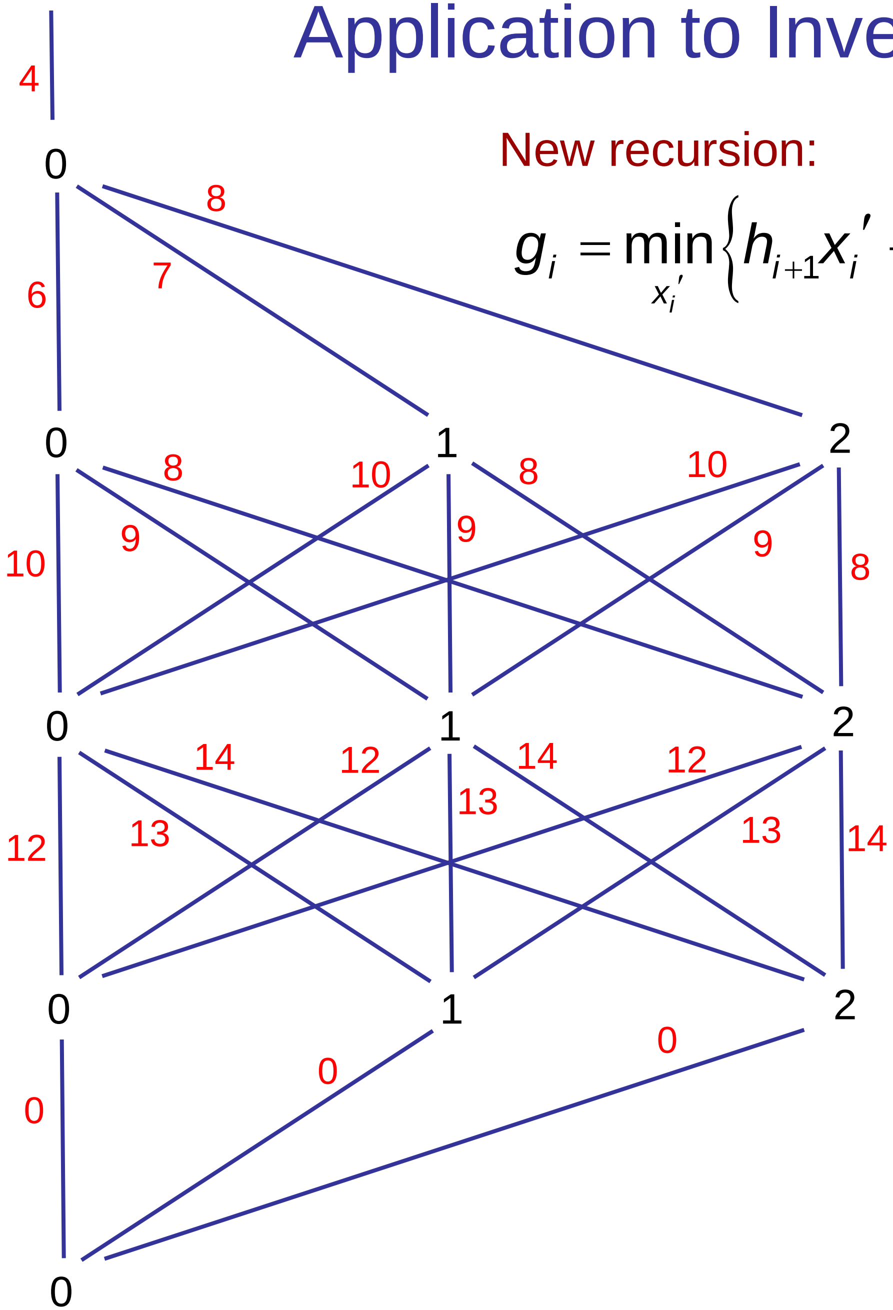


$$g_i(s_i) = \min_{x_i'} \left\{ h_i s_i + c_i(x_i' - s_i + d_i) + g_{i+1}(x_i') \right\}$$

These are canonical costs with

$$\alpha_i = \min_{s_{i+1}} \left\{ \bar{c}_i(s_{i+1}) \right\}$$

Application to Inventory Problem



New recursion:

$$g_i = \min_{x_i'} \left\{ h_{i+1} x_i' + c_i (x_i' - m + d_i) + c_{i+1}(m - x_i') + g_{i+1} \right\}$$

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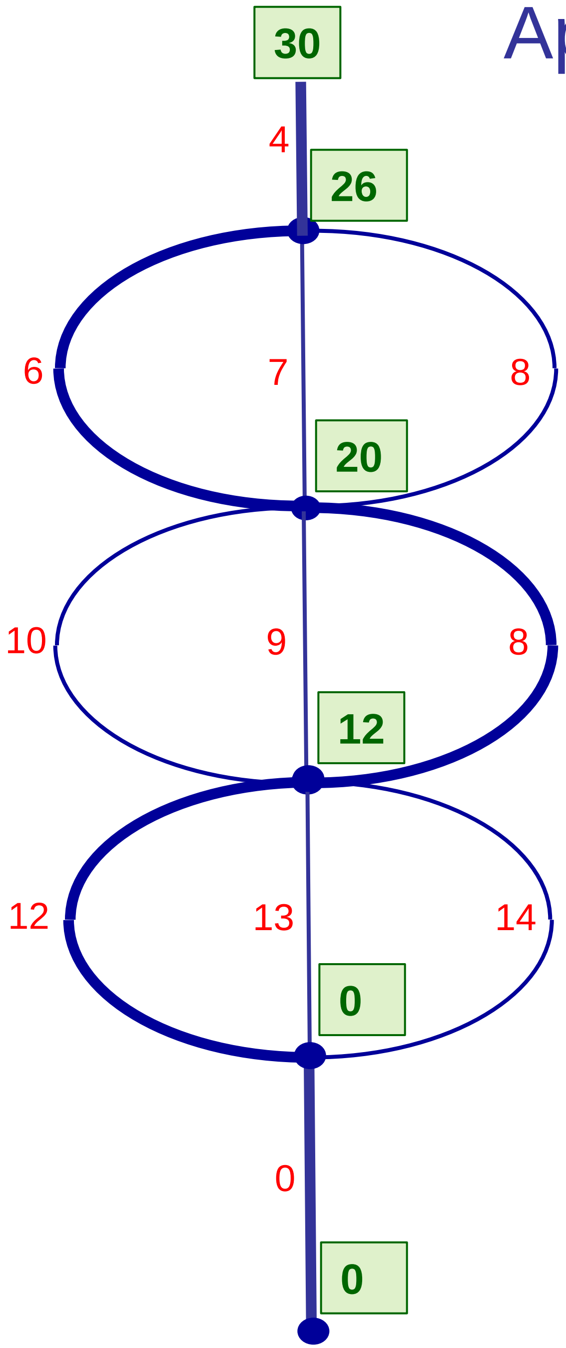
Application to Inventory Problem



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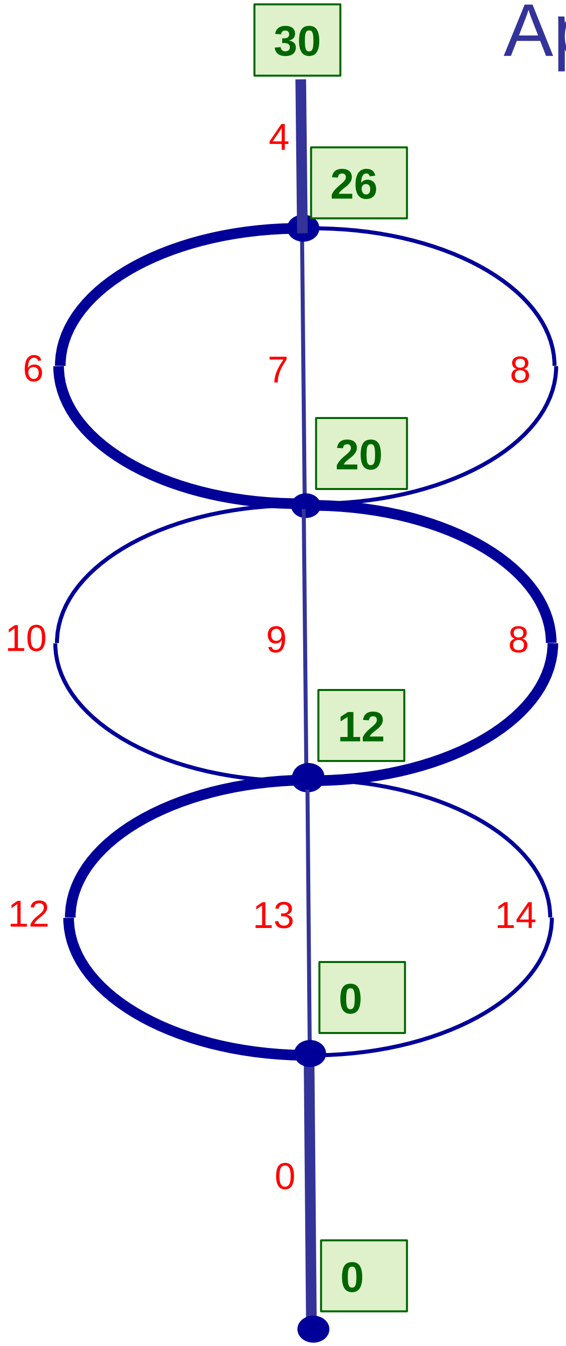
Now there is only one state per period.



Application to Inventory Problem

New recursion:

$$g_i = \min_{x_i'} \left\{ h_{i+1} x_i' + c_i (x_i' - m + d_i) + c_{i+1} (m - x_i') + g_{i+1} \right\}$$

Now there is only one state per period.

Note that computational tests are not necessary.

We immediately see the speedup from the reduction in the state space.

Ongoing Research

- DP model simplification
  - Go through the classical DP models and see under what conditions they can be simplified.

Ongoing Research

- DP model simplification
  - Go through the classical DP models and see under what conditions they can be simplified.
- DP models for optimization based on decision diagrams
  - Use DP model as basis for building **relaxed** decision diagram.
  - Relaxed decision diagram provides bounds and branching mechanism.