Two Schemes for Combining Mixed Integer Programming with Constraint Programming

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General Form of Conditional Model

When there are no *x*'s

min	$f(x)$ [or $r(y)$]	objective function
s.t.	$p_i(y), \quad i \in I_1$	checkable constraints
	$g_i(x), \quad i \in I_2$	soluble constraints
	$q_i(y) \rightarrow h_i(x), \quad i \in I_3$	conditional constraints
	$d_i(x, y), \quad i \in I_4$	defined constraints
	$x \in X$	solution variables
	$y_j \in D_j, \quad \text{all } j$	search variables

Soluble constraint

Set of checkable constraints

Relaxation of Conditional Model at a Given Node of the Search Tree

$$
\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & g_i(x), \quad i \in I_2 \\
& h_i(x), \quad \text{all } i \in I_3 \text{ for which } q_i(y) \text{ is true} \\
& \text{relaxation of } d_i(x, y), \quad i \in I_4 \\
& x \in X
\end{aligned}
$$

At each node of search tree:

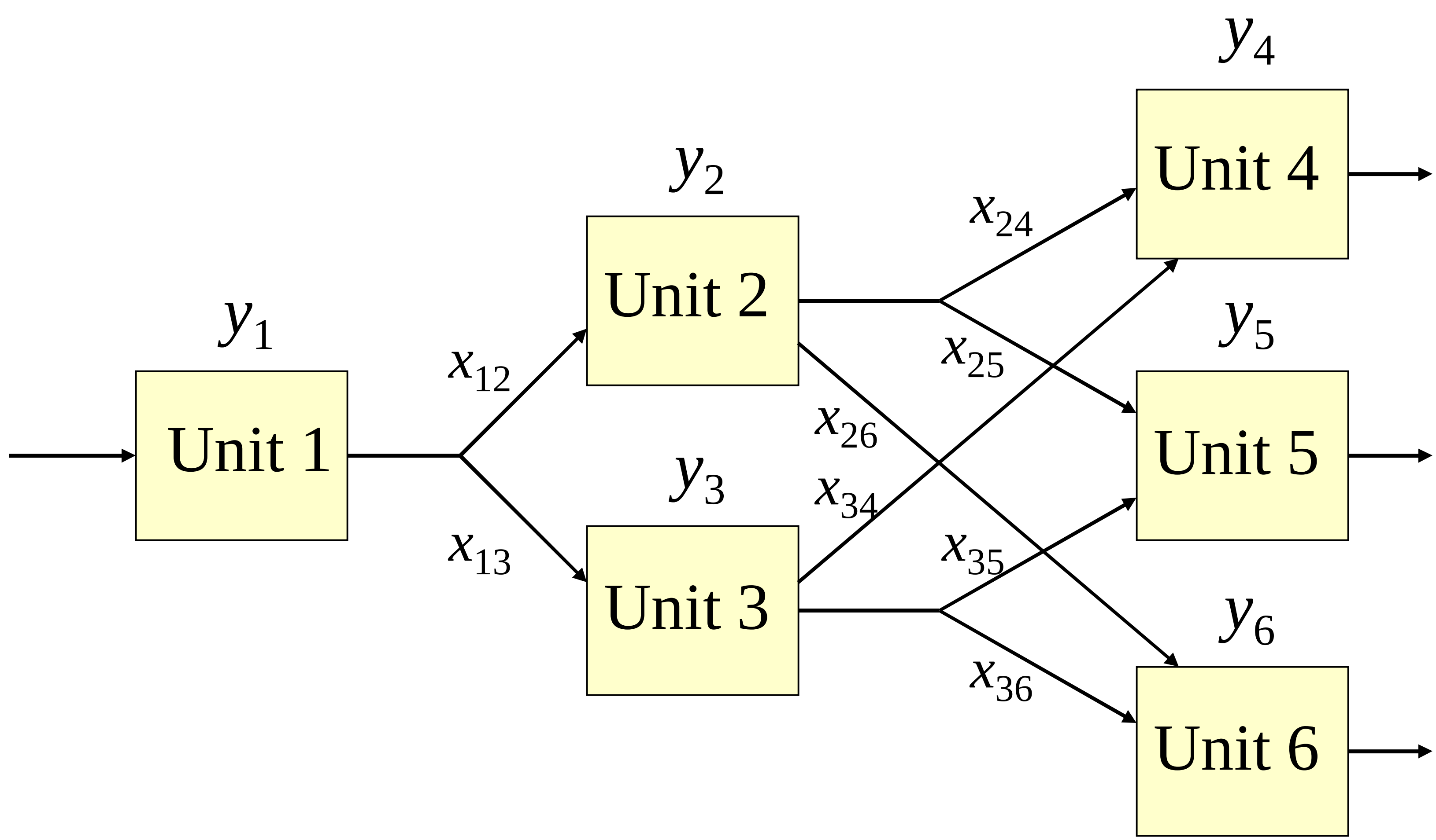
- Perform constrain propagation to reduce domains of search variables y_i and help determine truth or falsehood of $q_i(y)$'s.

- Formulate and solve relaxation, which consists of soluble constraints.

- If relaxation is feasible, try to find values for search variables that are consistent with solution variables. If this succeeds, one has feasible solution.

- Use optimal value of relaxation to prune tree if possible.

Processing Network Design



Processing Network Design

- The model uses search variables y_i to indicate the presence or absence of a unit.
- It uses conditional constraints to require that the fixed cost be incurred or the unit shut down.

$$
\begin{aligned}
0.6u_2 &= x_{24} + x_{25} \\
0.4u_2 &= x_{26} \\
0.7u_3 &= x_{34} \\
0.3u_3 &= x_{35} + x_{36}
\end{aligned}
$$

$$
\begin{array}{lll}
\max & \sum_i r_i u_i^{1/2} - \sum_i z_i & \\
\text{s.t.} & u = Ax & \text{flow thru units} \\
& bu = Bx & \text{flow balance} \\
& (y_i = \text{true}) \rightarrow (z_i = d_i), \text{ all } i & \text{unit is open} \\
& (y_i = \text{false}) \rightarrow (u_i = 0), \text{ all } i & \text{unit is closed} \\
& u \le c & \text{unit capacities} \\
& u, x \ge 0 &
\end{array}
$$

Processing Network Design

- Add don’t-be-stupid constraints to ensure that a unit is not opened unless downstream units are opened.

$$
\begin{array}{lll}
\max & \sum_i r_i u_i^{1/2} - \sum_i z_i & \\
\text{s.t.} & u = Ax & \text{flow thru units} \\
& bu = Bx & \text{flow balance} \\
& (y_i = \text{true}) \rightarrow (z_i = d_i), \text{ all } i & \text{unit is open} \\
& (y_i = \text{false}) \rightarrow (u_i = 0), \text{ all } i & \text{unit is closed} \\
& u \le c & \text{unit capacities} \\
& u, x \ge 0 & \\
& \left\{ \begin{array}{ll}
y_1 \rightarrow (y_2 \vee y_3) & y_3 \rightarrow y_4 \\
y_2 \rightarrow y_1 & y_3 \rightarrow (y_5 \vee y_6) \\
y_2 \rightarrow (y_4 \vee y_5) & y_4 \rightarrow (y_2 \vee y_3) \\
y_2 \rightarrow y_6 & y_5 \rightarrow (y_2 \vee y_3) \\
y_3 \rightarrow y_1 & y_6 \rightarrow (y_2 \vee y_3)
\end{array} \right\} & \text{don't - be - stupid}
\end{array}
$$

Processing Network Design

- Use an *inequality-or* global constraint to obtain good relaxation of disjunctive constraints.
- Use *cnf* global constraint to invoke resolution algorithm for don't-be-stupid constraints.

$$
\begin{array}{lll}
\max & \sum_i r_i u_i^{1/2} - \sum_i z_i & \\
\text{s.t.} & u = Ax & \text{flow thru units} \\
 & bu = Bx & \text{flow balance} \\
 & \text{inequality-or}\left(\begin{bmatrix} y_i \\ \neg y_i \end{bmatrix}, \begin{bmatrix} z_i \ge d_i \\ u_i = 0 \end{bmatrix} \right) & \text{global constraint} \\
 & u \le c & \text{unit capacities} \\
 & u, x \ge 0 & \\
 & \text{cnf}\begin{pmatrix} y_1 \rightarrow (y_2 \vee y_3) & y_3 \rightarrow y_4 \\ y_2 \rightarrow y_1 & y_3 \rightarrow (y_5 \vee y_6) \\ y_2 \rightarrow (y_4 \vee y_5) & y_4 \rightarrow (y_2 \vee y_3) \\ y_2 \rightarrow y_6 & y_5 \rightarrow (y_2 \vee y_3) \\ y_3 \rightarrow y_1 & y_6 \rightarrow (y_2 \vee y_3) \end{pmatrix} & \text{global constraint}
\end{array}
$$

Knapsack Problem with All-different

Original problem

$$
\begin{aligned}
\text{min} \quad & 5y_1 + 8y_2 + 4y_3 \\
\text{s.t.} \quad & 3y_1 + 5y_2 + 2y_3 \geq 30 \\
& \text{all-different}(y_1, y_2, y_3) \\
& y_j \in \{1,2,3,4\}, \quad \text{all } j
\end{aligned}
$$

Solved by branching and domain reduction only.

Knapsack Problem with All-different

The *continuous* predicate adds a continuous relaxation and any desired cutting planes.

$$
\begin{aligned}
\text{min} \quad & \text{continuous}(5y_1 + 8y_2 + 4y_3) \\
\text{s.t.} \quad & \text{continuous}(3y_1 + 5y_2 + 2y_3 \geq 30) \\
& \text{all-different}(y_1, y_2, y_3) \\
& y_j \in \{1,2,3,4\}, \quad \text{all } j
\end{aligned}
$$

Replace objective function with $5x_1 + 8x_2 + 4x_3$

Add $3x_1 + 5x_2 + 2x_3 \geq 30$ and link(y_j,x_j) and possibly knapsack cuts

Knapsack Problem with All-different

The *cut* predicate generates cuts in the search variables so that domain reduction is applied to cuts. *Continous* adds continuous relaxation of problem and cuts.

$$
\begin{aligned}
\min \quad & z \\
\text{s.t.} \quad & \text{continuous}\left(\text{cut}\begin{pmatrix} z \geq 5y_1 + 8y_2 + 4y_3 \\ 3y_1 + 5y_2 + 2y_3 \geq 30 \end{pmatrix} \right) \\
& \text{all-different}(y_1, y_2, y_3) \\
& y_j \in \{1,2,3,4\}, \quad \text{all } j
\end{aligned}
$$

Cumulative Global Constraint

Ensures that total resources consumed by jobs at any one time do not exceed C.

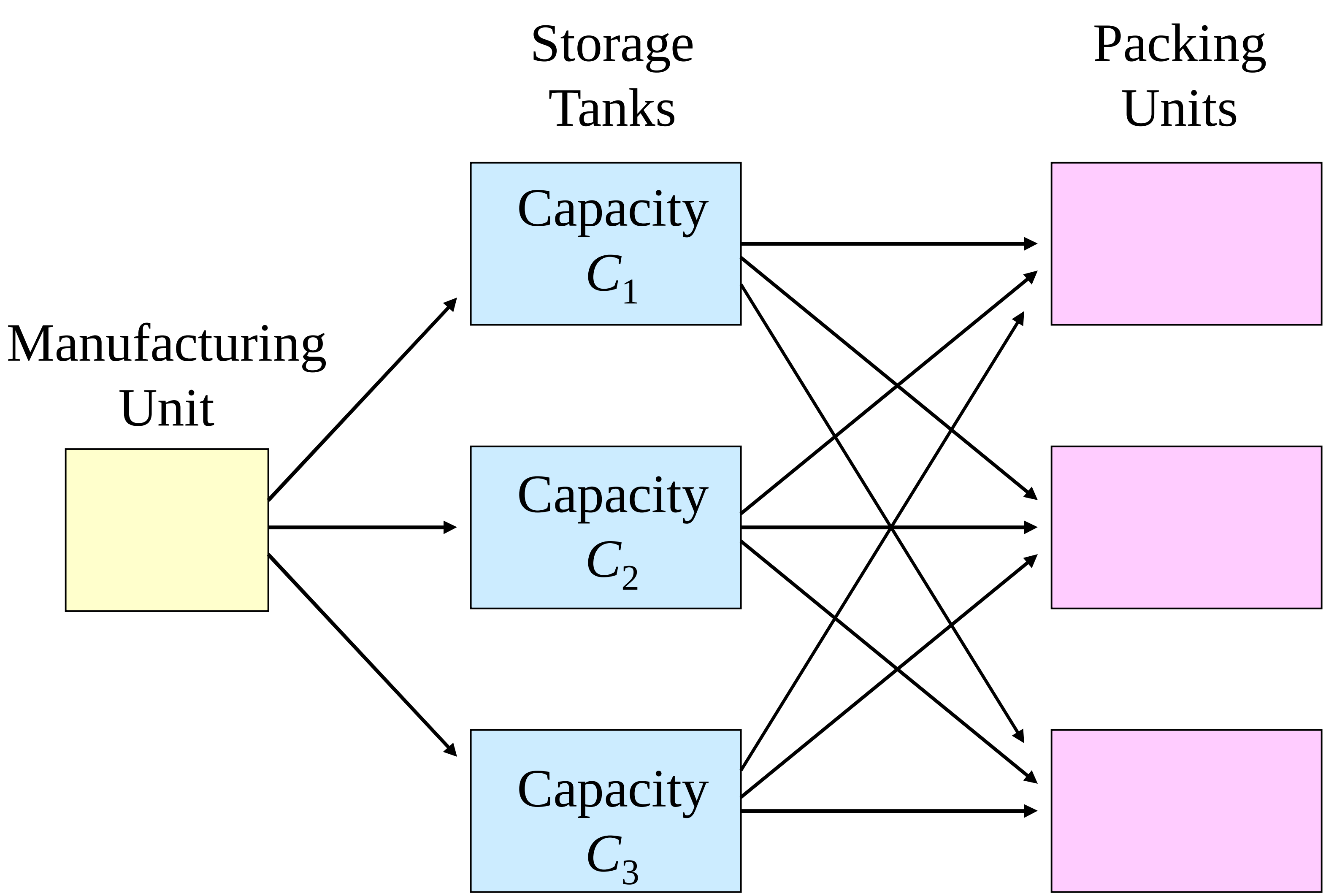
$$\text{cumulative}\big((t_1,\ldots,t_n),(d_1,\ldots,d_n),(r_1,\ldots,r_n),C\big)$$

Job start times

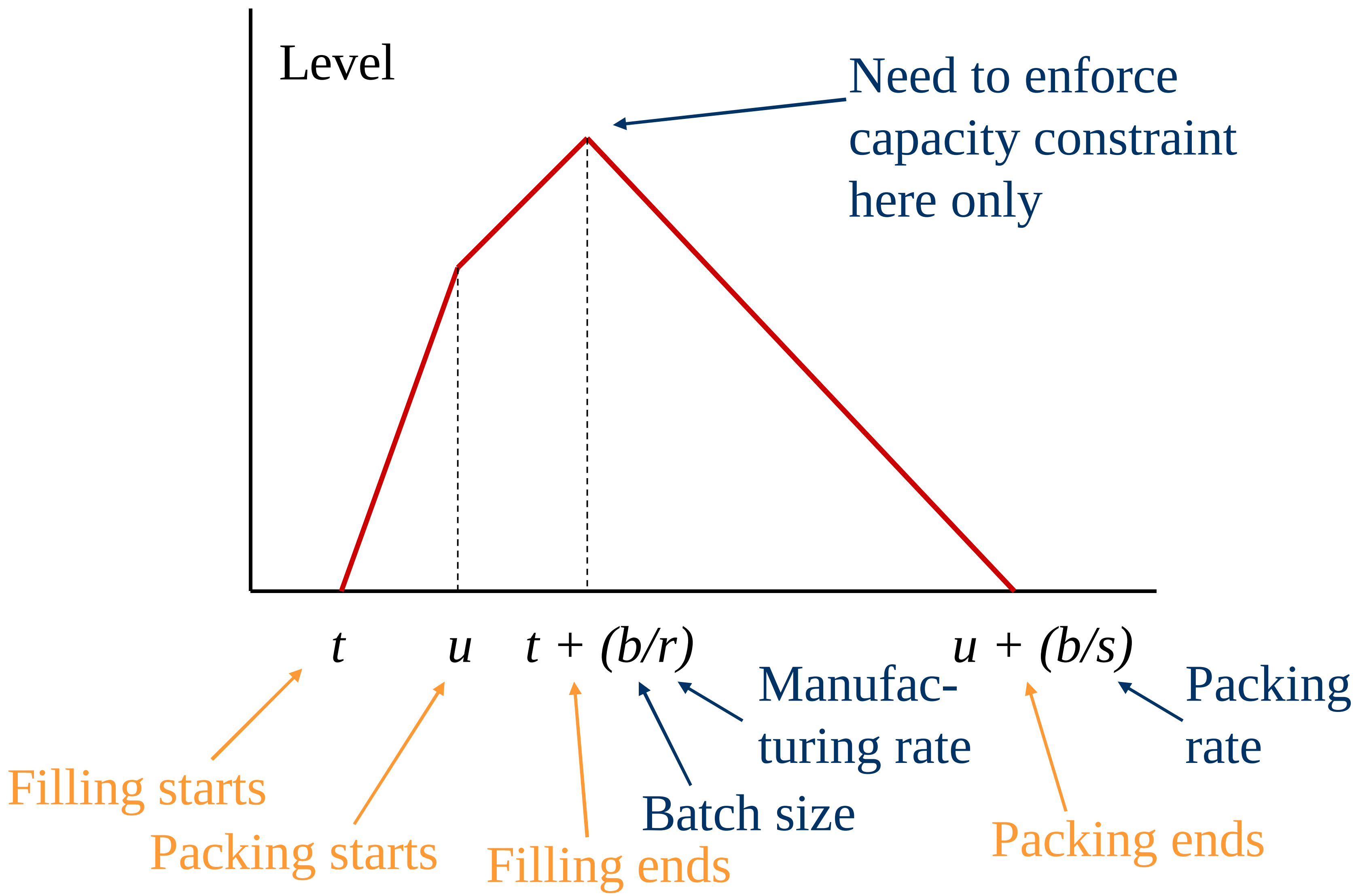
Job durations

Job resource requirements

Production Scheduling



Filling of StorageTank



$$\min\ T \quad \leftarrow \text{Makespan}$$

$$\text{s.t.}\quad T \geq u_j + \frac{b_j}{s_j},\quad \text{all } j$$

$$t_j \geq R_j,\quad \text{all } j \quad \leftarrow \text{Job release time}$$

$$\text{cumulative}\big(t, v, (1,\ldots 1), m\big) \quad \leftarrow m \text{ storage tanks}$$

$$v_i = u_i + \frac{b_i}{s_i} - t_i,\quad \text{all } i \quad \leftarrow \text{Job duration}$$

$$b_i\left(1 - \frac{s_i}{r_i}\right) + s_i u_i \leq C_i,\quad \text{all } i \quad \leftarrow \text{Tank capacity}$$

$$\text{cumulative}\left(u, \left(\frac{b_1}{s_1}, \ldots, \frac{b_n}{s_n}\right), e, p\right) \quad \leftarrow p \text{ packing units}$$

$$u_j \geq t_j \geq 0$$

Logic-Based Benders Model

- Master problem contains optimization problem, such as MILP.
- Subproblem contains constraint satisfaction problem.
- Conditional constraints determine which constraints go into the subproblem.

General Form of Benders Model

$$
\begin{aligned}
\min \quad & f(y) \\
\text{s.t.} \quad & p_i(y), \quad i \in I_1 \\
& g_i(x), \quad i \in I_2 \\
& q_i(y) \rightarrow h_i(x), \quad i \in I_3 \\
& y \in Y \\
& x_j \in D_j, \quad \text{all } j
\end{aligned}
$$

Master problem

Subproblem

Conditional constraints

Fixing y to $\bar{y}$ defines the subproblem (a feasibility problem):

$$\begin{aligned}
& g_i(x), \quad i \in I_2 \\
& h_i(x), \quad \text{all } i \in I_1 \text{ for which } q_i(\bar{y}) \text{ is true} \\
& x_j \in D_j, \quad \text{all } j
\end{aligned}$$

This produces a Benders cut $B_{\bar{y}}(y)$

The master problem is

$$\begin{aligned}
\min \quad & f(y) \\
\text{s.t.} \quad & p_i(y), \quad i \in I_1 \\
& B_{y^k}(y), \quad k = 1, \cdots, K \\
& y \in Y
\end{aligned}$$

Machine Scheduling
(Jain & Grossmann)

- Schedule jobs on parallel machines to minimize production cost.
- It costs C_{ij} to process job j on machine i.
- Assign jobs to machine in master problem (MILP).
- Schedule jobs on each machine in subproblem (constraint satisfaction).

Machine assigned to job j

Total cost

$$\begin{aligned} \min \quad & \sum_j C_{y_j j} \\ \text{s.t.} \quad & t_j \geq R_j, \quad \text{all } j \\ & t_j + D_{y_j j} \leq S_j, \quad \text{all } j \\ & \text{cumulative}\left((t_j | y_j = i), (D_{ij} \mid y_j = i), (1,\ldots,1), 1 \right) \end{aligned}$$

Release times

Deadlines

Start times of jobs assigned to machine i

Schedule jobs assigned to machine i

Problem in Benders Form

$$
\begin{aligned}
\min \quad & \sum_j C_{y_j j} \\
\text{s.t.} \quad & t_j \geq R_j, \quad \text{all } j \\
& t_j + D_{y_j j} \leq S_j, \quad \text{all } j \\
& \text{cumulative}\big((t_j | y_j = i), (D_{ij} \mid y_j = i), (1, \ldots, 1), 1\big) \\
& t'_j \geq R_j, \quad \text{all } j \\
& t'_j + D_{y_j j} \leq S_j, \quad \text{all } j \\
& \text{link}\big(t'_j, t_j\big)
\end{aligned}
$$

Finite domain variable linked to t_j

The subproblem decomposes into a scheduling problem for each machine *i*:

$$
\begin{aligned}
&t'_j \geq R_j, \quad \text{all } j \text{ with } \bar{y}_j = i \\
&t'_j + D_{\bar{y}_j j} \leq S_j, \quad \text{all } j \text{ with } \bar{y}_j = i \\
&\text{cumulative}\Big((t'_j | \bar{y}_j = i), (D_{ij} \mid \bar{y}_j = i), (1,\ldots,1), 1 \Big)
\end{aligned}
$$

If this problem is infeasible, then at least one joib assigned machine *i* must be assigned to some other machine. This gives the Benders cut,

Disjunction ⟶ $\bigcup_{\substack{j \\ \bar{y}_j = i}} \left(y_j \neq i \right)$

The master problem becomes,

$$\begin{aligned}
\min \quad & \sum_j C_{y_j j} \\
\text{s.t.} \quad & t_j \geq R_j, \quad \text{all } j \\
& t_j + D_{y_j j} \leq S_j, \quad \text{all } j \\
& \bigcup_{\substack{j \\ y_j^k = i}} \left(y_j \neq i \right), \quad i \in I^k,\ k = 1, \ldots, K
\end{aligned}$$

This is easily converted to an MILP model,

$$
\begin{aligned}
\min \quad & \sum_{ij} C_{ij} x_{ij} \\
\text{s.t.} \quad & t_j \geq R_j, \quad \text{all } j \\
& t_j + \sum_i D_{ij} x_{ij} \leq S_j, \quad \text{all } j \\
& \sum_{\substack{j \\ y_j^k = i}} (1 - x_{ij}) \geq 1, \quad i \in I^k,\ k = 1, \ldots, K \\
& \sum_j D_{ij} x_{ij} \leq \max_j \{S_j\} - \min_j \{R_j\}, \quad \text{all } i \\
& x_{ij} \in \{0,1\}
\end{aligned}
$$

Strengthens continuous relaxation