Optimization Bounds from Binary Decision Diagrams

J. N. Hooker

Joint work with
David Bergman, André Ciré, Willem-Jan van Hoeve

Carnegie Mellon University

Journal Presentation Track
CP 2014, Lyon, France

Based on…

D. Bergman, A.A. Ciré, W.-J. van Hoeve, J.N. Hooker,
Optimization bounds from binary decision diagrams,
INFORMS Journal on Computing **26** (2014) 253–268.

Binary Decision Diagrams

- **BDD**s historically used for circuit design and verification.
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- **Compact** graphical representation of **boolean** function.
  - Can also represent **feasible set** of problem with binary variables.
  - Slight generalization (MDDs) represents **finite domain** variables.
- **Reduced** BDD is result of superimposing isomorphic subtrees in a search tree.
  - Unique reduced BDD for given variable ordering.

A little history…

- First CP workshop was in 1993 (first *conference* 1995)
  - I was there! My paper *competed* with BDDs.
    - Logic circuit verification by Benders decomposition
      The Newport Papers: Proceedings of PPCP 1993

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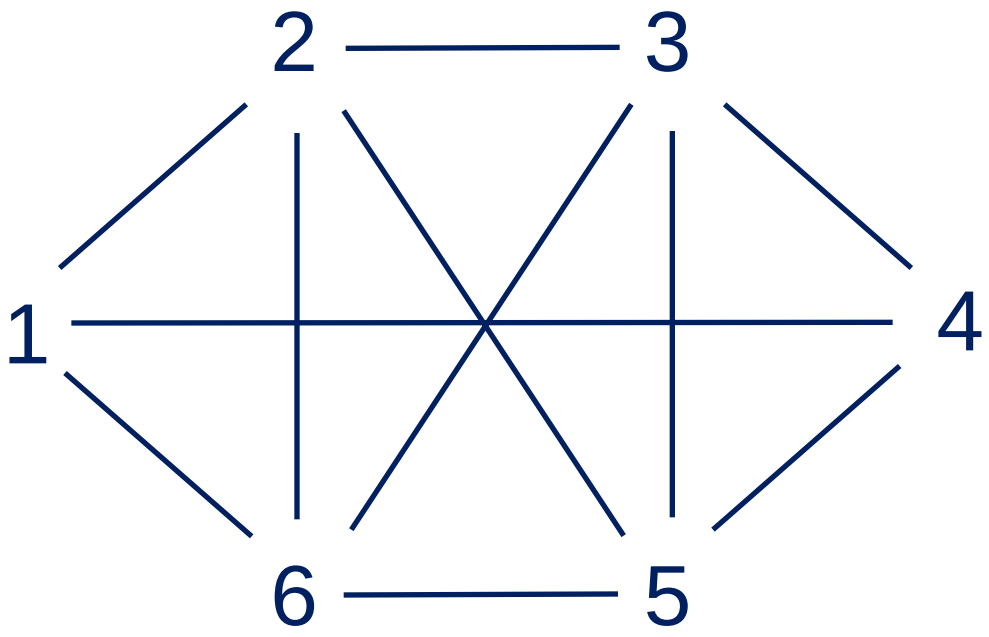
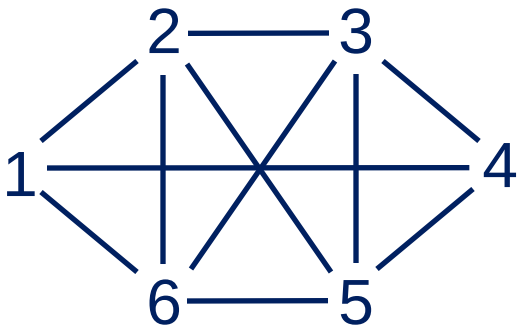
- We focus on **stable set problem** on a graph…

Stable Set Problem

Let each vertex have weight w_i

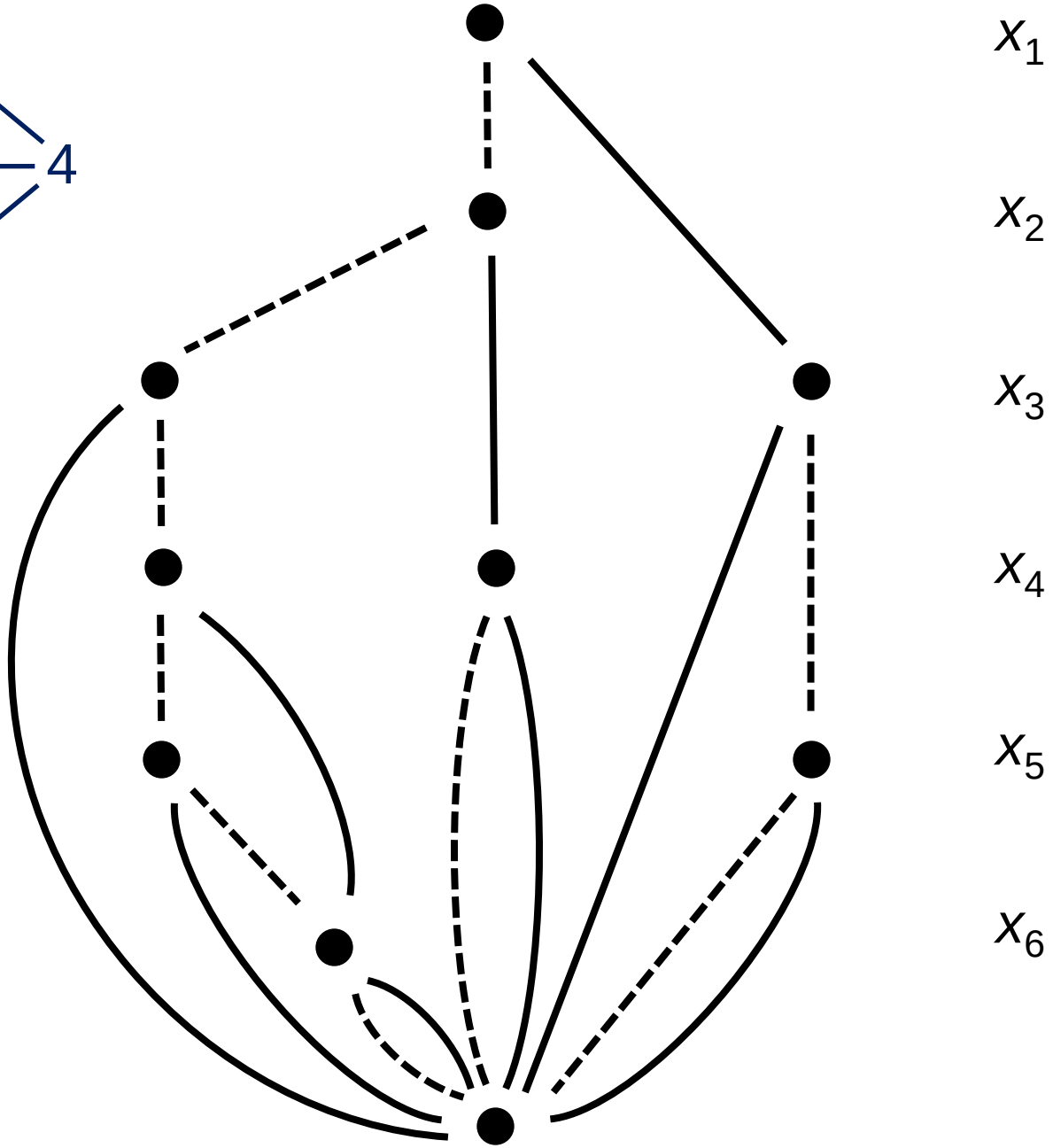
Select set of nonadjacent vertices to maximize $\sum_i w_i x_i$

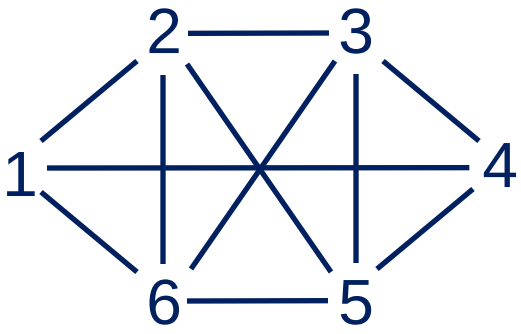




Exact BDD for stable set problem

"zero-suppressed" BDD





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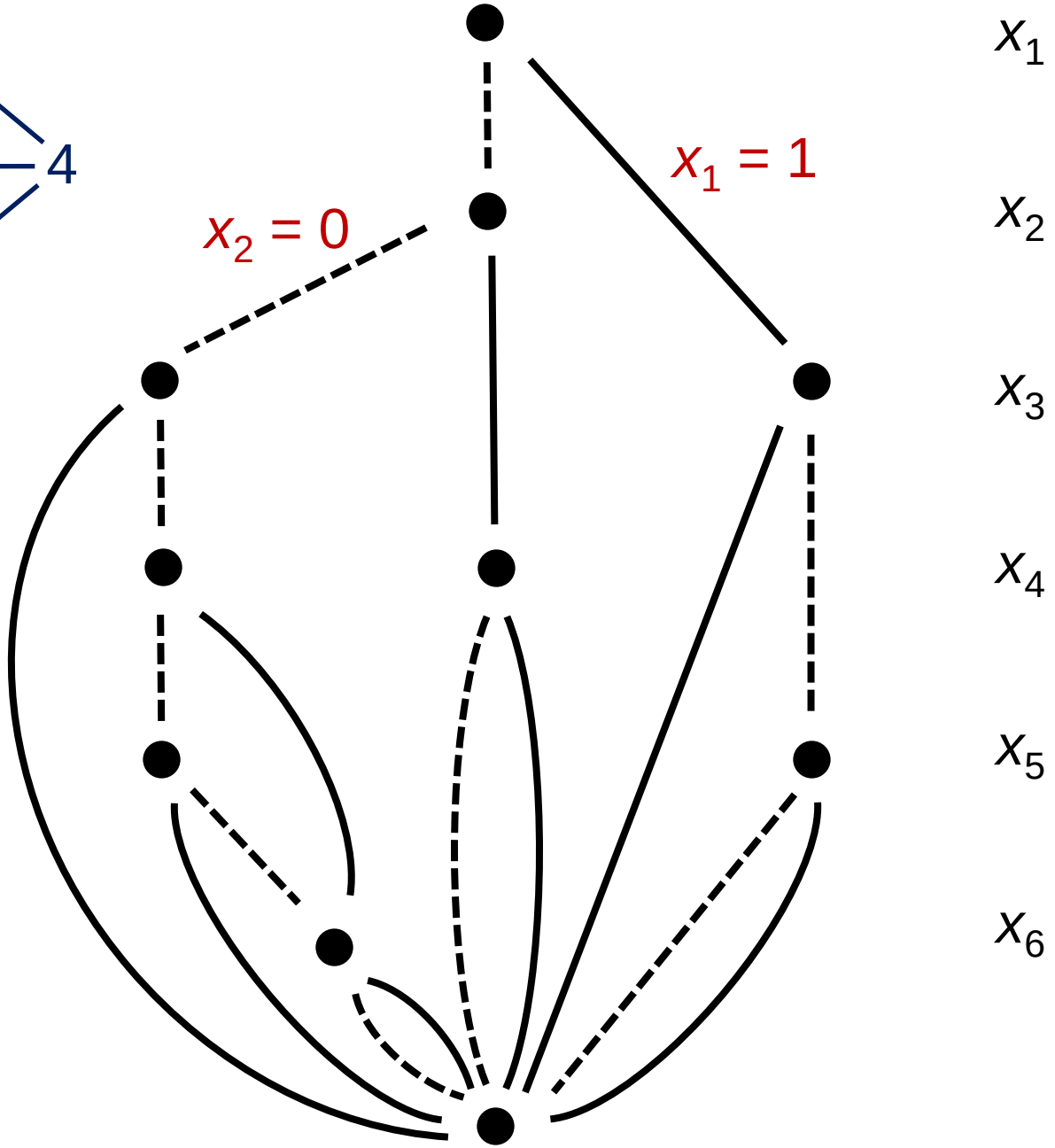
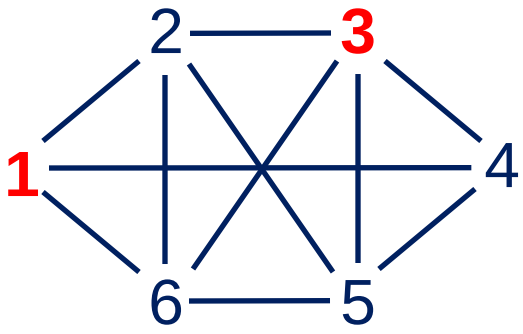
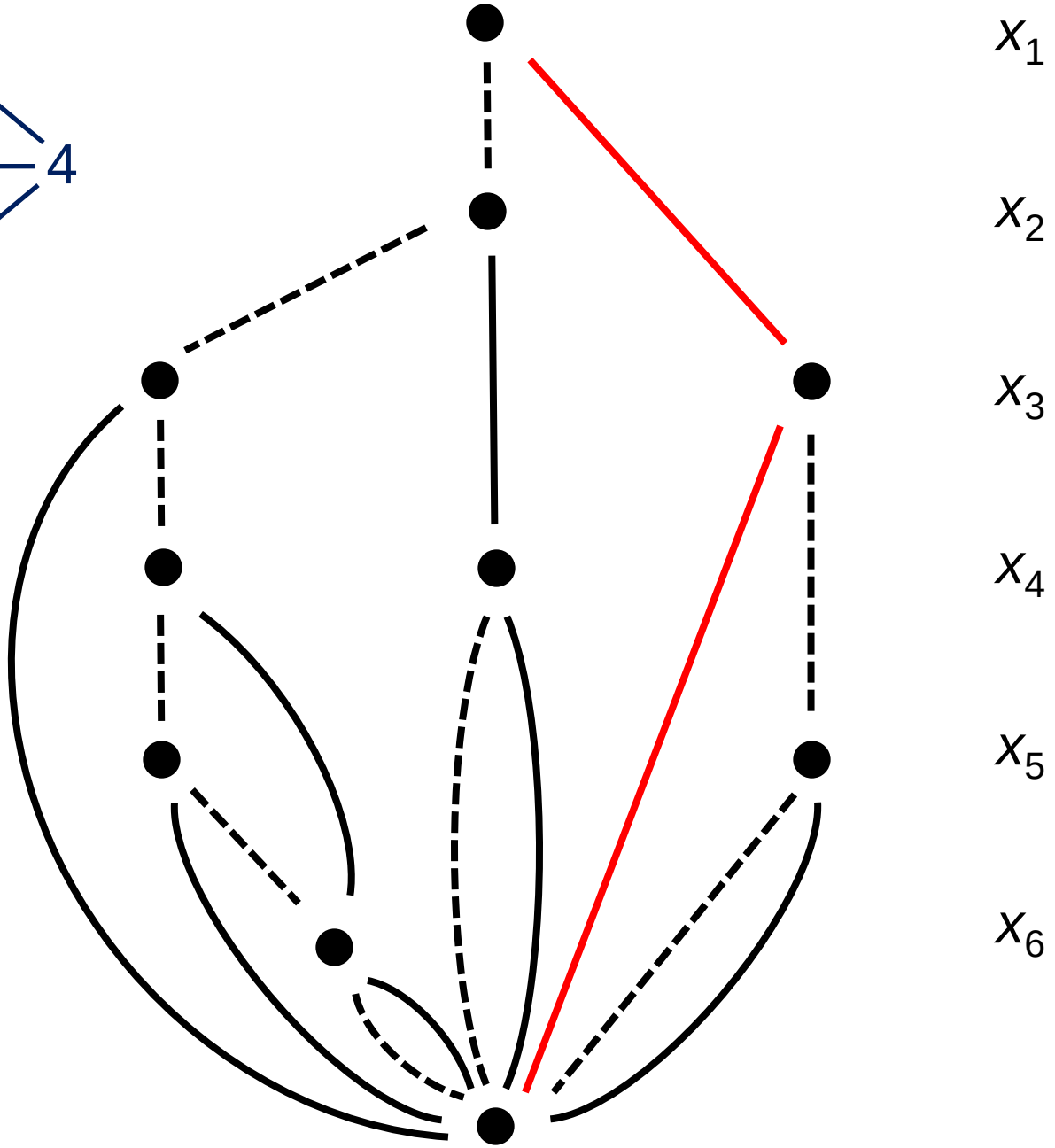
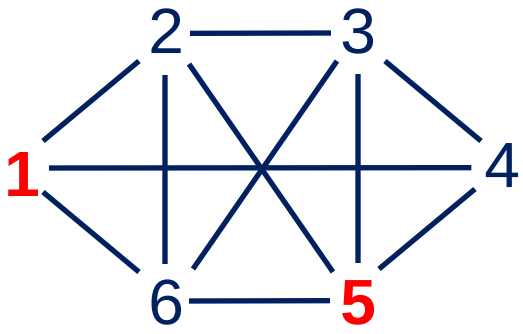




Paths from top to bottom correspond to the 11 feasible solutions





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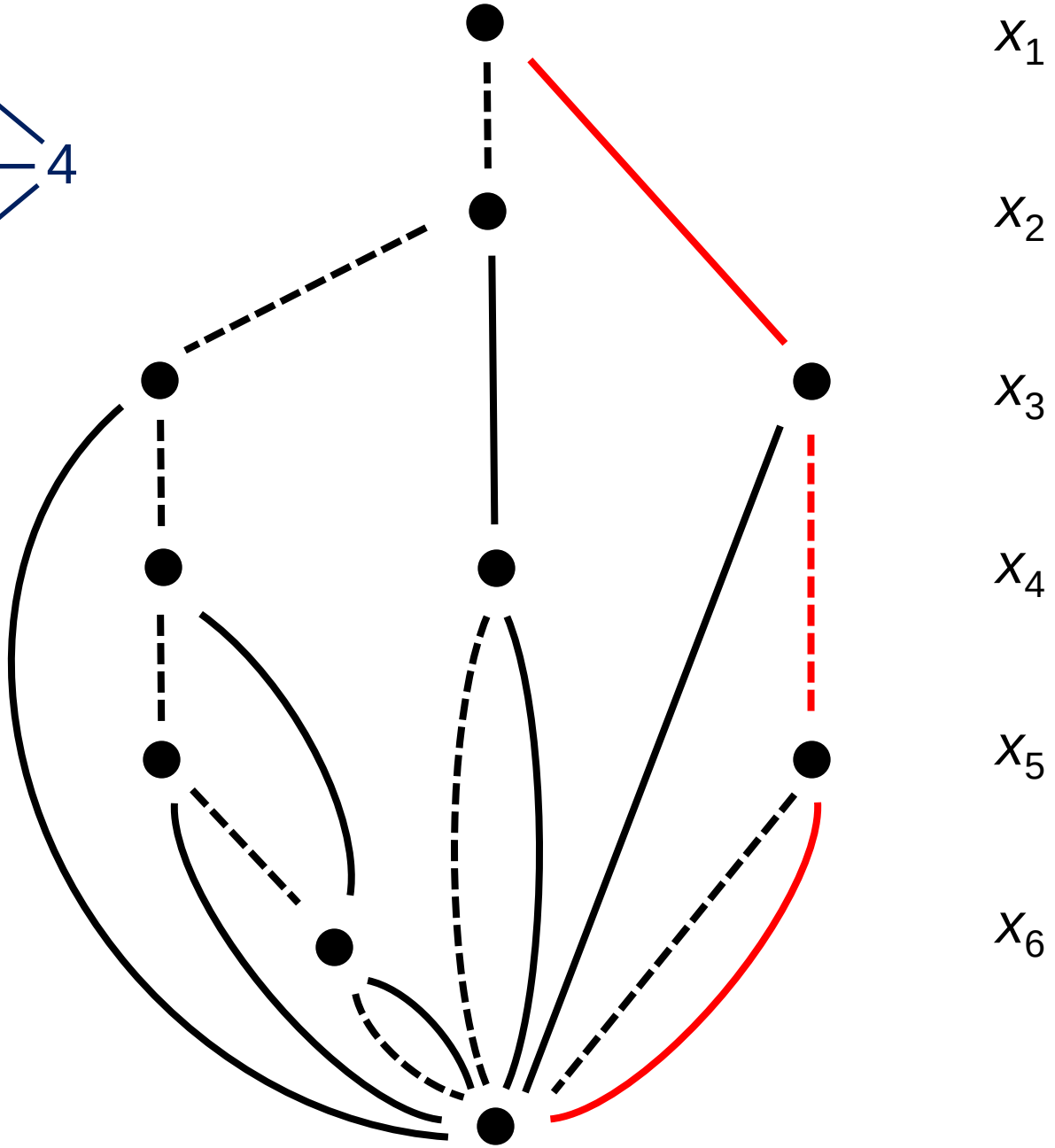

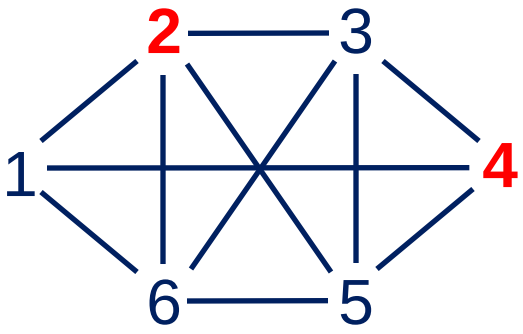

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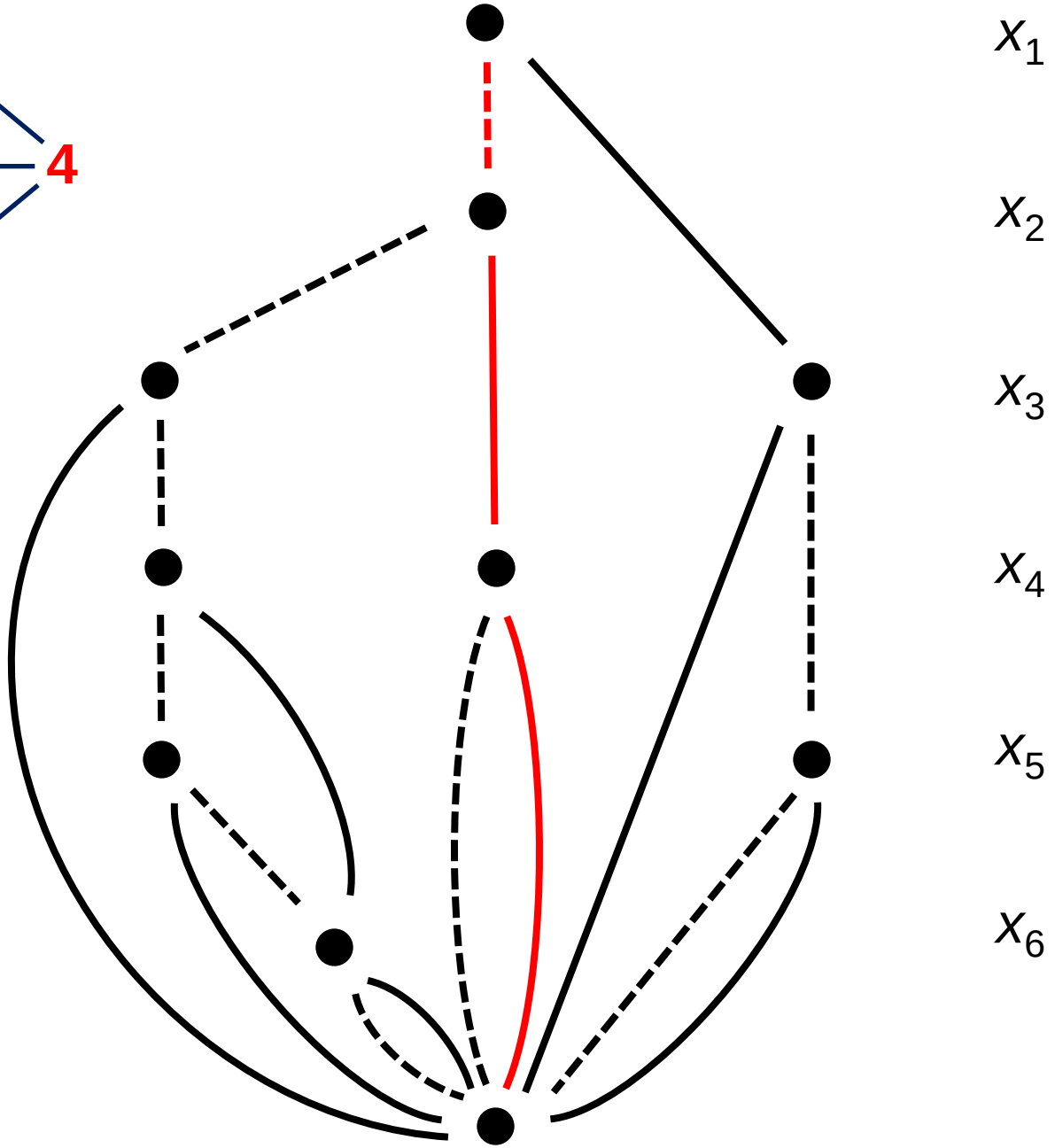

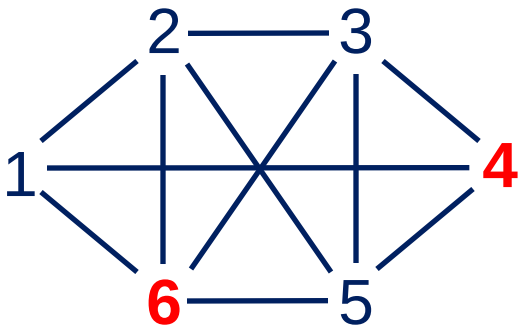

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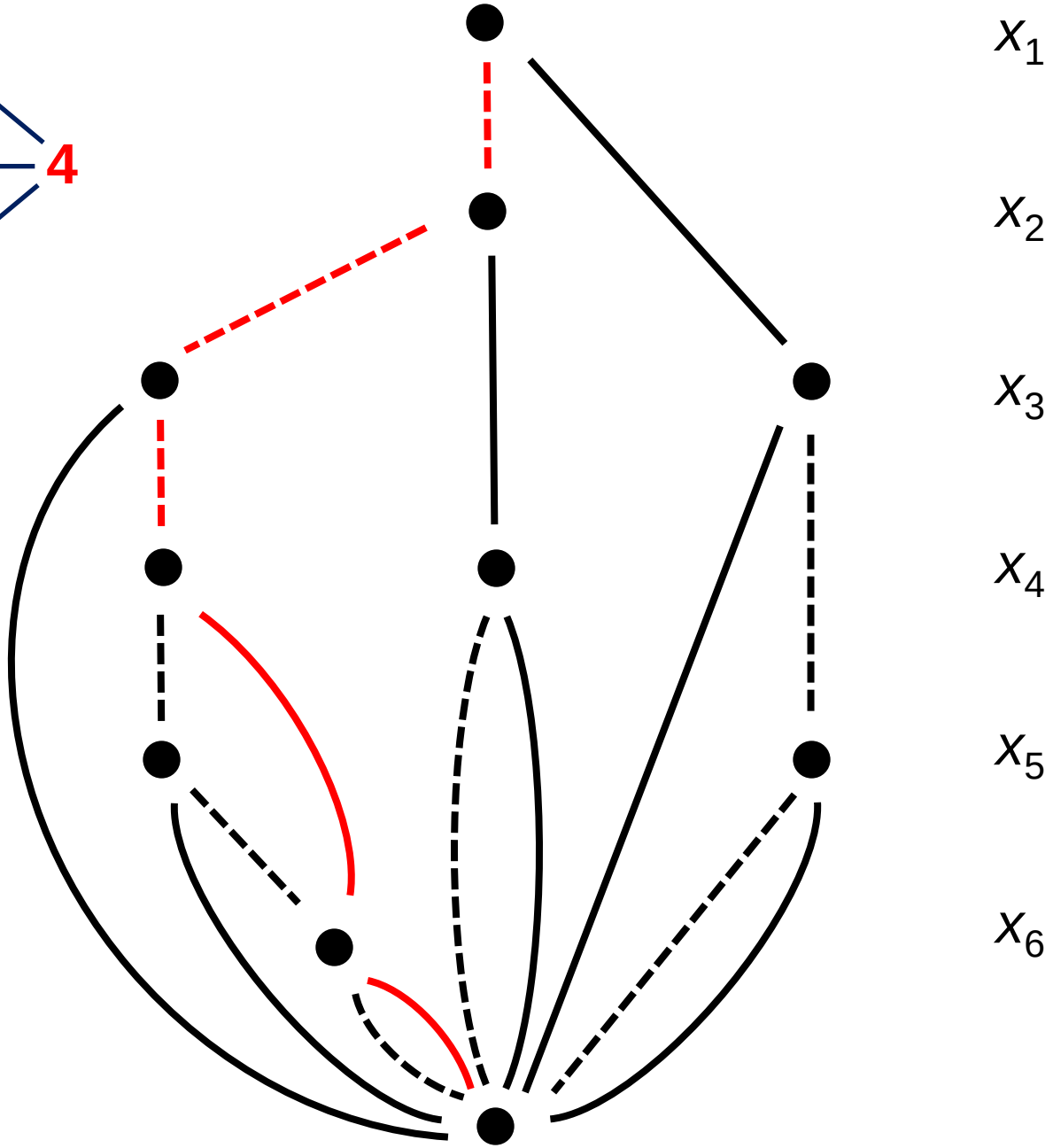

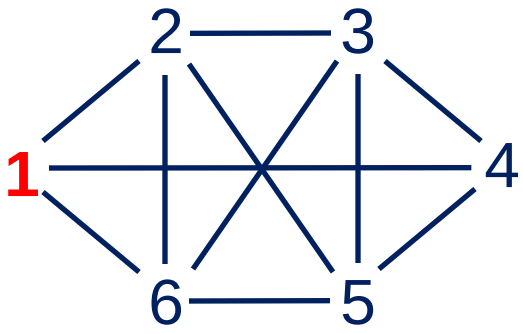

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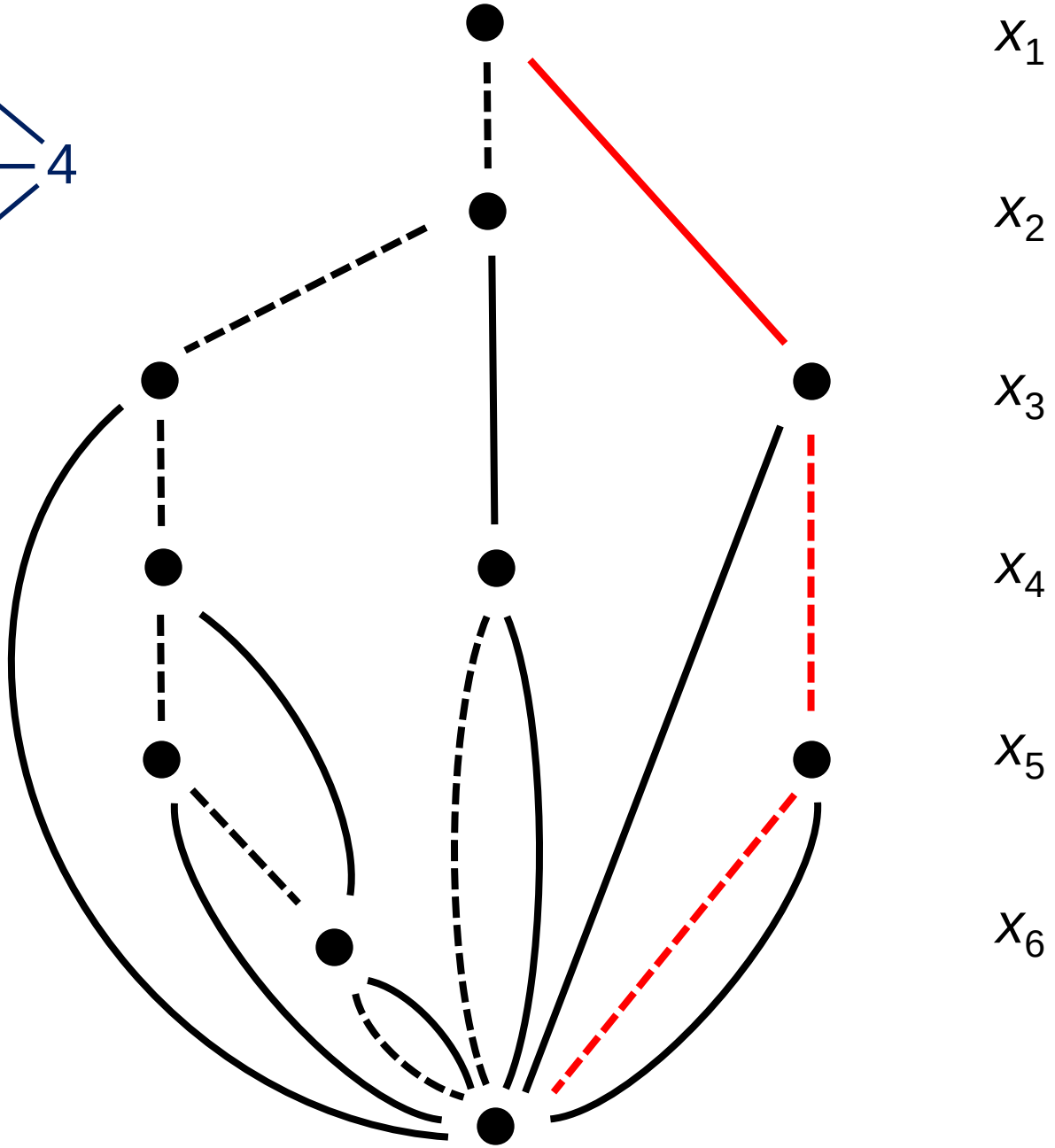
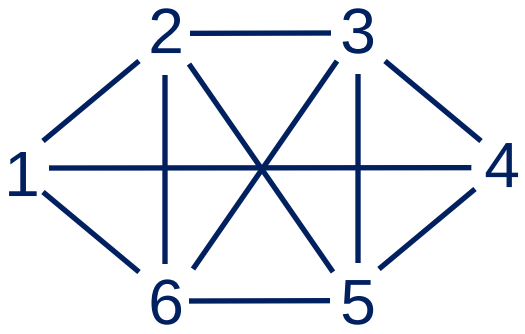
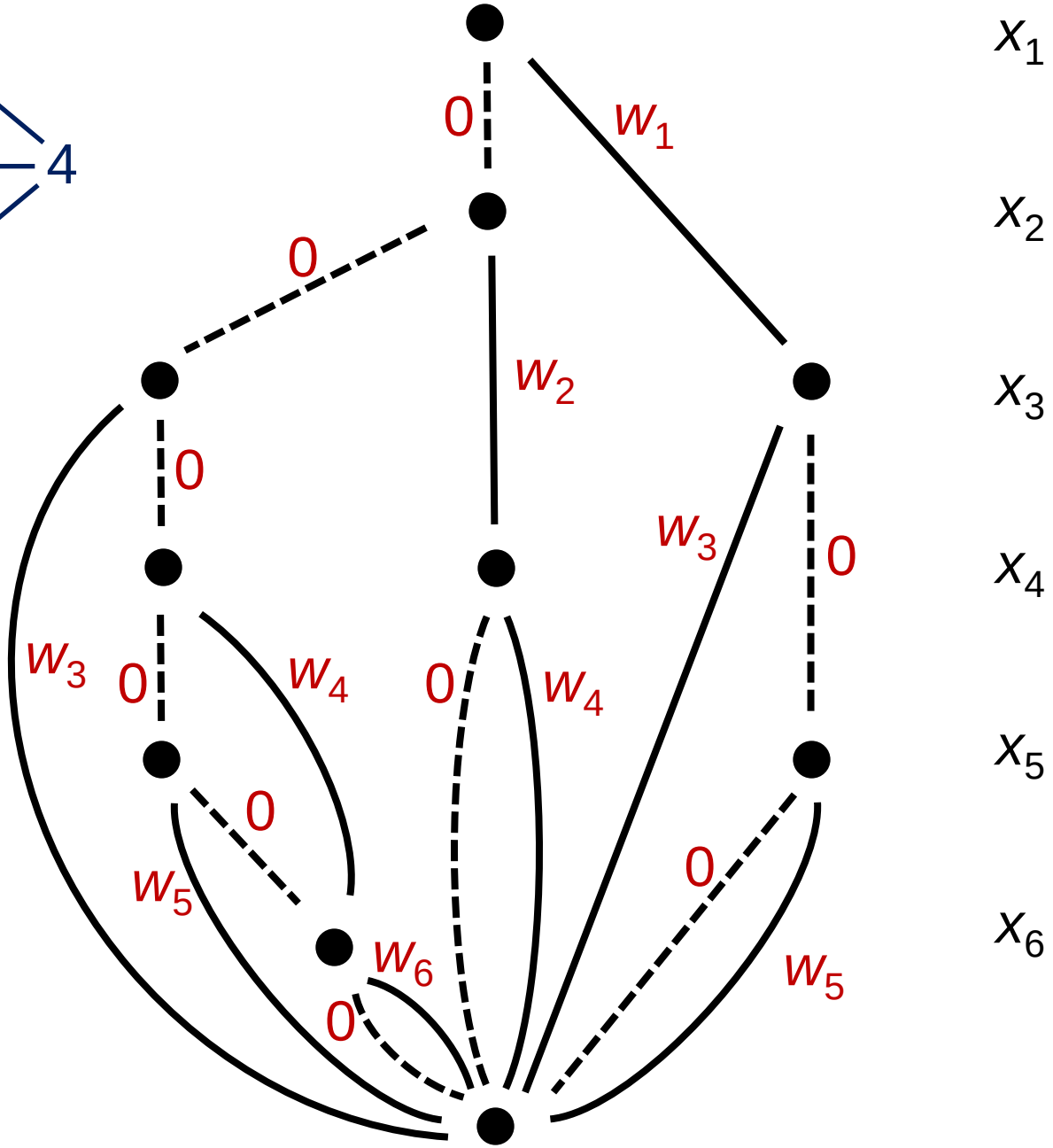
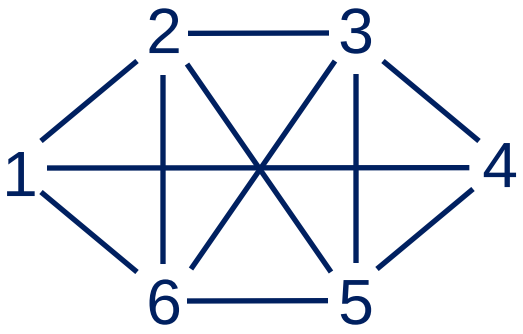
…and so forth





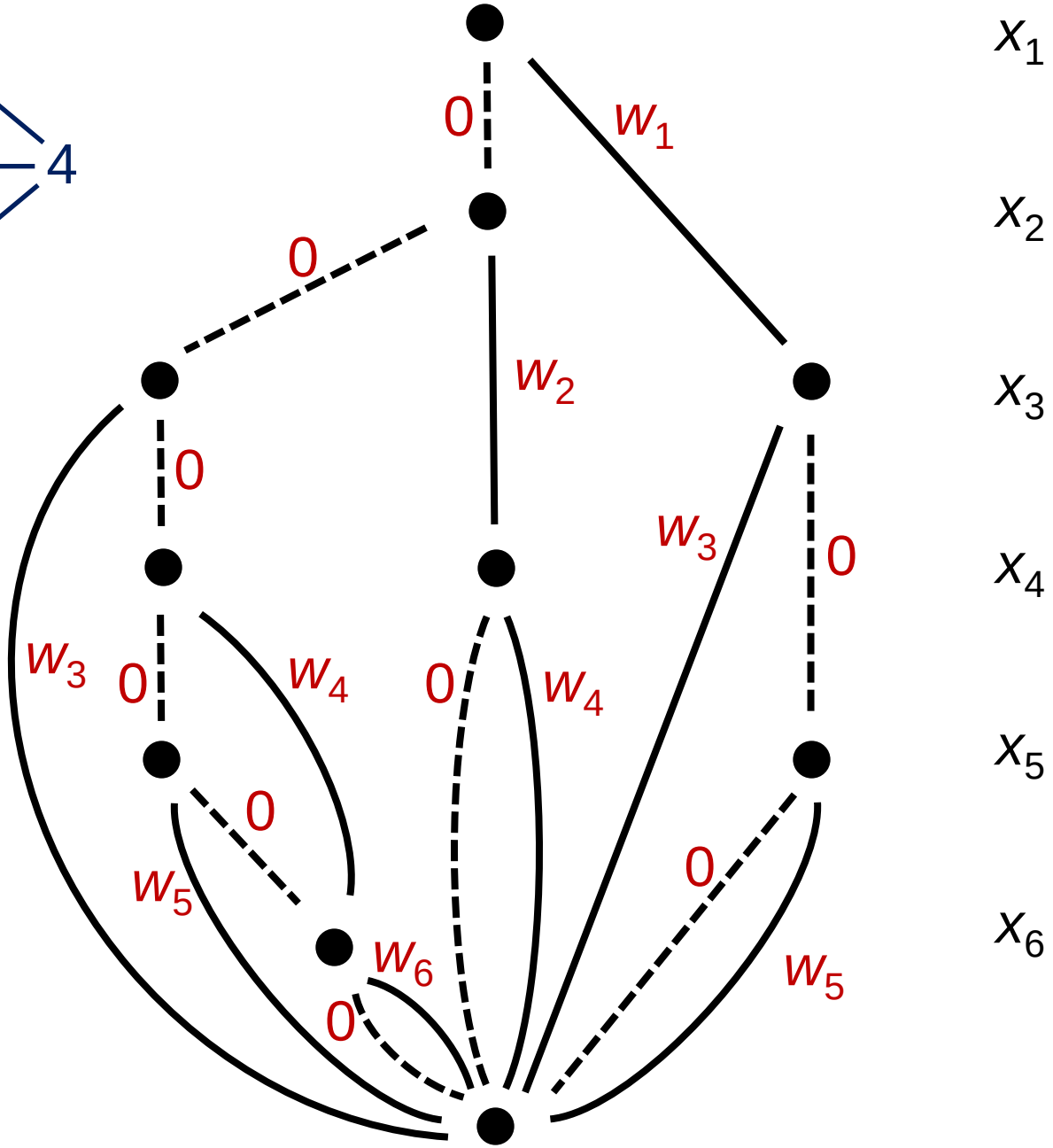
For objective function, associate weights with arcs

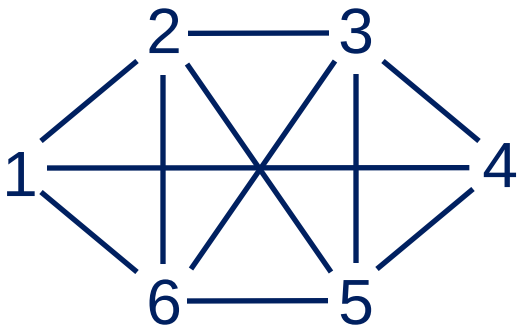


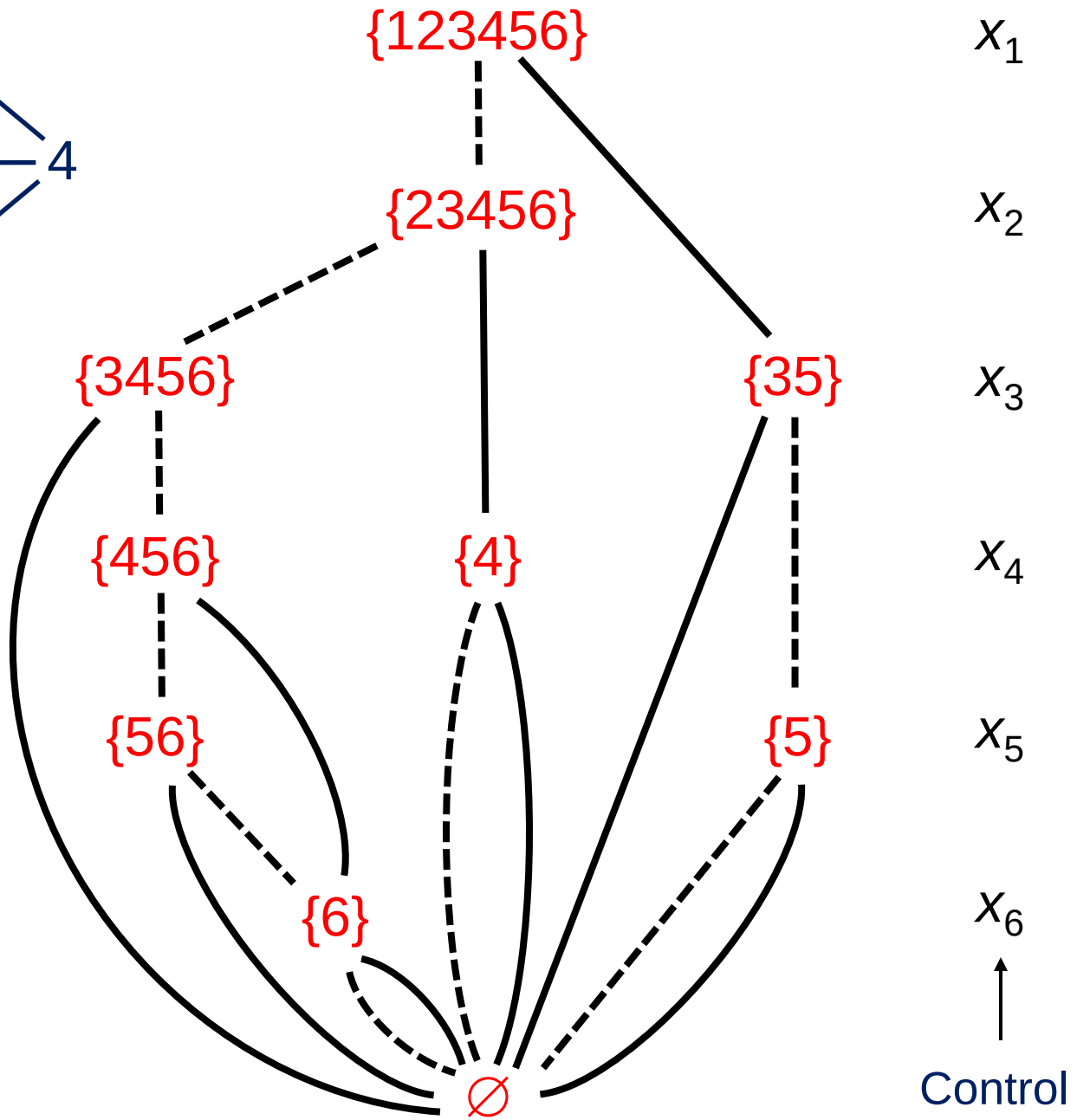


For objective function, associate weights with arcs

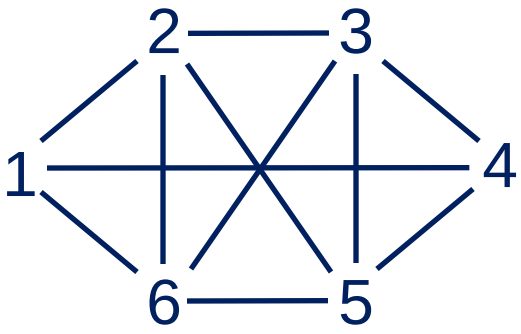
Optimal solution is **longest path**







To build BDD, associate **state** with each node as in dynamic programming



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x_1

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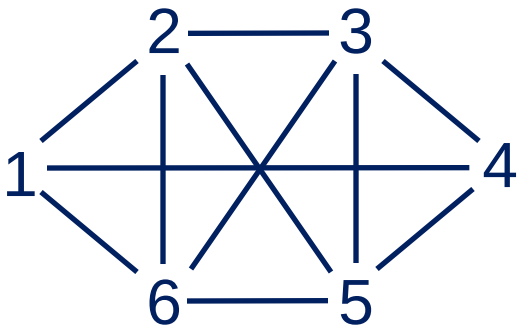
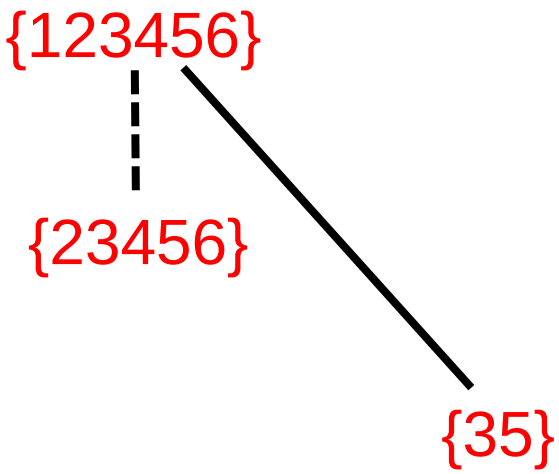
Control

To build BDD, associate **state** with each node





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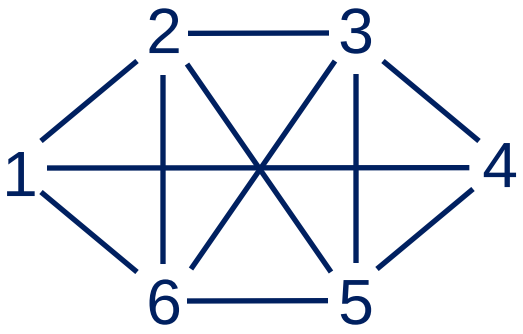

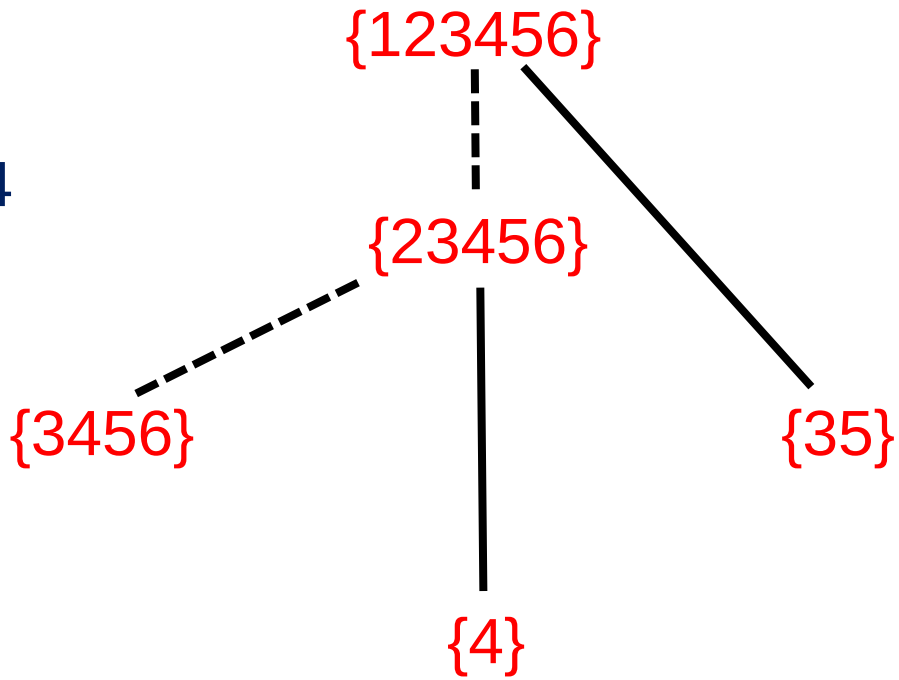

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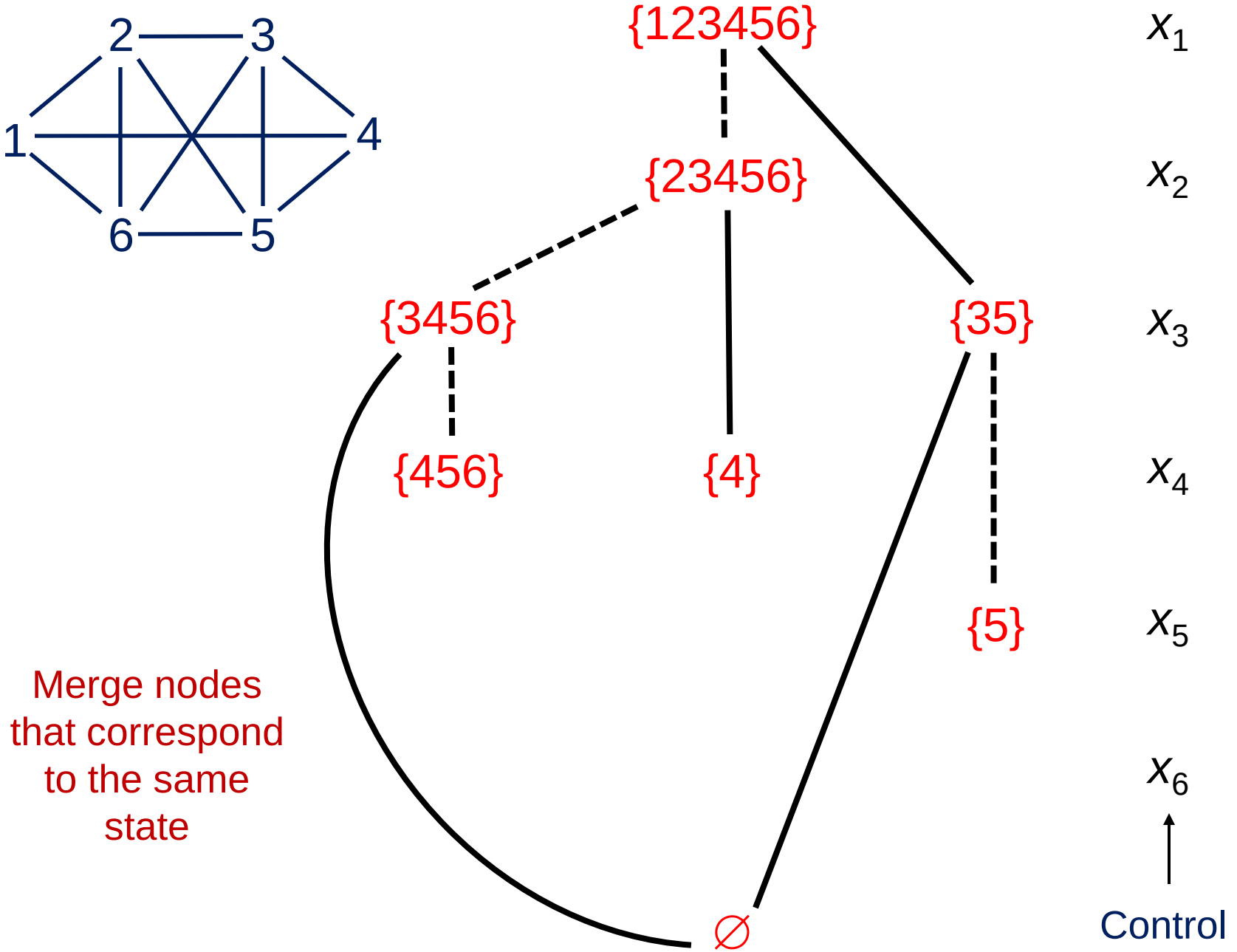
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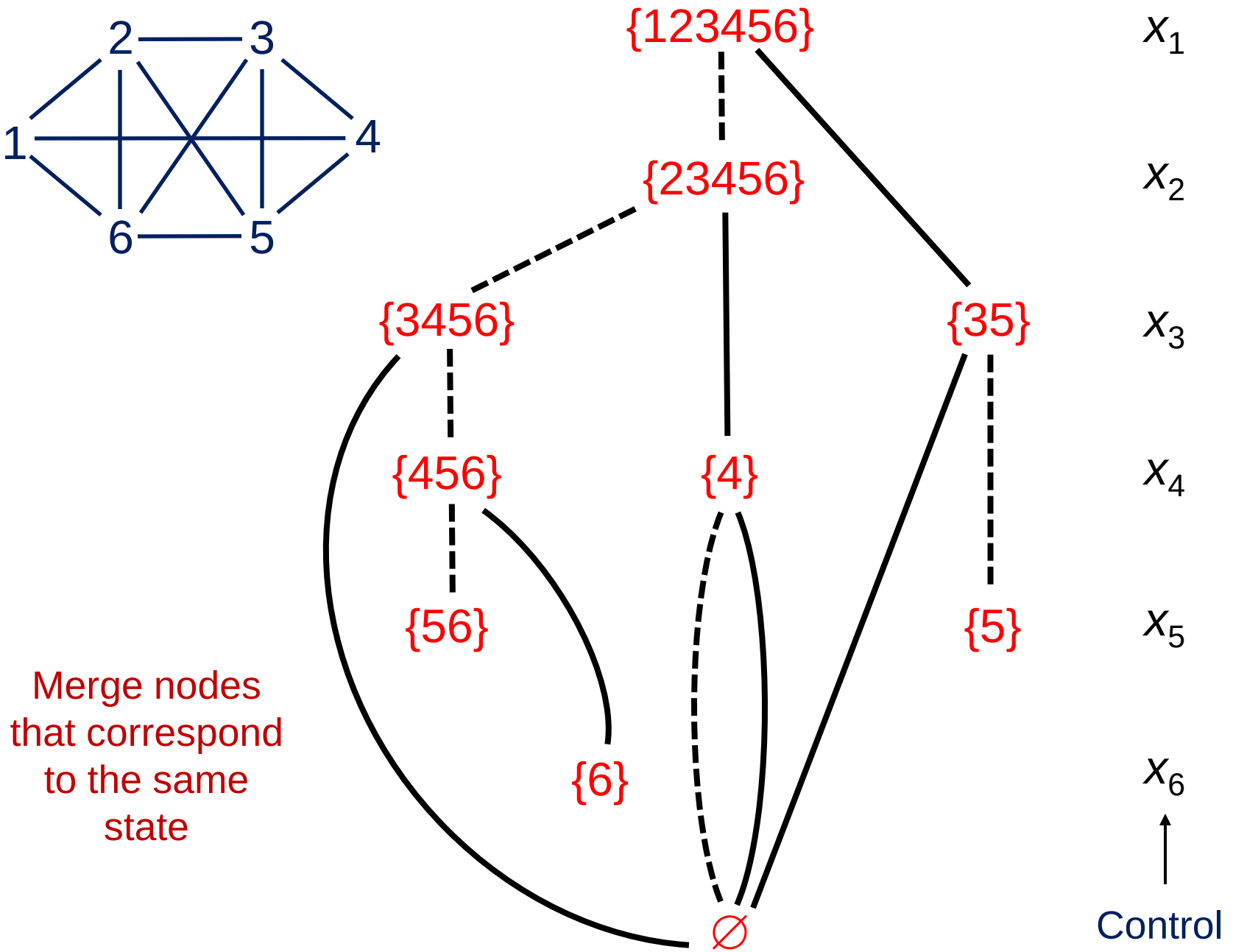
Control
Merge nodes that correspond to the same state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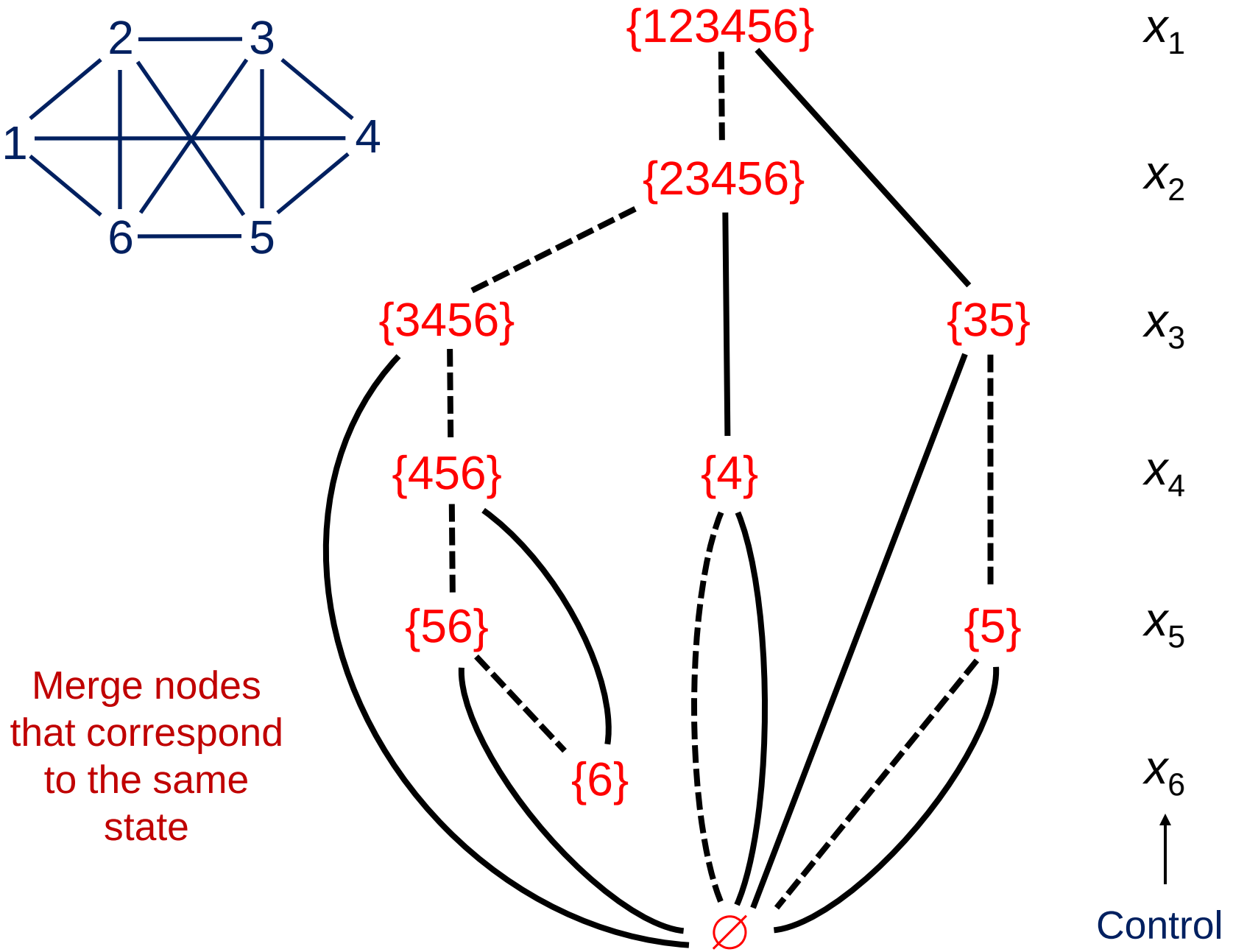
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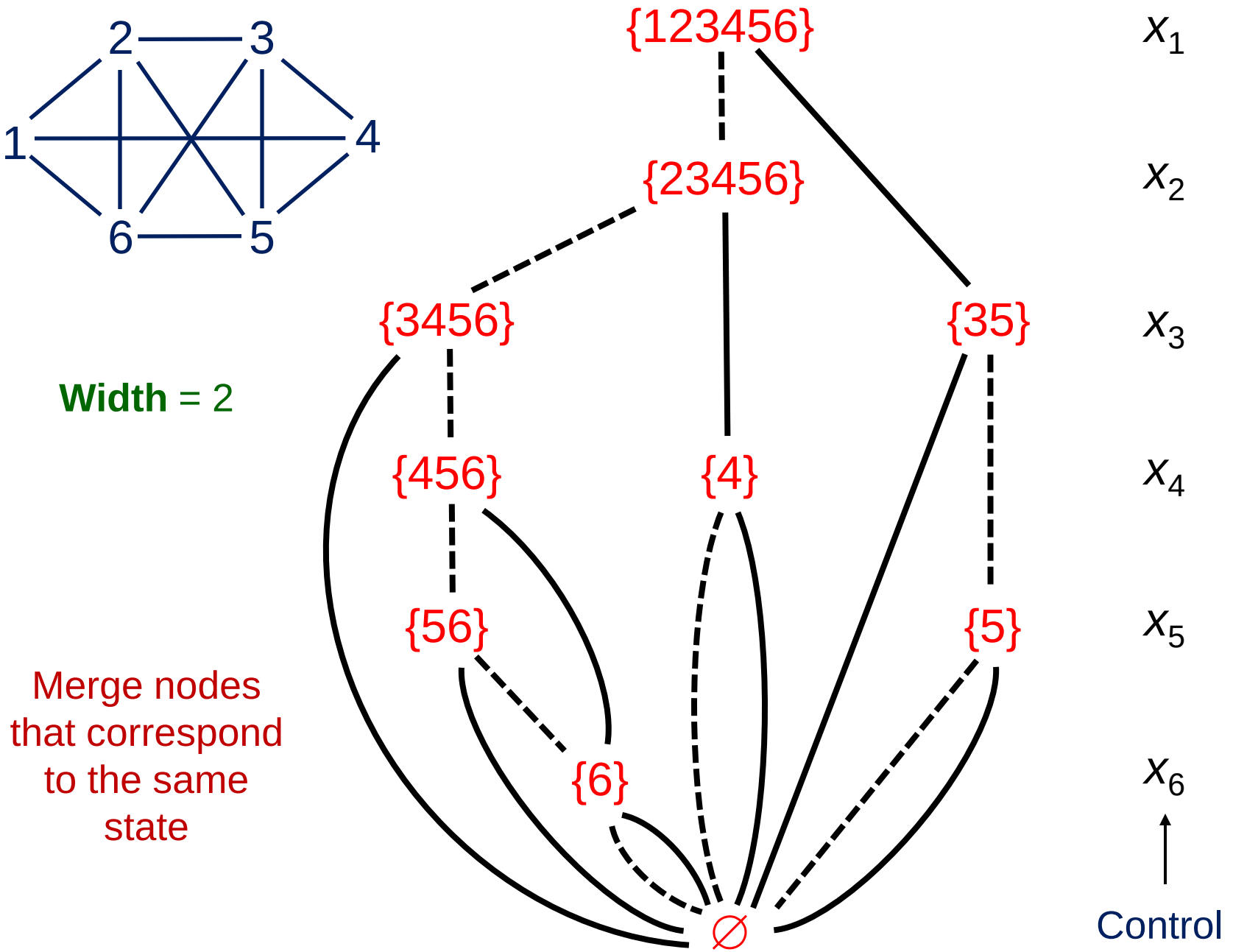
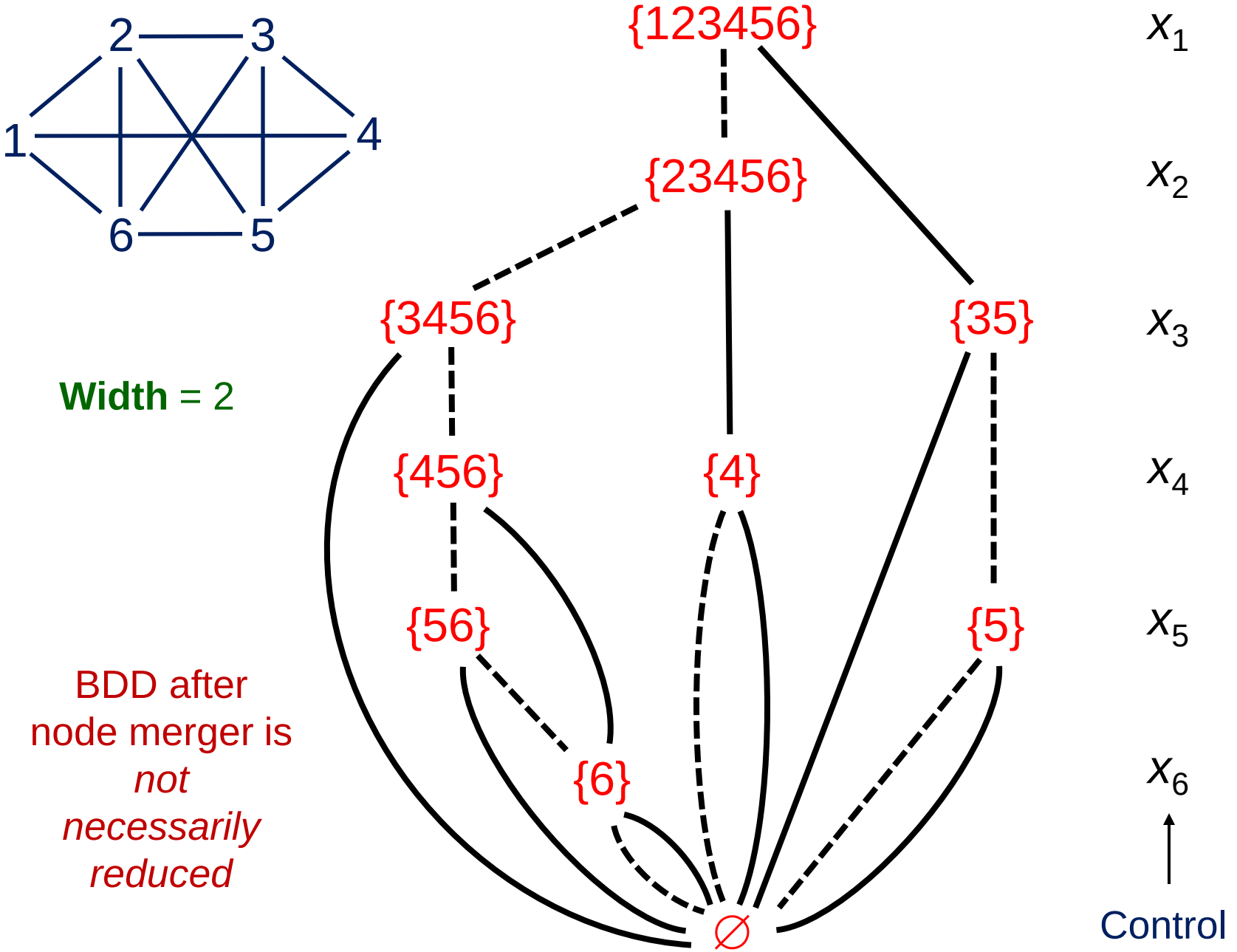


Width = 2

Merge nodes that correspond to the same state

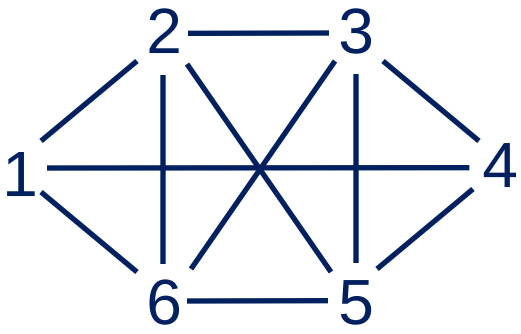


Width = 2

BDD after node merger is *not necessarily reduced*

Objective Function

- In general, objective function can be **any function of the current state.**
  - Linear or nonlinear, convex or nonconvex
  - There is a unique reduced BDD with **canonical** edge costs.



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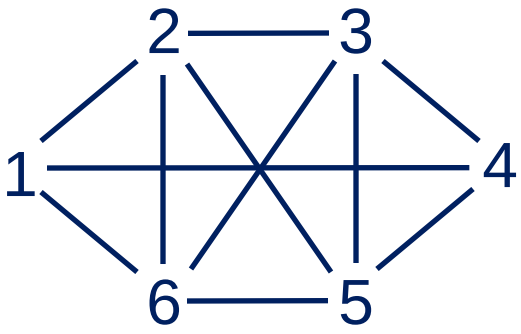

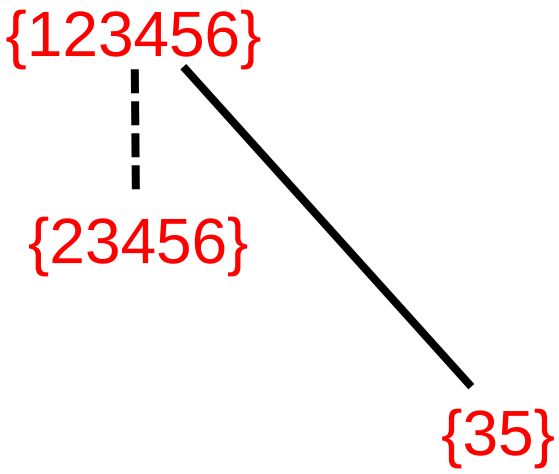

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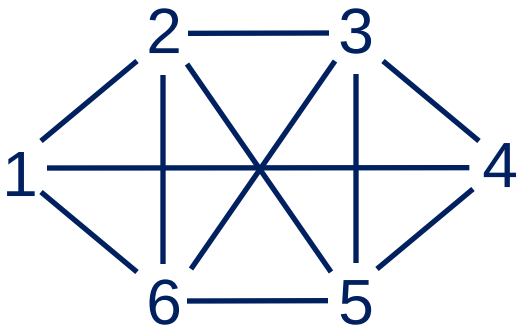

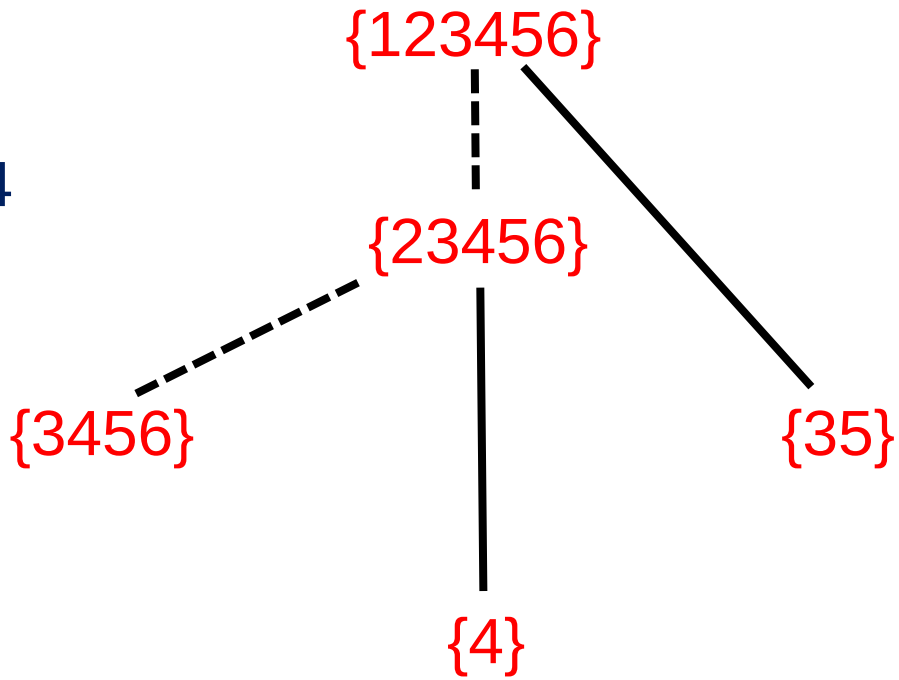

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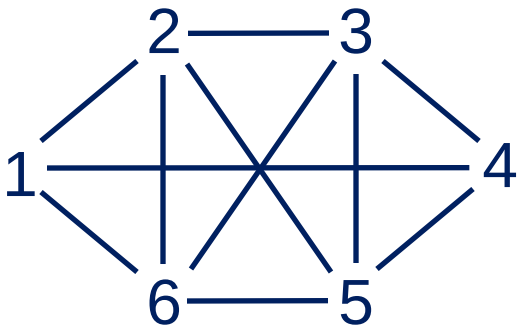

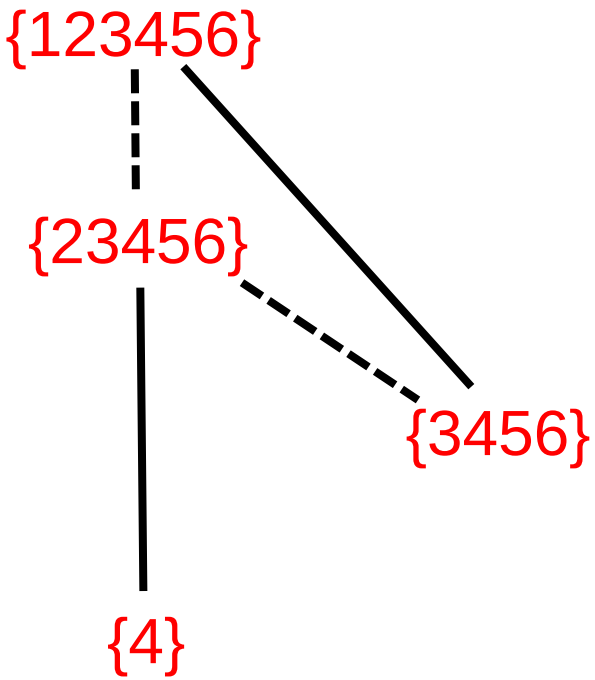

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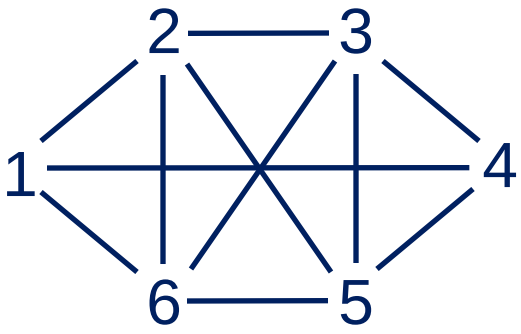

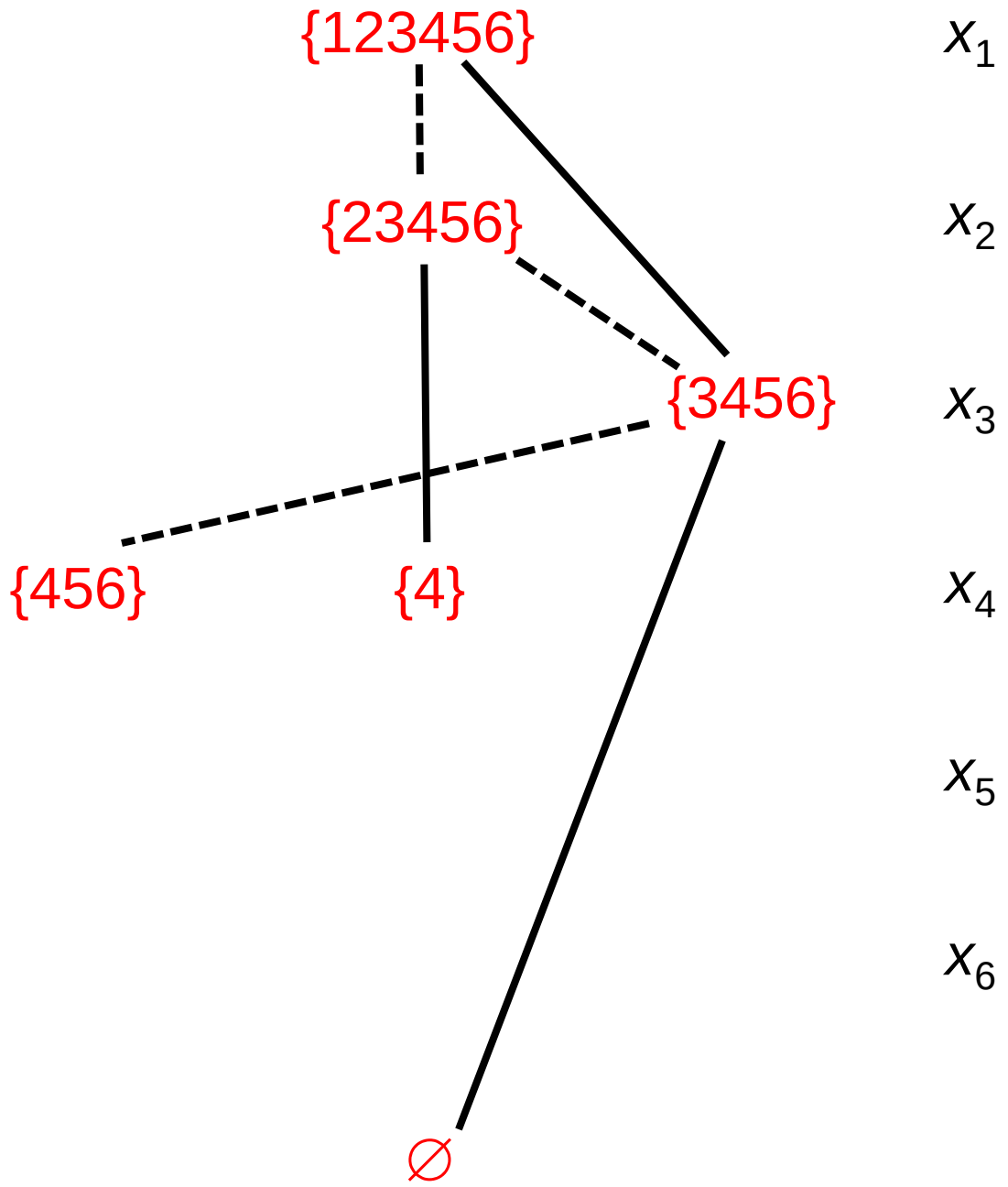

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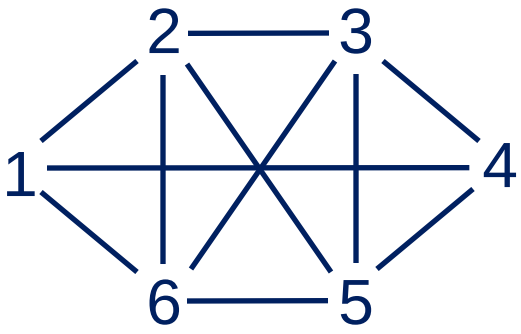

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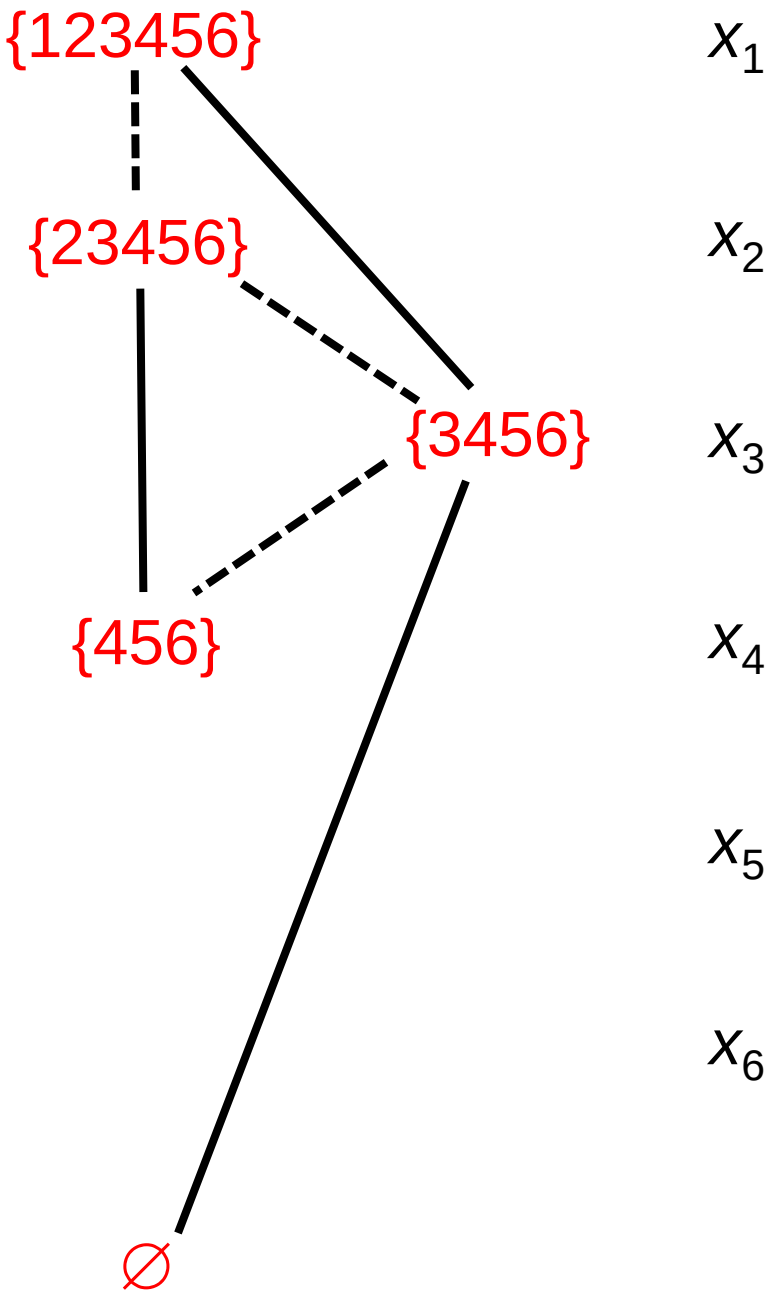

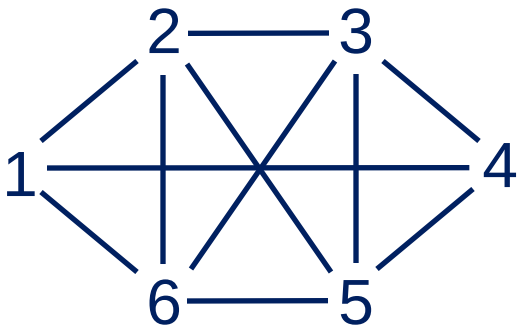

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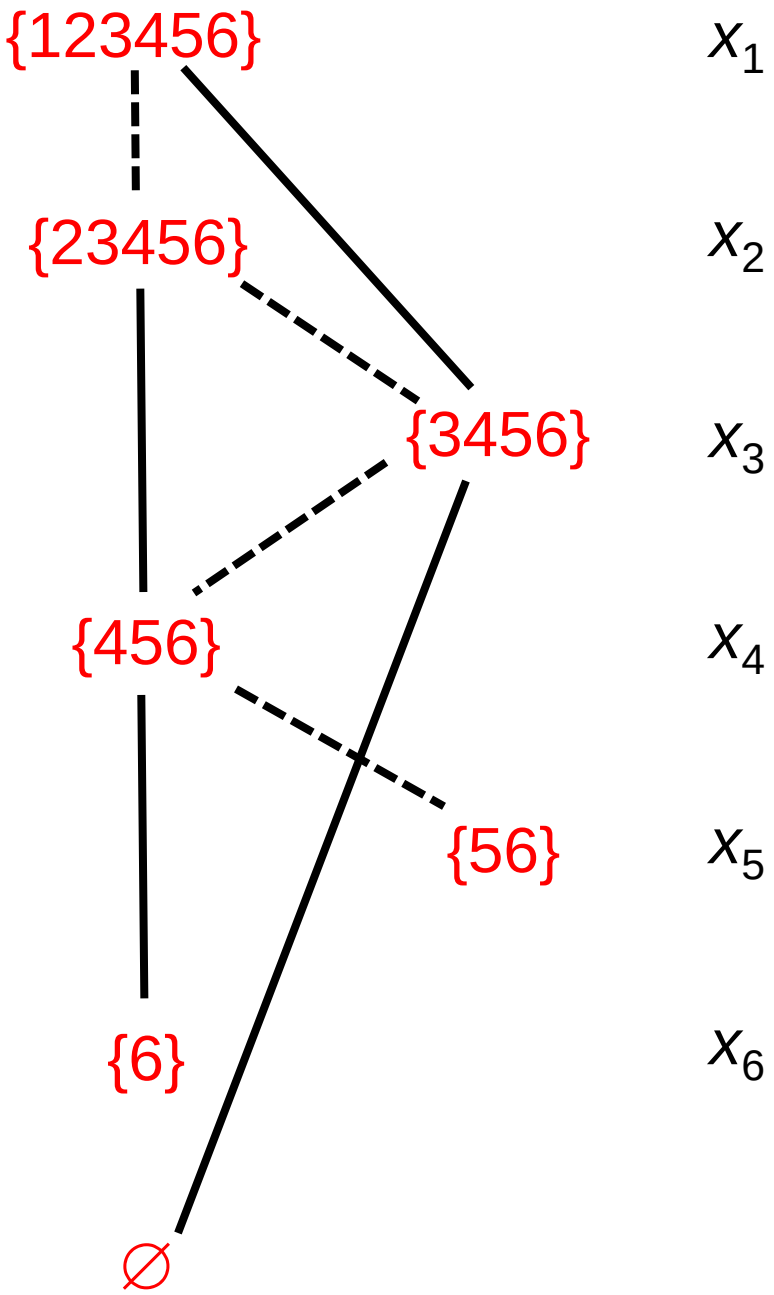

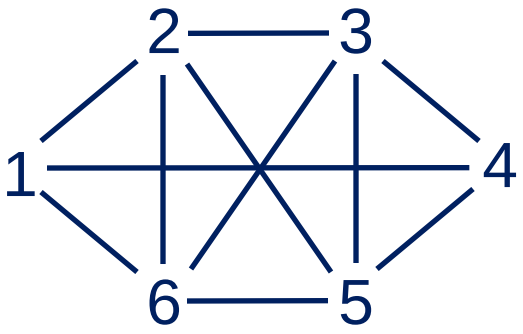

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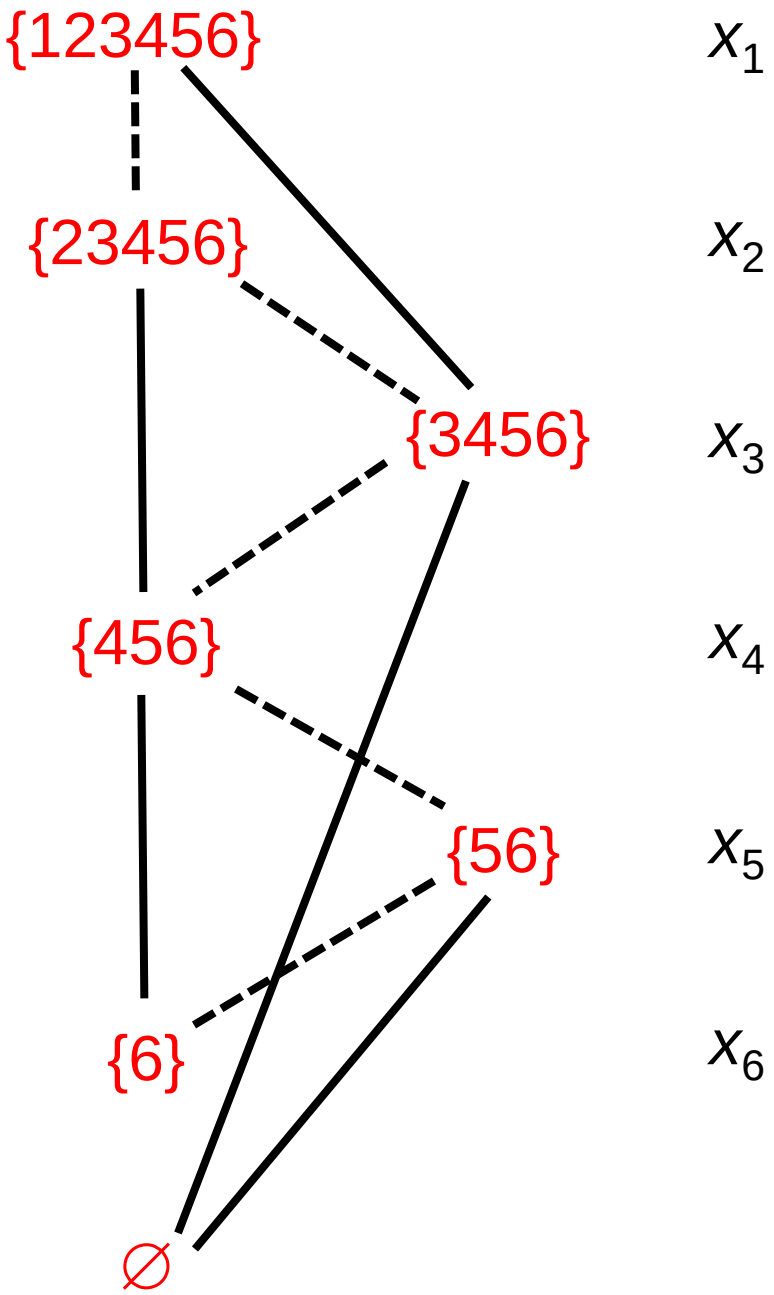

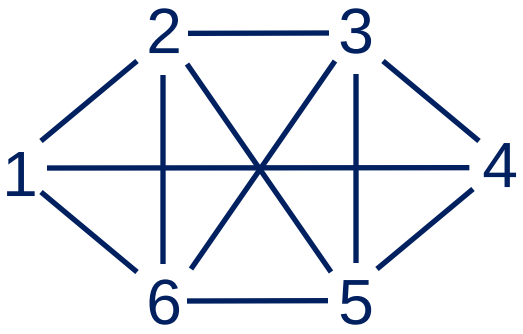

Width = 1

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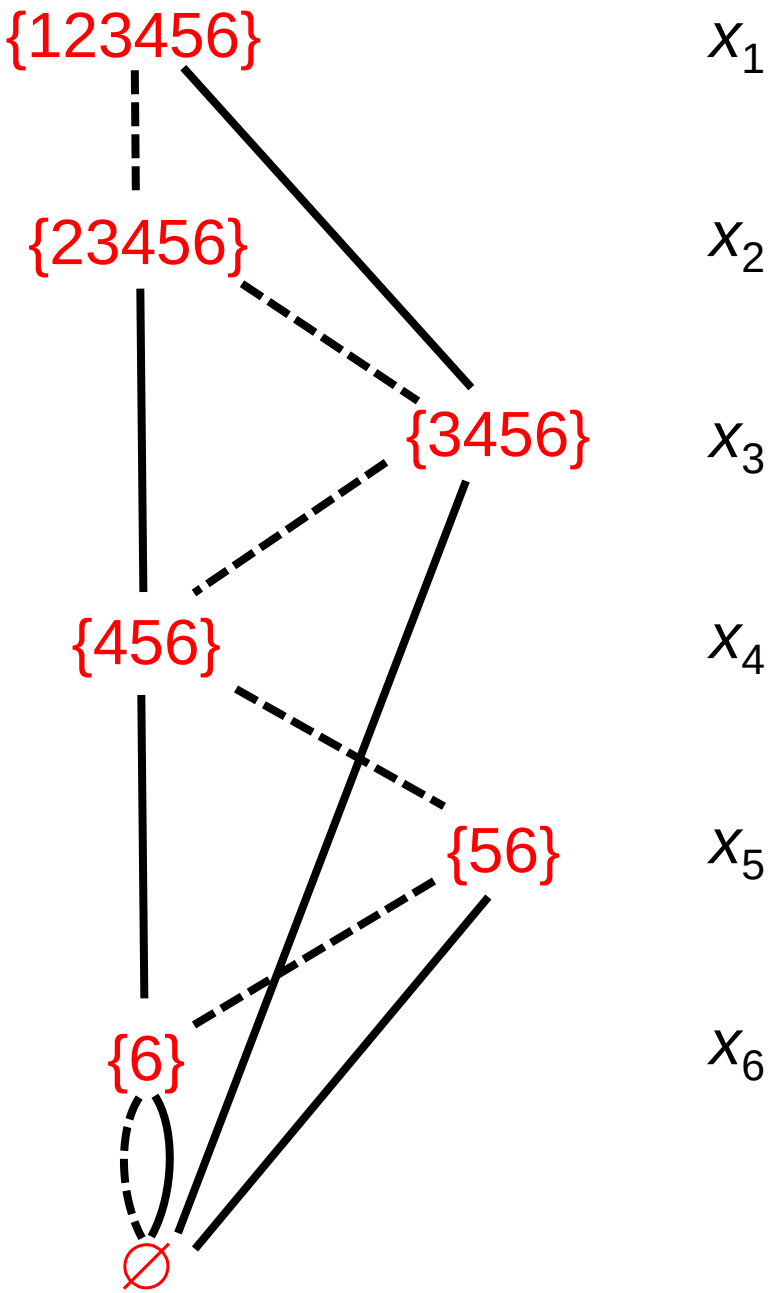
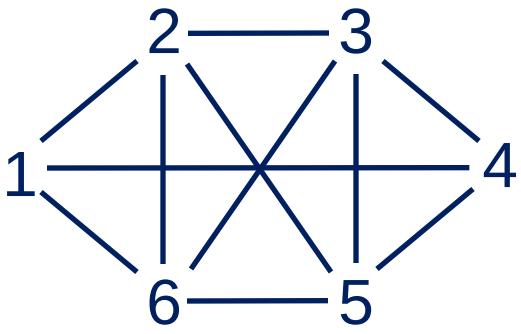
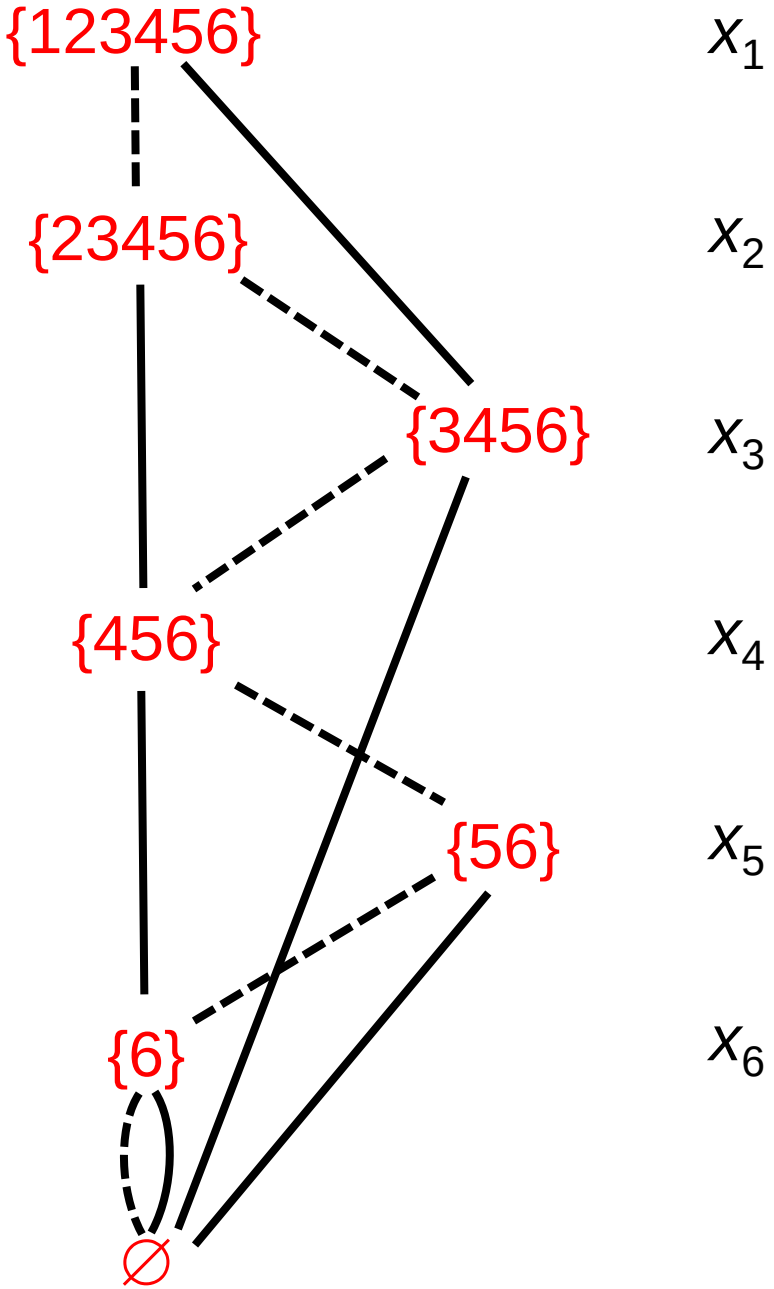
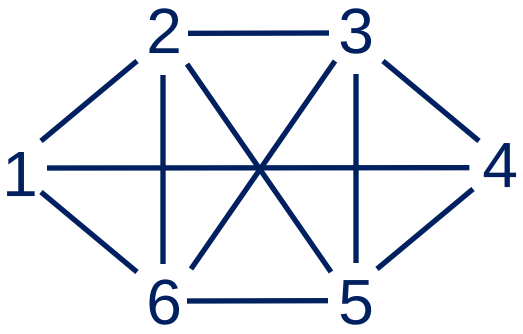




Width = 1

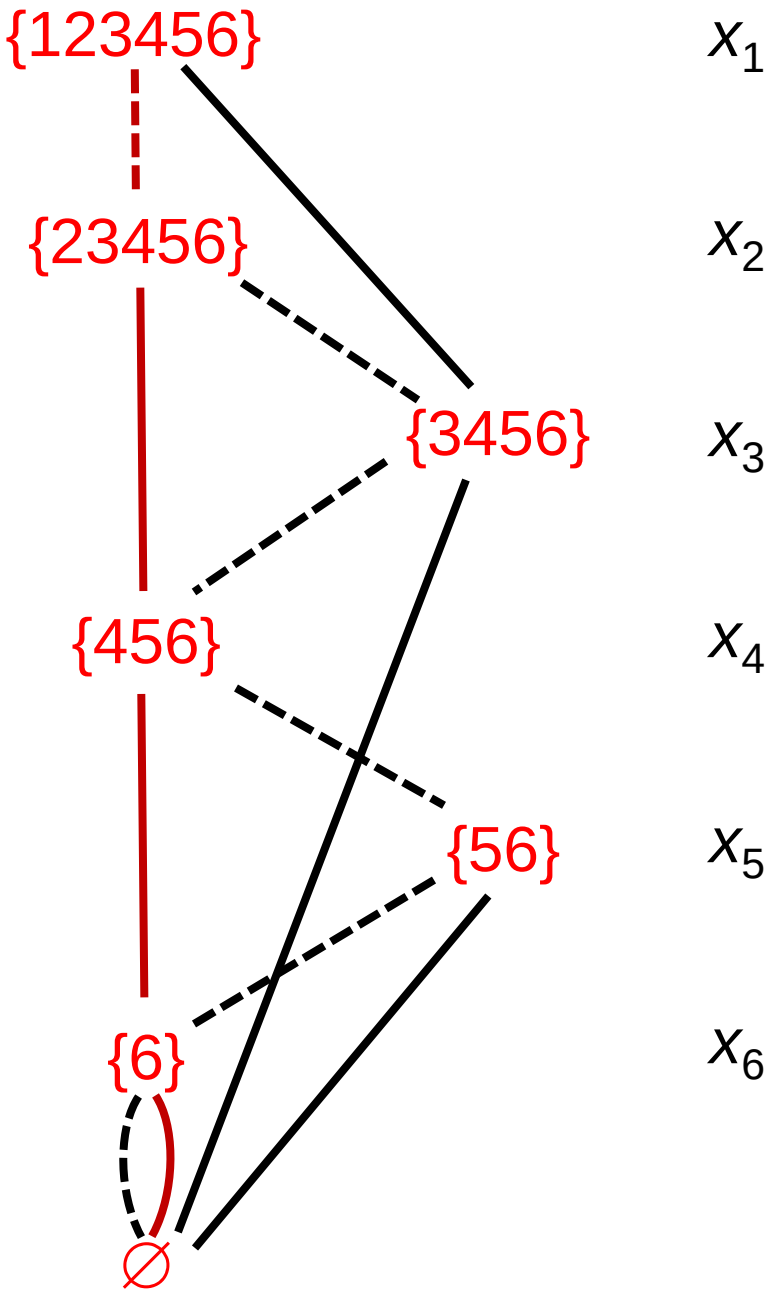
Represents 18 solutions, including 11 feasible solutions





Width = 1

Longest path
gives bound
of 3 on optimal
value of 2



Variable Ordering

- Variable ordering is important.
  - Just as in branching methods.
  - Some orderings provide tighter bounds.
  - Recursive model can dictate natural variable ordering.

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  - For an ordering induced by maximal path decomposition of the graph.

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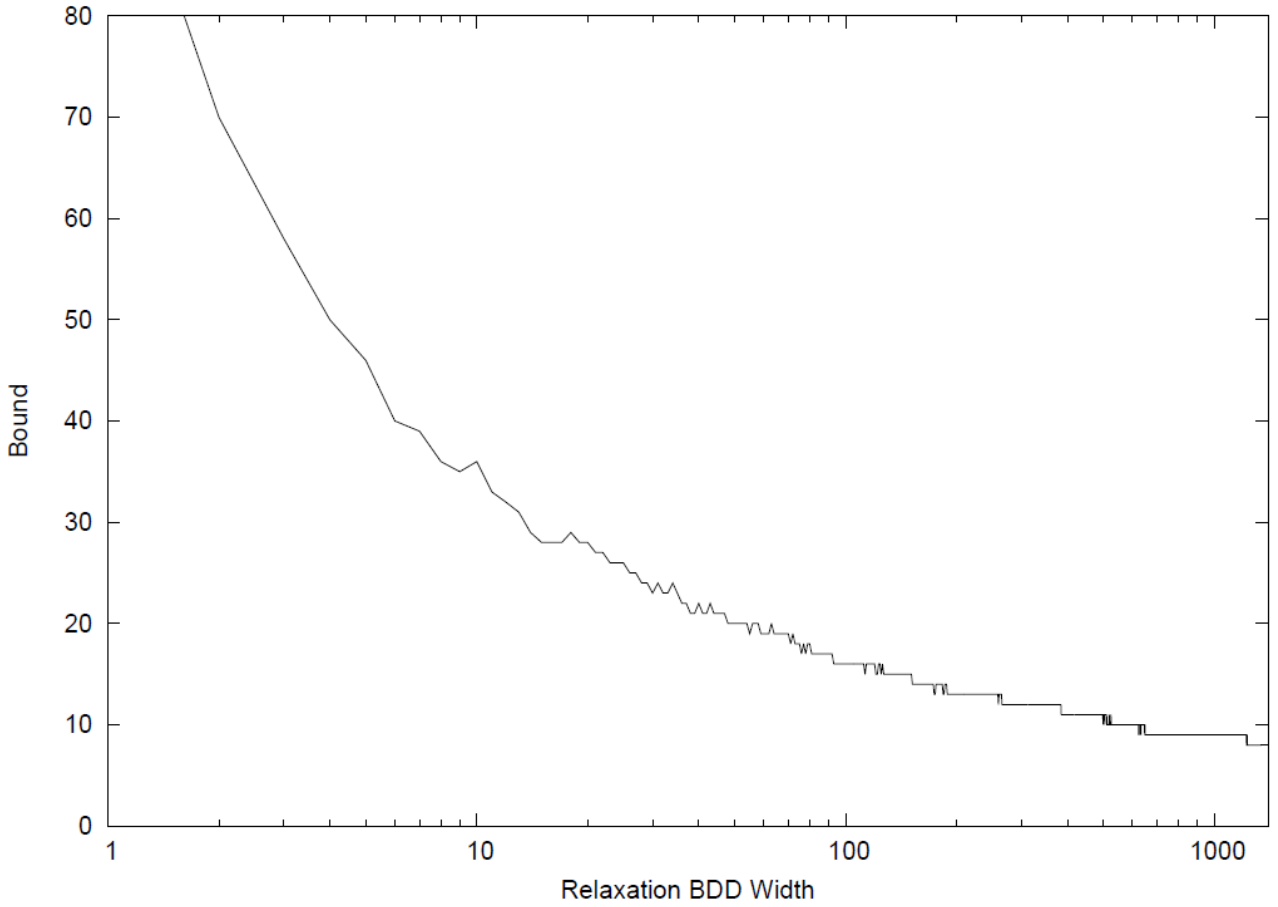
  - Recursive model can dictate natural variable ordering.
- For stable set problem, width of exact BDD is bounded by Fibonacci numbers.
  - For an ordering induced by maximal path decomposition of the graph.
- We used a dynamic ordering heuristic.
  - Next variable is the one appearing in the smallest number of states on the current level.
    - Better than maximal path ordering.

Merging Heuristic

- Which nodes to merge when building relaxation?
- A quality-of-solution heuristic seems by far the best.
  - Define a state variable that indicates solution quality.
  - Merge nodes associated with poor solution quality
  - In this case, solution quality = longest path into node.
  - We lose information in areas not likely to be part of the solution.

Width of BDD

- Wider BDDs yield tighter bounds.
  - But take longer to build.



Comparison with LP Bound

- Random and benchmark instances
- Compare with LP bound at root node in CPLEX
  - Use clique cover IP formulation
    - Requires precomputing clique cover.

$$\max \sum_i w_i y_i$$

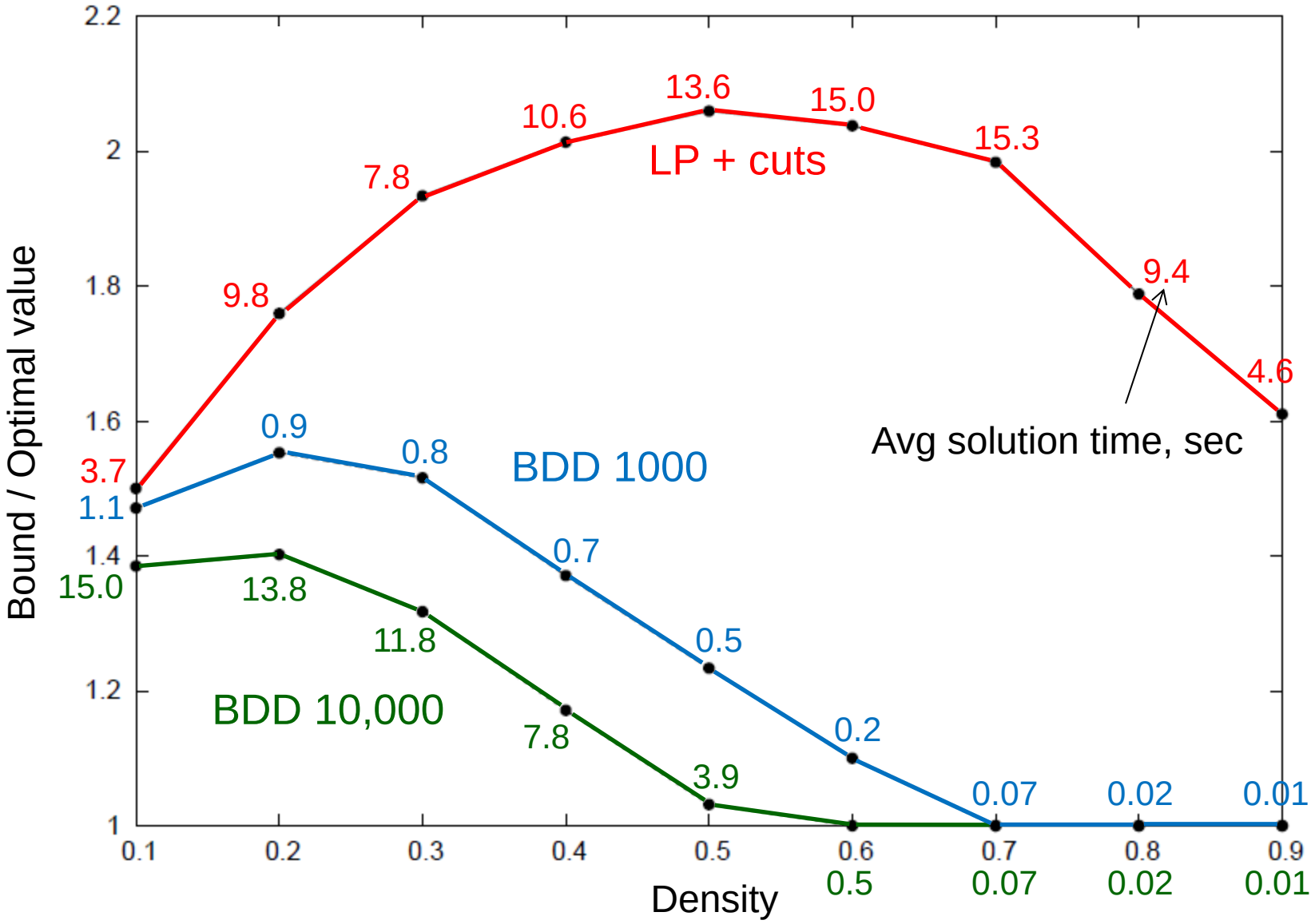
$$\sum_{i \in C_k} y_i \leq 1, \quad \text{all cliques } C_k \text{ in clique cover}$$

$$y_i \in \{0,1\}, \quad \text{all } i$$

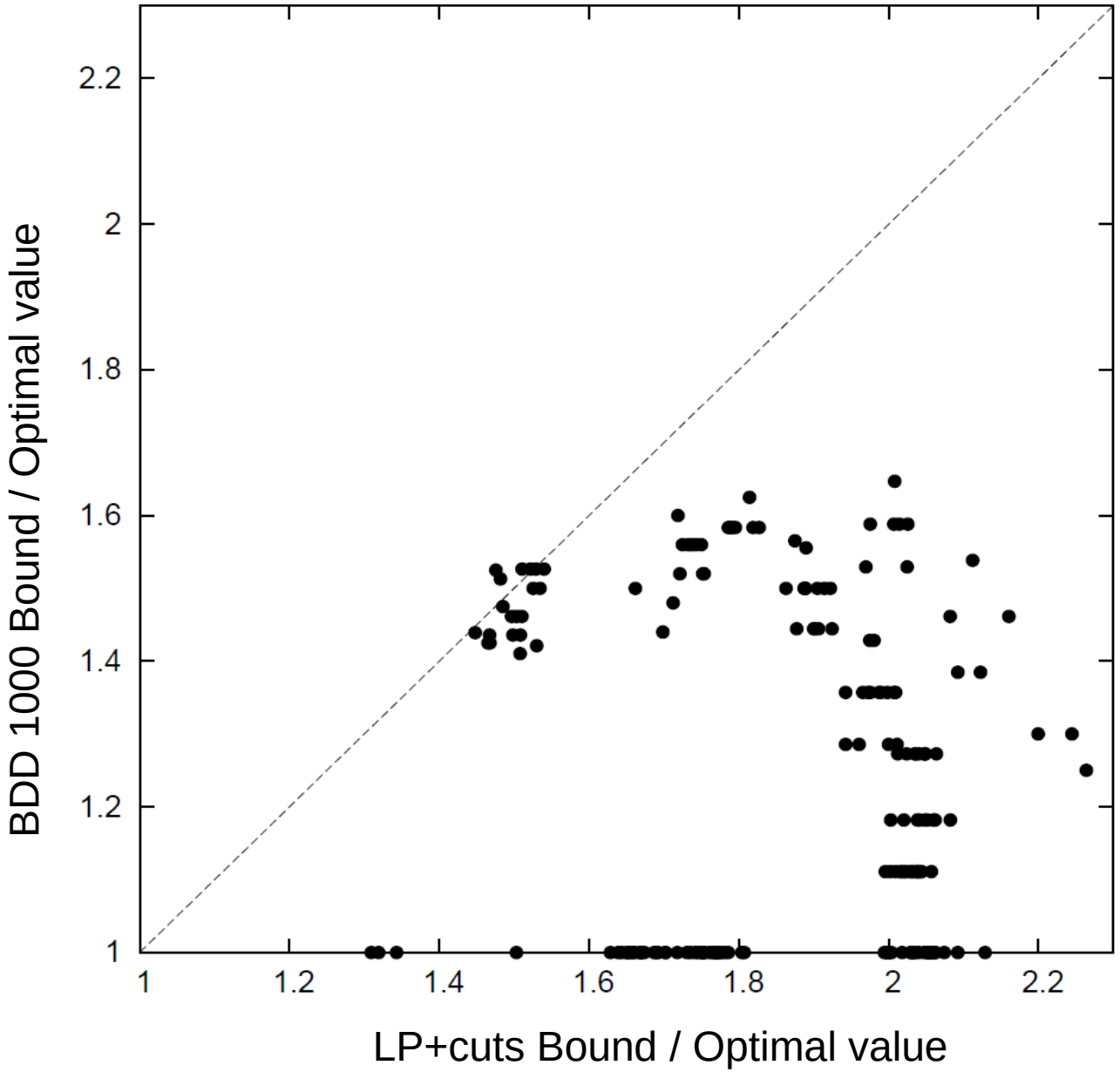
Comparison with LP Bound

- CPLEX settings
  - Turn off presolve
    - It makes CPLEX bound worse or at most 1 better.
    - BDD bounds don’t use presolve.
  - Use full cutting plane resources (50 years of development)
  - Use interior point (barrier) LP solver
    - Faster than simplex on these instances.

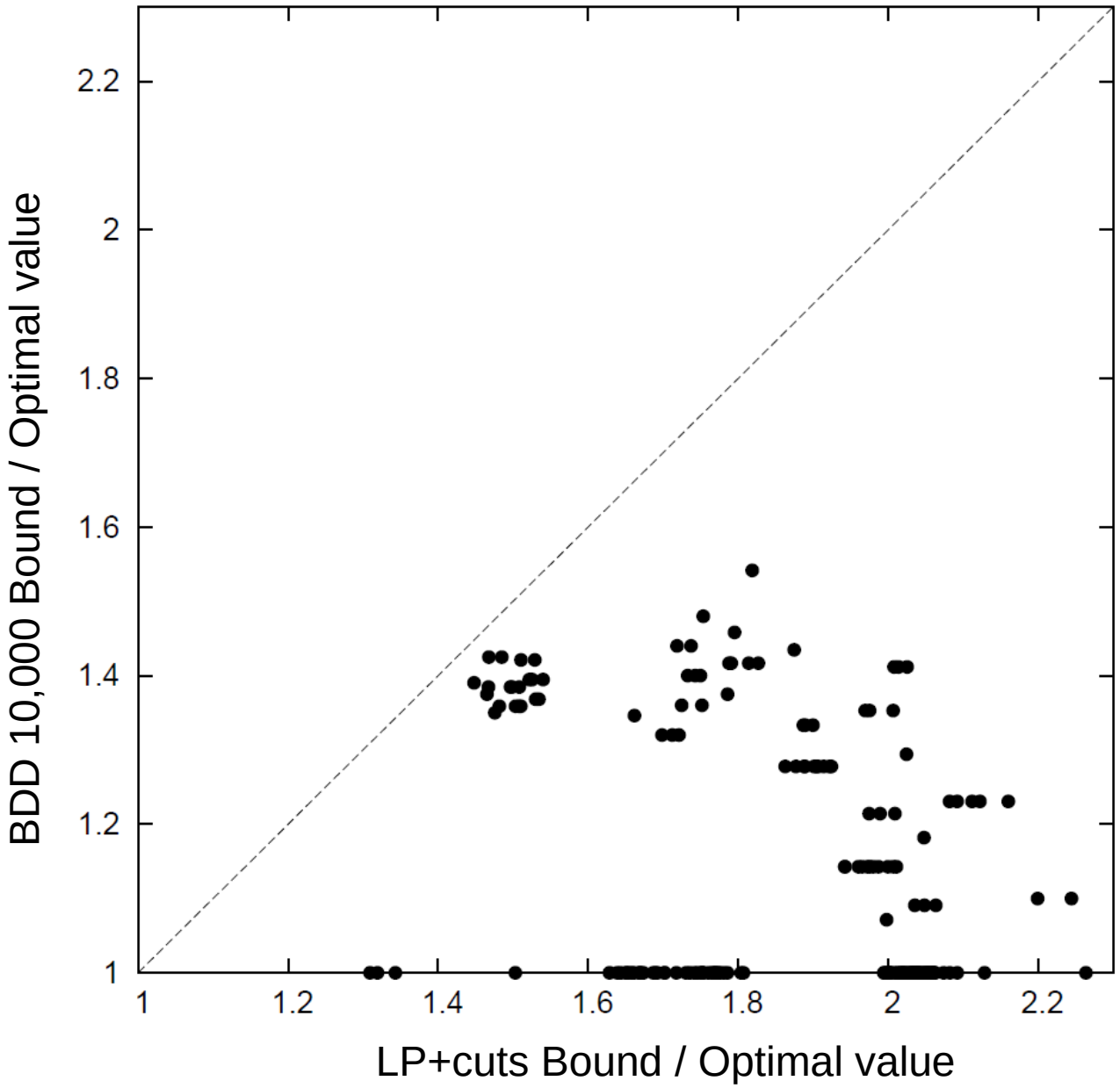
Random instances



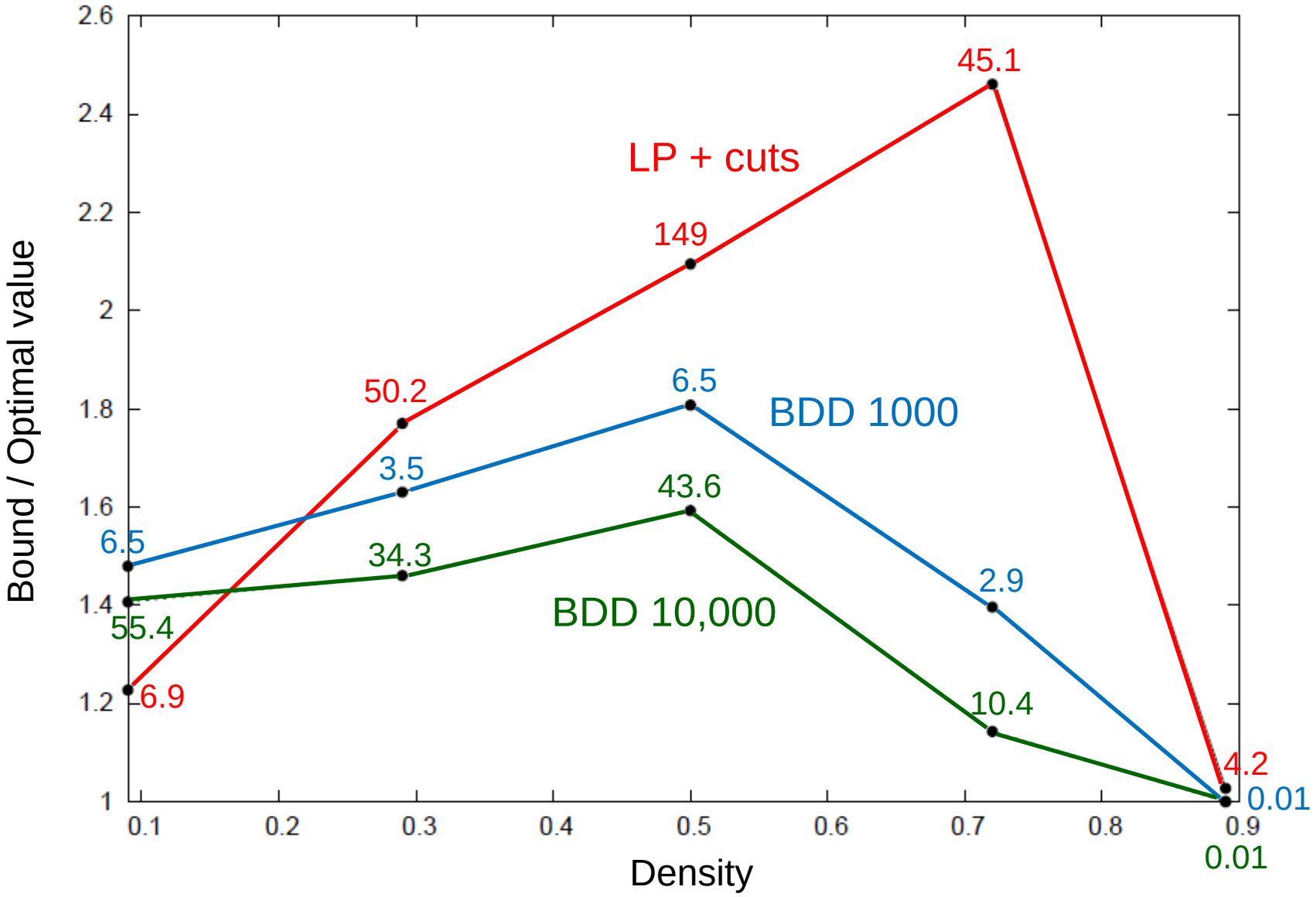
Random instances



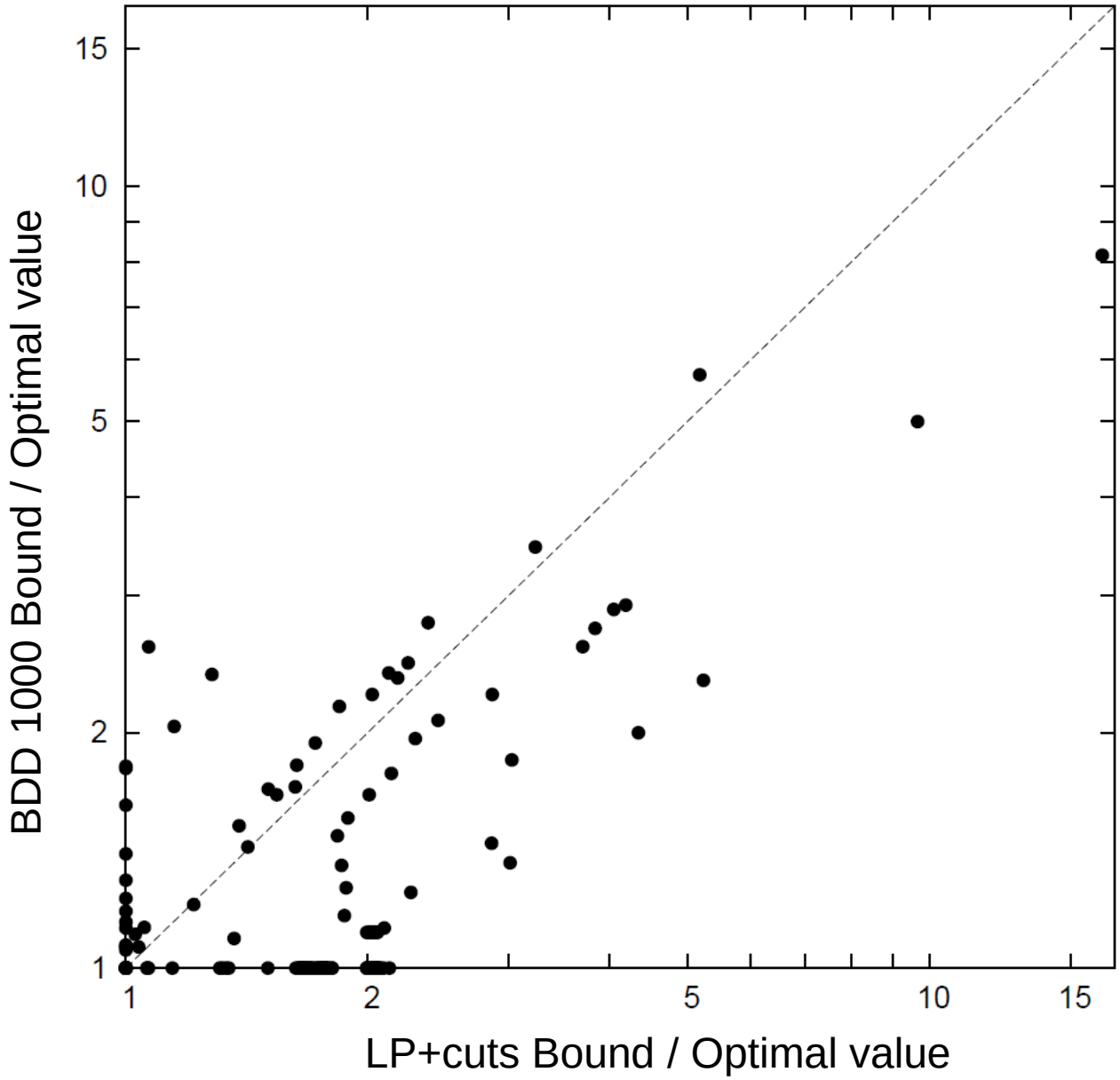
Random instances



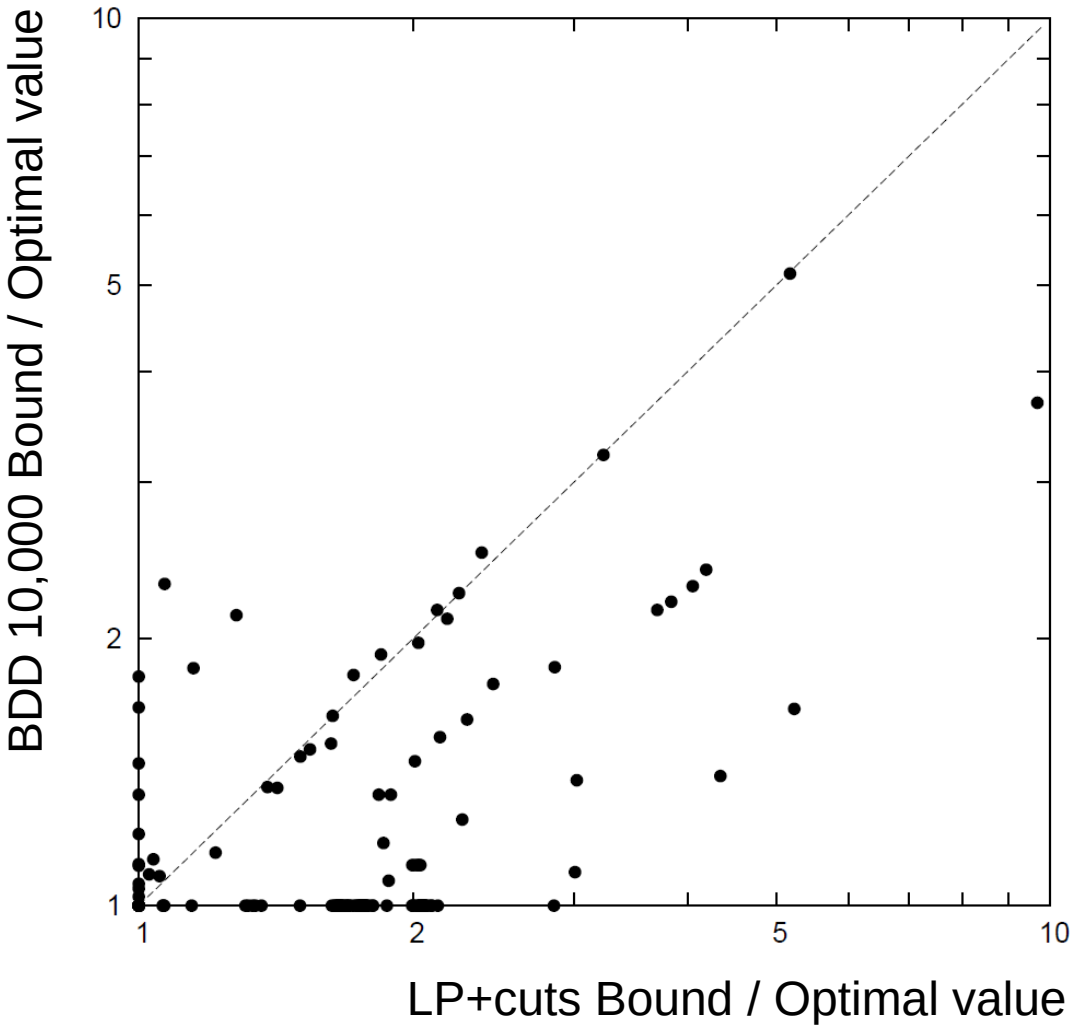
DIMACS instances



DIMACS instances



DIMACS instances



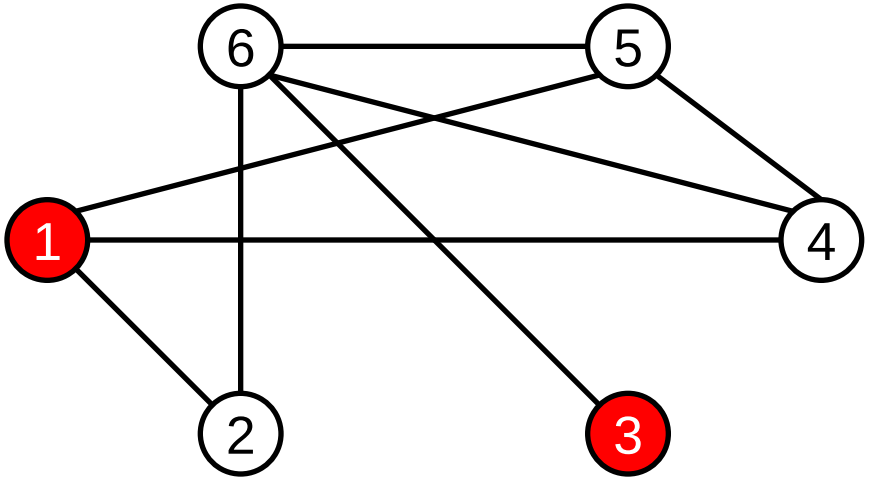
Incrementality

- Fast incremental calculation of bound.
  - Modify relaxed BDD after variable is fixed in branching tree (fast).
  - Recompute longest path (fast).
- However, may be better to rebuild relaxation from scratch.
  - Research issue.

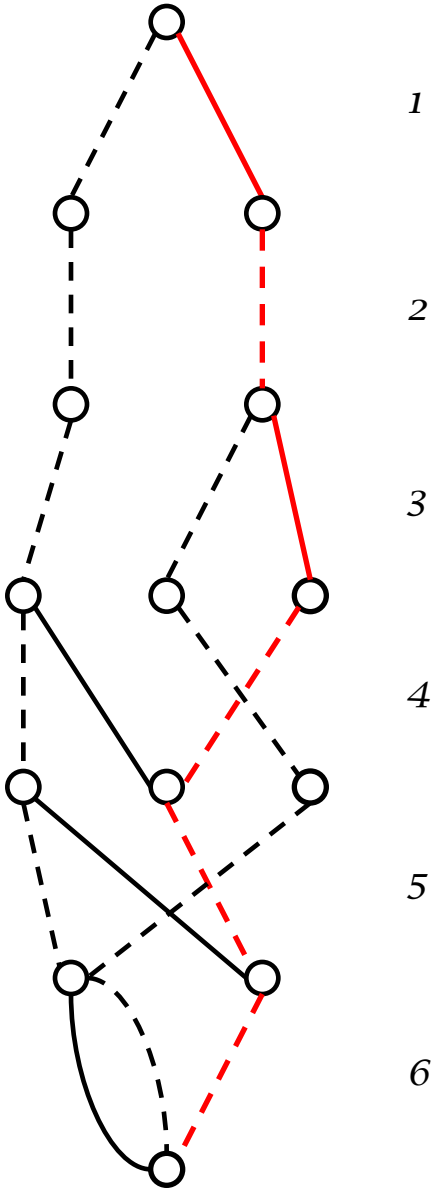
Restricted BDDs

- **Restricted** BDDs as **primal heuristic**
  - All paths are feasible.
  - Longest path is a good feasible solution
  - Reduce width by intelligently removing nodes.
- Apply to stable set problem
  - DIMACS instances
  - Compare with lower bound obtained by CPLEX primal heuristics at rote node.
  - Details and updates in Bergman, Ciré, van Hoeve, Yunes, *J. of Heuristics* 2014.

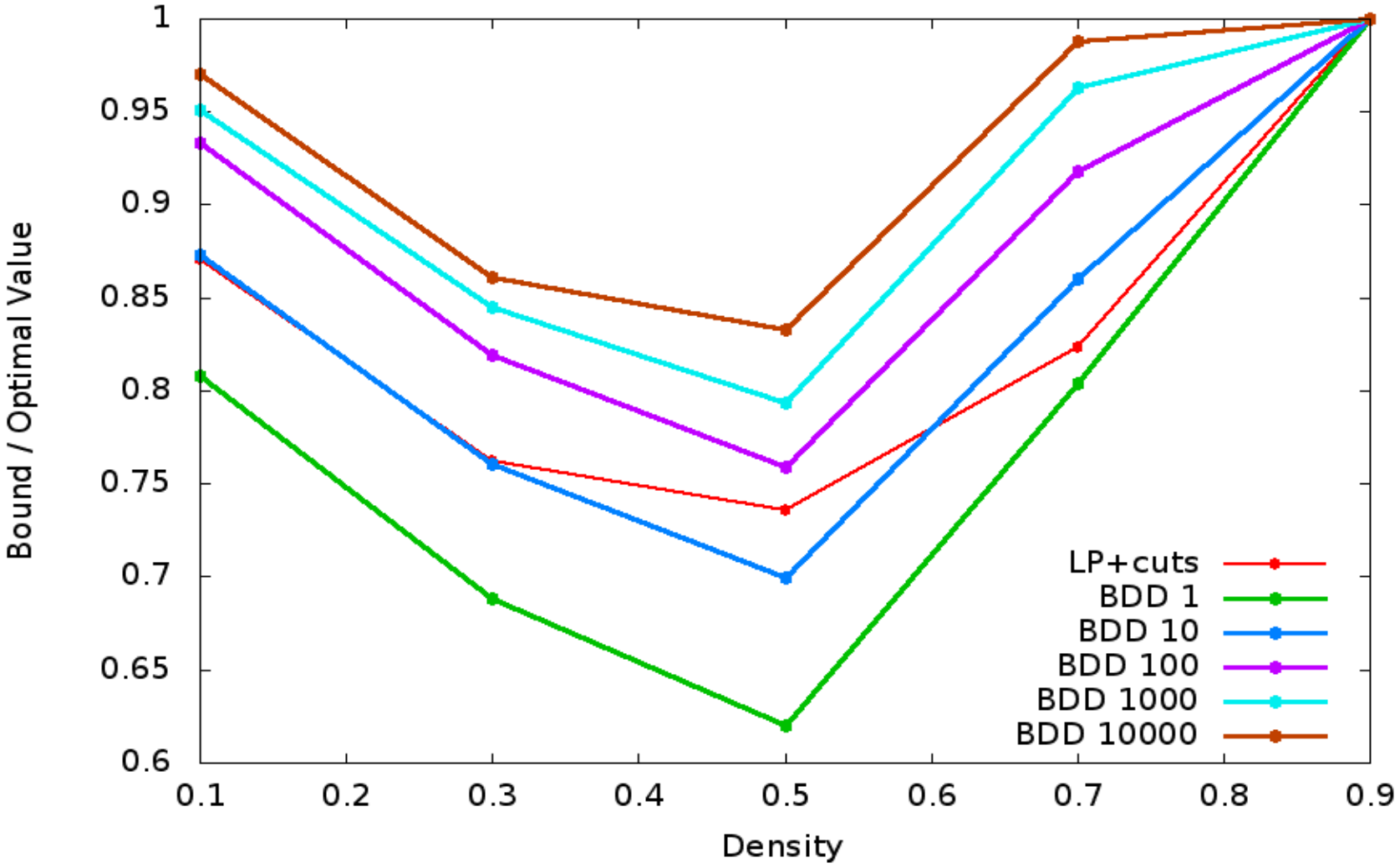
Restricted BDD
Width = 3



Longest path is
feasible solution
and gives upper bound
of **2** on optimal value of **3**



DIMACS instances



How to Use BDD Bounds

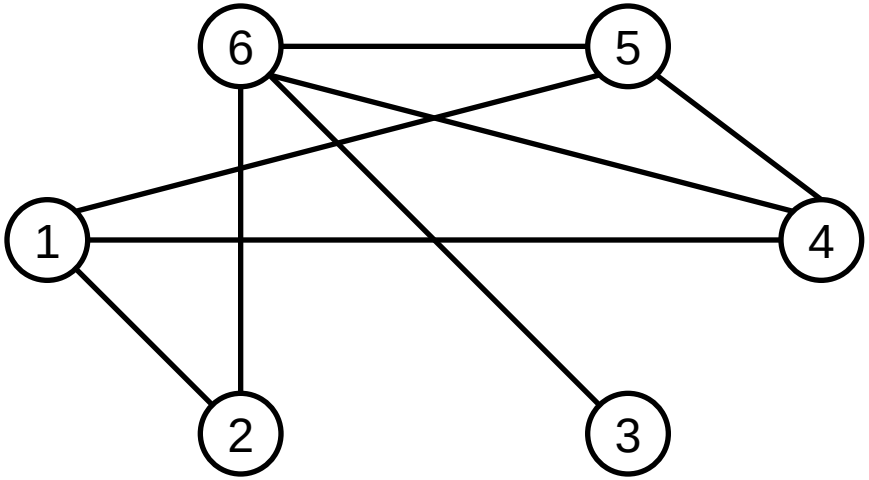
- Assist **existing solvers**
  - MIP solvers.
    - Bound at root node.
    - Can generate BDD bounds at nodes below root node, due to speed of BDD processing.
  - CP solvers.
    - Bound at root node.

How to Use BDD Bounds

- Part of general **BDD-based solver**
  - **Branch in the relaxed BDD**.
  - Combine with BDD-based propagation
    - BDD plays role of LP relaxation, cutting planes
  - BDD-based primal heuristic
  - Dynamic programming style formulation
    - Specify state variable for BDDs
    - Linearity, convexity irrelevant
  - Details in talk by David Bergman tomorrow (ACP Thesis Award).

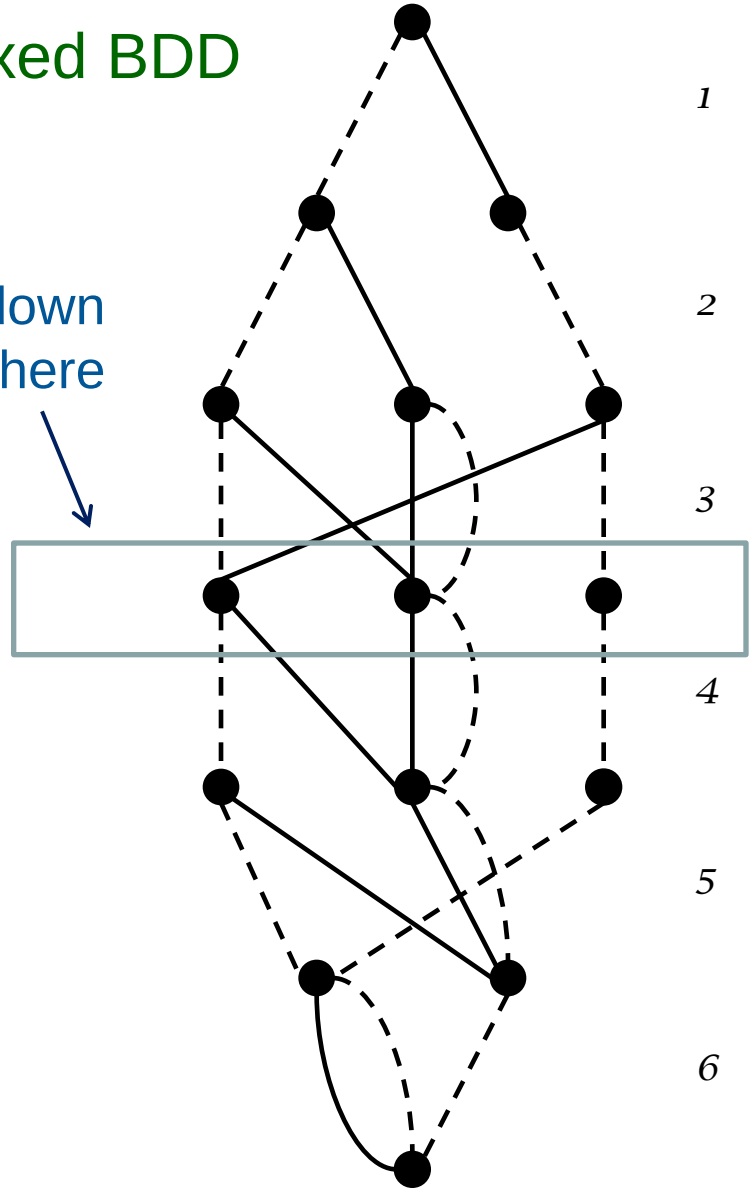
Branching in Relaxed BDD

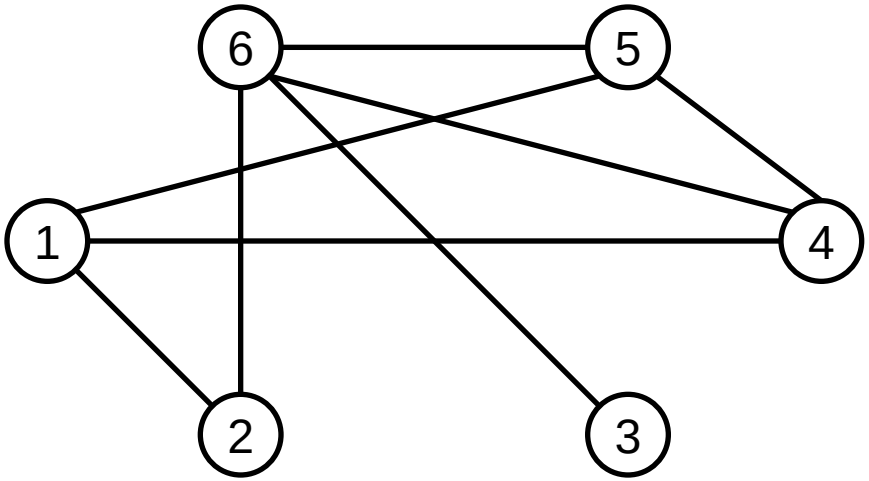
- Novel branching scheme
  - Branch on nodes of relaxed BDD.
  - …rather than on variables.



Relaxed BDD

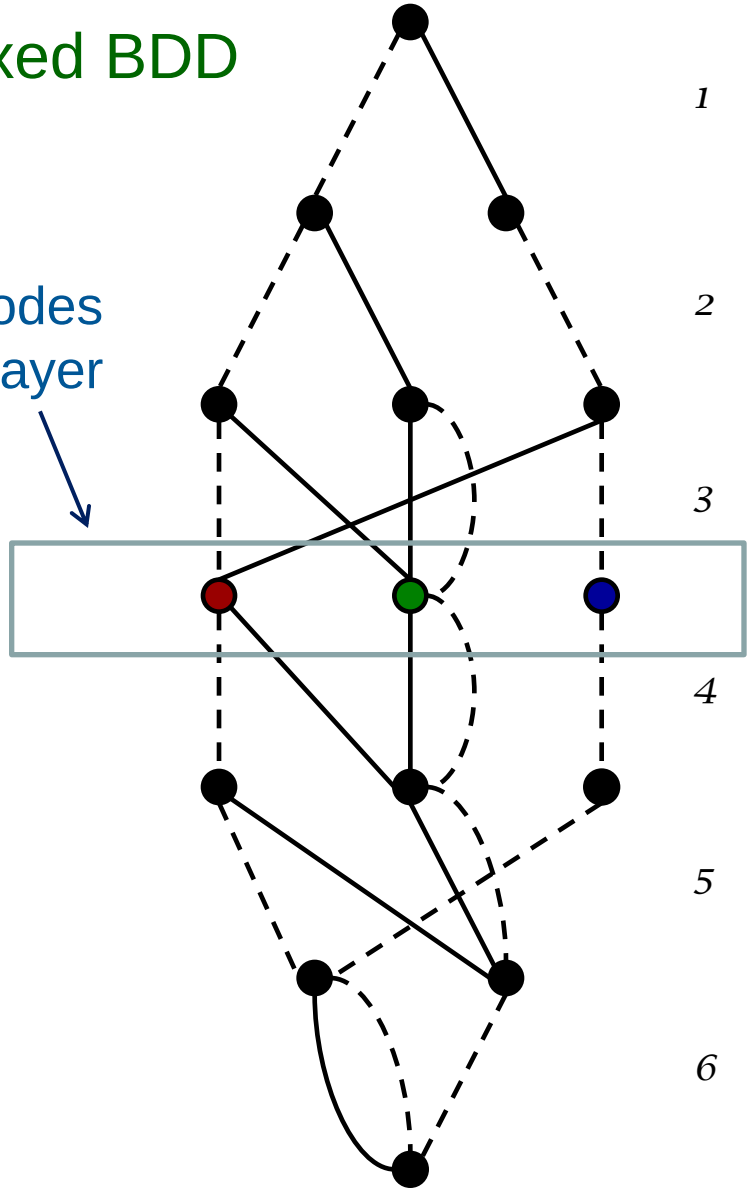
Exact down to here

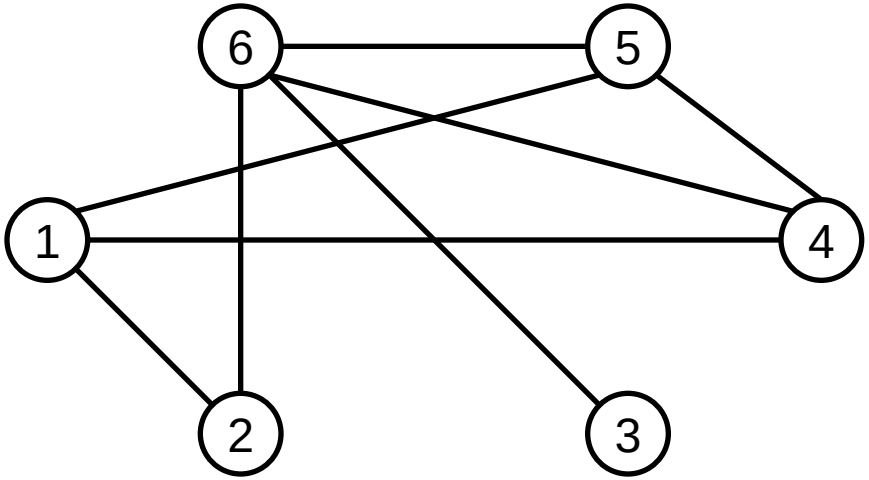




Relaxed BDD

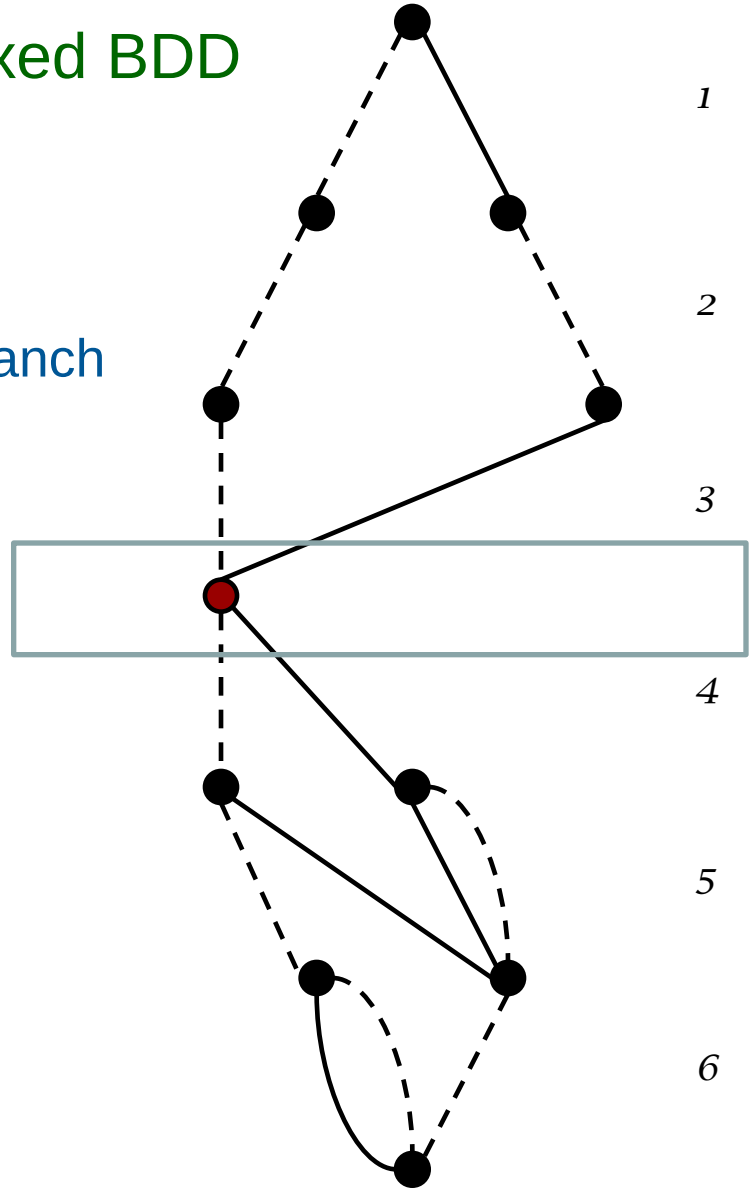
Branch on nodes
in this layer





Relaxed BDD

First branch



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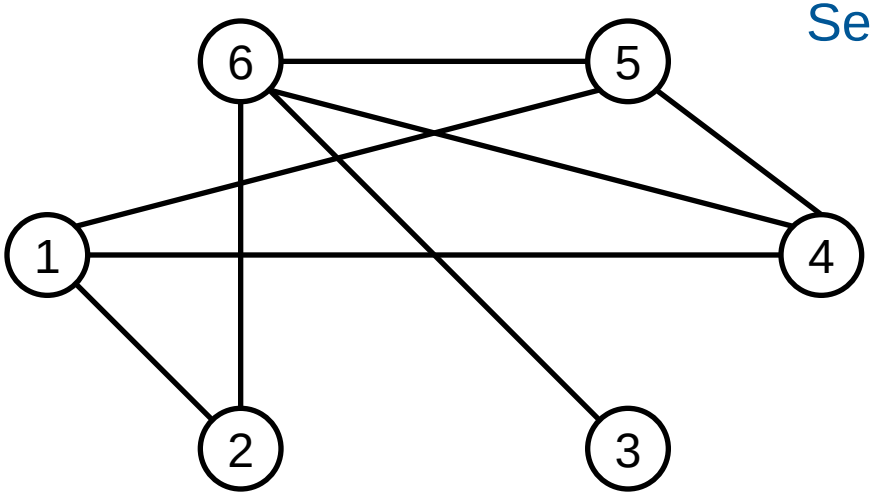
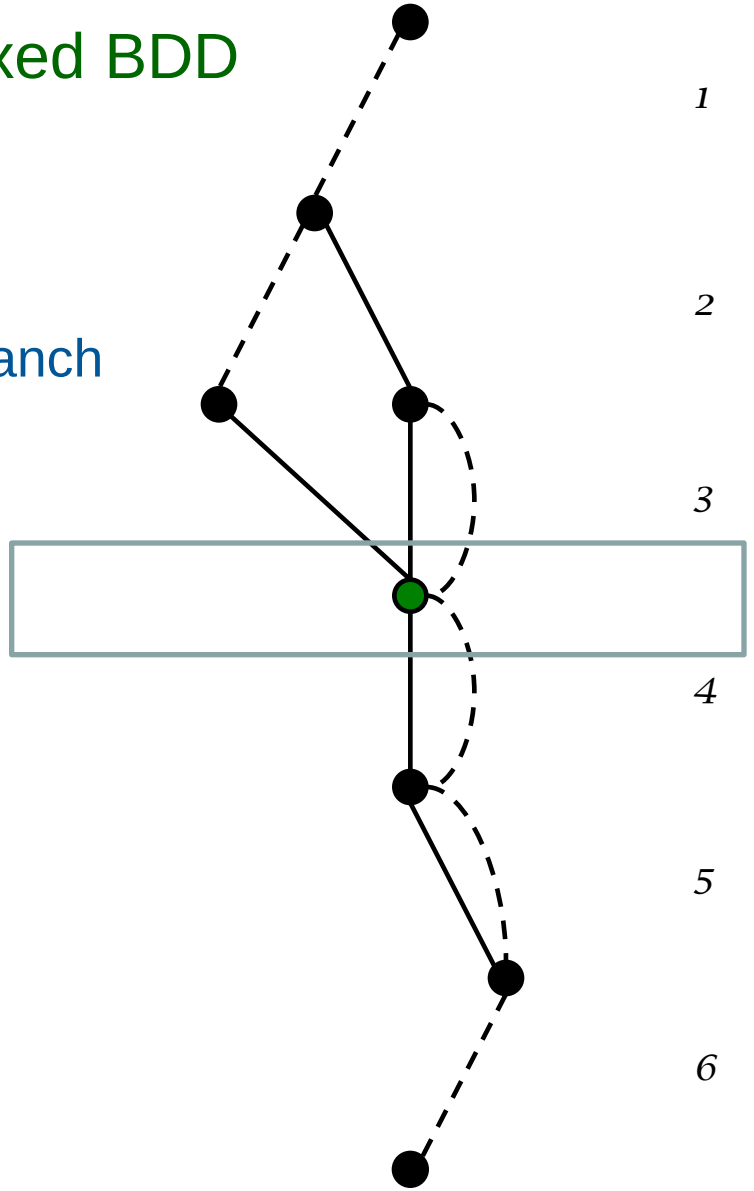
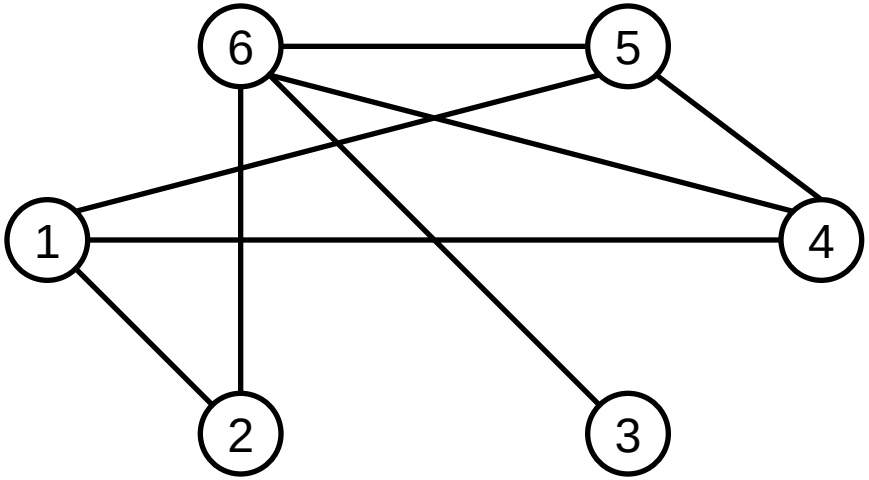


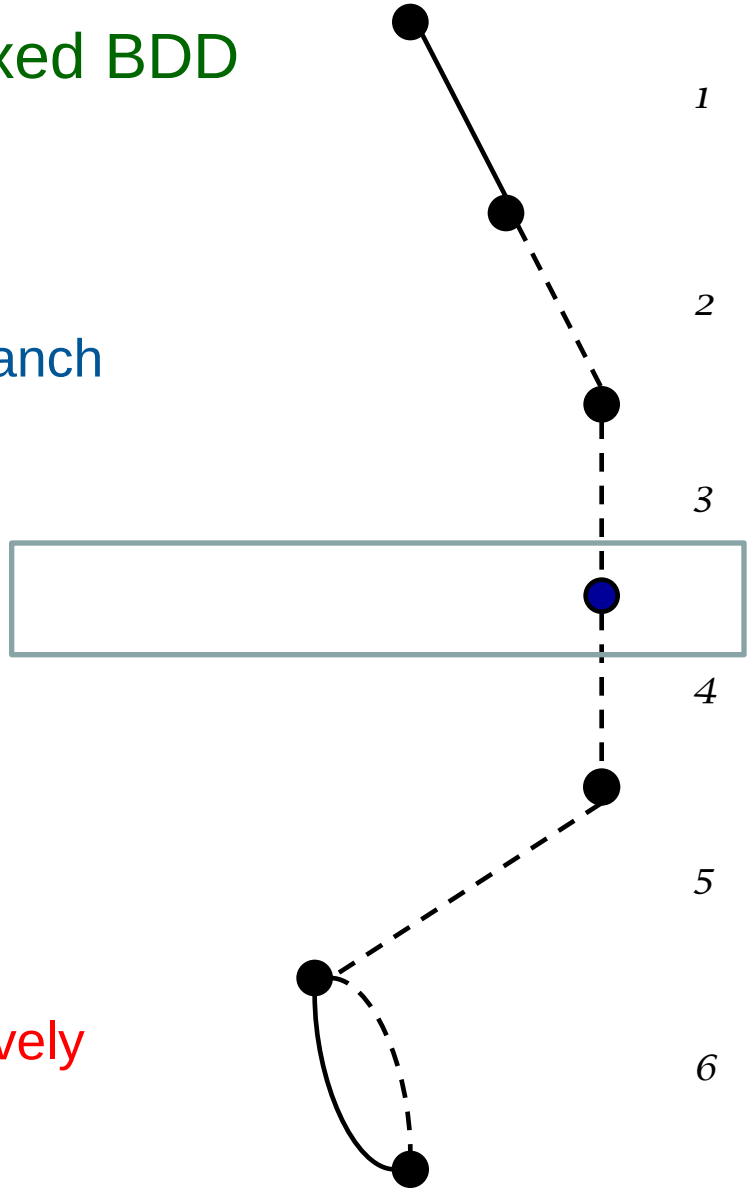
Second branch



Relaxed BDD



Third branch



Continue recursively

Parallelization

- BDD-based optimization parallelizes very efficiently
  - Near-linear speedups.
  - Details in Bergman, Ciré, Sabharwal, Samulowitz, Saraswat, van Hoeve, CPAIOR 2014.

Recursive Modeling

- Flexibility of dynamic programming
  - Without having to search an exponential state space.
- Potential for “big data” optimization
  - No need to “load” a huge model, as in integer programming.
  - Recursive model is essentially constant size.

Recursive Modeling

- Example: simple one-machine sequencing problem.
  - At least 6 integer programming models.
  - Very simple DP state
    - set of jobs not yet sequenced
    - finish time of last job processed.

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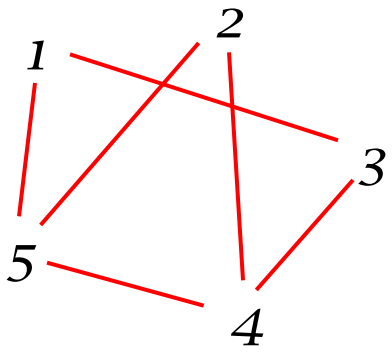
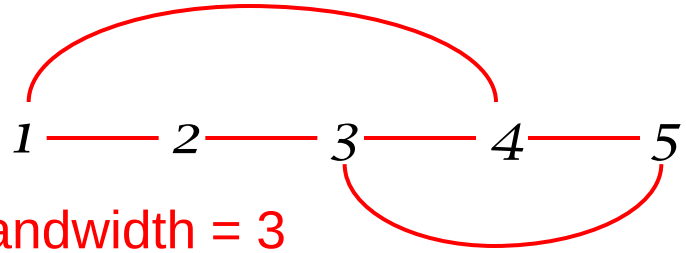
  - Maintenance schedule, conditional time windows, etc.
  - Easily written in terms of current state.
- Nonstandard objective functions.
  - Job processing cost depends on which components available from earlier jobs.
  - Can be table lookup function.

Recursive Modeling

- Min bandwidth and linear arrangement problems.
  - Sequence electronic components.
  - Some pairs are connected by a wire.
  - Minimize longest wire, or sum of wire lengths.





Bandwidth = 3

sum = 9

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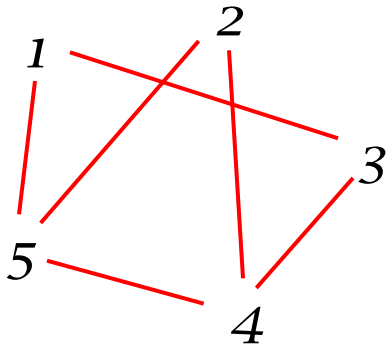

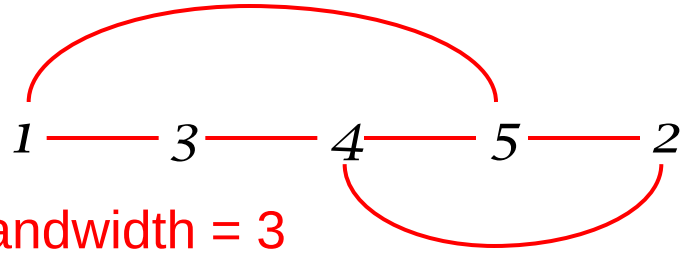

Bandwidth = 3
sum = 9

- No usable IP model
- Recursive model
  - Control = length of wire connecting pair of components
  - State = (for each component) remaining positions it can be assigned

Future Work

- Continuous global optimization.
  - Recursive models.
  - Massive discretization of continuous variables.
  - Node merger heuristics control discretization density.
  - No need for convexification and Lipschitz bounds.
  - Use “big data” approach of applying simple model to huge dataset (of discrete values).